THE MINIMUM OF A BILINEAR FORM.

By

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1. In this paper I investigate the lower bound $M(B)$ of a bilinear form

$$B(x, y, z, t) = \alpha x z + \beta x t + \gamma y z + \delta y t, \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are real, and x, y, z, t take all integral values subject to

$$x t - y z = \pm 1. \tag{1.2}$$

We say that two bilinear forms are equivalent if one may be transformed into the other by a substitution

$$\begin{pmatrix} x & z \\ y & t \end{pmatrix} = \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} x' & z' \\ y' & t' \end{pmatrix} \quad or \quad \begin{pmatrix} p & r \\ q & s \end{pmatrix} \begin{pmatrix} z' & x' \\ t' & y' \end{pmatrix}, \tag{1.3}$$

where p, q, r, s are integers and $ps - qr = \pm 1$. It is clear that equivalent forms assume the same set of values for integral x, y, z, t subject to (1.2), and so have the same lower bound $M(B)$. Further, if we set

$$\Delta = \Delta(B) = \alpha\delta - \beta\gamma,$$
$$\theta = \theta(B) = |\beta - \gamma|,$$

then Δ and θ are invariants of B under equivalence transformation, of weights two and one respectively.

Associated with a bilinear form B is the quadratic form

$$Q(x, y) = B(x, y, x, y) = \alpha x^2 + (\beta + \gamma) x y + \delta y^2, \tag{1.4}$$

of discriminant

$$D = (\beta + \gamma)^2 - 4\alpha\delta = \theta^2 - 4\Delta. \tag{1.5}$$

If two bilinear forms are equivalent under a transformation (1.3), then, putting $x = z$, $y = t$, we see that the associated quadratic forms are also equivalent. Conversely, a quadratic form $Q(x, y) = a x^2 + b x y + c y^2$ is associated with the two bilinear forms

$$B(x, y, z, t) = a\,x\,z + \tfrac{1}{2}(b \pm \theta)\,x\,t + \tfrac{1}{2}(b \mp \theta)\,y\,z + c\,y\,t \tag{1.6}$$

for arbitrary $\theta \geq 0$, and these two forms are equivalent under the transformation $x \to z,\ y \to t,\ z \to x,\ t \to y$. Thus there is a one-one correspondence between classes of equivalent bilinear forms with given invariants Δ, θ and classes of equivalent quadratic forms with discriminant $D = \theta^2 - 4\,\Delta$.

We shall suppose throughout that $D \neq 0$, and also, when $D > 0$, that $Q(x, y)$ is a non-zero form, i.e. that $Q(x, y)$ does not represent zero for integral $x, y \neq 0, 0$. Under these conditions, the problem of finding best possible estimates for $M(B)$ has been studied in detail for two particular classes of forms.

First, if $\theta = 0$, then $\beta = \gamma$ in (1.1) and B is symmetrical; Schur[1] has then proved:

Theorem (Schur). *If B is symmetrical and $D > 0$, then*

$$M(B) \leq \frac{\sqrt{D}}{2\sqrt{5}}, \tag{1.7}$$

with equality if and only if B is equivalent to a multiple of $x\,z + \frac{1}{2}(x\,t + y\,z) - y\,t$.

Secondly, if $D > 0$ and $\theta = \sqrt{D}$, then $\Delta = 0$, and B may be factorized as $B = (\lambda\,x + \mu\,y)(\lambda' z + \mu' t)$, where $\lambda, \mu, \lambda', \mu'$ are real. Davenport and Heilbronn[2] have then proved:

Theorem (Davenport and Heilbronn). *If B is factorizable and $D > 0$, then*

$$M(B) \leq \frac{3 - \sqrt{5}}{2\sqrt{5}}\sqrt{D}, \tag{1.8}$$

with equality if and only if B is equivalent to a multiple of

$$\left(x + \frac{1 + \sqrt{5}}{2}\,y\right)\left(z + \frac{1 - \sqrt{5}}{2}\,t\right).$$

Now if we define $\omega = \omega(B) \geq 0$ by

$$\omega = \frac{\theta}{\sqrt{|D|}},$$

then ω is an absolute invariant of B, and the forms considered by these authors are characterised by the relations $D > 0$, $\omega = 0$ and 1 respectively. Their results therefore suggest the problems of finding for $M(B)$:

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[1] *Sitz.-Bericht Akad. Wiss. Berlin* (1913), 212—231.

[2] *Quart. Journal of Math.* 18 (1947), 107—123. The author has given an alternative proof of their complete results, based on the ideas of this paper, in *Acta Math.*

(1) an estimate independent of the value of ω;

(2) the best possible estimate in terms of ω and D.

The first problem is easily solved, and I establish the following theorem:

Theorem 1. (i) *If* $D > 0$, *then*

$$M(B) \leq \frac{\sqrt{D}}{2\sqrt{5}}, \tag{1.9}$$

with equality if and only if B *is equivalent to a multiple of one of the forms*

$$xz + \tfrac{1}{2}(1 - \omega\sqrt{5})\,xt + \tfrac{1}{2}(1 + \omega\sqrt{5})\,yz - yt,$$

where $\omega = 2k/\sqrt{5}$ $(k = 0, 1, 2, \ldots)$.

(ii) *If* $D < 0$, *then*

$$M(B) \leq \frac{\sqrt{|D|}}{2\sqrt{3}}, \tag{1.10}$$

with equality if and only if B *is equivalent to a multiple of one of the forms*

$$xz + \tfrac{1}{2}(1 - \omega\sqrt{3})\,xt + \tfrac{1}{2}(1 + \omega\sqrt{3})\,yz + yt,$$

where $\omega = 2k/\sqrt{3}$ $(k = 0, 1, 2, \ldots)$.

The second problem is more difficult. I give here the complete solution, in the case $D > 0$, for the range

$$0 \leq \omega \leq \omega_0 = \tfrac{1}{2}\left(\sqrt{2} + \frac{5}{4\sqrt{2} - 1}\right) = 1{\cdot}2439\ldots$$

of values of ω, which may be of interest since it includes the values $\omega = 0$ and $\omega = 1$ considered above.

We define the numbers $\varkappa_i$ $(i = 0, 1, \ldots, 8)$ and τ_i $(i = 1, 2, \ldots, 9)$ (the significance of which will become clearer in Lemma 2 below) by

$$\left.\begin{array}{lllll}
\varkappa_0 = -\dfrac{1}{\sqrt{5}}, & \varkappa_1 = 0, & \varkappa_2 = \dfrac{1}{\sqrt{17}}, & \varkappa_3 = \dfrac{1}{\sqrt{15}}, & \varkappa_4 = \dfrac{1}{\sqrt{13}}, \\
\varkappa_5 = \dfrac{\sqrt{3}}{4}, & \varkappa_6 = \dfrac{1}{\sqrt{5}}, & \varkappa_7 = \dfrac{1}{\sqrt{2}}, & \varkappa_8 = \dfrac{5}{4\sqrt{2} - 1}; & \\
\tau_1 = \dfrac{1}{\sqrt{5}}, & \tau_2 = \dfrac{1}{\sqrt{2}}, & \tau_3 = \dfrac{3}{\sqrt{17}}, & \tau_4 = \dfrac{3}{\sqrt{15}}, & \tau_5 = \dfrac{3}{\sqrt{13}}, \\
\tau_6 = \dfrac{\sqrt{3}}{2}, & \tau_7 = \dfrac{3}{\sqrt{5}}, & \tau_8 = \sqrt{2}, & \tau_9 = \sqrt{2}; &
\end{array}\right\} \tag{1.11}$$

Each of the sequences $(\varkappa_i)$ and (τ_i) is increasing (strictly, except that $\tau_8=\tau_9$), and $\varkappa_i<\tau_i$ for each i. In terms of these numbers we define a function $\chi(\omega)$ by

$$\left.\begin{aligned} \chi(\omega)&=\tau_i-\omega \quad \text{for} \quad \tfrac{1}{2}(\varkappa_{i-1}+\tau_i)\le\omega\le\tfrac{1}{2}(\varkappa_i+\tau_i) \qquad (1.12)\\ \chi(\omega)&=\omega-\varkappa_i \quad \text{for} \quad \tfrac{1}{2}(\varkappa_i+\tau_i)\le\omega\le\tfrac{1}{2}(\varkappa_i+\tau_{i+1}) \qquad (1.13)\end{aligned}\right\}\quad (i=1,2,\ldots,8).$$

This defines $\chi(\omega)$ for the range $0=\frac{1}{2}(\varkappa_0+\tau_1)\le\omega\le\frac{1}{2}(\varkappa_8+\tau_9)=\omega_0$ as a continuous function of ω. The graph of $\chi(\omega)$ is easily seen to be a zig-zag line, with turning points at $\omega=\frac{1}{2}(\varkappa_{i-1}+\tau_i)$ and $\omega=\frac{1}{2}(\varkappa_i+\tau_i)$ $(i=1,2,\ldots,8)$.

Finally we define the quadratic forms $Q_i(x,y)$ for $i=0,1,\ldots,8$ by

$$\left.\begin{aligned} Q_0&=Q_6=x^2+xy-y^2\\ Q_1&=Q_7=x^2+2xy-y^2\\ Q_2&=2x^2+3xy-y^2\\ Q_3&=3x^2+6xy-2y^2\\ Q_4&=x^2+3xy-y^2\\ Q_5&=4x^2+12xy-3y^2\\ Q_8&=x^2+3xy-(6-\sqrt{8})y^2\end{aligned}\right\}\tag{1.14}$$

With these definitions, we can state the results obtained as

Theorem 2. *If* $D>0$ *and* $0\le\omega\le\omega_0=1{\cdot}2439\ldots$, *then*

$$M(B)\le\tfrac{1}{2}\sqrt{D}\,\chi(\omega).\tag{1.15}$$

For any such value of ω, *there exists a form* B *for which equality holds in* (1.15). *More precisely, if the quadratic form* Q *associated with* B *is equivalent to a multiple of some* Q_i $(i=0,1,\ldots,8)$, *then equality holds in* (1.15) *when* ω *satisfies*

$$\begin{aligned} &\tfrac{1}{2}(\varkappa_i+\tau_i)\le\omega\le\tfrac{1}{2}(\varkappa_i+\tau_{i+1}) && (i=1,2,\ldots,8)\\ \textit{or}\quad &\tfrac{1}{2}(\varkappa_i+\tau_{i+1})\le\omega\le\tfrac{1}{2}(\varkappa_{i+1}+\tau_{i+1}) && (i=0,1,\ldots,7);\end{aligned}\tag{1.16}$$

and every ω *satisfying* $0\le\omega\le\omega_0$ *lies in one of the intervals* (1.16).

The result (1.15) reduces of course to (1.7) and (1.8) when $\omega=0$ and $\omega=1$ respectively. In fact, since $\frac{1}{2}(\varkappa_0+\tau_1)=0$, (1.12) gives $\chi(0)=\tau_1=1/\sqrt{5}$; and, since

$$\tfrac{1}{2}(\varkappa_6+\tau_7)=2/\sqrt{5}<1<\tfrac{1}{2}(\varkappa_7+\tau_7)=\tfrac{1}{2}\left(\frac{1}{\sqrt{2}}+\frac{3}{\sqrt{5}}\right),$$

(1.12) gives $\chi(1)=\tau_7-1=(3-\sqrt{5})/\sqrt{5}$.

2. The first step in the proof of these results is to reduce the problem to one in the theory of quadratic forms, by means of the following lemma.

Lemma 1. *Let Q be the quadratic form associated with B, and suppose that $D \neq 0$. Then*

$$M(B) = \tfrac{1}{2} \sqrt{|D|}\ \text{l.b.} \left| \frac{b}{\sqrt{|D|}} - \omega \right|, \tag{2.1}$$

where b runs through the coefficients of xy in the forms $ax^2 + bxy + cy^2$ equivalent to Q.

Proof. If $ax^2 + bxy + cy^2$ is any form equivalent to Q, then B is equivalent to the form $axz + \frac{1}{2}(b-\theta)xt + \frac{1}{2}(b+\theta)yz + cyt$ [cf. (1.6)], which assumes the value $\frac{1}{2}(b-\theta)$ for $(x, y, z, t) = (1, 0, 0, 1)$. Conversely, if $B(p, q, r, s) = \beta$, where $ps - qr = \pm 1$, then, applying the first of the transformations (1.3), we obtain an equivalent form B' for which $B'(1, 0, 0, 1) = \beta$. Hence $\beta = \frac{1}{2}(b-\theta)$, where $ax^2 + bxy + cy^2 = Q'(x, y)$ is the quadratic form associated with B' and which is therefore equivalent to Q.

Thus all numbers represented by B are of the form $\beta = \frac{1}{2}(b-\theta)$, whence

$$M(B) = \tfrac{1}{2}\ \text{l.b.}\ |b-\theta| = \tfrac{1}{2}\sqrt{|D|}\ \text{l.b.} \left| \frac{b}{\sqrt{|D|}} - \omega \right|.$$

Proof of Theorem 1. Let a_0 be any value assumed by the associated quadratic form $Q(x, y)$ for coprime integers x, y. Then there exists a form $\varphi(x, y) = a_0 x^2 + b_0 xy + c_0 y^2$, say, equivalent to Q. Applying the substitution $x \to x + py$ (p integral), we see that Q is equivalent to the form $a_0 x^2 + (b_0 + 2pa_0) xy + (a_0 p^2 + b_0 p + c_0) y^2$ for any integer p. Given any ω, we can now chose p so that

$$\left| \frac{b_0 + 2pa_0}{\sqrt{|D|}} - \omega \right| \le \frac{|a_0|}{\sqrt{|D|}},$$

since $a_0 \neq 0$ by our hypothesis that Q does not represent zero. Lemma 1 now gives

$$M(B) \le \tfrac{1}{2} |a_0|. \tag{2.2}$$

Suppose now that $D > 0$. Then, by a well-known theorem of Markoff, we can choose a_0 so that

$$|a_0| \le \sqrt{\frac{D}{5}}, \tag{2.3}$$

where the equality sign is necessary if and only if Q is equivalent to a multiple of $x^2 + xy - y^2$. The required result (1.9) now follows at once from (2.2) and (2.3).

If equality holds in (1.9), then equality must hold in (2.3), and Q is equivalent to a multiple of $x^2 + xy - y^2$. For this form, $b/\sqrt{D}$ takes the values $(2k+1)/\sqrt{5}$

($k=0, \pm 1, \pm 2, \ldots$) and these values only. Hence, by (2.1), equality holds in (1.9) if and only if $Q \sim a(x^2+xy-y^2)$ and

$$\text{l. b.} \left| \frac{2k+1}{\sqrt{5}} - \omega \right| = \frac{1}{\sqrt{5}} \quad (k=0, \pm 1, \pm 2, \ldots),$$

i.e. $\omega = 2k/\sqrt{5}$ for some integer k. This completes the proof of Theorem 1 (i).

Suppose next that $D<0$. Then we can choose a_0 so that

$$|a_0| \leq \sqrt{\frac{|D|}{3}}, \tag{2.4}$$

where the equality sign is necessary if and only if Q is equivalent to a multiple of x^2+xy+y^2. (1.10) now follows from (2.2), (2.4). The forms for which equality occurs in (1.10) may be established by an argument precisely similar to that used above.

3. We now show that Theorem 2 may be deduced from Lemma 1 and the following result, which will be proved in § 4. (The notation is that of § 1, (1.11) and (1.14)).

Lemma 2. (i) *Suppose that $Q(x, y)$ is a non-zero quadratic form of discriminant $D>0$. Then, for each $i=1, 2, \ldots, 8$, there exists a form $\varphi(x, y) = ax^2+bxy+cy^2$ equivalent to Q for which*

$$\varkappa_i \leq \frac{b}{\sqrt{D}} \leq \tau_i. \tag{3.1}$$

(ii) *If Q is equivalent to a multiple of Q_i for some $i=0, 1, \ldots, 8$, then $b/\sqrt{D}$ assumes no value lying strictly between $\varkappa_i$ and τ_{i+1}. The equality sign is therefore necessary on the left of (3.1) when Q is equivalent to a multiple of Q_i, and on the right when Q is equivalent to a multiple of Q_{i-1}.*

Deduction of Theorem 2. (i) By Lemma 2, given any ω we can satisfy

$$\left| \frac{b}{\sqrt{D}} - \omega \right| \leq \max\{\tau_i - \omega, \omega - \varkappa_i\}$$

for each $i=1, 2, \ldots, 8$ with an appropriate $\varphi(x, y) = ax^2+bxy+cy^2$ equivalent to Q; i.e.

$$\left.\begin{aligned} \left| \frac{b}{\sqrt{D}} - \omega \right| &\leq \tau_i - \omega \quad \text{when} \quad \omega \leq \tfrac{1}{2}(\varkappa_i + \tau_i) \qquad (3.2)\\ \left| \frac{b}{\sqrt{D}} - \omega \right| &\leq \omega - \varkappa_i \quad \text{when} \quad \omega \geq \tfrac{1}{2}(\varkappa_i + \tau_i) \qquad (3.3) \end{aligned}\right\} \quad (i=1, 2, \ldots, 8).$$

Hence, from the definition (1.12), (1.13) of $\chi(\omega)$, we can satisfy

$$\left|\frac{b}{\sqrt{D}}-\omega\right| \leq \chi(\omega) \tag{3.4}$$

for any ω with $0 \leq \omega \leq \omega_0$. The required result (1.15) now follows from (2.1) and (3.4).

(ii) Suppose that Q is equivalent to a multiple of Q_i for some $i=0, 1, \ldots, 8$. Then, by Lemma 2 (ii),

$$\left|\frac{b}{\sqrt{D}}-\omega\right| \geq \min\{\tau_{i+1}-\omega, \omega-\varkappa_i\}$$

for all b and ω satisfying $\varkappa_i \leq \omega \leq \tau_{i+1}$, so that, in particular,

$$\left|\frac{b}{\sqrt{D}}-\omega\right| \geq \tau_{i+1}-\omega \quad \text{when} \quad \tfrac{1}{2}(\varkappa_i+\tau_{i+1}) \leq \omega \leq \tfrac{1}{2}(\varkappa_{i+1}+\tau_{i+1}) \quad (i=0, 1, \ldots, 7), \tag{3.5}$$

$$\left|\frac{b}{\sqrt{D}}-\omega\right| \geq \omega-\varkappa_i \quad \text{when} \quad \tfrac{1}{2}(\varkappa_i+\tau_i) \leq \omega \leq \tfrac{1}{2}(\varkappa_i+\tau_{i+1}) \quad (i=1, 2, \ldots, 8). \tag{3.6}$$

Comparing (3.5), (3.6) with the definition of $\chi(\omega)$ (replacing i by $i+1$ in (1.12)), we see that when Q is equivalent to a multiple of Q_i for some $i=0, 1, \ldots, 8$ we have

$$\left|\frac{b}{\sqrt{D}}-\omega\right| \geq \chi(\omega)$$

for all b and all ω lying in the intervals specified in (1.16). Lemma 1 now gives

$$M(B) \geq \tfrac{1}{2}\sqrt{D}\,\chi(\omega),$$

and so equality holds in (1.15).

4. This and the following two sections will be devoted to the proof of Lemma 2.

Since Q does not represent zero, it is equivalent to a form $\varphi(x, y)=ax^2+bxy+cy^2$ for which

$$0<\sqrt{D}-b<2|c|<\sqrt{D}+b, \tag{4.1}$$

and which we call a reduced form.[1] The reduced forms equivalent to Q may be ordered as an infinite chain

$$\ldots, \varphi_{-2}, \varphi_{-1}, \varphi_0, \varphi_1, \varphi_2, \ldots$$

where

$$\varphi_\nu=\varphi_\nu(x, y)=(-1)^{\nu-1}a_\nu x^2+b_\nu xy+(-1)^\nu a_{\nu+1}y^2 \quad (\nu=0, \pm 1, \pm 2, \ldots), \tag{4.2}$$

and all the coefficients a_ν, b_ν may be assumed positive. Each φ_ν is transformed into its right neighbour $\varphi_{\nu+1}$ by the substitution $\begin{pmatrix}0 & -1\\ 1 & (-1)^\nu k_\nu\end{pmatrix}$, where

[1] For these results on reduced forms, see I. Schur, *loc. cit.*, 214—216.

$$k_\nu = \frac{b_\nu + b_{\nu+1}}{2\, a_{\nu+1}} \tag{4.3}$$

is a positive integer. We set

$$r_\nu = \frac{\sqrt{D} + b_\nu}{2\, a_{\nu+1}}, \quad s_\nu = \frac{\sqrt{D} - b_\nu}{2\, a_{\nu+1}}, \tag{4.4}$$

whence

$$\frac{a_\nu}{a_{\nu+1}} = r_\nu\, s_\nu, \quad \frac{b_\nu}{a_{\nu+1}} = r_\nu - s_\nu, \quad \frac{\sqrt{D}}{a_{\nu+1}} = r_\nu + s_\nu. \tag{4.5}$$

In the usual notation for simple continued fractions, we then have

$$r_\nu = (k_\nu, k_{\nu+1}, k_{\nu+2}, \ldots), \quad s_\nu = (0, k_{\nu-1}, k_{\nu-2}, \ldots). \tag{4.6}$$

We also write

$$r'_\nu = r_\nu - k_\nu = (0, k_{\nu+1}, k_{\nu+2}, \ldots), \tag{4.7}$$

so that

$$0 < r'_\nu,\ s_\nu < 1. \tag{4.8}$$

We denote by (K) the infinite series of positive integers

$$\ldots, k_{-2}, k_{-1}, k_0, k_1, k_2, \ldots.$$

(K) is then determined by Q, and, conversely, (K) determines to within an arbitrary multiple the class of forms equivalent to Q.

Finally we define the numbers $T_\nu(p)$ for all integral ν and p by

$$T_\nu(p) = \frac{b_\nu - 2\, p\, a_{\nu+1}}{\sqrt{D}} = \frac{k_\nu + r'_\nu - s_\nu - 2\, p}{k_\nu + r'_\nu + s_\nu}, \tag{4.9}$$

where the identity of the last two expressions follows from (4.5) and (4.7). The connection between the numbers $T_\nu(p)$ and Lemma 2 is shown by the following result:

Lemma 3. (*i*) *For any integers ν, p, there exists a form $\varphi(x, y) = a x^2 + b x y + c y^2$ equivalent to Q for which*

$$\pm \frac{b}{\sqrt{D}} = T_\nu(p). \tag{4.10}$$

(ii) *Suppose that Q is equivalent to a form $\varphi(x, y)$ for which*

$$\frac{|b|}{\sqrt{D}} \le \sqrt{2}. \tag{4.11}$$

Then (4.10) *holds for some integers ν, p.*

Proof. (i) Q is equivalent to the reduced form φ_ν for any ν, and so also to $\varphi(x, y) = \varphi_\nu(x, (-1)^\nu p x + y)$ for any integer p. This gives $b = b_\nu - 2\, p\, a_{\nu+1}$, whence

$b/\sqrt{D} = T_\nu(p)$, by (4.9). Since the forms $ax^2 \pm bxy + cy^2$ are equivalent, we may take either sign in (4.10).

(ii) If (4.11) holds for some $\varphi = ax^2 + bxy + cy^2$, then

$$0 \le \frac{b^2}{D} = 1 + \frac{4ac}{D} \le 2,$$

whence

$$|ac| \le \tfrac{1}{4} D.$$

By interchanging x and y, if necessary, we may take $|a| \ge |c|$, so that $|c| \le \frac{1}{2}\sqrt{D}$. Applying a transformation $y \to qx + y$, we obtain an equivalent form $\varphi' = a'x^2 + b'xy + cy^2$ for which $b' = b + 2qc$. We choose the integer q so that

$$0 \le \sqrt{D} - b - 2qc < 2|c|,$$

whence

$$0 < \sqrt{D} - b' < 2|c|,$$

since $b' \ne \sqrt{D}$ by our hypothesis that Q is a non-zero form. Since $2|c| \le \sqrt{D}$, this gives $b' > 0$, and so

$$0 < \sqrt{D} - b' < 2|c| < \sqrt{D} + b'.$$

Now this is just the condition (4.1) that φ' be a reduced form. Hence $\varphi' = \varphi_\nu$, $c = (-1)^\nu a_{\nu+1}$, $b' = b_\nu$ for some ν, and $b = b' - 2qc = b_\nu - 2(-1)^\nu q a_{\nu+1}$. Thus $b/\sqrt{D} = T_\nu(p)$ for $p = (-1)^\nu q$.

From the definitions (1.11), we see that $0 \le \varkappa_i, \tau_i \le \sqrt{2}$ $(i = 1, 2, \ldots, 8)$, so that, by Lemma 3, any value of $b/\sqrt{D}$ satisfying (3.1) must be of the form $|T_\nu(p)|$ for some integers ν, p. Thus Lemma 2 (i) is equivalent to the assertion that, for any sequence (K) and any $i = 1, 2, \ldots, 8$, we can find a $T_\nu(p)$ satisfying

$$\varkappa_i \le |T_\nu(p)| \le \tau_i. \tag{4.12}$$

We now establish some general results (Lemmas 4, 5, 6, 7) which will help to simplify later calculations.

Lemma 4. (i) *If n is a positive integer, and $k_\nu \ge n$ for any ν, then there exists a $T_\nu(p)$ lying in any interval of length $2(n+1)/(n^2+n+2)$.*

(ii) *If $k_\nu \ge 5$ for any ν, there exists a $T_\nu(p)$ lying in any interval of length $3/8$, and so in any of the intervals $(\varkappa_i, \tau_i)$ for $i = 1, 2, \ldots, 7$.*

Proof. (i) We have

$$T_\nu(p-1) - T_\nu(p) = \frac{2}{k_\nu + r'_\nu + s_\nu}.$$

If now any $k_\nu \geq n+1$, this gives, using also (4.8),

$$0 < T_\nu(p-1) - T_\nu(p) < \frac{2}{n+1} \leq \frac{2(n+1)}{n^2+n+2}.$$

Otherwise we have $\max\{k_\nu\} = n$; then, for some ν, $k_\nu = n$, $k_{\nu-1} \leq n$, $k_{\nu+1} \leq n$, so that r'_ν, $s_\nu > 1/(n+1)$ and

$$0 < T_\nu(p-1) - T_\nu(p) < \frac{2(n+1)}{n^2+n+2}.$$

Part (i) of the lemma follows immediately.

(ii) The first statement of (ii) is merely the case $n=5$ of (i); and it is easily verified that, for $i=1, 2, \ldots, 7$, we have

$$\tau_i - \varkappa_i \geq \tau_5 - \varkappa_5 = 0{\cdot}399 \cdots > \tfrac{3}{8}.$$

Lemma 5. (i) *For all* ν,

$$|T_\nu(0)|,\ |T_\nu(k_\nu)| > \frac{k_\nu - 1}{k_\nu + 1}.$$

(ii) *If* $k_\nu \geq 3$,

$$|T_\nu(1)|,\ |T_\nu(k_\nu - 1)| < \frac{k_\nu - 1}{k_\nu + 1}.$$

Proof. (i) We have

$$|T_\nu(0)| = \frac{k_\nu + r'_\nu - s_\nu}{k_\nu + r'_\nu + s_\nu},$$

and this is an increasing function of r'_ν and a decreasing function of s_ν, since $k_\nu \geq 1 > r'_\nu$, $s_\nu > 0$. Hence

$$|T_\nu(0)| > \frac{k_\nu - 1}{k_\nu + 1},$$

and a similar argument shows that

$$|T_\nu(k_\nu)| = \frac{k_\nu - r'_\nu + s_\nu}{k_\nu + r'_\nu + s_\nu} > \frac{k_\nu - 1}{k_\nu + 1}.$$

(ii) We have, for $k_\nu \geq 3$,

$$|T_\nu(1)| = \frac{k_\nu + r'_\nu - s_\nu - 2}{k_\nu + r'_\nu + s_\nu},$$

and this is an increasing function of r'_ν and a decreasing function of s_ν for $0 < r'_\nu$, $s_\nu < 1$. Hence

$$|T_\nu(1)| < \frac{k_\nu + 1 - 2}{k_\nu + 1} = \frac{k_\nu - 1}{k_\nu + 1};$$

and a similar argument shows that

$$|T_\nu(k_\nu-1)| = \frac{k_\nu - r'_\nu + s_\nu - 2}{k_\nu + r'_\nu + s_\nu} < \frac{k_\nu - 1}{k_\nu + 1}.$$

Lemma 6. (i) *Suppose that* $0 < \tau < 1$ *and that* $|T_\nu(p)| > \tau$ *for* $p = 0$ *and* $p = k_\nu$. *Then*

$$r'_\nu,\ s_\nu < \frac{k_\nu(1-\tau)}{2\tau}. \tag{4.13}$$

(ii) *Suppose that* $0 < \varkappa < 1$, $k_\nu \geq 3$, *and* $|T_\nu(p)| < \varkappa$ *for* $p = 1$ *and* $p = k_\nu - 1$. *Then*

$$r'_\nu,\ s_\nu > \frac{k_\nu(1-\varkappa) - 2}{2\varkappa}. \tag{4.14}$$

Proof. (i) We have

$$|T_\nu(0)| = \frac{k_\nu + r'_\nu - s_\nu}{k_\nu + r'_\nu + s_\nu} > \tau, \quad |T_\nu(k_\nu)| = \frac{k_\nu - r'_\nu + s_\nu}{k_\nu + r'_\nu + s_\nu} > \tau,$$

i.e.

$$(1-\tau) r'_\nu > (1+\tau) s_\nu - k_\nu(1-\tau),$$
$$(1-\tau) s_\nu > (1+\tau) r'_\nu - k_\nu(1-\tau).$$

From these two inequalities we derive

$$(1-\tau)^2 r'_\nu > (1+\tau)\{(1+\tau) r'_\nu - k_\nu(1-\tau)\} - k_\nu(1-\tau)^2,$$

or

$$k_\nu(1-\tau) > 2\tau r'_\nu.$$

This establishes (4.13) for r'_ν; and, from the symmetry in r'_ν, s_ν of the above inequalities, the same result holds for s_ν.

(ii) For $k_\nu \geq 3$, we have

$$|T_\nu(1)| = \frac{k_\nu + r'_\nu - s_\nu - 2}{k_\nu + r'_\nu + s_\nu} < \varkappa, \quad |T_\nu(k_\nu - 1)| = \frac{k_\nu - r'_\nu + s_\nu - 2}{k_\nu + r'_\nu + s_\nu} < \varkappa,$$

i.e.

$$(1-\varkappa) r'_\nu < (1+\varkappa) s_\nu + 2 - k_\nu(1-\varkappa),$$
$$(1-\varkappa) s_\nu < (1+\varkappa) r'_\nu + 2 - k_\nu(1-\varkappa),$$

whence we derive (4.14) precisely as above.

Lemma 7. *If* $\max\{k_\nu\} = 2$, *there exists a* $T_\nu(p)$ *satisfying*

$$0{\cdot}527 < |T_\nu(p)| \leq \frac{1}{\sqrt{2}} = 0{\cdot}707\ldots \tag{4.15}$$

Proof. Since $1 \leq k_\nu \leq 2$ for all ν, we have

$$r'_\nu,\ s_\nu \leq (0,\ 1,\ 2,\ 1,\ 2,\ \ldots) = \sqrt{3} - 1,$$
$$r'_\nu,\ s_\nu \geq (0,\ 2,\ 1,\ 2,\ 1,\ \ldots) = \tfrac{1}{2}(\sqrt{3} - 1).$$

Take now any ν with $k_\nu = 2$. Then

$$|T_\nu(0)| = \frac{2 + r'_\nu - s_\nu}{2 + r'_\nu + s_\nu} \geq \frac{2 + \frac{1}{2}(\sqrt{3}-1) - (\sqrt{3}-1)}{2 + \frac{1}{2}(\sqrt{3}-1) + (\sqrt{3}-1)} = \frac{5 - \sqrt{3}}{1 + 3\sqrt{3}} > 0{\cdot}527,$$

and similarly

$$|T_\nu(2)| > 0{\cdot}527.$$

Hence if (4.15) is not satisfied, we must have $|T_\nu(0)|, |T_\nu(2)| > 1/\sqrt{2}$. Lemma 6 (i) (with $k_\nu = 2$, $\tau = 1/\sqrt{2}$) now shows that

$$r'_\nu,\ s_\nu < \sqrt{2} - 1 = (0, \overset{*}{2}). \tag{4.16}$$

This implies in particular that $k_{\nu-1} \geq 2$, $k_{\nu+1} \geq 2$, whence $k_{\nu-1} = k_{\nu+1} = 2$. Repeating the argument with ν replaced by $\nu - 1$ and $\nu + 1$, and so on, we see that $k_\nu = 2$ for all ν. But then $r'_\nu = s_\nu = (0, \overset{*}{2})$, contradicting (4.16); this contradiction establishes the lemma.

5. We now proceed to a systematic discussion of the inequality (3.1) or, what is the same thing, (4.12), for $i = 1, 2, \ldots, 7$. (The case $i = 8$ will be treated in the following section.)

By Lemma 4 (ii), we may suppose throughout that $k_\nu \leq 4$ for all ν. We note also that if every $k_\nu = 1$, then $r'_\nu = s_\nu = (0, \overset{*}{1}) = \frac{1}{2}(\sqrt{5} - 1)$, and so

$$T_\nu(p) = \frac{1 - 2p}{\sqrt{5}}; \quad |T_\nu(0)| = \frac{1}{\sqrt{5}} = 0{\cdot}447 \ldots; \quad |T_\nu(2)| = \frac{3}{\sqrt{5}}\cdot \tag{5.1}$$

Case I: $i = 1$. Here (4.12) is

$$\varkappa_1 = 0 \leq |T_\nu(p)| \leq \tau_1 = \frac{1}{\sqrt{5}} = 0{\cdot}447 \ldots \tag{5.2}$$

If any $k_\nu \geq 2$, then, by Lemma 4 (i), there exists a $T_\nu(p)$ in any interval of length $3/4$, and so in the interval $(-1/\sqrt{5}, 1/\sqrt{5})$; thus (5.2) is certainly satisfied. Otherwise every $k_\nu = 1$, and then, by (5.1), (5.2) holds with $p = 0$.

Case II: $i = 2$. Here (4.12) is

$$\varkappa_2 = \frac{1}{\sqrt{17}} = 0{\cdot}242 \cdots \leq |T_\nu(p)| \leq \tau_2 = \frac{1}{\sqrt{2}} = 0{\cdot}707 \cdots \tag{5.3}$$

If $\max\{k_\nu\} = 1$, (5.1) shows that (5.3) is satisfied with $p = 0$; if $\max\{k_\nu\} = 2$, Lemma 7 gives the result; and if any $k_\nu \geq 4$, Lemma 4 (i) with $n = 4$ shows that there is a $T_\nu(p)$ in any interval of length $5/11 = 0{\cdot}45 \ldots$, and so in the interval $(\varkappa_2, \tau_2)$.

We may therefore suppose now that $\max\{k_\nu\}=3$. Take any ν with $k_\nu=3$. Then, by Lemma 5, $|T_\nu(0)|,\ |T_\nu(3)|>\frac{1}{2}>\varkappa_2;\ |T_\nu(1)|,\ |T_\nu(2)|<\frac{1}{2}<\tau_2$. Lemma 6 (with $k_\nu=3,\ \tau=\tau_2=1/\sqrt{2},\ \varkappa=\varkappa_2=1/\sqrt{17}$) then gives

$$r'_\nu,\ s_\nu<\tfrac{3}{2}(\sqrt{2}-1)=(0,\overset{*}{1},1,1,1,1,\overset{*}{3}),$$

$$r'_\nu,\ s_\nu>\tfrac{1}{2}(\sqrt{17}-3)=(0,\overset{*}{1},1,\overset{*}{3}). \tag{5.4}$$

From these two inequalities for $r'_\nu=(0,k_{\nu+1},k_{\nu+2},k_{\nu+3},\ldots)$, we see at once that $k_{\nu+1}=k_{\nu+2}=1,\ 1\le k_{\nu+3}\le 3$. Suppose now that $k_{\nu+3}\ne 3$, so that $k_{\nu+3}\le 2$. Then

$$k_{\nu+2}=1,\quad r'_{\nu+2}=(0,k_{\nu+3},\ldots),\quad s_{\nu+2}=(0,1,3,k_{\nu-1},\ldots),$$

and so

$$(0,3)=\tfrac{1}{3}<r'_{\nu+2}<1;\quad (0,1,3)=\tfrac{3}{4}<s_{\nu+2}<(0,1,4)=\tfrac{4}{5}.$$

Hence

$$|T_{\nu+2}(1)|=\frac{1-r'_{\nu+2}+s_{\nu+2}}{1+r'_{\nu+2}+s_{\nu+2}}>\frac{1-1+\frac{3}{4}}{1+1+\frac{3}{4}}=\frac{3}{11}>\varkappa_2,$$

$$|T_{\nu+2}(1)|<\frac{1-\frac{1}{3}+\frac{4}{5}}{1+\frac{1}{3}+\frac{4}{5}}=\frac{11}{16}<\tau_2,$$

and (5.3) is satisfied by $|T_{\nu+2}(1)|$. Thus if (5.3) is not satisfied, the only remaining possibility is that $k_{\nu+3}=3$. But then we may repeat the above argument, with ν replaced by $\nu+3$, to show that $k_{\nu+4}=k_{\nu+5}=1,\ k_{\nu+6}=3$. In this way we see that $r'_\nu=(0,1,1,3,1,1,3,\ldots)=(0,\overset{*}{1},1,\overset{*}{3})$, which contradicts (5.4).

Case III: $i=3$. Here (4.12) is

$$\varkappa_3=\frac{1}{\sqrt{15}}=0{\cdot}2581\cdots\le|T_\nu(p)|\le\tau_3=\frac{3}{\sqrt{17}}=0{\cdot}7276\cdots \tag{5.5}$$

The initial argument of case II shows that we need consider only sequences (K) for which $\max\{k_\nu\}=3$. Then, taking any ν with $k_\nu=3$, we must have

$$|T_\nu(0)|,\ |T_\nu(3)|>\tau_3;\quad |T_\nu(1)|,\ |T_\nu(2)|<\varkappa_3;$$

if (5.5) is not satisfied. Lemma 6 (with $k_\nu=3,\ \tau=\tau_3=3/\sqrt{17},\ \varkappa=\varkappa_3=1/\sqrt{15}$) now gives

$$r'_\nu,\ s_\nu<\tfrac{1}{2}(\sqrt{17}-3)=(0,\overset{*}{1},1,\overset{*}{3}), \tag{5.6}$$

$$r'_\nu,\ s_\nu>\tfrac{1}{2}(\sqrt{15}-3)=(0,\overset{*}{2},\overset{*}{3}). \tag{5.7}$$

From (5.7) we have $r'_\nu=(0,k_{\nu+1},\ldots)>(0,2,\ldots)$, whence $k_{\nu+1}\le 2$. We distinguish the two cases $k_{\nu+1}=1$ and $k_{\nu+1}=2$.

Suppose first that $k_{\nu+1}=1$. Then, by (5.6),

$$r'_\nu=(0,\,1,\,k_{\nu+2},\,k_{\nu+3},\,\ldots)<(0,\,1,\,1,\,3,\,1,\,\ldots).$$

Comparing successive partial quotients in this inequality, we find in turn: $k_{\nu+2}\le 1$, whence $k_{\nu+2}=1$; $k_{\nu+3}\ge 3$, whence $k_{\nu+3}=3$; $k_{\nu+4}\le 1$, whence $k_{\nu+4}=1$. But now we may repeat this argument with ν replaced by $\nu+3$, since $k_{\nu+3}=3$, $k_{\nu+4}=1$, and deduce that $k_{\nu+5}=1$, $k_{\nu+6}=3$, $k_{\nu+7}=1$. Thus, repeating the argument indefinitely, we find that $r'_\nu=(0,\,\overset{*}{1},\,1,\,\overset{*}{3})$, which contradicts (5.6).

Suppose next that $k_{\nu+1}=2$. By (5.7),

$$r'_\nu=(0,\,2,\,k_{\nu+2},\,k_{\nu+3},\,\ldots)>(0,\,2,\,3,\,2,\,\ldots),$$

whence $k_{\nu+2}\ge 3$, and so $k_{\nu+2}=3$; and then $k_{\nu+3}\le 2$. Also, $k_{\nu+3}\ne 1$, since otherwise we should have $k_{\nu+2}=3$, $k_{\nu+3}=1$, and so should arrive at a contradiction as in the preceding paragraph. Thus $k_{\nu+3}=2$. From $k_\nu=3$, $k_{\nu+1}=2$, we have therefore deduced that $k_{\nu+2}=3$, $k_{\nu+3}=2$. Repeating this argument indefinitely, with ν replaced by $\nu+2$, $\nu+4$, ..., we find that $r'_\nu=(0,\,\overset{*}{2},\,\overset{*}{3})$, which contradicts (5.7).

Case IV: $i=4$. Here (4.12) is

$$\varkappa_4=\frac{1}{\sqrt{13}}=0{\cdot}2773\cdots\le|T_\nu(p)|\le\tau_4=\frac{3}{\sqrt{15}}=0{\cdot}7745\cdots \tag{5.8}$$

The initial argument of case II applies again to show that we need consider only sequences (K) for which $\max\{k_\nu\}=3$. And then, taking any ν with $k_\nu=3$, we must have

$$|T_\nu(0)|,\,|T_\nu(3)|>\tau_4;\quad |T_\nu(1)|,\,|T_\nu(2)|<\varkappa_4;$$

if (5.8) is not satisfied. Lemma 6 (with $k_\nu=3$, $\tau=\tau_4=3/\sqrt{15}$, $\varkappa=\varkappa_4=1/\sqrt{13}$) now gives

$$r'_\nu,\ s_\nu<\tfrac{1}{2}(\sqrt{15}-3)=(0,\,\overset{*}{2},\,\overset{*}{3}), \tag{5.9}$$

$$r'_\nu,\ s_\nu>\tfrac{1}{2}(\sqrt{13}-3)=(0,\,\overset{*}{3}). \tag{5.10}$$

From (5.9), $r'_\nu=(0,\,k_{\nu+1},\,\ldots)<(0,\,2,\,\ldots)$, and so $k_{\nu+1}\ge 2$. We distinguish the two cases $k_{\nu+1}=2$ and $k_{\nu+1}=3$.

Suppose first that $k_{\nu+1}=3$. Then, by (5.10),

$$r'_\nu=(0,\,3,\,k_{\nu+2},\,\ldots)>(0,\,3,\,3,\,\ldots),$$

whence $k_{\nu+2}\ge 3$, $k_{\nu+2}=3$. We may now repeat this argument with ν replaced by $\nu+1$, since $k_{\nu+1}=k_{\nu+2}=3$, and deduce that $k_{\nu+3}=3$. Repeating the argument indefinitely, we see that $r'_\nu=(0,\,\overset{*}{3})$, which contradicts (5.10).

Suppose next that $k_{\nu+1}=2$. Then, by Lemma 5 (i),

$$|T_{\nu+1}(0)|,\ |T_{\nu+1}(2)| > \tfrac{1}{3} > \varkappa_4,$$

and so, if (5.8) is not satisfied, we require

$$|T_{\nu+1}(0)|,\ |T_{\nu+1}(2)| > \tau_4.$$

Lemma 6 (i) (with $k_{\nu+1}=2$, $\tau=\tau_4=3/\sqrt{15}$) now gives

$$r'_{\nu+1},\ s_{\nu+1} < \tfrac{1}{3}(\sqrt{15}-3) = (0, \overset{*}{3}, \overset{*}{2}),$$

whence

$$r'_\nu = \frac{1}{r_{\nu+1}} = \frac{1}{2+r'_{\nu+1}} > (0, \overset{*}{2}, \overset{*}{3}),$$

which contradicts (5.9).

Case V: $i=5$. Here (4.12) is

$$\varkappa_5 = \frac{\sqrt{3}}{4} = 0{\cdot}4330\cdots \le |T_\nu(p)| \le \tau_5 = \frac{3}{\sqrt{13}} = 0.8320\ldots \tag{5.11}$$

If $\max\{k_\nu\}=1$, (5.11) is satisfied with $p=0$, by (5.1); and if $\max\{k_\nu\}=2$, Lemma 7 gives the result. We may therefore confine ourselves to sequences (K) for which $\max\{k_\nu\}=3$ or 4.

(a) Suppose that $\max\{k_\nu\}=3$. Take any ν with $k_\nu=3$. Then, by Lemma 5 (i), $|T_\nu(0)|,\ |T_\nu(3)| > \tfrac{1}{2} > \varkappa_5$, and so if (5.11) is not satisfied we must have

$$|T_\nu(0)|,\ |T_\nu(3)| > \tau_5.$$

Lemma 6 (i) (with $k_\nu=3$, $\tau=\tau_5=3/\sqrt{13}$) now gives

$$r'_\nu,\ s_\nu < \tfrac{1}{2}(\sqrt{13}-3) = (0, \overset{*}{3}). \tag{5.12}$$

Thus $r'_\nu=(0, k_{\nu+1}, \ldots) < (0, 3, \ldots)$, whence $k_{\nu+1}\ge 3$, $k_{\nu+1}=3$. Repeating the argument with ν replaced by $\nu+1, \nu+2, \ldots$, we see that $r'_\nu=(0, \overset{*}{3})$, which contradicts (5.12).

(b) Suppose that $\max\{k_\nu\}=4$. Take any ν with $k_\nu=4$. Then, by Lemma 5,

$$|T_\nu(0)|,\ |T_\nu(4)| > \tfrac{3}{5} > \varkappa_5; \quad |T_\nu(1)|,\ |T_\nu(3)| < \tfrac{3}{5} < \tau_5;$$

hence, if (5.11) is not satisfied, we must have

$$|T_\nu(0)|,\ |T_\nu(4)| > \tau_5; \quad |T_\nu(1)|,\ |T_\nu(3)| < \varkappa_5.$$

Lemma 6 (with $k_\nu=4$, $\tau=\tau_5=3/\sqrt{13}$, $\varkappa=\varkappa_5=\sqrt{3}/4$) now gives

$$r'_\nu,\ s_\nu < \tfrac{2}{3}(\sqrt{13}-3) = (0, 2, 2, 10, \ldots), \tag{5.13}$$

$$r'_\nu,\ s_\nu > \tfrac{1}{3}(4\sqrt{3}-6) = (0, \overset{*}{3}, \overset{*}{4}). \tag{5.14}$$

From these two inequalities for r'_ν, we see that $2 \le k_{\nu+1} \le 3$.

Suppose now that $k_{\nu+1}=2$. Then (5.13) gives

$$r'_\nu=(0,2,k_{\nu+2},k_{\nu+3},\ldots)<(0,2,2,10,\ldots),$$

whence $k_{\nu+2}\leq 2$. Also, if $k_{\nu+2}=2$, this last inequality gives $k_{\nu+3}\geq 10$, which is impossible. Hence $k_{\nu+2}=1$. We therefore have

$$k_{\nu+1}=2,\ r'_{\nu+1}=(0,1,\ldots),\ s_{\nu+1}=(0,4,\ldots),$$

so that

$$(0,2)=\tfrac{1}{2}<r'_{\nu+1}<(0,1,5)=\tfrac{5}{6};\quad (0,5)=\tfrac{1}{5}<s_{\nu+1}<(0,4)=\tfrac{1}{4}.$$

But then

$$|T_{\nu+1}(2)|=\frac{2-r'_{\nu+1}+s_{\nu+1}}{2+r'_{\nu+1}+s_{\nu+1}}>\frac{2-\frac{5}{6}+\frac{1}{5}}{2+\frac{5}{6}+\frac{1}{5}}=\frac{41}{91}=0\cdot45\cdots>\varkappa_5,$$

$$|T_{\nu+1}(2)|<\frac{2-\frac{1}{2}+\frac{1}{4}}{2+\frac{1}{2}+\frac{1}{4}}=\frac{7}{11}=0\cdot63\cdots<\tau_5,$$

and (5.11) is satisfied by $|T_{\nu+1}(2)|$.

Thus if (5.11) is not satisfied, we must have $k_{\nu+1}=3$. Then, by (5.14),

$$r'_\nu=(0,3,k_{\nu+2},\ldots)>(0,3,4,\ldots),$$

whence $k_{\nu+2}\geq 4$, $k_{\nu+2}=4$. We may now repeat the above argument with ν replaced by $\nu+2$, and deduce that $k_{\nu+3}=3$, $k_{\nu+4}=4$. Repeating the argument indefinitely, we see that $r'_\nu=(0,\overset{*}{3},\overset{*}{4})$, which contradicts (5.14).

Case VI: $i=6$. Here (4.12) is

$$\varkappa_6=\frac{1}{\sqrt{5}}=0\cdot447\cdots\leq|T_\nu(p)|\leq\tau_6=\frac{\sqrt{3}}{2}=0\cdot866\cdots \tag{5.15}$$

If $\max\{k_\nu\}=1$, (5.1) shows that (5.15) is satisfied with $p=0$; and if $\max\{k_\nu\}=2$, Lemma 7 gives the result. We need therefore consider only sequences (K) for which $\max\{k_\nu\}=3$ or 4.

If now $k_\mu=3$ for some μ, Lemma 5 (i) gives

$$|T_\mu(0)|,\ |T_\mu(3)|>\tfrac{1}{2}>\varkappa_6,$$

and so, if (5.15) is not satisfied, we must have

$$|T_\mu(0)|,\ |T_\mu(3)|>\tau_6.$$

Lemma 6 (i) (with $k_\mu=3$, $\tau=\tau_6=\sqrt{3}/2$) then shows that

$$r'_\mu,\ s_\mu<\tfrac{1}{2}(2\sqrt{3}-3)=(0,\overset{*}{4},\overset{*}{3}), \tag{5.16}$$

whence $k_{\mu+1}\geq 4$, and so $k_{\mu+1}=4$.

We may therefore suppose that max $\{k_\nu\}=4$. Take any ν with $k_\nu=4$. Then, by Lemma 5,

$$|T_\nu(0)|,\ |T_\nu(4)|>\tfrac{3}{5}>\varkappa_6;\quad |T_\nu(1)|,\ |T_\nu(3)|<\tfrac{3}{5}<\tau_6;$$

and so, if (5.15) is not satisfied, we require

$$|T_\nu(0)|,\ |T_\nu(4)|>\tau_6;\quad |T_\nu(1)|,\ |T_\nu(3)|<\varkappa_6.$$

Lemma 6 (with $k_\nu=4$, $\tau=\tau_6=\sqrt{3}/2$, $\varkappa=\varkappa_6=1/\sqrt{5}$) now gives

$$r'_\nu,\ s_\nu<\tfrac{1}{3}(4\sqrt{3}-6)=(0,\overset{*}{3},\overset{*}{4}), \tag{5.17}$$

$$r'_\nu,\ s_\nu>\sqrt{5}-2=(0,\overset{*}{4}). \tag{5.18}$$

From these two inequalities for r'_ν, we see that $3\le k_{\nu+1}\le 4$. We distinguish the two cases $k_{\nu+1}=3$ and $k_{\nu+1}=4$.

Suppose first that $k_{\nu+1}=4$. Then (5.18) gives

$$r'_\nu=(0,4,k_{\nu+2},\ldots)>(0,4,4,\ldots),$$

whence $k_{\nu+2}\ge 4$, $k_{\nu+2}=4$. Repeating the argument with ν replaced by $\nu+1$, $\nu+2$, ..., we see that $r'_\nu=(0,\overset{*}{4})$, which contradicts (5.18).

Suppose next that $k_{\nu+1}=3$. Then (5.16), with $\mu=\nu+1$, gives

$$r'_{\nu+1}=(0,k_{\nu+2},k_{\nu+3},\ldots)<(0,4,3,\ldots),$$

whence $k_{\nu+2}\ge 4$, $k_{\nu+2}=4$; $k_{\nu+3}\le 3$, whence $k_{\nu+3}=3$ (since, as was shown above, any element 4 must be followed by 3 or 4). Repeating the argument with ν replaced by $\nu+2$, $\nu+4$, ..., we see that $r'_\nu=(0,\overset{*}{3},\overset{*}{4})$, which contradicts (5.17).

Case VII: $i=7$. Here (4.12) is

$$\varkappa_7=\frac{1}{\sqrt{2}}=0{\cdot}707\cdots\le|T_\nu(p)|\le\tau_7=\frac{3}{\sqrt{5}}=1.341\cdots \tag{5.19}$$

If any $k_\nu\ge 3$, Lemma 4 shows that there exists a $T_\nu(p)$ in any interval of length $4/7=0{\cdot}57\cdots<\tau_7-\varkappa_7$, so that (5.19) is certainly satisfied. Also, if every $k_\nu=1$, (5.1) gives $|T_\nu(2)|=3/\sqrt{5}=\tau_7$. We need therefore consider only sequences (K) for which max $\{k_\nu\}=2$.

Take any ν with $k_\nu=2$. Then

$$|T_\nu(0)|=\frac{2+r'_\nu-s_\nu}{2+r'_\nu+s_\nu}<1<\tau_7;\quad |T_\nu(2)|=\frac{2-r'_\nu+s_\nu}{2+r'_\nu+s_\nu}<1<\tau_7; \tag{5.20}$$

$$|T_\nu(3)|=\frac{4-r'_\nu+s_\nu}{2+r'_\nu+s_\nu}>1>\varkappa_7. \tag{5.21}$$

Suppose now that (5.19) is not satisfied. Then, by (5.20), $|T_\nu(0)|$, $|T_\nu(2)| < \varkappa_7$, whence (by the analysis of Lemma 6 (i), reversing the inequality sign throughout)

$$r'_\nu,\ s_\nu > \frac{k_\nu(1-\varkappa_7)}{2\varkappa_7} = \sqrt{2}-1 = (0,\dot{2}). \tag{5.22}$$

If $k_{\nu+1}=2$, (5.22) gives

$$r'_\nu = (0, 2, k_{\nu+2}, \ldots) > (0, 2, 2, \ldots),$$

whence $k_{\nu+2} \geq 2$, $k_{\nu+2}=2$. Repeating the argument with ν replaced by $\nu+1$, $\nu+2$, ..., we see that $r'_\nu = (0, \dot{2})$, which contradicts (5.22). Similarly, if $k_{\nu-1}=2$, we find $s_\nu = (0, \overset{*}{2})$, again contradicting (5.22). Thus $k_{\nu-1} \neq 2$, $k_{\nu+1} \neq 2$, and so $k_{\nu-1} = k_{\nu+1} = 1$,

$$r'_\nu = (0, 1, \ldots) > \tfrac{1}{2}, \quad s_\nu = (0, 1, \ldots) > \tfrac{1}{2}.$$

But then, since $|T_\nu(3)|$ is a decreasing function of r'_ν and of s_ν,

$$|T_\nu(3)| < \frac{4-\frac{1}{2}+\frac{1}{2}}{2+\frac{1}{2}+\frac{1}{2}} = \frac{4}{3} < \frac{3}{\sqrt{5}} = \tau_7,$$

and so, by (5.21), (5.19) is satisfied by $|T_\nu(3)|$.

This completes the proof of Lemma 2 (i) for $i=1, 2, \ldots, 7$. Before proceeding to the case $i=8$, we establish the assertion made in Lemma 2 (ii) for $i=0, 1, \ldots, 7$; namely that when Q is equivalent to a multiple of Q_i, $b/\sqrt{D}$ assumes no value lying strictly between $\varkappa_i$ and τ_{i+1}.

Since the numbers $\varkappa_i$, τ_i satisfy $-1/\sqrt{5} \leq \varkappa_i, \tau_i \leq \sqrt{2}$, it suffices, after Lemma 3, to consider only the values of $T_\nu(p) = (b_\nu - 2pa_{\nu+1})/\sqrt{D}$ corresponding to the reduced forms φ_ν equivalent to each Q_i. As is easily verified, each of the forms Q_i is reduced, and so may be taken to be $\varphi_1(x, y) = a_1x^2 + b_1xy - a_2y^2$.

(i) $i=0$ or 6. $Q_0 = Q_6 = x^2+xy-y^2$, and so

$$\varphi_1 = x^2+xy-y^2, \quad \varphi_2 = -x^2+xy+y^2,$$

these being the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p) = T_2(p) = \frac{1-2p}{\sqrt{5}},$$

and the only values of $|T_\nu(p)| \leq \sqrt{2}$ are: $1/\sqrt{5} = -\varkappa_0 = \tau_1 = \varkappa_6$, $3/\sqrt{5} = \tau_7$.

(ii) $i=1$ or 7. $Q_1 = Q_7 = x^2+2xy-y^2$, and so

$$\varphi_1 = x^2+2xy-y^2, \quad \varphi_2 = -x^2+2xy+y^2,$$

these being the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p) = T_2(p) = \frac{1-p}{\sqrt{2}},$$

and the only values of $|T_\nu(p)| \leq \sqrt{2}$ are: $0 = \varkappa_1$; $1/\sqrt{2} = \tau_2 = \varkappa_7$; $\sqrt{2} = \tau_8$.

(iii) $i = 2$. $Q_2 = 2x^2 + 3xy - y^2$, and so

$$\varphi_1 = 2x^2 + 3xy - y^2, \quad \varphi_2 = -x^2 + 3xy + 2y^2, \quad \varphi_3 = 2x^2 + xy - 2y^2;$$

$\varphi_4, \varphi_5, \varphi_6$ are derived from these by changing the signs of the extreme coefficients, and these are the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p) = T_4(p) = \frac{3-2p}{\sqrt{17}},$$

$$T_2(p) = T_5(p) = \frac{3-4p}{\sqrt{17}},$$

$$T_3(p) = T_6(p) = \frac{1-4p}{\sqrt{17}},$$

and the only values of $|T_\nu(p)| \leq \tau_3 = 3/\sqrt{17}$ are: $1/\sqrt{17} = \varkappa_2$, $3/\sqrt{17} = \tau_3$.

(iv) $i = 3$. $Q_3 = 3x^2 + 6xy - 2y^2$, and so

$$\varphi_1 = 3x^2 + 6xy - 2y^2, \quad \varphi_2 = -2x^2 + 6xy + 3y^2,$$

these being the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p) = \frac{3-2p}{\sqrt{15}}, \quad T_2(p) = \frac{3-3p}{\sqrt{15}},$$

and the only values of $|T_\nu(p)| \leq \tau_4 = 3/\sqrt{15}$ are: 0, $1/\sqrt{15} = \varkappa_3$, $3/\sqrt{15} = \tau_4$.

(v) $i = 4$. $Q_4 = x^2 + 3xy - y^2$, and so

$$\varphi_1 = x^2 + 3xy - y^2, \quad \varphi_2 = -x^2 + 3xy + y^2,$$

these being the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p) = T_2(p) = \frac{3-2p}{\sqrt{13}},$$

and the only values of $|T_\nu(p)| \leq \tau_5 = 3/\sqrt{13}$ are: $1/\sqrt{13} = \varkappa_4$, $3/\sqrt{13} = \tau_5$.

(vi) $i = 5$. $Q_5 = 4x^2 + 12xy - 3y^2$, and so

$$\varphi_1 = 4x^2 + 12xy - 3y^2, \quad \varphi_2 = -3x^2 + 12xy + 4y^2,$$

these being the only distinct elements of the chain $\{\varphi_\nu\}$. Hence

$$T_1(p)=\frac{6-3p}{4\sqrt{3}},\quad T_2(p)=\frac{3-2p}{2\sqrt{3}},$$

and the only values of $|T_\nu(p)|\leq\tau_6=\sqrt{3}/2$ are: $0, 1/2\sqrt{3}, \sqrt{3}/4=\varkappa_5, \sqrt{3}/2=\tau_6$.

This completes the proof of Lemma 2 for $i\leq 7$.

6. For the case $i=8$ of Lemma 2, we must first consider the inequality

$$\varkappa_8=\frac{5}{4\sqrt{2}-1}=1{\cdot}073\cdots\leq|T_\nu(p)|\leq\tau_8=\sqrt{2}=1{\cdot}414\cdots \tag{6.1}$$

We note first that we need consider only sequences (K) with $k_\nu\leq 5$ for all ν; for if any $k_\nu\geq 6$, Lemma 4 (i) shows that there exists a $T_\nu(p)$ lying in any interval of length $7/22=0{\cdot}318\cdots<\tau_8-\varkappa_8$, so that (6.1) may certainly be satisfied.

(i) If $\max\{k_\nu\}=1$, then by (5.1), (6.1) is satisfied with $p=2$.

(ii) If $\max\{k_\nu\}=2$, take any ν with $k_\nu=2$. Then

$$r'_\nu,\ s_\nu\leq(0,\overset{*}{1},\overset{*}{2})=\sqrt{3}-1,$$

and so

$$|T_\nu(-1)|,\ |T_\nu(3)|\geq\frac{4}{2+2(\sqrt{3}-1)}=\frac{2}{\sqrt{3}}=1{\cdot}15\cdots>\varkappa_8.$$

If now $k_{\nu-1}$, $k_{\nu+1}$ are not both equal to 2, we may suppose, by symmetry, that $k_{\nu+1}=1$. Then

$$r'_\nu=(0,1,\ldots)>\tfrac{1}{2},\ s_\nu>0,$$

whence

$$|T_\nu(3)|<\frac{4-\frac{1}{2}}{2+\frac{1}{2}}=\frac{7}{5}<\sqrt{2},$$

and (6.1) is satisfied by $T_\nu(3)$.

Otherwise we must have $k_{\nu-1}=k_{\nu+1}=2$ whenever $k_\nu=2$. This implies that $k_\nu=2$ for all ν, and so

$$r'_\nu=s_\nu=(0,\overset{*}{2})=\sqrt{2}-1,\quad T_\nu(-1)=\sqrt{2}=\tau_8.$$

(iii) If $\max\{k_\nu\}=3$, take any ν with $k_\nu=3$. Then

$$r'_\nu,\ s_\nu\leq(0,\overset{*}{1},\overset{*}{3})=\tfrac{1}{2}(\sqrt{21}-3),$$

and so

$$|T_\nu(-1)|,\ |T_\nu(4)|\geq\frac{5}{3+(\sqrt{21}-3)}=\frac{5}{\sqrt{21}}=1{\cdot}09\cdots>\varkappa_8.$$

If now $k_{\nu-1}$, $k_{\nu+1}$ are not both equal to 3, we may suppose, by symmetry, that $k_{\nu+1} \leq 2$, $k_{\nu-1} \leq 3$. Then

$$r'_\nu = (0, k_{\nu+1}, \ldots) > \tfrac{1}{3}, \quad s_\nu > \tfrac{1}{4},$$

whence

$$|T_\nu(4)| < \frac{5 + \frac{1}{4} - \frac{1}{3}}{3 + \frac{1}{4} + \frac{1}{3}} = \frac{59}{43} = 1{\cdot}37 \cdots < \sqrt{2},$$

and (6.1) is satisfied by $T_\nu(4)$.

Otherwise we must have $k_{\nu-1} = k_{\nu+1} = 3$ whenever $k_\nu = 3$. This implies that $k_\nu = 3$ for all ν, and so

$$r'_\nu = s_\nu = (0, \overset{*}{3}) = \tfrac{1}{2}(\sqrt{13} - 3), \quad T_\nu(-1) = \frac{5}{\sqrt{13}} = 1{\cdot}38 \cdots$$

(iv) If max $\{k_\nu\} = 4$, take any ν with $k_\nu = 4$. Then

$$r'_\nu, \; s_\nu > \tfrac{1}{5},$$

$$|T_\nu(-1)|, \; |T_\nu(5)| < \frac{6 + \frac{1}{5} - \frac{1}{5}}{4 + \frac{1}{5} + \frac{1}{5}} = \frac{15}{11} < \tau_8. \tag{6.2}$$

We now show that, if (6.1) is not satisfied,

$$k_{\nu-1} = k_{\nu+1} = 1, \quad k_{\nu-2} \geq 3, \quad k_{\nu+2} \geq 3. \tag{6.3}$$

For if $k_{\nu-1} = 1$ and $k_{\nu-2} \leq 2$, we have

$$s_\nu = (0, k_{\nu-1}, \ldots) \leq (0, 1, 2, \overset{*}{1}, \overset{*}{4}) = \frac{2(4 - \sqrt{2})}{7}; \tag{6.4}$$

while if $k_{\nu-1} \geq 2$, then

$$s_\nu < \tfrac{1}{2} < \frac{2(4 - \sqrt{2})}{7};$$

so that (6.4) holds in either case. Also, since max $\{k_\nu\} = 4$,

$$r'_\nu \leq (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2} - 1). \tag{6.5}$$

From (6.4) and (6.5) we see that

$$|T_\nu(-1)| \geq \frac{6 + 2(\sqrt{2} - 1) - \frac{2}{7}(4 - \sqrt{2})}{4 + 2(\sqrt{2} - 1) + \frac{2}{7}(4 - \sqrt{2})} = \frac{2}{7}(1 + 2\sqrt{2}) = 1{\cdot}093 \cdots > \varkappa_8;$$

and it follows from (6.2) that (6.1) is satisfied by $T_\nu(-1)$. In the same way (interchanging the roles of r'_ν and s_ν in the above), we may show that (6.1) is satisfied by $T_\nu(5)$ if either $k_{\nu+1} \geq 2$ or $k_{\nu+1} = 1$ and $k_{\nu+2} \leq 2$.

We may therefore suppose now that (6.3) holds whenever $k_\nu = 4$. If we suppose further that $k_\nu \neq 3$ for any ν, it follows from (6.3) and our assumption that max $\{k_\nu\} = 4$ that (K) is the periodic sequence

$$\ldots, 1, 4, 1, 4, 1, 4, \ldots \tag{6.6}$$

For the sequence (6.6) we have, whenever $k_\nu = 4$,

$$r'_\nu = s_\nu = (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2}-1), \quad T_\nu(-2) = \sqrt{2} = \tau_8,$$

so that (6.1) is satisfied.

If (K) is not the sequence (6.6), it follows from (6.3) that, for some ν, $k_\nu = 4$ and either $k_{\nu-2} = 3$ or $k_{\nu+2} = 3$; by symmetry, we may take $k_{\nu+2} = 3$. Thus

$$k_{\nu-1} = 1, \quad k_\nu = 4, \quad k_{\nu+1} = 1, \quad k_{\nu+2} = 3.$$

Then, since

$$r'_{\nu+2}, \; s_{\nu+2} \leq (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2}-1),$$

we have

$$|T_{\nu+2}(4)| = \frac{5 - r'_{\nu+2} + s_{\nu+2}}{3 + r'_{\nu+2} + s_{\nu+2}} \geq \frac{5}{3 + 4(\sqrt{2}-1)} = \varkappa_8;$$

and, since

$$r'_{\nu+2} = (0, k_{\nu+3}, \ldots) > (0, 5) = \tfrac{1}{5},$$

$$s_{\nu+2} = (0, 1, 4, \ldots) > (0, 1, 4) = \tfrac{4}{5},$$

we have

$$|T_{\nu+2}(4)| < \frac{5 - \frac{1}{5} + \frac{4}{5}}{3 + \frac{1}{5} + \frac{4}{5}} = \frac{7}{5} < \tau_8;$$

so that (6.1) is satisfied by $T_{\nu+2}(4)$.

(v) If max $\{k_\nu\} = 5$, take any ν with $k_\nu = 5$. Then, since $r'_\nu, s_\nu > 0$,

$$|T_\nu(-1)|, \; |T_\nu(6)| < \tfrac{7}{5} < \tau_8;$$

and, since $r'_\nu, s_\nu < 1$,

$$|T_\nu(-2)|, \; |T_\nu(7)| > \frac{9}{5+1+1} = \frac{9}{7} > \varkappa_8.$$

Hence, if (6.1) is not satisfied, we require

$$|T_\nu(-1)| < \varkappa_8; \quad |T_\nu(6)| < \varkappa_8; \quad |T_\nu(-2)| > \tau_8; \quad |T_\nu(7)| > \tau_8.$$

On substituting the values of $T_\nu(p)$, we obtain from the first three of these inequalities, respectively,

$$r'_\nu(\varkappa_8 - 1) + s_\nu(\varkappa_8 + 1) > 7 - 5\varkappa_8; \tag{6.7}$$

$$s_\nu(\varkappa_8 - 1) + r'_\nu(\varkappa_8 + 1) > 7 - 5\varkappa_8; \tag{6.8}$$

$$r'_\nu(\tau_8 - 1) + s_\nu(\tau_8 + 1) < 9 - 5\tau_8. \tag{6.9}$$

Since $r'_\nu < 1$, (6.7) gives

$$s_\nu > \frac{8-6\varkappa_8}{\varkappa_8+1} = \frac{1\cdot 55\ldots}{2\cdot 07\ldots} > 0\cdot 7;$$

and, since $s_\nu < 1$, (6.8) gives similarly

$$r'_\nu > 0\cdot 7.$$

Substituting these inequalities in (6.9), we obtain

$$0\cdot 7 < \frac{9-5\tau_8}{2\tau_8} = \frac{9-5\sqrt{2}}{2\sqrt{2}} = 0\cdot 68\ldots,$$

which is clearly false. It follows that (6.1) may always be satisfied when $k_\nu = 5$.

We have now established the first assertion of Lemma 2 for $i=8$. It remains to show that, when Q is equivalent to a multiple of the form

$$Q_8(x, y) = x^2 + 3xy - (6-\sqrt{8})y^2,$$

$b/\sqrt{D}$ assumes no value lying strictly between $\varkappa_8$ and τ_9. Since the sequence (K) corresponding to this special form is

$$\ldots, 4, 1, 4, 1, 3, 1, 4, 1, 4, \ldots, \tag{6.10}$$

it suffices, after Lemma 3 (ii), to show that for the sequence (6.10) we have *either*

$$|T_\nu(p)| \le \varkappa_8 = \frac{5}{4\sqrt{2}-1} = 1\cdot 073\ldots, \tag{6.11}$$

or

$$|T_\nu(p)| \ge \tau_9 = \sqrt{2} = 1\cdot 414\ldots, \tag{6.12}$$

for all integral ν, p.

(a) Suppose that $k_\nu = 1$, so that

$$T_\nu(p) = \frac{1+r'_\nu - s_\nu - 2p}{1+r'_\nu + s_\nu}.$$

For $p=0$ or 1, we have clearly

$$|T_\nu(p)| < 1,$$

and (6.11) holds. Also, since $k_{\nu-1} \ge 3$, $k_{\nu+1} \ge 3$, we have

$$r'_\nu,\ s_\nu < \tfrac{1}{3},\quad |T_\nu(-1)|,\ |T_\nu(2)| > \frac{3+\frac{1}{3}-\frac{1}{3}}{1+\frac{1}{3}+\frac{1}{3}} = \frac{9}{5} = 1\cdot 8,$$

and (6.12) therefore holds for $p \le -1$ and $p \ge 2$.

(b) Suppose that $k_\nu = 3$. Then

$$r'_\nu = s_\nu = (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2}-1),$$

and so

$$T_\nu(p) = \frac{3-2p}{4\sqrt{2}-1}.$$

For $p \leq -2$ and $p \geq 5$, we therefore have

$$|T_\nu(p)| \geq \frac{7}{4\sqrt{2}-1} = 1{\cdot}50 \cdots > \tau_9;$$

and, for $-1 \leq p \leq 4$,

$$|T_\nu(p)| \leq \frac{5}{4\sqrt{2}-1} = \varkappa_8.$$

(c) Suppose that $k_\nu = 4$, so that

$$T_\nu(p) = \frac{4+r'_\nu - s_\nu - 2p}{4+r'_\nu + s_\nu}.$$

By symmetry, we may suppose that $k_\nu = 4$ occurs to the left of the element 3 in (6.10). Then

$$s_\nu = (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2}-1),$$

$$r'_\nu = (0, 1, 4, 1, 4, \ldots, 1, 4, 1, 3, \overset{*}{1}, \overset{*}{4}),$$

so that

$$r'_\nu < (0, \overset{*}{1}, \overset{*}{4}) = 2(\sqrt{2}-1),$$

$$r'_\nu \geq (0, 1, 3, \overset{*}{1}, \overset{*}{4}) = \tfrac{1}{2}(3-\sqrt{2}).$$

For $p \leq -2$ and $p \geq 6$, we therefore have

$$|T_\nu(p)| > \frac{8}{4+4(\sqrt{2}-1)} = \sqrt{2},$$

and (6.12) holds. Also

$$|T_\nu(5)| \leq \frac{6-\frac{1}{2}(3-\sqrt{2})+2(\sqrt{2}-1)}{4+\frac{1}{2}(3-\sqrt{2})+2(\sqrt{2}-1)} = \frac{5}{4\sqrt{2}-1} = \varkappa_8,$$

$$|T_\nu(-1)| \leq \frac{6+\frac{1}{2}(3-\sqrt{2})-2(\sqrt{2}-1)}{4+\frac{1}{2}(3-\sqrt{2})+2(\sqrt{2}-1)} < \varkappa_8,$$

and so (6.11) holds for $-1 \leq p \leq 5$.

This completes the proof of Lemma 2 (ii), so that Theorem 2 is now established.

7. It is clear, after Lemma 1, that the best possible estimate for $M(B)$ in terms of its two fundamental invariants always takes the form

$$M(B) \leq \tfrac{1}{2}\sqrt{D}\,\chi(\omega)$$

where

$$\chi(\omega) = \underset{Q}{\text{u.b.}}\ \underset{b}{\text{l.b.}} \left|\frac{b}{\sqrt{D}} - \omega\right|.$$

$\chi(\omega)$ has been evaluated above, for the range $0 \leq \omega \leq \omega_0 = 1{\cdot}2439\ldots$, by an inductive process, each 'critical' form $Q_i(x, y)$ providing the link between successive intervals $(\varkappa_i, \tau_i)$, $(\varkappa_{i+1}, \tau_{i+1})$ in Lemma 2.

This process, however, breaks down for values of ω slightly larger than ω_0. For it is easily seen from the analysis of section 6 that, for the form $Q_8(x, y)$, $b/\sqrt{D}$ assumes values arbitrarily near to (but greater than) $\sqrt{2}$. Thus the lower bound of the values of $b/\sqrt{D} > \varkappa_8$ assumed for $Q_8(x, y)$ is precisely $\sqrt{2} = \tau_8$, and we must take $\tau_9 = \tau_8$. Then $\varkappa_9$, defined as the upper bound of numbers $\varkappa$ for which the inequality

$$\varkappa \leq \frac{b}{\sqrt{D}} \leq \tau_9$$

may always be satisfied, is clearly equal to $\varkappa_8$.

I have been unable to find any method of evaluating $\chi(\omega)$ for $\omega > \omega_0$. This appears to be a difficult problem; for it may be shown that, for any $\varepsilon > 0$, $\chi(\omega)$ has an infinity of turning points in the interval $\omega_0 < \omega < \omega_0 + \varepsilon$, and it is probable that the set of turning points has points of accumulation other than ω_0.

I should like to note finally that the methods of this paper may be used to evaluate the 'successive minima'[1] of $M(B)$ for any particular value of ω. Complete results have been found for $\omega = 0$, by Schur, and for $\omega = 1$, by Davenport and Heilbronn, in the papers referred to in section 1.

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[1] Cf. the author's paper "The minimum of a factorizable bilinear form", to appear in *Acta Mathematica.*