

ON CAPILLARY FREE SURFACES IN A GRAVITATIONAL FIELD

BY

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We present here the second part of our study of the equation

$$\operatorname{div} \left(\frac{1}{W} \nabla u \right) \equiv \sum_{i=1}^n \frac{\partial}{\partial x_i} \left(\frac{1}{W} \frac{\partial u}{\partial x_i} \right) = n \mathcal{H}(\mathbf{x}; u), \quad W = (1 + |\nabla u|^2)^{\frac{1}{2}}, \quad (1)$$

for a scalar function $u(\mathbf{x})$ over an n -dimensional domain Ω with bounding surface Σ . For information on physical background and smoothness hypotheses we refer the reader to the Introduction in [7], where we study the case \mathcal{H} independent of u . The interest for the present work, in which \mathcal{H} is permitted to depend on u explicitly in certain ways, centers on the capillary equation

$$Nu \equiv \operatorname{div} \left(\frac{1}{W} \nabla u \right) = \kappa u \quad (2)$$

where $\kappa \neq 0$ is a constant, under a boundary condition

$$\mathbf{T}u \cdot \mathbf{v} \equiv \frac{1}{W} \mathbf{v} \cdot \nabla u = \cos \gamma \quad (3)$$

where \mathbf{v} is unit exterior normal on Σ and γ is prescribed (see [7]). However, we shall discuss considerably more general situations to which our methods apply.

§ 1

We impose on $\mathcal{H}(\mathbf{x}; u)$ in (1) a single requirement:

A : For any $\delta > 0$ there exists $M_\delta < \infty$, such that at least one of the conditions

$$A_1: \{ \mathcal{H} \leq \delta^{-1} \Rightarrow u \leq M_\delta \}$$

$$A_2: \{ \mathcal{H} \geq -\delta^{-1} \Rightarrow u \geq -M_\delta \}$$

$$A_3: \{ \mathcal{H} \leq \delta^{-1} \Rightarrow u \geq -M_\delta \}$$

$$A_4: \{ \mathcal{H} \geq -\delta^{-1} \Rightarrow u \leq M_\delta \}$$

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holds for all $x \in \Omega$, that is, $\lim_{u \rightarrow \infty} \mathcal{H}(\mathbf{x}; u) = \infty$ uniformly for $\mathbf{x} \in \Omega$ in the case A_1 , with analogous expressions in the other cases.

Remark. The capillary equation (2) is contained in A_1 and A_2 if $\kappa > 0$, and in A_3 and A_4 if $\kappa < 0$.

THEOREM 1. *Suppose A_1 (resp. A_2) is satisfied by $\mathcal{H}(\mathbf{x}; u)$, and let $u(\mathbf{x})$ be a solution of (1) in an n -ball B_δ^n of radius δ . Then $u(\mathbf{x}) < M_\delta + \delta$ (resp. $u(\mathbf{x}) > -M_\delta - \delta$) for all $\mathbf{x} \in B_\delta^n$.*

COROLLARY: *If there is a fixed $\delta > 0$ such that Ω can be covered by an interior family $\{B_\delta^n\}$, then the conclusion holds uniformly in Ω , for any solution $u(\mathbf{x})$ over Ω . If Ω is arbitrary and r is distance to Σ , there still holds $u(\mathbf{x}) < M_r$ (resp. $u(\mathbf{x}) > -M_r$).*

Proof of Theorem 1. Suppose first that A_1 holds; let $x \in B_\delta^n$. Choose δ' , $0 < \delta' < \delta$, and let $S_{\delta'}^n$ be an n -sphere (boundary of an $(n+1)$ -ball) of radius δ' whose center (\mathbf{x}_0, u_0) lies on the vertical through the center $(\mathbf{x}_0, 0)$ of B_δ^n . If u_0 is sufficiently large, $S_{\delta'}^n$ will lie above the solution surface $\mathcal{S}: u = u(\mathbf{x})$. Let \bar{u}_0 be the largest value of u_0 for which $S_{\delta'}^n$ contacts \mathcal{S} , and let $\mathbf{p}_1 = (\mathbf{x}_1, u_1)$ be a point of contact. Then \mathbf{p}_1 lies on the lower hemisphere of $S_{\delta'}^n$ (it cannot lie on the equatorial sphere since $|\nabla u| < \infty$ at these points), and $S_{\delta'}^n$ shares with \mathcal{S} a common normal at \mathbf{p}_1 . None of the normal curvatures (at \mathbf{p}_1) of curves on \mathcal{S} through \mathbf{p}_1 (considered as positive when the curvature vector is directed into $S_{\delta'}^n$) can exceed $1/\delta'$, for otherwise there would be points of \mathcal{S} interior to $S_{\delta'}^n$, contrary to the construction. It follows that the mean curvature of \mathcal{S} at \mathbf{p}_1 , as defined by the left side of (1), cannot exceed $1/\delta'$. Thus $\mathcal{H}(\mathbf{x}_1, u_1) \leq 1/\delta'$, from which, by A_1 , $u_1 \leq M_{\delta'}$. Hence $u(\mathbf{x}) < M_{\delta'} + \delta'$ in B_δ^n . The proof is completed by letting $\delta' \rightarrow \delta$.

Similarly, if A_2 holds, one finds $u(\mathbf{x}) > -(M_\delta + \delta)$.

Remarks.

(i) A somewhat stronger (geometrical) theorem could have been stated. It would have sufficed to know that $u(\mathbf{x}) < M_\delta$ (resp. $u(\mathbf{x}) > -M_\delta$) whenever the maximum κ_μ of the principal normal curvatures of \mathcal{S} satisfies $\kappa_\mu < \delta^{-1}$ (resp. $\kappa_\mu > -\delta^{-1}$).

(ii) For the capillary equation $n\mathcal{H}(\mathbf{x}; u) \equiv \kappa u$, $\kappa > 0$, we may take $M_\delta = n/(\kappa\delta)$ for the case A_1 or A_2 . Thus, in this case there holds $|u| < n/(\kappa\delta) + \delta$ for any solution in B_δ^n .

(iii) We note that the corollary holds without explicit hypotheses on the boundary Σ of Ω , and that no hypothesis is introduced on the boundary behavior of $u(\mathbf{x})$. Under hypotheses A_1 and A_2 , it provides an a priori bound on $|u(\mathbf{x})|$ in any compact subdomain of an arbitrary Ω .

(iv) If $\mathcal{H}_u \geq 0$ (as in the capillary equation with $\kappa > 0$) the result can be improved somewhat by using as comparison surface a rotationally symmetric solution $\varphi(r; \delta)$ of the equation, determined by the condition $\varphi_r(\delta; \delta) = \infty$. It is not hard to show the existence of such solutions (cf. § 2) and to estimate them from above. The result $u < \varphi(r; \delta)$ on $|x| = r$ then follows from the general comparison principle of § 3.6. In the case of the capillary equation the improvement obtainable in this way has the order $O(\delta)$ as $\delta \rightarrow 0$.

(v) It should be noted that the proof of Theorem 1 does not use the maximum principle, and the result holds for many equations for which the maximum principle does not apply, either for a solution of (1) or for a difference of solutions. It is this fact that suggests that a corresponding result be sought for the case of the capillary equation (2) with reversed gravitational field, $\kappa < 0$. The best result of this sort we can offer is the following:

THEOREM 2. *Suppose A_3 (resp. A_4) is satisfied by $\mathcal{H}(x; u)$. Then if $u(x)$ is a solution of (1) in a ball B_δ^n , there is a point $x \in B_\delta^n$ for which $u(x) \geq -M_\delta$ (resp. $u(x) \leq M_\delta$).*

The proof is analogous to that of Theorem 1, the point x being in each case the projection onto the base hyperplane of a point of last contact of \mathcal{S} with S_δ^n . The method yields, however, no global bound throughout the domain of definition, even if (as in the capillary equation with $\kappa < 0$) A_3 and A_4 hold simultaneously.

§ 2

The extent to which a global estimate can be obtained under hypotheses such as A_3 or A_4 remains open. We have stated such estimates in § 11 of [5] and in § 3 of [6]; although there is evidence that the statements given there have physical meaning, we have since found that our demonstrations of these results are in the full generality indicated incomplete. The following remarks bear on this point, and show that at least for $n = 2$, the result of Theorem 1 still holds, with $M_\delta = n/|\kappa|\delta$, for the rotationally symmetric solutions of (2) with $\kappa < 0$.

2.1. Consider solutions of

$$\operatorname{div} \frac{1}{W} \nabla u = -u \quad (4)$$

in $n = 2$ variables, that are rotationally symmetric about a vertical axis. Such solutions are functions $u(r)$ of a single variable, and satisfy the equation

$$Nu \equiv \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{ru_r}{\sqrt{1+u_r^2}} \right) = -u. \quad (5)$$

We study solutions $u(r)$ of (5) such that $u(0) = u_0 < 0$, $u_r(0) = 0$. The local existence of such a solution can be proved by the method of Lohnstein [11, 12] or of Johnson and Perko [10], although this case does not seem to have been explicitly studied in those papers.⁽¹⁾ We summarize here the global behavior of $u(r)$, in its functional dependence on u_0 . Some of these features were already known to Bashforth and Adams [2] and to W. Thomson [13], although perhaps not in mathematical rigor. We state here only the results we have established; complete details will appear in a later paper.

2.2. *If, in the initial value problem of 2.1, $u_0^4 < 1/3$, then the solution $u(r)$ exists for all positive r . It has an infinity of zeros. For any two successive extrema r_a, r_b of $u(r)$ there holds $|u(r_b)| < |u(r_a)|$. Asymptotically as $u_0 \rightarrow 0$ the first zero r_1 is the first zero of $J_0(r)$, $r_1 \cong 2.405$.*

2.3. *As u_0 decreases from 0 to $-\infty$, there is a first value $u_0 = u_{01}$, such that the corresponding function $u(r)$ cannot be continued for all r as a solution of (5). The continuation is possible only in an interval $0 < r < r^{(1)}$, and $\lim_{r \rightarrow r^{(1)}} u'(r) = \infty$. There holds $-3^{-1/4} > u_{01} > -3^{5/4}$; the point of infinite slope on the solution curve is precisely the second point of intersection of this curve with the hyperbola $ru = -1$ (see Fig. 1). (We note that numerical computations [4] yield the value $u_{01} \approx -2.6$.)*

2.4. *If $u_0 = u_{01}$, the solution can be continued indefinitely as a solution of the parametric system*

$$\left. \begin{aligned} \frac{d\psi}{ds} &= -u - \frac{1}{r} \sin \psi \\ \frac{du}{ds} &= \sin \psi \\ \frac{dr}{ds} &= \cos \psi. \end{aligned} \right\} \quad (6)$$

Here s is arc length on the solution curve, ψ is the angle measured counterclockwise from the positively directed r -axis to the tangent line. There holds everywhere $-(\pi/2) < \psi \leq (\pi/2)$; the function $u(s(r))$ is a solution of (5) except at the single point $r = r^{(1)}$.

2.5. *As u_0 decreases past u_{01} , the solution continues to exist everywhere as a parametric solution of (6); however, the point $(r^{(1)}, u^{(1)})$ moves below the hyperbola $ru = -1$, and the solution continues, with decreasing r , to a second point $(r^{(2)}, u^{(2)})$ of infinite slope, lying above the hyperbola (see Fig. 2)⁽²⁾. On this branch, $u(s(r))$ is a solution of the equation*

⁽¹⁾ We have been unable to obtain a copy of Lohnstein's dissertation, and we have had to infer its content from his later papers and from the general report by Bakker [1].

⁽²⁾ This step in the discussion is based partly on numerical computation; we have not yet established the result formally in the full strength indicated here.

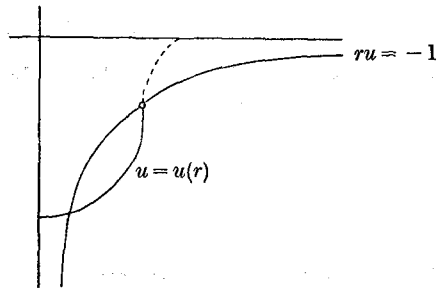


Fig. 1

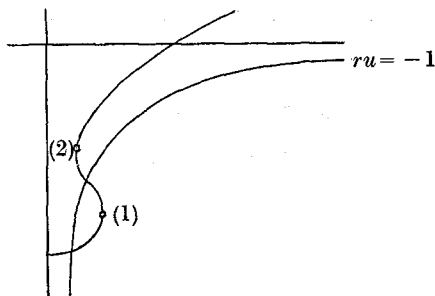
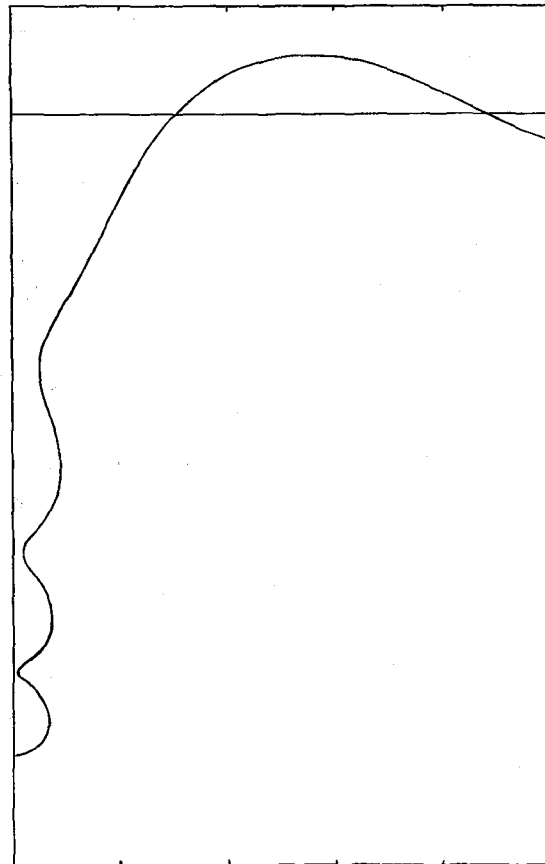


Fig. 2



$u_0 = -6$

Fig. 3

$$Nu = \frac{1}{r} \frac{\partial}{\partial r} \left(\frac{ru_r}{\sqrt{1+u_r^2}} \right) = +u \tag{7}$$

in which the sign of the right side is the reverse of that in (5). At $(r^{(2)}, u^{(2)})$, the curve reverses again, and continues indefinitely as a solution of (5).

2.6. As u_0 continues to decrease, the procedure repeats, leading to the formation of repeated "bubbles" (Fig. 3). All inflection points on the meridional curve lie between the two hyperbolas $ru = \pm 1$. Denoting by k_m the meridional curvature referred to the u -axis, we have $k_m(u^{(j)}) \geq 0$ according as $r^{(j)}u^{(j)} \geq -1$. The entire curve lies above the hyperbola $ru = -2$. Asymptotically for large $|u_0|$,

$$-\frac{2}{u_0} < r^{(1)} < -\frac{2}{u_0} - \frac{3}{u_0^3},$$

while for all $u_0 < -10$ there holds

$$u_0 - \frac{2}{u_0} - \frac{4}{u_0^2} < u^{(1)} < u_0 - \frac{2}{u_0} + \frac{4}{u_0^2}.$$

2.7. There is evidence to support the assertion that as $u_0 \rightarrow -\infty$, the "bubble" solutions converge, uniformly in compacta excluding the origin, to a new nonparametric solution $u = U(r)$ of (5), satisfying

$$-\frac{1}{r} < U(r) < -\frac{1}{r} \frac{1-r^4}{(1+r^4)^{\frac{1}{2}}} \quad (8)$$

for $0 < r < r_{(1)}$, where $r_{(1)}$ is the first zero of $U(r)$.

This new solution is defined for all $r > 0$, and yields a solution of (5) with an isolated singularity at the origin. We have not yet demonstrated this convergence, nor have we shown the existence of $U(r)$, but we hope to return to these matters in a subsequent paper. Here we note in passing that the right side of (8) is precisely the negative of the mean curvature of the surface defined by the left side of (8).

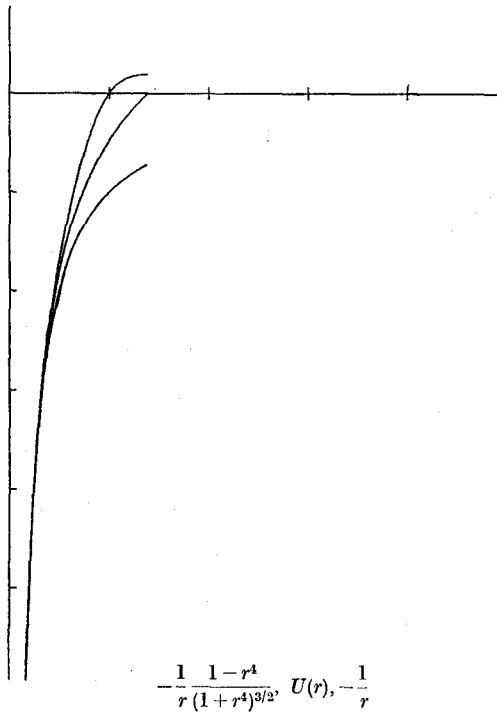
2.8. Numerical calculations of $U(r)$ were performed on the CDC 6600 computer using a variable-step-length fourth-order Adams-Moulton method. The first two terms of the formal asymptotic expansion

$$U(r) \sim -\frac{1}{r} + \frac{5}{2}r^3 + O(r^7), \quad (r \rightarrow 0)$$

provided the initial height and slope with which to begin the numerical integration at small values of r . The result behaved stably with respect to changes in the initial value of r and appears to support the conjecture of § 2.7. The numerical solution for $U(r)$ is compared with the left and right sides of (8) and with bubble solutions corresponding to several choices of u_0 , in Figs. 4, 5.⁽¹⁾

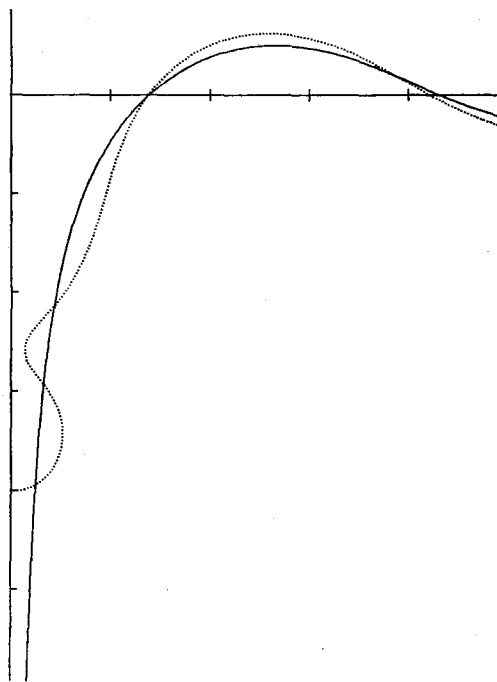
2.9. The surmised existence of $U(r)$ indicates that the result of Theorem 2 cannot be extended to a global bound for solutions that are not rotationally symmetric, that is, Theorem 1 would be qualitatively incorrect under the hypotheses A_3 or A_4 . To see this, we consider the solution $U(r)$, $r^2 = x_1^2 + x_2^2$, and a ball B_δ^2 (as in Theorem 2) which lies close to but does not include the origin. As indicated in Theorem 2, B_δ^2 will contain points at

⁽¹⁾ We wish to thank W. H. Benson and F. C. Gey for making available their computer programs and carrying out some of the calculations.



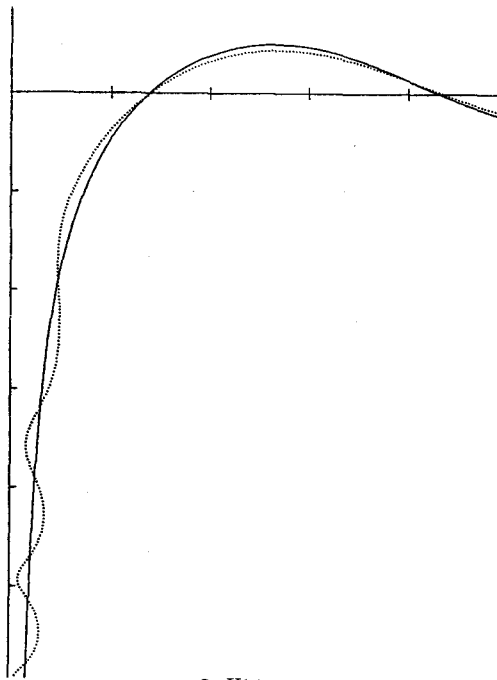
$$-\frac{1}{r} \frac{1-r^4}{(1+r^4)^{3/2}}, U(r), -\frac{1}{r}$$

Fig. 4



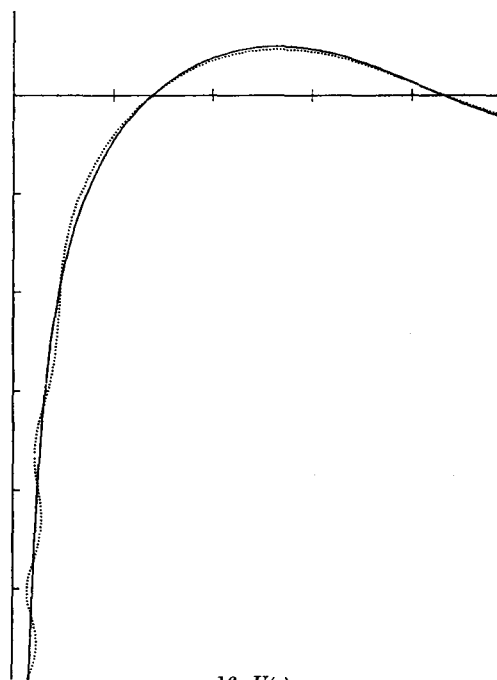
$$u_0 = -4, U(r)$$

Fig. 5a



$$u_0 = -8, U(r)$$

Fig. 5b



$$u_0 = -16, U(r)$$

Fig. 5c

which $|U(r)| < M_\delta$, but it can also be made to include points at which $|U(r)|$ is as large as desired.

2.10. By restricting attention to a wedge-shaped region formed by two lines through the origin and meeting with angle 2α , we realize the situation studied in [5, 6]. If the solution $U(r)$ exists, then the surface it defines meets the boundary walls of the corresponding cylindrical wedge in an angle $\gamma = \pi/2$; thus $\alpha + \gamma > \pi/2$, yet the solution surface is not bounded, in apparent conflict with the result stated in § 11, (ii) of [5].

2.11. The construction of §§ 2.1, 2.2 yields as corollary the nonuniqueness of the solution of the capillary problem, (2, 3) in Ω , when $\kappa < 0$. For example, in the case $\gamma = \pi/2$, the horizontal plane $u = 0$ yields one solution for any choice of Ω ; if Ω is the disk $r < r_a$, where r_a is the first maximum of $u(r)$, then $u(r)$ yields a second solution for this domain. Calculations indicating criteria for uniqueness of rotationally symmetric solutions, with $\kappa < 0$, are given in [4].

§ 3

If information is known on the boundary behavior of $u(\mathbf{x})$, then Theorems 1 and 2 can be sharpened. Write $\Sigma = \Sigma' + \Sigma^0$, Σ' being the set of points $\mathbf{x} \in \Sigma$ which lie interior to $(n-1)$ dimensional surface neighborhoods of class $C^{(1)}$. Consider a surface S defined by a solution $u(\mathbf{x})$ of (1) in Ω , such that $u(\mathbf{x}) \in C^{(1)}$ up to Σ' . The angle $\gamma = \gamma(\mathbf{x})$ between S and the bounding cylinder walls Z' over Σ' is then well defined. Denote by Z^0 the bounding cylinder walls over Σ^0 .

THEOREM 3. *Suppose A_1 is satisfied by $\mathcal{H}(\mathbf{x}; u)$ and suppose there is a lower hemisphere S_δ^n , lying partly (or entirely) over Ω , that does not meet Z^0 and that meets Z' (if at all) in angles γ_s satisfying $0 \leq \gamma_s \leq \gamma(\mathbf{x})$ at each contact point that projects onto $\mathbf{x} \in \Sigma'$. Letting B_δ^n be the projection of S_δ^n onto the hyperplane $u = 0$, there holds $u(\mathbf{x}) \leq M_\delta + \delta$ for all $\mathbf{x} \in B_\delta^n \cap \Omega$. If A_2 holds and if there is an upper hemisphere for which $\gamma(\mathbf{x}) \leq \gamma_s \leq \pi$, then $u(\mathbf{x}) \geq -M_\delta - \delta$ in $B_\delta^n \cap \Omega$.*

COROLLARY. *If for fixed $\delta > 0$, Ω can be covered by a family of such balls B_δ^n , then the indicated bounds hold uniformly in Ω .*

It suffices to prove the theorem for the case A_1 , as the other case is analogous. If $\gamma_s < \gamma(\mathbf{x})$ at all contact points, the proof is formally identical to that of Theorem 1; we need only note that because of the condition $\gamma_s < \gamma(\mathbf{x})$, none of the contact points corresponding to $u = \bar{u}_0$ can lie in Z . If we are given only $\gamma_s \leq \gamma(\mathbf{x})$, consider a concentric sphere S_δ^n , $0 < \delta' < \delta$, and the corresponding projection B_δ^n . One verifies easily that $\gamma_s < \gamma_s$ at any points of

contact that lie on a corresponding generator of Z' . Thus, the proof follows for $\mathbf{x} \in B_\delta^n \cap \Omega$ again as it did for Theorem 1; the result in the general case is obtained by letting $\delta' \rightarrow \delta$.

We also have:

THEOREM 4. *Suppose A_3 (resp. A_4) is satisfied by $\mathcal{H}(\mathbf{x}; u)$, and that Ω satisfies the hypothesis of the Corollary to Theorem 3. Then in each B_δ^n there is a point $\mathbf{x} \in B_\delta^n$ for which $u(\mathbf{x}) \geq -M_\delta$ (resp. $u(\mathbf{x}) \leq M_\delta$).*

3.1. We illustrate the above theorems by considering, as in [7, § 3.5], a wedge region \mathcal{W} with boundary Σ defined by

$$r = x \sec \alpha, \quad r^2 = x^2 + \sum_{j=1}^{m-1} y_j^2, \quad 2 \leq m \leq n. \tag{9}$$

Here Σ is smooth except at the $(n - m)$ dimensional "vertex" continuum $\Sigma^0: r = 0$. If $\gamma \geq (\pi/2) - \alpha$, $R/(1 + \sin \alpha) > \delta > R/2$, the spherical cylinder

$$S_\delta^m: (x - \delta)^2 + \sum_{j=1}^{m-1} y_j^2 + u^2 = (R - \delta)^2$$

will meet the walls of the cylinder Z' over $\Sigma' = \Sigma - \Sigma^0$ in an angle $\gamma_s < \gamma$, and will lie interior to the cylinder $r = R$.

Suppose condition A_1 is satisfied by $\mathcal{H}(\mathbf{x}; u)$. Let $u(\mathbf{x})$ be a solution of (1) in $\mathcal{W}^R = \mathcal{W} \cap \{r < R\}$, and suppose that on the part Σ' of the boundary of this domain, the solution surface meets Z' in an angle $\gamma(\mathbf{x}) \geq (\pi/2) - \alpha$. We shall show that $u(\mathbf{x})$ is bounded above as the vertex is approached in any way from within \mathcal{W} .

To do so, consider first the sphere, for arbitrary $\{b_j\}$,

$$\hat{S}_\delta^n: (x - \delta)^2 + \sum_{j=1}^{m-1} y_j^2 + \sum_{j=m}^n (y_j - b_j)^2 + u^2 = (R - \delta)^2$$

and its projection \hat{B}_δ^n on the base space $u = 0$. Clearly \hat{S}_δ^n again meets Z' in the angle $\gamma_s < \gamma$. Theorem 3 then yields immediately that $u(\mathbf{x}) < M_\delta + \delta$ in $W \cap \hat{B}_\delta^n$.

We now observe that the $\{b_j\}$ are arbitrary, and it follows that the same result holds in $W \cap B_\delta^n$, with B_δ^n the projection of \hat{S}_δ^n on the base space $y_j = 0$, $j \geq m$. Letting $\delta \rightarrow R/2$, we find that the bound holds uniformly in $B_{R/2}^n$ up to the vertex. Finally, we note that if $R^* = (2/(1 + \sqrt{5}))R$, then W^{R^*} can be covered by balls of radius $R/2$, each of which lies in the set $r \leq R$ and meets Σ' in an angle not exceeding γ ; thus the same method yields a uniform bound in \mathcal{W}^{R^*} . An analogous discussion holds under the condition A_2 .

We summarize the result:

Let $\mathcal{H}(\mathbf{x}; u)$ satisfy A_1 (resp. A_2). Let $u(\mathbf{x})$ be a solution of (1) in \mathcal{W}^R and suppose the solution surface meets the part Z' of the bounding walls in angles $\gamma(\mathbf{x})$ for which $\gamma \geq (\pi/2) - \alpha$ (resp. $\gamma \leq (\pi/2) + \alpha$). Then, without regard to the conditions on the remainder of the boundary, there holds $u(\mathbf{x}) < M_{R/2} + R/2$ (resp. $u(\mathbf{x}) > -M_{R/2} - R/2$) in \mathcal{W}^{R^*} , $R^* = (2/(1 + \sqrt{5}))R$.

In fact, a bound holds in any $\mathcal{W}^{R'}$ with $R' < R$. The value R^* was chosen because of the simple explicit nature of the estimate in this case.

3.2. Now suppose $\alpha + \gamma < (\pi/2) - \varepsilon_0$ for some $\varepsilon_0 > 0$ at all $x \in \Sigma'$. Let S_δ^n be a sphere of radius δ , with center on the line of symmetry at distance ρ from the vertex Σ^0 . If S_δ^n meets Z' in an angle $\leq \gamma_0 \leq \gamma$, there follows $\delta \leq \rho \sin \alpha \sec \gamma_0 < \rho$. Thus, no set of these spheres of fixed radius can cover all points in the corner. The method yields only the growth estimate $u(\mathbf{x}) \leq M_{\rho'} + \rho'$ (resp. $u(\mathbf{x}) \geq -M_{\rho'} - \rho'$) with $\rho' = \rho \sin \alpha \sec \gamma_0$, for points \mathbf{x} at distance $\geq \rho$ from Σ' , as $\rho \rightarrow 0$. We proceed to show in a particular case that this estimate is qualitatively the best that can be expected.

3.3. We consider, for $\kappa > 0$, a solution $u(x, \mathbf{y}) \equiv u(x, y_1, \dots, y_{n-1})$ of the equation $Nu = \kappa u$, defined in a region \mathcal{K}_R bounded between parts of the spherical surface $r^2 = x^2 + \sum_{j=1}^{n-1} y_j^2 = R^2$, the conical surface $\Sigma': r = x \sec \alpha$, $0 < \alpha < \pi/2$, $0 < x < R$, and the vertex $\Sigma^0: r = 0$. Let $\gamma_0 = \text{glb}_{\Sigma'} \gamma$. If $\gamma_0 \geq (\pi/2) - \alpha$, then $u(x, \mathbf{y})$ is bounded above near Σ^0 , by 3.1. Suppose $\gamma_0 < (\pi/2) - \alpha$, and set $k_0 = \sin \alpha \sec \gamma_0$. The function

$$\varphi(x, \mathbf{y}; \gamma_0) = \frac{n-1}{k_0 \kappa r} \left(\frac{x}{r} - t \right), \quad t = \sqrt{k_0^2 - 1 + (x/r)^2} \quad (10)$$

is then defined and positive in \mathcal{K}_R . We assert that for any $R' < R$, there is a constant C , depending only on the geometry and on R' (and not on the particular solution considered), such that

$$u(x, \mathbf{y}) < \varphi(x, \mathbf{y}; \gamma_0) + C \quad (11)$$

uniformly in \mathcal{K}_R .

The proof can be obtained, with minor modifications, from a similar result, given in our earlier paper [6] for the case $n = 2$. We present here an alternative, more geometrical proof, modeled on the considerations of § 1 of this paper.

We suppose first $\gamma_0 > 0$, and consider $\hat{\gamma}_0$ in the range $0 < \hat{\gamma}_0 < \gamma_0$, $\hat{\gamma}_0$ to be determined later. We introduce a function

$$f(\lambda) = \begin{cases} \lambda, & 0 \leq \lambda \leq M \\ 2M - \sqrt{2M^2 - \lambda^2}, & M \leq \lambda \leq \sqrt{2} M \end{cases}$$

and set

$$\varphi_M(x, \mathbf{y}; \hat{\gamma}_0) = f[\varphi(x, \mathbf{y}; \hat{\gamma}_0)].$$

Thus, φ_M is defined in the subdomain \mathcal{K}_R^M in which $\varphi \leq \sqrt{2}M$. We note that on the spherical cap $\varphi = \sqrt{2}M$ the normal derivative $(\partial/\partial\nu)\varphi_M = \infty$.

The calculation of $N\varphi_M$ in \mathcal{K}_R^M and of $\mathbf{T}\varphi_M \cdot \mathbf{v}$ on Σ' is facilitated by the observation that the level surfaces of φ_M are spheres that meet Σ' in the constant angle $\hat{\gamma}_0$. We find

$$N\varphi_M \leq \kappa\varphi_M + \eta(x, y), \quad |\eta(x, y)| < Cr^3 \tag{12}$$

$$\mathbf{T}\varphi_M \cdot \mathbf{v}]_{\Sigma'} = \cos \hat{\gamma}_0 + \mu(r), \quad -Cr^4 < \mu(r) \leq 0 \tag{13}$$

uniformly as r tends to its minimum, for all $\hat{\gamma}_0$ in the range considered, and for $M > M_0 > 0$.

We now choose $\hat{\gamma}_0 < \gamma_0$, and $r_0 < R$, so that for $r < r_0$ there holds $\cos \hat{\gamma}_0 + \mu(r) > \cos \gamma_0$. Clearly, by (13), it suffices to choose $\hat{\gamma}_0$ so that $\cos \hat{\gamma}_0 > \cos \gamma_0 + Cr_0^4$, for r_0 sufficiently small that this inequality is possible.

Now set $\omega_M = \varphi_M + C$, and choose C to be the smallest value for which $\omega_M \geq u$ in $\mathcal{K}_{r_0}^M$. For this choice of C , there must be at least one point $\mathbf{p} \in \mathcal{K}_{r_0}^M$ at which $u(\mathbf{p}) = \omega_M(\mathbf{p})$. The point \mathbf{p} cannot lie on the inner cap $\varphi = \sqrt{2}M$, since $(\partial/\partial\nu)\varphi_M = \infty$ on this cap; similarly, since $\mathbf{T}\varphi_M \cdot \mathbf{v} > \cos \gamma_0$, the surface φ_M meets the cylinder walls Z' over Σ' in an angle smaller than γ_0 ; thus \mathbf{p} cannot lie on Σ' unless it lies on the outer cap $r = r_0$.

If \mathbf{p} is an interior point of $\mathcal{K}_{r_0}^M$, then $u(\mathbf{p}) = \omega_M(\mathbf{p}) = \varphi_M(\mathbf{p}) + C$, and since at \mathbf{p} , the mean curvature of the surface $u(x, y)$ cannot exceed that of the surface $\omega_M(x, y)$, there holds $\kappa u(\mathbf{p}) \leq \kappa\varphi_M(\mathbf{p}) + \eta(\mathbf{p})$; thus, $\kappa C \leq \eta$ in this case.

If \mathbf{p} lies on the outer cap $r = r_0$, then from $u(\mathbf{p}) = \varphi_M(\mathbf{p}) + C = \varphi(\mathbf{p}) + C$ we find $C \leq \max_{r=r_0} [u(x, y) - \varphi(x, y)]$. Theorem 1 provides an a priori bound for $u(x, y)$ on the arc $r = r_0$, and a bound for φ on this arc is known explicitly.

In both events, C is bounded a priori independent of M , and we are free to let $M \rightarrow \infty$. This yields a bound of the form

$$u(x, y) \leq \varphi(x, y; \hat{\gamma}_0) + C \tag{14}$$

in any \mathcal{K}_R , and it remains only to investigate the transition $\hat{\gamma}_0 \rightarrow \gamma_0$.

Choose $r_0 < \min\{1, R\}$ and sufficiently small that $\hat{\gamma}_0$ can be chosen to satisfy $\cos \hat{\gamma}_0 > \cos \gamma_0 + Cr_0^4$, as above. If we choose $\hat{\gamma}_0$ sufficiently close to γ_0 that also $\cos \hat{\gamma}_0 < \cos \gamma_0 + 2Cr_0^4$ we note, using the explicit form of $\varphi(x, y; \gamma)$, that there is a constant C_0 such that

$$\varphi(x, y; \hat{\gamma}_0) < \varphi(x, y; \gamma_0) + C_0 r_0^4 / r \tag{15}$$

for all points (x, y) for which $r \leq r_0$, and with C_0 independent of r_0 in the range considered. In particular, in the range $r_0^2 \leq r \leq r_0$, there holds

$$\varphi(x, y; \hat{\gamma}_0) < \varphi(x, y; \gamma_0) + C_0 r_0^2$$

and hence by (14)

$$u(x, \mathbf{y}) \leq \varphi(x, \mathbf{y}; \hat{\gamma}_0) + C < \varphi(x, \mathbf{y}; \gamma_0) + C + C_0 r_0^2$$

in this range.

Using again (15), we may choose $\hat{\gamma}_1, \hat{\gamma}_0 \leq \hat{\gamma}_1 < \gamma_0$, such that for $r \leq r_0^2$ there holds

$$\varphi(x, \mathbf{y}; \hat{\gamma}_1) < \varphi(x, \mathbf{y}; \gamma_0) + \dot{C}_0 r_0^8 / r$$

so that in the range $r_0^4 \leq r \leq r_0^2$

$$\varphi(x, \mathbf{y}; \hat{\gamma}_1) < \varphi(x, \mathbf{y}; \gamma_0) + C_0 r_0^4$$

We note $(\partial/\partial(\cos \gamma))\varphi > 0$; thus $\varphi(x, \mathbf{y}; \hat{\gamma}_1) > \varphi(x, \mathbf{y}; \gamma_0)$ and it follows that on the sphere $r = r_0^2$

$$u(x, \mathbf{y}) - \varphi(x, \mathbf{y}; \hat{\gamma}_1) < C + C_0 r_0^2$$

Applying the above proof of (14) in $\mathcal{K}_{r_0^2}$, with $\hat{\gamma}_0$ replaced by $\hat{\gamma}_1$, we obtain

$$\begin{aligned} u(x, \mathbf{y}) &< \varphi(x, \mathbf{y}; \hat{\gamma}_1) + C + C_0 r_0^2 \\ &< \varphi(x, \mathbf{y}; \gamma_0) + C + C_0 (r_0^2 + r_0^4) \end{aligned}$$

in the range $r_0^4 < r < r_0^2$. Iteration of this procedure, with r_0^2 replaced successively by r_0^4, r_0^8, \dots , yields the estimate, for all $r \leq r_0$,

$$u(x, \mathbf{y}) < \varphi(x, \mathbf{y}; \gamma_0) + C + r_0^2 C_0 / (1 - r_0^2)$$

which completes the proof of (11) in the case $\gamma_0 \neq 0$.

Finally, suppose $\gamma_0 = 0$. In this case, $\varphi(x, \mathbf{y}; 0)$ satisfies the boundary condition exactly, that is, $\mu(r) \equiv 0$ in (13). We consider first an interior region \mathcal{K}'_R obtained by translating the cone Σ slightly along the axis of symmetry. On the new conical wall Σ' there holds $\gamma > \gamma_0 > 0$, hence the above proof can be repeated, yielding the stated result in \mathcal{K}'_R . Since $\mu(r) \equiv 0$, the estimate is in this case independent of the amount of shifting; thus we are free to let Σ' slide back to Σ , and the assertion follows again in \mathcal{K}_R .

3.4. We obtain from 3.3. a universal a priori bound above, for all solutions of (2) defined in \mathcal{K}_R . A corresponding bound holds of course from below, and is obtained from the given one under the transformation $u \rightarrow -u, \gamma \rightarrow \pi - \gamma$. Under this generality, little more can be said. However, if it is known that $\gamma \leq \gamma_1 < (\pi/2) - \alpha$ on Σ' , then there holds also

$$\varphi(x, \mathbf{y}; \gamma_1) - C < u(x, \mathbf{y})$$

uniformly in any $\mathcal{K}_{R'}$, $R' < R$, where again C depends only on the geometry and on R' . We do not at present have a geometrical proof of this fact, and we refer the reader to § 3.7, where it is obtained under weaker conditions as a consequence of a much more general result. Alternatively the analytical proof given in [6] for the case $n = 2, \gamma \equiv \gamma_1$, can be modified, using the results of § 1, to yield the assertion.

3.5. We collect the above results, together with others that are proved analogously, in a general statement. To do so, it is convenient to introduce a function

$$\Phi(x, y; \lambda) = \begin{cases} \varphi(x, y; \gamma) & \text{if } k^2 = \sin^2 \alpha \sec^2 \gamma < 1, \\ 0 & \text{if } k^2 \geq 1, \end{cases}$$

where $\varphi(x, y; \gamma)$ is defined by (10).

In terms of $\Phi(x, y; \gamma)$, we then have:

THEOREM 5. *There is a constant C , depending only on $\kappa, R, R' < R$ (and not on the particular solution $u(x, y)$), such that if $u(x, y)$ satisfies $Nu = \kappa u, \kappa > 0$, in \mathcal{K}_R and $\gamma_0 \leq \gamma(\mathbf{x}) \leq \gamma_1$ on Σ' , then*

$$\Phi(x, y; \gamma_1) - C \leq u(x, y) \leq \Phi(x, y; \gamma_0) + C \tag{16}$$

in \mathcal{K}_R .

We have immediately:

COROLLARY: *Under the above hypotheses, if $\gamma = \text{const.}$ on Σ' , then*

- (i) *if $\alpha \geq |(\pi/2) - \gamma|$, then $-C \leq u(x, y) \leq C$ in \mathcal{K}_R ;*
- (ii) *if $\alpha < |(\pi/2) - \gamma|$, then $\varphi(x, y; \gamma) - C \leq u(x, y) \leq \varphi(x, y; \gamma) + C$ in \mathcal{K}_R .*

Thus, if $\gamma \equiv \text{const.}$, the asymptotic behavior of $u(x, y)$ is characterized completely to within a (universal) additive constant. More precisely, all solutions for which $(\pi/2) - \alpha \leq \gamma(\mathbf{x}) \leq (\pi/2) + \alpha$ are bounded in \mathcal{K}_R , while if $\gamma_1 < (\pi/2) - \alpha$ or if $\gamma_0 > (\pi/2) + \alpha$, then all solutions are unbounded. Irrespective of boundary behavior, no solution in \mathcal{K}_R can grow faster in magnitude than $O(r^{-1})$ at the vertex.

3.6. Theorem 5 can be obtained alternatively, and under weaker smoothness hypotheses, as a consequence of a maximum principle, which is closely related to, but not equivalent to, the result of Theorem 6 in [7].

THEOREM 6. *Let $\Sigma = \Sigma^0 + \Sigma^\alpha + \Sigma^\beta$ be a decomposition of Σ , such that Σ^β is either a null set or of class $C^{(1)}$ and Σ^0 is small in the sense introduced in § 3.5 of [7]. Let $u(\mathbf{x}), v(\mathbf{x})$ be of class $C^{(2)}$ in Ω , and suppose*

- (i) *$Nu > Nv$ at all $\mathbf{x} \in \Omega$ for which $u - v > 0$.*
- (ii) *for any approach to Σ^α from within Ω , $\limsup [u - v] \leq 0$.*
- (iii) *on Σ^β , $Tu \cdot \mathbf{v} \leq Tv \cdot \mathbf{v}$ almost everywhere as a limit⁽¹⁾ from points of Ω .*

Then $u(\mathbf{x}) \leq v(\mathbf{x})$ in Ω .

(1) A somewhat weaker hypothesis suffices; cf. the remarks in the preceding note [9].

The proof is identical to the proof of Theorem 6 of [7] for the case $\Sigma^\alpha \not\subset \Sigma^0$, as in that proof no use was made of the set for which $u - v < 0$. We note that the conclusion of Theorem 6 of [7] for the case $\Sigma^\alpha \subset \Sigma^0$ can not be expected to hold in the present situation.

3.7. We are now in position to give an independent proof of Theorem 5, without smoothness conditions or bounds for $u(\mathbf{x})$ on Σ . We suppose only that $Tu \cdot \nu$ exists as a limit almost everywhere on Σ' . It suffices to prove the right-hand inequality, as the left inequality follows analogously.

We choose Σ^0 to be the vertex of \mathcal{K}_R , Σ^β the conical walls, and Σ^α the set $r = R' < R$ in \mathcal{K}_R .

Suppose first $0 < \gamma_0 < (\pi/2) - \alpha$. Since $\gamma_0 \leq \gamma(\mathbf{x})$, (13) implies that for any constant C the function $v(x, \mathbf{y}) = \varphi(x, \mathbf{y}; \hat{\gamma}_0) + C$ satisfies (iii) for sufficiently small R' , for any $\hat{\gamma}_0 < \gamma_0$. From (12) we see that C may be chosen so that, (a) $v(x, \mathbf{y}) \geq u(x, \mathbf{y})$ on Σ^α , and (b) $Nv \leq Nu = \kappa u$ at all points where $u > v$, uniformly as $\hat{\gamma}_0 \rightarrow \gamma_0$. Thus, (i) and (ii) hold, and since Σ^α can be covered by balls of radius bounded from zero, it follows from Theorem 3 that C can be chosen independent of the particular solution $u(x, \mathbf{y})$ considered. Since (iv) obviously holds in this situation, Theorem 6 yields $u(x, \mathbf{y}) \leq v(x, \mathbf{y})$ in \mathcal{K}_R . The transition $\hat{\gamma}_0 \rightarrow \gamma_0$ is effected as in § 3.3.

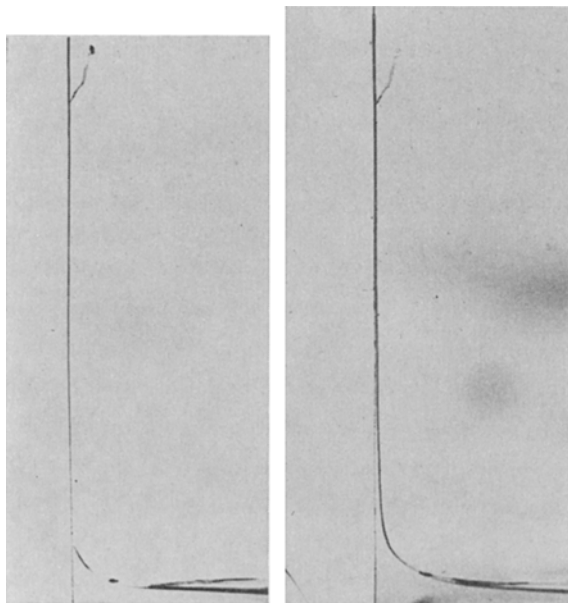
If $\gamma_0 = 0$, then (13) holds with $\mu(r) \equiv 0$ and one may choose $\hat{\gamma}_0 = \gamma_0$; no transition is then needed.

If $\gamma_0 > (\pi/2) + \alpha$, the transformation $u \rightarrow -u, \gamma \rightarrow \pi - \gamma$ reduces the problem to the previous case.

Finally, if it is known only that $(\pi/2) - \alpha \leq \gamma_0 \leq \gamma(\mathbf{x})$, then a uniform bound above for any solution in \mathcal{K}_R is obtained by the methods of § 3.1.

3.8. Theorem 5 was verified physically in an ad hoc experiment by Mr. Tim Coburn in the Medical School at Stanford University. Mr. Coburn constructed a wedge by machining one edge of a 1/2 inch thick rectangular piece of acrylic plastic 4" high, then placing the edge in contact with a face of a similar piece of the same plastic; the configuration was placed on a flat horizontal plastic surface, and the bottom of the wedge was then covered by a small amount of distilled water. The results corresponding to the half angles α of approximately 12° and 9° are shown in figure 6. Both photographs are on approximately the same scale, the scratch mark on each corresponding to a height at the vertex of about 7 cm.

We interpret the result with the aid of Theorem 3. Since for distilled water $\sigma = 78$ dynes/cm, we find $\kappa = \rho g / \sigma = 980 / 73 > 13$; hence, choosing $\delta = \sqrt{2} / 13$ and using the remark (ii) following the proof of Theorem 1, we obtain $|u(x)| < 0.8$ cm. whenever $\alpha + \gamma \geq \pi/2$. This is above the observed rise height for $\alpha = 12^\circ$ but less than 1/10 the height observed for

Fig. 6a. $\alpha = 12^\circ$.Fig. 6b. $\alpha = 9^\circ$.

$\alpha = 9^\circ$. (The liquid failed to rise the entire height of the plates in the latter case presumably because uniform contact between the two plates could not be maintained all the way to the top). We conclude, in particular, that the contact angle of distilled water with acrylic plastic (under the given conditions) has a value between approximately 78° and 81° .

3.9. We consider now the case $\kappa < 0$. The material of § 2 suggests that Theorem 5 cannot be expected to hold as stated in this situation. We show, however, that a weaker form of the result, suggesting the qualitative behavior when $\alpha < |(\pi/2) - \gamma|$, still holds when $\kappa < 0$.

THEOREM 7. *Let $u(x, y)$ satisfy $Nu = \kappa u$, $\kappa < 0$, in \mathcal{K}_R . If, on Σ' , $\gamma \geq \gamma_0 > (\pi/2) + \alpha$ (resp. $\gamma \leq \gamma_1 < (\pi/2) - \alpha$), then there is a sequence of points in \mathcal{K}_R tending to the vertex Σ^0 , along which $u(x, y) > \lambda/r$ (resp. $u(x, y) < -(\lambda/r)$), with*

$$\lambda > \frac{n-1}{\kappa} \frac{\sin \alpha + \cos \gamma_0}{(\sin \alpha)^{n-1}} \quad \left(\text{resp. } \lambda > \frac{n-1}{\kappa} \frac{\sin \alpha - \cos \gamma_1}{(\sin \alpha)^{n-1}} \right)$$

Proof. Suppose $\gamma \geq \gamma_0 > (\pi/2) + \alpha$. Consider the region Δ_r cut off from \mathcal{K}_R by the plane $x=r$. Integrating the equation over Δ_r and using the estimates $\mathbf{T}u \cdot \mathbf{v} > -1$ on the plane, $\mathbf{T}u \cdot \mathbf{v} = \cos \gamma \leq -\cos \gamma_0$ on Σ' , we find

$$\kappa \int_{\Delta_r} u(x, y) dy dx < w_{n-1} (r \sec \alpha)^{n-1} (\sin \alpha + \cos \gamma_0)$$

where ω_{n-1} is the volume of a unit $(n-1)$ -ball. If there were to hold $u(x, y) < \lambda/\rho$ for some constant λ when $x = \rho < r$, we would have

$$\kappa \int_{\Delta_r} u(x, y) dy dx > \kappa \lambda \int_0^r \frac{1}{\rho} (\rho \tan \alpha)^{n-1} \omega_{n-1} d\rho = \frac{\kappa \lambda}{n-1} (r \tan \alpha)^{n-1}$$

from which the result follows.⁽¹⁾ The other case is proved analogously.

3.10. We remark that the derivatives of $H(\mathbf{x})$ were at no time used for any of the results of the first part of our study [7]; thus all the results of that paper apply equally to any situation discussed in the present work, whenever the solution $u(\mathbf{x})$ is known to be bounded. In particular they apply whenever the hypotheses of the Corollary to Theorem 1 are satisfied, as these hypotheses yield an a priori bound on any possible solution in Ω . As an example, we note that if $u(\mathbf{x})$ is a solution of $Nu = \kappa u$, $\kappa > 0$, in a star-shaped domain Ω with smooth boundary Σ , then, in the notation of [7], (10) of [7] implies

$$\frac{\kappa}{n} \bar{u} = \widehat{\cos \gamma} \bar{H}^\Sigma.$$

3.11. The criterion in the corollary to Theorem 5 relates closely to recent work of Emmer [8] on the existence of solutions of a variational problem associated with (2, 3). For a domain \mathcal{K}_R , Emmer's criterion $|\nu| \sqrt{1+L^2} < 1$, when applied to the boundary at the vertex, is equivalent to the condition $\alpha > |(\pi/2) - \gamma|$; thus, his criterion is almost identical to the one ensuring boundedness of all solutions in \mathcal{K}_R . If $n = 2$ and $\alpha < |(\pi/2) - \gamma|$, Theorem 5 shows there is no solution in the regularity class considered by Emmer. M. Miranda has pointed out that the variational expression admits no finite lower bound in this case.

It should be noted, however that if $\kappa > 0$, solutions of (2, 3) regular in Ω and in the class $\mathcal{L}_1^{\text{loc}}(\Sigma^\beta)$ can be shown to exist in (essentially) the class of domains considered in § 3.6, whenever $\Sigma^\alpha = \phi$. This follows from the method of Emmer in conjunction with the results of § 1 and general a-priori interior estimates for the derivatives of the solution, cf. [3].

We remark that Emmer's condition $\kappa > 0$ is necessary, even for smooth boundaries; this follows from Corollary 3.1 in [7].

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⁽¹⁾ A somewhat better estimate for λ could have been obtained by using more carefully the bound $-u(x, y) < \lambda/r$. In view of the apparently tentative nature of the result, this improvement does not seem to us at this time to be of great value.

fortable and pleasing milieu afforded us privacy and anonymity that were important for our efforts. We enjoyed also the concomitant privilege, during moments of relaxation, of observing the many fascinating and changing vignettes of everyday life that passed before us in the lobby of what is certainly one of the world's more splendid hotels.

Notes added in proof:

1. The surmised (§ 2.7) existence of the singular solution $U(r)$ is now proved; details will appear in a work by these authors, now in preparation.
2. We have been informed that the solution $U(r)$ was encountered independently in a computational study by C. Huh (Capillary Hydrodynamics... Dissertation, Dept. of Chem. Eng., Univ. of Minnesota, 1969).
3. We note a particular choice for the two constants C in (16). Define by \mathcal{K}^j , $j=0, 1$, the part of \mathcal{K}_R between the vertex Σ^0 and the outer cap of a sphere, of radius $\delta_j = \min\{\delta, \delta \sin \alpha |\sec \gamma_j|\}$, centered on the axis at distance $\delta < R/2$ from Σ^0 . We may then choose $C = n(\kappa\delta_j)^{-1} + \varepsilon_j(\delta)$ in \mathcal{K}^j on (respectively) the right and the left of (16). Here $\varepsilon_j(\delta) \rightarrow 0$ with δ , $\varepsilon_j \equiv \delta_j$ if $\sin \alpha |\sec \gamma_j| \geq 1$.

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