BERNOULLI MEASURE ALGEBRAS

BY

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1. Introduction

We make a detailed study of certain L-subalgebras of the algebra $M(\mathbf{T})$ of bounded regular Borel measures on the circle. These algebras constitute perhaps the simplest case in which one can investigate the interplay between the convolution measure algebra structure of $M(\mathbf{T})$ and the arithmetic of the underlying group $\mathbf{T}$. Of course there is nothing new in considering $M(\mathbf{T})$, more generally $M(G)$ for any locally compact abelian group G, as a Banach algebra and as an L-space, but the study of the blend of these structures (i.e. the *convolution measure algebra* approach) has gained considerable impetus in the last few years. In particular J. L. Taylor has, in a brilliant sequence of papers [19], [20], [21], [22], [23], [24], [25], located the "good" subalgebras of $M(G)$ (crudely those with group maximal ideal spaces) in terms of the so called critical points of the maximal ideal space $\Delta(G)$ of $M(G)$. However, for non-discrete G, the residual structure of $\Delta(G)$ is largely unexplored and a cardinal objective of work of the present kind is to obtain more information in this area.

The work of Yu. A. Šreider in [17] leads to a description of the elements of $\Delta(G)$ as *generalized characters* (see § 2). In particular $M(G)$ can be exhibited as an inductive limit of certain single generator L-subalgebras, then, by duality $\Delta(G)$ appears as a projective limit of simpler maximal ideal spaces. Moreover Taylor shows in [19], that, given any convolution measure algebra N (e.g. a single generator L-subalgebra of $M(G)$), there exists a compact abelian jointly continuous semigroup $\Sigma(N)$, the *structure semigroup* of N, such that N is embedded as a weak $*$ dense L-subalgebra of the measure algebra $M(\Sigma(N))$ and the complex homomorphisms of N are induced by the continuous semicharacters of $\Sigma(N)$. So far these general tools have had little impact on the discussion of the fine arithmetical structure of $\Delta(G)$, indeed Taylor exclaims in [25] that "generalized characters are clearly impossible to understand," thus we believe that the time is appropriate for specific local studies.

Of course there is no difficulty in describing the maximal ideal spaces of the subalgebras which we have dubbed "good". Alongside these good algebras we should consider the L-subalgebras introduced by R. Arens and I. Singer in [1] because their maximal ideal spaces are straightforward to find (loc. cit.) and a good deal is known about their structure semigroups (see [19]). Apart from these cases we know of no previous descriptions of structure semigroups of L-subalgebras and the few successful descriptions of particular generalized characters have depended on the study of local behaviour on either (a) measures whose support set is independent (so that arithmetic is legislated out) or (b) measures with strong arithemetical properties on special groups (so that the arithmetic is forced to be amenable). The prototype for all this is another Šreider paper, [18]. Under (a) the basic source is [11] by E. Hewitt and S. Kakutani. This was extended by A. B. Simon, [16], and his arguments simplified in [4]. As far as we know that is the only case (additional to the examples mentioned above) where the maximal ideal space of an L-subalgebra has already been described. This paper is firmly under the heading (b) and we follow another Hewitt-Kakutani paper, [12], its elaboration by R. Kaufman, [14], and the work of B. E. Johnson in [13].

In fact by a *Bernoulli measure algebra* we mean an L-subalgebra of $M(G)$ generated by a probability measure μ which is itself an infinite product of discrete probability measures ($\mu = \ast_{n=1}^{\infty}\delta_n$, the limit being in the weak $\ast$ topology). Some of our techniques remain valid in that generality (cf. [7]) but in the present paper we restrict severely the class under discussion. In the first place we impose a strong arithmetical constraint by demanding that $G = \mathbf{T}$ and that each δ_n has two point support $\{0, d_n\}$ where d_n/d_{n+1} is an integer (which may vary with n)—in familiar terminology μ is generated by a Cantor dissection process with varying ratio of dissection a reciprocal integer. The constraint that d_n/d_{n+1} be integral admits reformulation for other choices of G, but, in all cases, the absence of such a condition ensures that many of the questions answered explicitly here become number-theoretic problems of great difficulty. We impose also the less essential restriction that the mass distribution is uniform (i.e. $\delta_n = \frac{1}{2}\delta(0) + \frac{1}{2}\delta(d_n)$) to define the class $\mathbf{B}$ of measures μ which will be discussed. Our methods and results admit adaptation to the case where $\delta_n = p\,\delta(0) + (1-p)\delta(d_n)$, with $0 < p < 1$, $p \neq \frac{1}{2}$, p independent of n. But in the case where $p = p(n)$ some additional subtleties arise (cf. [2], [15]).

The second section is devoted to fixing terminology and obtaining preliminary results. In § 3 we discuss the $\sigma(L^\infty(\mu), L^1(\mu))$ closure of the continuous characters for μ in $\mathbf{B}$. This is a prerequisite for the description of the maximal ideal spaces and Šilov boundaries of the Bernoulli measure algebras in §§ 4, 5. Naturally we discuss also the restriction of $\Delta(\mathbf{T})$, and of $\partial M(\mathbf{T})$ the Šilov boundary of $M(\mathbf{T})$, i.e. we discuss the extension problem for the local algebras involved. The final section is given over to the identification of the structure semigroups which arise.

We omit the study of the involution closed L-subalgebra generated by a measure μ which belongs to **B**. Although this can be partly simplified by the observation that, in this case, $\tilde{\mu}$ is a translate of μ, an adequate discussion would take too long.

2. Notation and preliminary definitions

T denotes the circle group realised as $\mathbf{R}/\mathbf{Z}$ and $M(\mathbf{T})$ the algebra of bounded regular Borel measures on **T** with convolution multiplication. A subalgebra N of $M(\mathbf{T})$ closed with respect to the total variation norm is an L-subalgebra if $\mu \in N$ whenever $\mu \in M(\mathbf{T})$, $\nu \in N$, and $\mu \ll \nu$ (μ is absolutely continuous with respect to ν.)

By a *generalized character* of an L-subalgebra N of $M(\mathbf{T})$ we shall mean an element $\chi = (\chi_\mu)_{\mu \in N} \in \prod_{\mu \in N} L^\infty(\mu)$ which satisfies:

GC(i) if $\mu \ll \nu$, then $\chi_\mu = \chi_\nu \qquad (\mu \text{ a.e.})$,

GC(ii) $\chi_{\mu * \nu}(x+y) = \chi_\mu(x)\chi_\nu(y) \quad (\mu \times \nu \text{ a.e.})$,

GC(iii) $\sup \{\|\chi_\mu\|_\infty : \mu \in N\} > 0$.

The third condition is imposed to exclude the trivial element of $\prod L^\infty(\mu)$. Note that condition (ii) is different from the corresponding condition in Šreider's original formulation where he was at pains to pursue the formal analogy with the continuous characters of **T**. When the algebra N contains the identity $\delta(0)$ of $M(\mathbf{T})$ the supremum in (iii) is equal to 1. In general the supremum is not greater than 1.

Every generalized character χ of N gives rise to a complex homomorphism of N according to the formula

$$\mu \mapsto \int \chi_\mu d\mu \, (= \mu\hat{\ } (\chi) = \chi(\mu)), \quad (\mu \in N),$$

and in this way the maximal ideal space $\Delta(N)$ of N can be realised as the set of all generalized characters of N with the topology induced from the $\sigma(L^\infty(\mu), L^1(\mu))$-topology on each factor in the product space.

For any continuous $\mu \in M(\mathbf{T})$, we write $N(\mu)$ to indicate the L-subalgebra of $M(\mathbf{T})$ generated by μ, and write $S(\mu)$ for the space

$$\{\chi_\mu : \chi \in \Delta(N(\mu))\},$$

with the $\sigma(L^\infty(\mu), L^1(\mu))$-topology. It will also be convenient to denote by $N^0(\mu)$ the result of adjoining the identity $\delta(0)$ to $N(\mu)$. Then, of course, one has

$$\{\chi_\mu : \chi \in \Delta(N^0(\mu))\} = S(\mu) \cup \{0\},$$

where $\mathbf{0}$, the null function in $L^\infty(\mu)$, arises from that generalized character which is trivial on (powers of) μ but 1 on $\delta(0)$. The first step in providing a description of the maximal ideal space of $N(\mu)$ is given by the following observation:

(2.1) LOCALIZATION LEMMA. *The maximal ideal space $\Delta(N^0(\mu))$ of the convolution measure algebra $N^0(\mu)$ is homeomorphic to $S(\mu)$.*

Proof. Let χ be a generalized character of $N^0(\mu)$. Property (ii) of generalized characters implies that χ_μ uniquely determines χ_{μ^n} $(n=1, 2, 3, \ldots)$. However every member of $N^0(\mu)$ is absolutely continuous with respect to $\delta(0)+\sum_{n=1}^\infty 2^{-n}\|\mu\|^{-n}|\mu|^n$, and $\chi_{\delta(0)}(0)=1$. Using GC(i) we see that χ_μ uniquely determines $\chi \in \Delta(N^0(\mu))$. It follows that the restriction map $\chi \mapsto \chi_\mu\colon \Delta(N^0(\mu)) \to S(\mu) \cup \{0\}$ is a bijection and that the union on the right hand side is disjoint. Since the Gelfand topology is induced by the $\sigma(L^\infty(\nu), L^1(\nu))$-topology on each factor of $\prod_{\nu\in\Delta(N^0(\mu))} L^\infty(\nu)$, this map is continuous, hence, by compactness of $\Delta(N^0(\mu))$ it is a homomorphism to $S(\mu) \cup \{0\} (\subseteq L^\infty(\mu))$. The trivial homomorphism maps to 0 and the result follows.

Observe that if, say μ^n and μ^m fail to be concentrated on disjoint sets, this imposes a condition on membership of $S(\mu)$. Even in the absence of all conditions of this kind the requirement that e.g. $(s+t) \mapsto \chi_\mu(s)\chi_\mu(t)$ is well—defined for almost all s, t in support (μ) and almost all $s+t$ in support (μ^2) is a non-trivial constraint on $S(\mu)$. This is a point where arithmetical considerations are crucial.

Where it is possible to restrict attention to constant functions there is, of course, no such problem of definition and it is easy to work direct from GC (i), (ii). In particular a necessary and sufficient condition for μ to have independent powers is, that $S(\mu)$ contains *some* constant (function) lying strictly between $\mathbf{0}$ and $\mathbf{1}$, or equivalently, that *all* constants of modulus less than or equal to 1 belong to $S(\mu)$.

It is clear that the generalized characters of an L-subalgebra N of $M(\mathbf{T})$ form a semigroup with multiplication defined by

$$(\chi\cdot\psi)_\nu = \chi_\nu\cdot\psi_\nu \ \ (\nu \text{ a.e.}) \ \ (\forall \nu \in N),$$

where the product on the right hand side of this equation is pointwise multiplication of $L^\infty(\nu)$ functions. Of course the homeomorphism between $\Delta(N(\mu))$ and $S(\mu)$ induces a semigroup isomorphism, where $S(\mu)$ is regarded as a subsemigroup of $L^\infty(\mu)$. However it is important to note that, in general, multiplication in $S(\mu)$ is not jointly continuous.

We will denote the Šilov boundary of an L-subalgebra N by $\partial(N)$ and make use of the following result. (See [4].)

(2.2) EXTENSION THEOREM. *If* $\chi \in \Delta(N)$ *satisfies* $|\chi_\nu| = 1$ (ν a.e.) *for all* $\nu \in N$, *then* $\chi \in \partial(N)$ *and hence there exists* $\psi \in \Delta(M(\mathbf{T}))$ *such that* $\psi_\nu = \chi_\nu$ (ν a.e.) *for all* $\nu \in N$.

In the particular case when $N = N(\mu)$, it is sufficient to check only that $|\chi_\mu| = 1$ (μ a.e.).

We have already indicated our intention to focus on Bernoulli measure algebras, $N(\mu)$, where μ belongs to a restricted class $\mathbf{B}$ of Bernoulli convolutions. In fact let (a_n) be a sequence of integers greater than one, write $p_n = \prod_{r=1}^{n} a_r$ for the nth partial product and write also $d_n = (p_n)^{-1}$. The countable subgroup D of $\mathbf{T}$ generated by $\{d_n : n = 1, 2, 3, \dots\}$ will play an important role in what follows. The class $\mathbf{B}$ comprises all measures of the form

$$\mu = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}\,(\delta(0) + \delta(d_n)),$$

where $\delta(x)$ denotes the probability atom at x. The infinite convolution product converges in consequence of Kolmogorov's Three Series Theorem, [10], and all such measures are continous.

It will soon become apparent that the measures μ in $\mathbf{B}$ should be classified into two types. If $\sup_n a_n = \infty$, we shall say that the measure is *fine*. In the contrary case we shall say that the measure is *coarse*. These epithets are suggested by the measure of thinness of a set provided by Hausdorff dimension. In fact the support of a coarse measure has positive Hausdorff dimension, whereas the support of a fine measure has zero Hausdorff dimension. Of course the support of μ belonging to $\mathbf{B}$ is given explicitly as

$$\left\{ \sum_{n=1}^{\infty} \varepsilon_n d_n : \varepsilon_n = 0 \text{ or } 1 \quad (n = 1, 2, 3 \dots) \right\}.$$

Observe that this set will have zero Lebesgue measure provided infinitely many a_n's are not equal to 2. In this case, of course, μ is a singular measure. On the other hand, if all but finitely many of the a_n's equal 2, then μ is a finite sum of translates of restrictions of Lebesgue measure to subintervals of $\mathbf{T}$. In this latter case the Bernoulli measure algebra, $N(\mu)$, is precisely $L^1(\mathbf{T})$ whose Gelfand space is well understood. For this reason we shall usually concentrate on the class $\mathbf{B}'$ comprising the singular measures in $\mathbf{B}$.

A glance at the definition of μ indicates that we should also study the L-subalgebra $A(\mu)$ generated by $N(\mu)$ and the atoms at points of D. We write $A_c(\mu)$ for the L-ideal of continuous measures in $A(\mu)$ and $A_d(\mu)$ for the L-subalgebra of discrete measures in $A(\mu)$. Thus

$$A(\mu) \cong A_c(\mu) \oplus A_d(\mu).$$

We close this section with some obvious statements about the mass distribution of μ in $\mathbf{B}$, the objective being to establish some useful notation.

We write, for $m, k \in \mathbf{N}$,

$$\mu_{m,k} = \mathop{\ast}_{n=m+1}^{k} \tfrac{1}{2}(\delta(0) + \delta(d_n)),$$

$$\mu_m = \mathop{\ast}_{n=m+1}^{\infty} \tfrac{1}{2}(\delta(0) + \delta(d_n)),$$

and
$$D_m = \left\{ \sum_{i=1}^{m} \varepsilon_i d_i : \varepsilon_i = 0 \text{ or } 1 \right\}.$$

Denote the characteristic function of the interval $[x, x + d_m)$ by $c_m(x)$. Then we have

$$\mu = 2^{-m}\Big(\sum_{d \in D_m} \delta(d) \ast \mu_m \Big) = \sum_{d \in D_m} c_m(d) \cdot \mu,$$

and, for each $d \in D_m$, $(m = 1, 2, 3, \dots)$,

$$c_m(d) \cdot \mu = 2^{-m}(\delta(d) \ast \mu_m).$$

This shows, in particular, that the space $L^1(\mu)$ is spanned by the measures

$$\{\delta(d) \ast \mu_m : d \in D_m;\ m = 1, 2, 3, \dots\}.$$

We write also
$$D_{n,m} = \left\{ \sum_{i=n+1}^{m} \varepsilon_i d_i : \varepsilon_i = 0 \text{ or } 1 \right\}$$

so that $L^1(\mu_n)$ is spanned by the measures

$$\{\delta(d) \ast \mu_m : d \in D_{n,m};\ m = n+1, n+2, \dots\}.$$

3. Constants in the closure of the characters

As a preliminary to describing the maximal ideal space of $N(\mu)$ we obtain some information about those constants which belong to the $\sigma(L^\infty(\mu), L^1(\mu))$-closure of (the restriction to $L^\infty(\mu)$ of) the continuous characters of $\mathbf{T}$. However Theorem (3.2) is of interest in its own right. In fact Hewitt and Kakutani, [12], proved that, for every $\mu \in \mathbf{B}$ satisfying $\sum_{n=1}^\infty 1/a_n < \infty$, the closure of the continuous characters contains the entire unit disc. (Actually these authors placed the measures on the real line but their proof gives the (formally stronger) result for the circle.) In (3.2) we obtain the same conclusion under the much weaker hypothesis that μ is fine. This is to be regarded as a generalization of Šreider's construction of "an unusual generalized character" in [18]. We note that Kaufman has extended the Hewitt-Kakutani result in a different direction. In [14] he considers classes of measures $\mu_\varepsilon = \ast_{n=1}^\infty (\tfrac{1}{2}\delta(0) + \tfrac{1}{2}\delta(\varepsilon_n b_n))$, where the only restriction on the b_n is that they be positive rationals such that $\sum_{n=1}^\infty b_n < \infty$ and $\varepsilon = (\varepsilon_n)$ is a sequence of 0's and 1's regarded as an element of the Cantor set $\mathbf{D}(2)=\Pi\mathbf{Z}(2)$. For a fixed choice of b_n's, μ_ε has the Hewitt-Kakutani property for all ε in a dense G_δ of $\mathbf{D}(2)$. Observe that the auxiliary sequence ε is a device for introducing lacunarity so that the strength of that result lies in the lack of arithmetical constraint on the b_n's.

Using these ideas we have shown in [12] that the Hewitt-Kakutani property holds for virtually all Bernoulli convolutions in the sense of Baire category. To be precise it follows from results in [12] that for all but a first category subset of $\{\mathfrak{b}: b_n \geqslant 0, \sum_{n=1}^{\infty} b_n \leqslant 1\} \subseteq [0, 1]^{\aleph_0}$ (the b_n are *not* supposed rational) the $\sigma(L^\infty(\mu_\mathfrak{b}), L^1(\mu_\mathfrak{b}))$-closure of the continuous characters contains all constants in the unit disc, where $\mu_\mathfrak{b} = \ast_{n=1}^{\infty} \frac{1}{2}(\delta(0)+\delta(b_n))$.

We make use of a simple but powerful criterion due to Johnson [13]. (It would also be possible to employ Theorem 3.1 of [12].)

(3.1) LEMMA. *Let $\mu \in \mathbf{B}$. Then $z^{n(k)} \to \zeta$ in the $\sigma(L^\infty(\mu), L^1(\mu))$-topology if and only if $\hat{\mu}(n(k)) \to \zeta$ and $z^{n(k)} \to 1$ pointwise on D. (ζ is the constant function in $L^\infty(\mu)$ with value ζ.)*

Proof. The linear span of the $c_m(d)$, $d \in D_m$, $m=1, 2, \dots$ is dense in $L^1(\mu)$ so that $z^{n(k)} \to \zeta$ in the $\sigma(L^\infty(\mu), L^1(\mu))$—topology if and only if

$$z^{n(k)}(d)\int c_m(0)\, z^{n(k)}\, d\mu = \int c_m(d)\, z^{n(k)}\, d\mu \to \int \zeta c_m(d)\, d\mu = \zeta \int c_m(0)\, d\mu,$$

for each $d \in D_m$, $m=1, 2, \dots$. But

$$\hat{\mu}(n(k)) = \sum_{d \in D_m} z^{n(k)}(d) \int c_m(0)\, z^{n(k)}\, d\mu,$$

so that the result follows.

It is now possible to state and prove.

(3.2) THEOREM. *Let $\mu \in \mathbf{B}$. Every complex constant of modulus not greater than 1 belongs to the $\sigma(L^\infty(\mu), L^1(\mu))$-closure of the continuous characters of $\mathbf{T}$ if and only if μ is fine.*

Proof of necessity. Suppose μ is coarse, so that $a=\sup_n a_n$ is finite. For every integer k,

$$\hat{\mu}(k) = \prod_{n=1}^{\infty} \left(\tfrac{1}{2} + \tfrac{1}{2} \exp(2\pi i k d_n)\right),$$

so that

$$|\hat{\mu}(k)| \leqslant |\cos(\pi k d_n)|, \quad n=1, 2, \dots$$

Since the sequence (p_n) is monotonic increasing and unbounded, there exists, for large enough $|k|$, a positive integer n such that $p_{n-2} \leqslant |k| < p_{n-1}$. Hence $(a_n a_{n-1})^{-1} \leqslant |k| d_n < a_n^{-1} \leqslant \frac{1}{2}$. Now $|\hat{\mu}(k)| \leqslant \cos(k\pi d_n) \leqslant \cos(\pi/a^2)$. Thus $\overline{\lim}_{|k|\to\infty} |\hat{\mu}(k)| < 1$.

Proof of sufficiency. Having dispensed with the easy implication we give the main proof in three stages. First we construct an auxiliary measure ν on $\mathbf{R}$, then we show that certain values of the Fourier-Stieltjes transform of ν are admissible constants, and finally we check that enough constants arise in this way. The reader will observe that under the relatively mild hypothesis that $\liminf a_n = \infty$ this proof can be drastically shortened. It should therefore be useful to bear in mind a "hard" case e.g. where the sequence (a_n) comprises long strings of 2's interrupted by the subsequence 3, 4, 5, ...

Choose a sequence $(n(j))$ such that

$$a_{n(j)} \geqslant j, \quad j = 1, 2, 3, \ldots \tag{1}$$

Define $\nu_j = \ast_{n=n(j)}^{n(j+1)-1} \frac{1}{2}(\delta(0) + \delta(d_n/d_{n(j)}))$, as a measure on $\mathbf{R}$—in fact ν_j is a positive measure on $[0, 2]$ with $\|\nu_j\| = 1$. It follows that there exists a subsequence $(\nu_{j(k)})$ which has a $\sigma(M(\mathbf{R}), C(\mathbf{R}))$-limit ν, say. Without loss of generality we assume that $\nu_j \to \nu$, and we clearly have $\|\nu\| = 1 = \nu[0, 2]$. Also ν is certainly not the point measure $\delta(0)$.

Let θ be any real number in $]0, \frac{1}{2}[$. In view of (1), it is possible to find a sequence (θ_j) of rationals in $]0, \frac{1}{2}[$ such that $\theta_j \to \theta$ and

$$p_{n(j)}\theta_j \text{ is an integer divisible by } p_{n(j)-1} \tag{2}$$

(e.g. make the eventual choice $p_{n(j)}\theta_j = [\theta a_{n(j)}] p_{n(j)-1}$, where [ ] denotes integer part). It is an easy deduction from (2) that

$$z^{p_{n(j)}\theta_j} \to 1 \text{ pointwise on } D, \quad \text{as } j \to \infty. \tag{3}$$

We now seek to prove that

$$|\hat{\nu}_j(\theta_j) - \hat{\mu}(p_{n(j)}\theta_j)| \to 0, \quad \text{as } j \to \infty. \tag{4}$$

To cut down the formulae let us write $w_n = \theta_j p_{n(j)} d_{n-1+n(j+1)}$, $n = 1, 2, \ldots$, so that $w_n < 2^{-n}(j+1)^{-1}$ and the quantity to be estimated becomes

$$\left|1 - \prod_{n=1}^{\infty} \tfrac{1}{2}(1 + \exp(2\pi i w_n))\right| \leqslant \sum_{n=1}^{\infty} |1 - \tfrac{1}{2}(1 - \exp(2\pi i w_n))| = \sum_{n=1}^{\infty} \sin \pi w_n < \pi/(j+1).$$

Thus (4) has been established. Now since $\theta_j \to \theta$, $\exp(2\pi i \theta_j x)$ converges uniformly on $[0, 2]$ to $\exp(2\pi i \theta x)$. It follows that

$$\begin{aligned}|\hat{\nu}_j(\theta_j) - \hat{\nu}(\theta)| &\leqslant |\hat{\nu}_j(\theta_j) - \hat{\nu}_j(\theta)| + |\hat{\nu}_j(\theta) - \hat{\nu}(\theta)| \\ &\leqslant \sup_{0 \leqslant x \leqslant 2} |\exp(2\pi i \theta_j x) - \exp(2\pi i \theta x)| + |\hat{\nu}_j(\theta) - \hat{\nu}(\theta)|,\end{aligned}$$

and this tends to zero as $j \to \infty$. Combining the last statement with (4), we deduce that $\hat{\mu}(p_{n(j)}\theta_j) \to \hat{\nu}(\theta)$, and using (3) and Lemma (3.1) we see that

$$z^{p_{n(j)}\theta_j} \to \zeta \text{ (where } \zeta = \hat{\nu}(\theta)) \quad \text{in the } \sigma(L^\infty(\mu), L^1(\mu)) -$$

topology.

The constants in the closure of the continuous characters evidently form a closed sub-semigroup of the unit disc (under the usual multiplication for $\mathbf{C}$). Let us call it $C(\mu)$. We have just shown that $\hat{\nu}(\theta) \in C(\mu)$ for all $0 < \theta < \frac{1}{2}$. Now for any positive real α,

$$\begin{aligned} \hat{\nu}(\alpha/n)^n &= \left(\int \exp(2\pi i \alpha x/n)\, d\nu(x)\right)^n \\ &= (1 + (2\pi i \alpha/n)\int x d\,\nu(x) + O(n^{-2}))^n \\ &= (1 + (2\pi i \alpha E(\nu)/n + O(n^{-2}))^n. \end{aligned}$$

Since ν is a positive measure with compact support in $\mathbf{R}^+$, we see that $\lim_{n\to\infty} \hat{\nu}(\alpha/n)^n = \exp(2\pi i \alpha E(\nu))$ and that $C(\mu)$ contains the unit circle. Since $\nu \neq \delta(0)$, there is $\theta \in]0, \frac{1}{2}[$ with $|\hat{\nu}(\theta)| = r < 1$. $\hat{\nu}(0) = 1$, so by the continuity of $\hat{\nu}$ and the rotational invariance already established, we see that $C(\mu)$ contains the annulus $\{\zeta: r \leqslant |\zeta| \leqslant 1\}$. Since $C(\mu)$ is a multiplicative semigroup this implies that $C(\mu)$ is the whole disc and the proof of the theorem is complete.

Although Theorem (3.2) gives only negative information for coarse measures it is the case that $C(\mu)$ is never trivial for μ in $\mathbf{B}'$ and this is important for the discussion of the maximal ideal spaces in the next section. For that reason we quote the next two results from [5] with brief proofs.

(3.3) PROPOSITION. *For every μ in $\mathbf{B}'$, $C(\mu)$ contains a constant with modulus strictly between* 0 *and* 1. *In particular μ in $\mathbf{B}$ is singular if and only if μ belongs to $\mathbf{B}'$.*

Proof. Suppose that $\mu = \mathbf{B}'$ so that (a_n) contains a subsequence $(a_{n(m)})$ which contains no 2's. Then

$$\begin{aligned} |\hat{\mu}(p_{n(m)-1})| &\geqslant \cos(\pi/a_{n(m)}) \cos(\pi/2a_{n(m)}) \dots \cos(\pi/2^r a_{n(m)}) \dots \\ &\geqslant \prod_{r=1}^{\infty} \cos(2\pi/3\cdot 2^r) = 3\sqrt{3}/4\pi. \end{aligned}$$

Since $z^{p_n} \to 1$ pointwise on D the first assertion follows from (3.1). (In view of (3.2) we need this only in the coarse case although the above argument is more generally valid). The second assertion is immediate via the Riemann-Lebesgue lemma.

(3.4) PROPOSITION. *Suppose μ belongs to $\mathbf{B}'$, x belongs to $\mathbf{T}$, and m, n are distinct integers, then*

$$\delta(x) * \mu^m \perp \mu^n.$$

Proof. Since the $\sigma(L^\infty(\mu), L^1(\mu))$-topology is consistent with the compact Gelfand topology of $\Delta(M(\mathbf{T}))$, Proposition (3.3) guarantees the existence of a generalized character $\chi \in \Delta(M(\mathbf{T}))$ such that χ_μ is a non-zero constant with modulus strictly less than one. Now apply defining property (ii) of generalized characters.

(3.5) *Examples*. We are unable to make any more general statement concerning $C(\mu)$ when μ is coarse and singular than that $C(\mu)$ is a non-trivial proper subsemigroup of the unit disc. However for any measure μ in $\mathbf{B}$ and integer m, the argument of $\hat{\mu}(m)$ is congruent modulo 2π to the sum $m\pi \sum_{n=1}^\infty d_n = 2\pi m E(\mu)$. Therefore if $E(\mu)$ is rational the semigroup $C(\mu)$ can have elements on only a finite number of radii. (In view of (3.2) this proves $E(\mu)$ irrational for fine measures μ but naturally this is easy to check directly.) One case in which $E(\mu)$ is certainly rational is the constant ratio case i.e. when (a_n) is a constant sequence. For example in the triadic Cantor case with $a_n \equiv 3$, every element of $C(\mu)$ lies on the lines $\theta = 0$, $\pi/2$, π, $3\pi/2$. More generally $E(\mu)$ is rational when (a_n) is formed according to a simple repetitive pattern e.g. for $a_{2n-1} = p$, $a_{2n} = q$ then $E(\mu) = (q+1)/2(pq-1)$.

It is not hard to make explicit construction of coarse measures μ for which $E(\mu)$ is irrational, but it is even easier to demonstrate the existence of such measures. For example consider the family of measures obtained by choosing each a_n from $\{p, q\}$ (where for definiteness $p > q \geqslant 2$). Suppose that μ_1 corresponds to $(a_n^{(1)})$ and μ_2 corresponds to $(a_n^{(2)})$ and that these sequences differ first at the mth term, where $a_m^{(1)} = q$, $a_m^{(2)} = p$. Then

$$2E(\mu_1) \geqslant \sum_{r=1}^{m-1} d_r + d_m(q^{-1} + q^{-1}p^{-1} + q^{-1}p^{-2} + \dots)$$

$$2E(\mu_2) \leqslant \sum_{r=1}^{m-1} d_r + d_m(p^{-1} + p^{-1}q^{-1} + p^{-1}q^{-2} + \dots),$$

so that
$$\begin{aligned} 2p_m(E(\mu_1) - E(\mu_2)) &\geqslant (p/q(p-1)) - (q/p(q-1)) \\ &= (p-q)\,p^{-1}q^{-1}(1 - (p-1)^{-1}(q-1)^{-1}) > 0. \end{aligned}$$

Accordingly there are uncountably many distinct sequences leading to uncountably many distinct values $E(\mu)$, and therefore for some measure in this family, $E(\mu)$ is irrational.

4. Maximal ideal spaces of $N(\mu)$, $A(\mu)$

We propose to describe the maximal ideal spaces of the various convolution measure algebras introduced in § 2. Since this description is necessarily complicated it would be natural to restrict attention in this section to the simplest case, $N(\mu)$. It soon becomes clear, however, that there is a close relationship between the complex homomorphisms of $N(\mu)$ and those of $A(\mu)$ so that we will consider both cases. A further restriction we shall make here is that we consider only $\mathbf{B}'$ (the singular measures in $\mathbf{B}$). This is justified by Proposition 3.3 which shows that we are ruling out only the case of a finite sum of disjoint translates of Lebesgue measure where the corresponding results are easy to obtain.

(4.1) Recall that $A(\mu)$ is obtained from $N(\mu)$ by adding discrete measures supported on D. In fact we have a direct sum decomposition $A(\mu)=A_c(\mu)\oplus A_d(\mu)$, where $A_c(\mu)$ is an ideal and $A_d(\mu)$ is a subalgebra isomorphic with $L^1(D)$. Using the properties of generalized characters one sees also that every non-zero complex homomorphism of $A_c(\mu)$ induces a *non-zero* complex homomorphism of $N(\mu)$, hence there is a restriction map $\boldsymbol{\rho}_N$ from $\Delta(A_c(\mu))$ to $\Delta(N(\mu))$.

Much more important is the existence of a cononical extension map $\boldsymbol{\lambda}$: $\Delta(N(\mu))\to\Delta(A(\mu))$. It is reasonably clear that, given $\boldsymbol{\lambda}$ and $\Delta(N(\mu))$ (equivalently $S(\mu)$), $\Delta(A(\mu))$ is determined. It turns out that the first step in finding both $\boldsymbol{\lambda}$ and $S(\mu)$ amounts to the definition of a suitable map $\boldsymbol{\gamma}$: $S(\mu)\to D^{\wedge}$.

The definition of $\boldsymbol{\gamma}$ is ensured by (4.2) and this leads to the definition of $\boldsymbol{\lambda}$ in (4.3). At this stage we have $\Delta(N(\mu))\approx S(\mu)$ and $\Delta(A(\mu))\approx S(\mu)\cup\hat{D}$. It remains to give an explicit description of $S(\mu)$—indeed, since $\boldsymbol{\gamma}$ is in general neither injective nor surjective, we must give an explicit description of the image $\Gamma(\mu)$ of $\boldsymbol{\gamma}$ and describe the fibres $\boldsymbol{\gamma}^{-1}\{\phi\}$ for $\varphi\in\hat{D}$. This is achieved in (4.5–(4.9) where we make heavy use of the results of § 3. One description of the topology of $S(\mu)\cup\hat{D}$ is quickly available (viz. $S(\mu)$ has the $\sigma(L^\infty(\mu),L^1(\mu))$-topology $\hat{D}$ the dual topology and the topology of the union is determined by the direct sum decomposition of $A(\mu)$) but it is possible to do better than this in (4.10).

(4.2) PROPOSITION. *Let χ be a complex homomorphism of $N(\mu)$. Then there is a unique character $\boldsymbol{\gamma}(\chi_\mu)=\phi$ of the discrete group D such that if $d\in D_n$ then*

$$\chi_\mu(d+x)=\phi(d)\chi_\mu(x)\qquad(\mu_n\text{ a.e.})\tag{1}$$

Proof. Note first that, in view of the localization lemma (2.1), $\chi_\mu\neq 0$. We start by showing that there exists a positive integer m_0 such that, if $m\geqslant m_0$, $\chi(\mu_m)\neq 0$. Suppose, on the contrary, that there is a sequence of distinct positive integers (m_i) such that $\chi(\mu_{m_i})=0$ $(i=1,2,\dots)$. If $d\in D_{m_i}$,

$$\chi(\delta(d)*\mu_{m_i})^{p_{m_i}}=\chi(\delta(p_{m_i}d)*\mu_{m_i}^{p_{m_i}})=\chi(\mu_{m_i})^{p_{m_i}}=0,$$

and so $\chi(\delta(d)*\mu_{m_i})=0$. Since the set of measures $\{\delta(d)*\mu_{m_i}\colon d\in D_{m_i},\ i=1,2,3,\dots\}$ spans $L^1(\mu)$, it follows that $\chi_\mu=0$ and this gives the required contradiction.

In fact we can prove that if m_0 is the *first* positive integer such that $\chi(\mu_{m_0}) \neq 0$, then $\chi(\mu_m) \neq 0$ for all $m \geqslant m_0$. This follows immediately from the observation that μ_{m_0} is a linear combination of measures of the form $\delta(d) * \mu_m$ with $d \in D_m$. As before if $\chi(\mu_m)$ were equal to zero, $\chi(\delta(d) * \mu_m)$ would equal zero, as would $\chi(\mu_{m_0})$.

We define ϕ on D by specifying its values on $\{d_m: m \geqslant m_0\}$ and verifying the properties required to ensure extension to a character of D. In fact for $m \geqslant m_0$, we define $\phi(d_m)$ by

$$\phi(d_m)\chi(\mu_m) = \chi(\delta(d_m) * \mu_m) \tag{2}$$

Now suppose that $d \in D_m (m \geqslant m_0)$ so that we can write $d = d' + d_{m_1} + d_{m_2} + \ldots + d_{m_s}$, where $d' \in D_{m_0 - 1}$ and the m_i's are distinct integers not less than m_0. Then

$$\begin{aligned}
&\chi(\delta(d) * \mu_m)\, \chi(\mu_{m_1})\, \chi(\mu_{m_2}) \ldots \chi(\mu_{m_s}) \\
&\qquad = \chi((\delta(d') * \mu_m) * (\delta(d_{m_1} * \mu_{m_1})) * (\delta(d_{m_2}) * \mu_{m_2}) * \ldots * (\delta(d_{m_s}) * \mu_{m_s})) \\
&\qquad = \chi(\delta(d') * \mu_m)\, \phi(d_{m_1})\, \chi(\mu_{m_1})\, \phi(d_{m_2})\, \chi(\mu_{m_2}) \ldots \phi(d_{m_s})\, \chi(\mu_{m_s}).
\end{aligned}$$

Consequently
$$\chi(\delta(d) * \mu_m) = \chi(\delta(d') * \mu_m)\, \phi(d_{m_1})\, \phi(d_{m_2}) \ldots \phi(d_{m_s}) \tag{3}$$

For $m \geqslant k \geqslant m_0$, d_k/d_m is a positive integer (which, for this proof, we label q_k) and we have

$$\phi(d_m)^{q_k} \chi(\mu_m)^{q_k} = \chi(\delta(d_m) * \mu_m)^{q_k} = \chi(\delta(d_k) * \mu_m)\, \chi(\mu_m)^{q_k - 1},$$

which gives
$$\chi(\delta(d_k) * \mu_m) = \phi(d_m)^{q_k} \chi(\mu_m). \tag{4}$$

Now for $n > m_0$, we deduce from (3) with $d = d_{m-1}$, $d' = 0$,

$$\chi(\delta(d_{m-1}) * \mu_m) = \phi(d_{m-1})\, \chi(\mu_m),$$

while, from (4) with $k = m - 1$, we obtain

$$\chi(\delta(d_{m-1}) * \mu_m) = \phi(d_m)^{a_m} \chi(\mu_m).$$

This pair of equations gives, for $m > m_0$,

$$\phi(d_m)^{a_m} = \phi(d_{m-1}). \tag{5}$$

Using (4), with the notational convention $d_0 = 1$, we obtain for $n \geqslant m_0$,

$$\chi(\mu_m) = \phi(d_m)^{p_m} \chi(\mu_m),$$

and hence
$$\phi(d_m)^{p_m} = 1. \tag{6}$$

Equations (5) and (6) guarantee that ϕ has a unique extension (denoted again by ϕ) to a character of D. Note, in particular, that for $k \leqslant m$, $\varphi(d_k) = \phi(d_m)^{q_k}$, so that (4) can now be rewritten as

$$\chi(\delta(d_k) * \mu_m) = \phi(d_k)\,\chi(\mu_m). \tag{7}$$

For $d' = d_{n_1} + d_{n_2} + \dots d_{n_t}$ in D_{m_0-1}, we have

$$\chi(\delta(d_{n_1}) * \mu_m)\,\chi(\delta(d_{n_2}) * \mu_m) \dots \chi(\delta(d_{n_t}) * \mu_m) = \chi(\delta(d') * \mu_m)\,\chi(\mu_m)^{t-1} \tag{8}$$

(where $m \geqslant m_0$, and both sides equal $\chi(\delta(d') * \mu_m^t))$.

The combination of (3), (7), and (8) at last gives, for $m \geqslant m_0$, $d \in D_m$,

$$\chi(\delta(d) * \mu_m) = \phi(d)\chi(\mu_m). \tag{9}$$

To obtain (1) we must prove that for all $\nu \ll \mu_n$ and $d \in D_n$,

$$\chi(\delta(d) * \nu) = \phi(d)\chi(\nu) \tag{10}$$

However, as we noted in § 2, the space $L^1(\mu_n)$ is spanned by the collection of all measures of the form $\delta(d') * \mu_m$, where $d' \in D_{n,m}$ and $m > m_0$. Taking this fact into account we need prove (10) only for these measures, i.e. we have to show that

$$\chi(\delta(d) * \delta(d') * \mu_m) = \varphi(d)\chi(\delta(d') * \mu_m)$$

for $d' \in D_{n,m}$, $d \in D_n$, $m > n$, $m > m_0$. But, by (9), both sides of this equation are equal to $\phi(d + d')\chi(\mu_m)$ and so (1) has been established.

To see the uniqueness of ϕ, integrate (1) to obtain (9) which, as we have seen, determines a character of D uniquely.

This completes the proof.

(4.3) Corollary. *If χ is a non-zero complex homomorphism of $N(\mu)$, then there exists a unique extension $\chi' = \boldsymbol{\lambda}(\chi)$ of χ to $A(\mu)$.*

Proof. Let φ be the character of D which satisfies (1) of (4.2) (i.e. $\phi = \boldsymbol{\gamma}(\chi_\mu))$. For every $d \in D$, define $\chi'(\delta(d)) = \phi(d)$, and write also

$$\chi'_{\delta(d) * \mu^n}(d + x) = \phi(d)\,\chi_{\mu^n}(x) \qquad (\mu^n \text{ a.e.}).$$

Since every member of $A(\mu)$ is a sum of measures absolutely continuous with respect to measures ν for which we have so far defined χ'_ν, χ'_λ must be defined for all $\lambda \in A(\mu)$ using GC (i). In view of (3.4) and the that ϕ is a character it is easy to check that this definition of χ' is indeed consistent and does lead to a generalized character of $A(\mu)$.

If χ'' is any complex homomorphism of $A(\mu)$ whose restriction to $N(\mu)$ is χ, then for sufficiently large positive integers m,

$$\phi(d_m)\chi(\mu_m) = \chi''(\delta(d_m) * \mu_m) = \chi''(\delta(d_m))\chi(\mu_m),$$

where $\chi(\mu_m) \neq 0$. It follows that $\gamma(\chi') = \gamma(\chi'')$ and it is then simple to check, from the point of view of generalized characters, that $\chi' = \chi''$. The corollary is proved.

Restriction induces a canonical map ρ_c: $\lambda(\Delta(N(\mu))) \to \Delta(A_c(\mu))$. It is clear that $\rho_c \circ \lambda$ is the inverse of ρ_N which is thus a homeomorphism. In fact we are now able to state

(4.4) THEOREM (i) $\Delta(A_c(\mu))$, $\Delta(N(\mu))$, $S(\mu)$ *are homeomorphic and isomorphic as semigroups and homeomorphic as topological spaces.*

(ii) $\Delta(A(\mu)) \approx S(\mu) \cup \hat{D}$, *where* $S(\mu)$ *has the* $\sigma(L^\infty(\mu), L^1(\mu))$*-topology,* $\hat{D}$ *has the topology of the dual group and* $f_n \to \phi$ *(with* $f_n \in S(\mu)$, $\varphi \in \hat{D}$*) if and only if* $f_n \to 0$ $\sigma(L^\infty(\mu), L^1(\mu))$ *and* $\lambda(f_n) \to \phi$ *in* $\hat{D}$. *Moreover* $S(\mu)$, $\hat{D}$ *have their usual semigroup structure, the group* $\hat{D}$ *being the minimal ideal of* $S(\mu) \cup \hat{D}$ *according to the linking formula:* $f \cdot \beta = \gamma(f) \cdot \phi$ $(f \in S(\mu), \varphi \in \hat{D})$.

Proof. Given the remarks in (4.1) this is now routine verification.

We must now set about calculating $S(\mu)$. The first step is a modification of an argument given by Johnson in [13].

(4.5) PROPOSITION. *Let* $f \in S(\mu)$ *and* $\phi = \lambda(f)$. *Let* $f_n(\phi)$ *be the member of* $L^\infty(\mu)$ *defined by*

$$f_n(\phi) = \sum_{d \in D_n} \phi(d)\, c_n(d),$$

where $c_n(d)$ *is the characteristic function of* $[d, d+d_n)$. *Then there is a sequence* (a_n) *of complex numbers* $(|a_n| \leqslant 1)$ *such that*

$$\|a_n f_n(\phi) - f\|_{L^2(\mu)} \to 0;$$

moreover it is possible to choose

$$a_n = \int_{\mathbf{T}} f(x)\, d\mu_n(x).$$

Proof. *Let* $\chi \in \Delta N$ be such that $f = \chi_\mu$. Let X_n be the (finite dimensional) subspace of $L^2(\mu)$ generated by $\{c_n(d) : d \in D_n\}$ and let P_n be the orthogonal projection onto X_n. Since $\bigcup_{n=1}^\infty X_n$ is dense in $L^2(\mu)$, $\|P_n(\chi_\mu) - \chi_\mu\|_{L^2(\mu)} \to 0$, so that it suffices to prove that

$$P_n(\chi_\mu) = \chi(\mu_n) \sum_{d \in D_n} \phi(d)\, c_n(d).$$

We note that $\{c_n(d) : d \in D_n\}$ is an orthogonal set and

$$\|c_n(d)\|_{L^2(\mu)} = \left(\int c_n(d)\, d\mu \right)^{\frac{1}{2}} = 2^{-n/2},$$

so that

$$P_n(\chi_\mu) = 2^n \sum_{d \in D_n} \langle \chi_\mu, c_n(d) \rangle\, c_n(d).$$

However

$$\langle \chi_\mu, c_n(d)\rangle = 2^{-n}\int \chi_\mu(x)\, d(\delta(d) * \mu_n)\,(x) = 2^{-n}\int \chi_\mu(d+x)\, d\mu_n(x),$$

which by (4.2) is equal to

$$2^{-n}\phi(d)\int \chi_\mu(x)\, d\mu_n(x) = 2^{-n}\phi(d)\,\chi(\mu_n),$$

and the result follows.

This result also gives information about the image of $\boldsymbol{\gamma}$.

(4.6) Corollary. *If ϕ is the character of D corresponding to some homomorphism χ of $N(\mu)$, then $\prod_{n=1}^\infty |\frac{1}{2}(1+\phi(d_n)|$ converges.*

(*Note*. We adopt the convention that an infinite product $\prod_{n=1}^\infty z_n$ *converges* to zero if there exists a least integer $m>1$ such that $\prod_{n=m}^\infty z_n$ converges to a non-zero limit.)

Proof. We choose m_0 such that $\chi(\mu_m)\neq 0$ for all $m\geqslant m_0$ (cf. the proof of (4.2)), and note that, since $\mu_m = (\frac{1}{2}\delta(0)+\frac{1}{2}\delta(d_{m+1}))*\mu_{m+1}$ and $|(\boldsymbol{\lambda}(\chi))\,(\delta(d_{m+1}))| = 1$, the sequence $(|\chi(\mu_m)|)$ is increasing to, say, a. Furthermore,

$$\int f_n(\phi)\, d\mu_{m_0} = 2^{-n}\sum_{d\in D_{m_0,n}} \phi(d) = \prod_{k=m_0+1}^{n} \tfrac{1}{2}(1+\phi(d_k)),$$

so that

$$\prod_{k=m_0+1}^{n} |\tfrac{1}{2}(1+\phi(d_k))| = \left|\int f_n(\phi)\, d\mu_{m_0}\right| \to a^{-1}|\chi(\mu_{m_0})| \neq 0,$$

and the result is proved.

In fact, the condition that $\prod_{n=1}^\infty |\frac{1}{2}+\frac{1}{2}\phi(d_n)|$ converges is also sufficient to guarantee the existence of a complex homomorphism χ of $N(\mu)$ such that $\phi = \boldsymbol{\gamma}(\chi_\mu)$. Anticipating this we define

$$\Gamma(\mu) = \left\{\phi\in \hat{D}: \prod_{n=1}^{\infty} \tfrac{1}{2}|1+\phi(d_n)| \text{ converges}\right\}.$$

Note that $\Gamma(\mu)$ is a subgroup of $\hat{D}$—for, writing $2\pi\alpha_n = \arg \phi(d_n)$, we see that the infinite product converges according as does the series, $\sum_{n=1}^\infty \log\cos \pi\alpha_n$. Thus ϕ in $\hat{D}$ belongs to $\Gamma(\mu)$ if and only if $\sum_{n=1}^\infty \alpha_n^2 < \infty$, and the rest is obvious.

We now give the essential "constructive" step in establishing the converse of (4.6).

(4.7) Lemma. *Let ϕ be a member of $\Gamma(\mu)$. Then there exists a subsequence of $(f_n(\varphi))$ which converges* (a.e. μ) *to a member f of $L^\infty(\mu)$.*

Proof. Choose m_0 such that $1+\phi(d_m)\neq 0$ for all $m\geqslant m_0$ and define $\beta_m = \prod_{n=m_0}^m (\frac{1}{2}+\frac{1}{2}\phi(d_n))$ for $m\geqslant m_0$. We define

$$x_m = \beta_m^{-1} f_m(\phi),$$

and claim that (x_m) is a martingale, i.e. that

$$\int_B x_{m+1} d\mu = \int_B x_m d\mu,$$

for all Borel sets B in the smallest σ-algebra $\mathcal{B}_m$ which makes $x_1, \ldots, x_m$ measurable. This σ-algebra is clearly generated by $\{c_m(d): d \in D_m\}$, and so it will suffice to prove that

$$\int x_{m+1} c_m(d)\, d\mu = \int x_m c_m(d)\, d\mu,$$

for all $d \in D_m$ and $m = m_0, m_0 + 1, m_0 + 2, \ldots$.

However

$$\begin{aligned}\int x_{m+1} c_m(d)\, d\mu &= \beta_{m+1}^{-1} \int \phi(d)\, c_{m+1}(d) + \phi(d + d_{m+1})\, c_{m+1}(d + d_{m+1})\, d\mu \\ &= \beta_{m+1}^{-1} \phi(d)\,(1 + \phi(d_{m+1})) \int c_{m+1}(0)\, d\mu \\ &= \beta_m^{-1} \phi(d)\, 2^{-m} = \beta_m^{-1} \int \phi(d) c_m(d)\, d\mu = \int x_m c_m(d)\, d\mu.\end{aligned}$$

Thus (x_m) is a martingale, and, moreover, the expectation $E(|x_m|)$ of $|x_m|$ is equal to $\int |x_m| d\mu = |\beta_m|^{-1}$. Since $(|\beta_m|)$ tends to a non-zero limit $\lim E(|x_m|) < \infty$ and hence, by the martingale convergence theorem (see [10] p. 319) $\lim x_m = x$ exists μ almost everywhere.

There exists a subsequence $(\beta_{m(i)})$ of (β_m) such that $\beta_{m(i)} \to \beta$, say, where $\beta \neq 0$. We now see that $f_{m(i)}(\phi) = \beta_{m(i)} x_{m(i)}$ converges to $\beta x (= f)$ almost everywhere with respect to μ.

(4.8) Lemma. *Let ϕ belong to $\Gamma(\mu)$. Then any limit point f of the sequence $(f_n(\phi))$ (with respect to pointwise convergence almost everywhere μ) belongs to $S(\mu)$.*

Proof. It is possible but a little troublesome to verify directly that f satisfies the appropriate consistency condition—viz. for $r \geq 1$, and $x_i \in$ support μ,

$$\prod_{i=1}^{r} f(x_i) \text{ is a well-defined function of } \sum_{i=1}^{r} x_i \,(\mu^r \text{ a.e.}). \tag{1}$$

We omit such verification of (1), since an indirect proof of this lemma is a corollary of the proofs of (5.2) and (5.4). Naturally these proofs have been arranged to be independent of the present lemma.

The last two results show that $\Gamma(\mu)$ is the image of $\boldsymbol{\gamma}$. We show now that the fibres are simple to describe.

(4.9) LEMMA. *Let ϕ be a character of D corresponding to two homomorphisms χ, χ' of $N(\mu)$. Then there exists a complex number a such that*

$$\chi'_\mu = a\chi_\mu.$$

Proof. It is obvious that $\boldsymbol{\gamma}(\chi'_\mu \bar{\chi}_\mu)$ is the constant character 1. Thus we need only show that if $f \in S(\mu)$ and $\boldsymbol{\gamma}(f) = 1$ then f is constant. But in this situation $f_n(1)$ is constant and equal to 1, so the result follows from Proposition (4.5).

The next theorem which summarizes the basic information concerning the maximal ideal spaces under discussion is to be read in conjunction with (4.4)—in particular $S(\mu) \cup \hat{D}$ is realized as a topological semigroup according to the topological isomorphism with $\Delta(A(\mu))$ described there. Recall that μ_m denotes the tail measure $\ast_{n=m+1}^{\infty} \frac{1}{2}(\delta(0) + \delta(d_n))$.

(4.10) THEOREM. *Let $\mu = \ast_{n=1}^{\infty} \frac{1}{2}(\delta(0) + \delta(d_n))$ belong to $\mathbf{B}'$. Then there exist continuous semigroup homomorphisms $\boldsymbol{\gamma}$: $S(\mu) \to \hat{D}$, $\boldsymbol{\tau}$: $S(\mu) \cup \hat{D} \to \hat{D}$ such that.*

(i) *the image of $\boldsymbol{\gamma}$ is the subgroup $\Gamma(\mu)$ of elements ϕ in $\hat{D}$ such that $\prod_{k=1}^{\infty} \frac{1}{2} |1 + \phi(d_k)|$ converges,*

(ii) *if $\varphi \in \Gamma(\mu)$, $\boldsymbol{\gamma}^{-1}(\phi) = \{af: 0 < |a| \leq 1, a \in \mathbf{C}\}$, where f is a member of $S(\mu)$ with constant unit modulus which is a pointwise limit point of the sequence $(\sum_{d \in D_n} \phi(d) c_n(d))$,*

(iii) *$f_n \to f$ in (the metrizable space) $S(\mu)$ if and only if $\boldsymbol{\gamma}(f_n) \to \boldsymbol{\gamma}(f)$ and there exists a positive integer m, such that $\int_{\mathbf{T}} f_n(x) d\mu_m(x) \to \int_{\mathbf{T}} f(x) d\mu_m(x) \neq 0$,*

(iv) *$\boldsymbol{\tau}$ is surjective and $\boldsymbol{\tau}^{-1}(\phi) = \boldsymbol{\gamma}^{-1}(\phi) \cup \{\phi\}$ for $\phi \in \hat{D}$.*

Proof. Suppose $\chi_\mu^{(n)} \to \chi_\mu$, (with $\chi^{(n)}$, χ in $\Delta(A_c(\mu))$) then

$$\chi^{(n)}(\mu_m \ast \delta(d)) \to \chi(\mu_m \ast \delta(d)) \text{ and } \chi^{(n)}(\mu_m) \to \chi(\mu_m)$$

for all $d \in D_m$, so that if m is chosen large enough to make $\chi(\mu_m) \neq 0$, $\boldsymbol{\gamma}(\chi_\mu^{(n)})(d) \to \boldsymbol{\gamma}(\chi_\mu)(d)$ proving the continuity of $\boldsymbol{\gamma}$. (i) follows from (4.6)–(4.8). In view of (4.7), (4.8) and the fact that $|f_n(\phi)| = 1$ for all $n(\phi \in \Gamma(\mu))$ there certainly exists $f \in S(\mu)$ such that $|f| = 1$ and $\boldsymbol{\gamma}(f) = \phi$. Since μ has independent powers, for every non-zero complex number a with $|a| \leq 1$, the constant $\mathbf{a} \in S(\mu)$. (4.9) now guarantees (ii).

For (iii) we consider $f^{(n)}$, $f \in S(\mu)$ and write $\phi_n = \boldsymbol{\gamma}(f^{(n)})$, $\varphi = \boldsymbol{\gamma}(f)$. Let $\chi^{(n)}$, χ denote the $\boldsymbol{\lambda}$-images of the elements of $\Delta N(\mu)$ determined by $f^{(n)}$, f. Suppose that $\phi_n \to \phi$ and there exists m such that $\hat{\mu}_m(\chi^{(n)}) \to \hat{\mu}_m(\chi) \neq 0$. Then for any $r \geq m$,

$$\hat{\mu}_m(\chi^{(n)}) = \hat{\mu}_r(\chi^{(n)}) \prod_{k=m+1}^{r} \tfrac{1}{2}(1 + \phi_n(d_k)) \to \hat{\mu}_m(\chi).$$

However,
$$\prod_{k=m+1}^{r} \tfrac{1}{2}(1 + \phi_n(d_k)) \to \prod_{k=m+1}^{r} \tfrac{1}{2}(1 + \phi(d_k)) \neq 0$$

and
$$\hat{\mu}_m(\chi) = \hat{\mu}_r(\chi) \prod_{k=m+1}^{r} \tfrac{1}{2}(1+\phi(d_k)),$$

so that
$$\hat{\mu}_r(\chi^{(n)}) \to \hat{\mu}_r(\chi).$$

Appealing once more to the convergence of φ_n, we see that

$$(\delta(d) * \mu_r)\hat{\ }(\chi^{(n)}) \to (\delta(d) * \mu_r)\hat{\ }(\chi),$$

for all $d \in D_r$, all $r \geqslant m$. Since the measures in question span $L^1(\mu)$, it follows that $\chi_\mu^{(n)} \to \chi_\mu$ in the $\sigma(L^\infty(\mu), L^1(\mu))$-topology. The converse is immediate so that (iii) is established.

To define $\boldsymbol{\tau}$, follow the isomorphism from $S(\mu) \cup \hat{D}$ to $\Delta(A(\mu))$ by the canonical map from $\Delta A(\mu)$ to $\hat{D}(\approx \Delta L^1(D))$. Surjectivity is immediate. Using the map $\boldsymbol{\rho}_c$ defined before (4.4) we see that $\boldsymbol{\tau}$ restricted to $S(\mu)$ coincides with $\boldsymbol{\gamma}$. It is now simple to verify (iv) that $\boldsymbol{\tau}$ is continuous. This completes the proof.

(4.10) has been formulated to correspond to a convenient visualization of $\Delta(N(\mu))$, $\Delta(A(\mu))$ where we think of $\hat{D}$ as a base space. In the case of $\Delta(A(\mu))$, at each point of $\Gamma(\mu)$ a unit disc is attached by its centre, while at points of $\hat{D} \setminus \Gamma(\mu)$ trivial discs are attached. Convergence in $\Delta(A(\mu))$ corresponds to convergence in the base space together with convergence in the appropriate position along the corresponding discs. $\Delta(N(\mu))$ is obtained by identifying all the centres of the discs to the point at infinity which is then removed, but the convergence respects the original indexing.

5. $N^0(\mu)$, $A(\mu)$ as subalgebras of $M(T)$

In this section we obtain further information on the maximal ideal spaces discussed in § 4 with particular emphasis on the question of restriction from $\Delta(M(\mathbf{T}))$. From this point of view it is more convenient to adjoin the identity $\delta(0)$ to $N(\mu)$ obtaining $N^0(\mu)$. The only difference this makes to the maximal ideal space is, of course, that $S(\mu)$ is replaced by $S^0(\mu) = S(\mu) \cup \{0\}$, the one-point compactification of $S(\mu)$, where 0 respresents the trivial homomorphism which takes the value one on $\delta(0)$ and zero on $N(\mu)$. We note at the outset that, while the close relationship which exists between $\Delta(A(\mu))$ and $\Delta(N^0(\mu))$ will often assist us to make statements concerning $S^0(\mu)$, we feel that no particular interest attaches to the exercise of reformulating for $A(\mu)$ each property of $N^0(\mu)$. Once more we restrict attention to $\mu \in \mathbf{B}'$ but now the subdivision of $\mathbf{B}'$ into coarse and fine measures becomes crucial.

We start by showing that, when μ is coarse, $\Gamma(\mu)$ takes a particularly simple form.

(5.1) PROPOSITION. *Suppose that $\mu \in \mathbf{B}'$ is coarse. Then $\Gamma(\mu)$ consists of all restriction to D of continuous characters of* $\mathbf{T}$.

Proof. Let ϕ belong to $\Gamma(\mu)$ and write $2\pi\alpha_n = \arg\phi(d_n)$. Let $a = \sup a_n$ and choose m such that

$$|\alpha_n| < a^{-2} \quad \text{for all } n \geqslant m. \tag{1}$$

Since D_m generates a finite subgroup of $\mathbf{T}$, there is some continuous character ψ of $\mathbf{T}$ such that $\psi|_{D_m} = \phi|_{D_m}$. Suppose that ψ is induced by the integer k. Since $\psi(d_m)$ evidently determines the value of ψ on D_m, we can (by adding or subtracting multiples of d_m^{-1}) suppose that $\alpha_m = kd_m$.

Now suppose that $\alpha_n = kd_n$ for some $n \geqslant m$. Then, since $\phi(d_n) = \phi(d_{n+1})^{a_{n+1}}$, we must have

$$\alpha_{n+1}a_{n+1} \equiv kd_n \,(\text{mod } 1). \tag{2}$$

Using (1),
$$|\alpha_{n+1}a_{n+1} - kd_n| \leqslant a^{-1} + a^{-2} < 1,$$

so that the congruence (2) can be replaced by the equality $\alpha_{n+1} = kd_{n+1}$, and it follows, by induction, that $\alpha_n = kd_n$ for all $n \geqslant m$. This, in turn, proves that $\phi = \psi$ on D. The converse is straightforward. For if $\psi \in \hat{\mathbf{T}}$ then $\hat{\mu}_n(\psi) \neq 0$ for some n and the definition of μ_n as a weak $*$ limit gives the stronger assertion that $\prod_{k=n+1}^{\infty} (\frac{1}{2} + \frac{1}{2}\psi(d_k))$ converges, and completes the proof.

Note that the last paragraph of the preceding proof applies equally well to fine measures, showing that $\Gamma(\mu)$ contains a copy of $\mathbf{Z}$.

(5.2) COROLLARY. *For coarse $\mu \in \mathbf{B}'$, $S^0(\mu) = \{af : |a| \leqslant 1$, f is the restriction to support μ of a continuous character of $\mathbf{T}\}$.*

Proof. Let $\phi \in \Gamma(\mu)$ then there is $\psi \in \hat{\mathbf{T}}$ with $\psi = \phi$ on D. Hence $(\sum_{d \in D_n} \phi(d) c_n(d))$ converges uniformly on support μ to ψ. This establishes Lemma (4.8) for coarse measures. The rest of the present assertion follows from the appropriate part of Theorem (4.10), provided we note the convention that e.g. $O \cdot \psi = 0$ denotes the trivial non-zero homomorphism of $N^0(\mu)$.

We are now in a position to give a substantial amount of information concerning the μ-coordinates of generalized characters of $M(\mathbf{T})$ when μ is coarse. It is convenient to adopt the notation that if X is a subset of the maximal ideal space of any L-subalgebra containing μ then $X_\mu = \{\chi_\mu : \chi \in X\}$, (for example $S(\mu) = \Delta(N(\mu))_\mu$). We write also $\mathbf{C}(\mu) = \{\mathbf{a} : a \in C(\mu)\}$ i.e. $\mathbf{C}(\mu)$ is the semigroup of constant functions in $S(\mu)$.

(5.3) THEOREM. *Let μ be a coarse measure in $\mathbf{B}'$. Then*

(i) $(\text{cl } \mathbf{Z})_\mu = \mathbf{C}(\mu) \cdot \mathbf{Z}_\mu$, *where $\mathbf{Z}$ denotes the continuous characters of $\mathbf{T}$ canonically embedded in $\Delta(M(\mathbf{T}))$.*

(ii) $\partial(M(\mathbf{T}))_\mu = \partial(N^0(\mu))_\mu$, *and both coincide with the circled hull of $\mathbf{C}(\mu) \cdot \mathbf{Z}_\mu$.*

(iii) $\Delta(M(\mathbf{T}))_\mu=\Delta(N^0(\mu))$, *and both coincide with the set* $\{a\psi\colon a\in\mathbf{C},\ |a|\leqslant 1,\ \psi\in\mathbf{Z}_\mu\}$.

(iv) $(\mathrm{cl}\ \mathbf{Z})_\mu\subset\partial(M(\mathbf{T}))_\mu\subset\Delta(M(\mathbf{T}))_\mu$.

Proof. Suppose $f\in(\mathrm{cl}\ \mathbf{Z})_\mu$. Then there is a sequence (ψ_m) of elements of $\mathbf{Z}_\mu$ such that $\psi_m\to f$ in $S^0(\mu)$. Thus, by (5.2), $f=a\psi$ with $a\in\mathbf{C}$, $|a|\leqslant 1$ and $\psi\in\mathbf{Z}_\mu$. Hence $\psi_m\bar{\psi}\to\mathbf{a}$, which is now seen to belong to $\mathbf{C}(\mu)$. This proves (i).

Let us write H for the circled hull of $(\mathrm{cl}\ \mathbf{Z})_\mu$. We check first that $H\subseteq\partial(N^0(\mu))$. This is, in fact, an application of the extension theorem (2.1), because any element of H is a limit of a sequence of elements of the form $(a\psi_m)$ with a fixed, $|a|=1$, and $\psi_m\in\mathbf{Z}_\mu$, and thus $|(a\psi_m)_\nu|=1$ (ν a.e.) for every $\nu\in N^0(\mu)$.

Since the inclusion $\partial(N^0(\mu))_\mu\subseteq\partial(M(\mathbf{T}))_\mu$ holds, it will suffice to prove that $\partial(M(\mathbf{T}))_\mu\subseteq H$. Note first that $\mathbf{0}\in\mathbf{C}(\mu)$ (e.g. use Proposition (3.3) although the present assertion is of more general validity) so that $\mathbf{0}\in H$. In conjunction with (5.2) this shows that if $\chi\in\Delta(M(\mathbf{T}))$ and $|\chi_\mu|^2=|\chi_\mu|$ then $\chi_\mu\in H$. But Taylor showed in [19] that the Šilov boundary of any convolution measure algebra is contained in the closure of the generalized characters with idempotent modulus. It follows that

$$H\subseteq\partial(N^0(\mu))_\mu\subseteq\partial(M(\mathbf{T}))_\mu\subseteq\mathrm{cl}\,H=H,$$

and (ii) is established.

In view of (5.2) and the obvious inclusion $\Delta(M(\mathbf{T}))_\mu\subseteq\Delta(N^0(\mu))_\mu$, (iii) is reduced to an extension problem. For this we use another observation by Taylor [19]—given a non-negative generalized character χ of a convolution measure algebra and $z\in\mathbf{C}$ with $\mathrm{Re}(z)>0$, then χ^z is also a generalized character of the algebra. For our present purposes we note, by Proposition (3.3), that there exists $0<b<1$ with $\mathbf{b}\in\mathbf{C}(\mu)\subseteq(\mathrm{cl}\ \mathbf{Z})_\mu$. Hence there is $\chi\in\Delta(M(\mathbf{T}))$ with $\chi_\mu=\mathbf{b}$. Now consider $|\chi|^z$ for varying z to obtain the required constant functions in $\Delta(M(\mathbf{T}))_\mu$.

Recall from the first part of the proof of Theorem 3.2 that $\mathbf{1}$ is the only constant function with unit modulus in $\mathbf{C}(\mu)$. This shows that the first inclusion in (iv) is proper. The fact that the second inclusion is proper also follows from the fact that the Fourier-Stieltjes transform of μ has range inside a proper subdisc of the unit disc together with the isolated point 1.

Theorem (5.3) must obviously be compared with Johnson's results in [13] for $\mu_J=\ast_{n=1}^{\infty}(\frac{2}{3}\delta(0)+\frac{1}{3}\delta(2^{-n}))$. While that example does not fit into our scheme of things and our results for coarse measures go much further than what was derived concerning μ_J in [13], it should be noted that Johnson was concerned only with obtaining the inclusion $\partial(M(\mathbf{T}))_{\mu_J}\subseteq\Delta(M(\mathbf{T}))_{\mu_J}$ and that several of the methods we have used are based on ideas to be found in [13].

The next project is to obtain the analogue for fine measures. The results are collected in Theorem (5.5), the meat is in the next proposition.

(5.4) PROPOSITION. *Let μ be a fine measure in* **B**. *Suppose that ϕ belongs to $\Gamma(\mu)$ and that f is a (pointwise) limit point of the sequence $(f_n(\phi))$. Then $f \in (\mathrm{cl}\,\mathbf{Z})_\mu$.*

Proof. We make a preliminary observation which will save some computation. $(\mathrm{cl}\,\mathbf{Z})_\mu$ is a semigroup which, as we have already proved (in Theorem (3.2)), contains all constant functions with modulus not greater than one. Accordingly it will suffice to show that $\gamma f \in (\mathrm{cl}\,\mathbf{Z})_\mu$, where γ is some constant of unit modulus.

To fix notation we suppose that the subsequence $(f_{n(i)}(\phi))$ converges to f and we choose m_0 such that $\beta_n = \prod_{k=m_0+1}^{n} \frac{1}{2}(1+\phi(d_k))$ is never zero for $n > m_0$. It follows from Lemma (4.7) that $(\beta_n^{-1} f_n(\phi))$ converges and hence that $\beta_{n(i)}$ converges to a non-zero limit which we call β.

(A cautionary remark is in order at this point. It is always possible to find $\phi \in \Gamma(\mu)$ and subsequences $(n(i))$, $(m(i))$ such that $f_{n(i)}(\phi) \to g \in S(\mu)$, $f_{m(i)}(\phi) \to h \in S(\mu)$, but $g \neq h$.)

Until further notice we fix the integer $n > m_0$. Note that for $d \in D_n$, $m \geqslant n$,

$$f_m(\varphi)\ (d+t) = \phi(d) f_m(\phi)\ (t) \quad (\mu_n \text{ a.e.}),$$

hence passing to the limit along the subsequence $(n(i))$,

$$f(d+t) = \phi(d) f(t) \quad (\mu_n \text{ a.e.}) \tag{1}$$

We now set about choosing the approximating sequence of continuous characters. In fact choose a sequence $(m(j))$ of positive integers such that $a_{m(j)+1} \to \infty$ as $j \to \infty$. (This is possible since μ is fine.) For each j, choose $\phi_{m(j)} \in \mathbf{Z}_\mu$ such that $\phi_{m(j)}$ coincides with ϕ on the finite subgroup generated by $D_{m(j)}$, noting that the integer defining $\phi_{m(j)}$ is to have modulus less than $p_{m(j)}$.

Now we apply (1), for $d \in D_m$, $m(j) > n$, to show

$$\int f \bar{\phi}_{m(j)} d(\delta(d) * \mu_n) = \phi(d) \bar{\phi}_{m(j)}(d) \int f(t)\, \bar{\phi}_{m(j)}(t)\, d\mu_n(t) = \int f(t)\, \bar{\phi}_{m(j)}(t)\, d\mu_n(t). \tag{2}$$

Also for $n(i) > m(j) > n$,

$$\int f_{n(i)}(\phi)\,(t)\, \bar{\phi}_{m(j)}(t)\, d\mu_n(t) = \left\{ \prod_{k=m(j)+1}^{n(i)} \tfrac{1}{2}(1 + \phi(d_k)\, \bar{\phi}_{m(j)}(d_k)) \right\} \int \bar{\phi}_{m(j)}(t)\, d\mu_{n(i)}(t).$$

Since $\mu_{n(i)} \to \delta(0)$ in the $\sigma(M(\mathbf{T}), C(\mathbf{T}))$ – topology, we deduce that

$$\int f(t)\, \bar{\phi}_{m(j)}(t)\, d\mu_n(t) = \lim_{i \to \infty} \prod_{k=m(j)+1}^{n(i)} \tfrac{1}{2}(1 + \phi(d_k)\, \bar{\phi}_{m(j)}(d_k)). \tag{3}$$

In order to estimate the expression on the right hand side of (3) we make use of an elementary lemma given by Hewitt and Kakutani in [12], viz.

$$\text{for } z_i, z_i' \in \mathbf{C}, |z_i|, |z_i'| \leqslant 1, \quad \text{then } \left|\prod_{i=1}^{k} z_i - \prod_{i=1}^{k} z_i'\right| \leqslant \sum_{i=1}^{k} |z_i - z_i'|.$$

Appropriate substitution leads to

$$\left|\prod_{k=m(j)+1}^{n(i)} \tfrac{1}{2}(1+\phi(d_k)\bar{\phi}_{m(j)}(d_k)) - \prod_{k=m(j)+1}^{n(i)} \tfrac{1}{2}(1+\phi(d_k))\right| \leqslant \sum_{k=m(j)+1}^{n(i)} \tfrac{1}{2}|1-\bar{\phi}_{m(j)}(d_k)|$$

$$< \sum_{k=0}^{\infty} \sin(\pi 2^{-k} a_{m(j)+1}^{-1}) < \pi/a_{m(j)+1}.$$

Combining this with (3), we deduce that

$$\left|\int f(t)\,\bar{\phi}_{m(j)}(t)\,d\mu_n(t) - (\beta/\beta_{m(j)})\right| < \pi/a_{m(j)+1}. \tag{4}$$

Now $|\beta_{m(j)}| \to |\beta|$ as $j \to \infty$, so by passing to a further subsequence which we may as well denote $(m(j))$, we can (and do) suppose that

$$\beta/\beta_{m(j)} \to \bar{\gamma}, \quad \text{for some } \gamma \in \mathbf{C},\ |\gamma| = 1. \tag{5}$$

Taking (2), (4) and (5), we now find

$$\lim_{j\to\infty} \int f\bar{\phi}_{m(j)}\, d(\delta(d) * \mu_n) = \bar{\gamma}. \tag{6}$$

Now free n to obtain a spanning set of measures $\delta(d) * \mu_n$ for $L^1(\mu)$ and deduce from (6) that

$$\bar{\phi}_{m(j)} f \to \bar{\gamma} \quad \sigma(L^\infty(\mu), L^1(\mu)).$$

Since f has constant modulus one, it follows that $\phi_{m(j)} \to \gamma f$ as required.

(5.5) THEOREM. *Let μ be a fine measure in* **B**. *Then*

$$(\mathrm{cl}\,\mathbf{Z})_\mu = \partial(N^0(\mu))_\mu = \partial(M(\mathbf{T}))_\mu = \Delta(M(\mathbf{T})_\mu = \Delta(N^0(\mu))_\mu.$$

Proof. The sets in question are totally ordered from left to right by inclusion. The only containment which requires comment is the first, which follows from (2.1). It suffices therefore to prove that $\Delta(N^0(\mu))_\mu \subseteq (\mathrm{cl}\,\mathbf{Z})_\mu$. This is immediate from (3.2), (4.10), (5.4). (5.6) At first glance it would appear that the situation is tidier for fine measures. However there appears to be no reasonable analogue of (5.2) for fine measures and one should interpret Theorem 5.5 as demonstrating that $(\mathrm{cl}\,\mathbf{Z})_\mu$ is extraordinarily complicated when μ is fine (rather than as showing that $\Delta(M(\mathbf{T}))_\mu$ takes a simple form in this case).

In fact it is possible to show by direct construction that, for every fine measure μ in $\mathbf{B}$, there exists $\chi \in \Delta(M(\mathbf{T}))$ such that χ_μ is not a constant multiple of an element of $\mathbf{Z}_\mu$; moreover χ_μ may be chosen continuous or discontinuous as desired.

6. Structure semigroups

In this section we find the structure semigroups of the L-subalgebras which we have already discussed. So far as we are aware this is the first explicit determination of structure semigroups apart from the cases mentioned by Taylor in the final section of his original paper [19]. As a corollary of the description we find a non-trivial example of an infinite family of non-isomorphic convolution measure algebras with the same structure semigroup. Such results suggest tentatively that, although Šreider's theory of generalized characters for an L-subalgebra N of $M(G)$ does not lead readily to the existence of the structure semigroup, it seems doubtful whether a consideration of the structure semigroup adds materially to one's knowledge of the structure of N. We start by recalling some definitions:

(6.1) *Terminology.* Naturally we follow Taylor [19], who introduced the concept of structure semigroup in the context of an abstract convolution measure algebra (CM-algebra). Since all the algebras we discuss are measure algebras in an obvious concrete way we have no need to recall the definition of a CM-algebra, but we shall call a map $\theta: N \to N'$ between measure algebras N, N' a *CM-morphism* if θ is an algebra homomorphism which satisfies also.

(i) $\mu \in N$, $\mu \geqslant 0 \Rightarrow \theta\mu \geqslant 0$, $\|\theta\mu\| = \|\mu\|$.
(ii) $\mu \in N$, $\omega \in N'$, $0 \leqslant \omega \leqslant \theta\mu \Rightarrow \exists \nu \in N$ such that $\theta\nu = \omega$.

((i) and (ii) are equivalent to the assertion that θ is an *L-homomorphism.*) For any L-subalgebra N of $M(\mathbf{T})$ the *structure semigroup*, $\Sigma(N)$, of N is a compact commutative jointly continuous semigroup uniquely determined by the fact that there exists a CM-morphism, $\theta: N \to M(\Sigma(N))$, such that

(i) $\theta(N)$ is $\sigma(M(\Sigma(N)), C(\Sigma(N)))$—dense in $M(\Sigma(N))$.
(ii) $\Sigma(N)^\wedge$ separates points of $\Sigma(N)$.
(iii) the complex homomorphisms of N are given by $\mu \mapsto \int f d\theta\mu$ for $f \in \Sigma(N)^\wedge$.

The existence of $\Sigma(N)$ is proved in [19] and, since N is semi-simple, it follows from results there that θ is isometric.

For any commutative topological semigroup S we write $\operatorname{ap}(S)$ for the almost periodic compactification of S (see [8], [9]). In other words given S there exists a compact abelian semigroup $\operatorname{ap}(S)$ and a continuous injection $k_S: S \to \operatorname{ap}(S)$ with the following property:

given a compact abelian semigroup S' and a continuous morphism $\phi\colon S\to S'$ there exists a continuous morphism $\phi\colon \mathrm{ap}(S)\to S'$ such that the diagram

$$\begin{tikzcd} S \arrow[r, "\phi"] \arrow[d, "k_S"'] & S' \\ ap(S) \arrow[ur, "\bar{\phi}"'] & \end{tikzcd}$$

commutes. In particular, for an LCA group G, $\mathrm{ap}(G)$ is the Bohr compactification of G.

We will make essential use of $\mathrm{ap}(\mathbf{N})$ where $\mathbf{N}$ is the additive semigroup of positive integers with the discrete topology. Therefore we note that $\mathrm{ap}(\mathbf{N})$ can be written in the form $\mathrm{ap}(\mathbf{N})=\mathrm{ap}(\mathbf{Z})\cup\mathbf{N}$ where the multiplication and topology are given by the following discussion which is implicit in [9]. The almost periodic functions $A(\mathbf{N})$ on $\mathbf{N}$ are the closed linear hull of the continuous semicharacters $\mathbf{N}^\wedge$. $\mathbf{N}^\wedge$ corresponds to the punctured disc $\{z\in C\colon 0<|z|\leqslant 1\}$ according to the formula $n\mapsto z^n (n\in\mathbf{N})$ so that decomposition of almost periodic functions of the form $f(n)=\sum_{i=1}^m \alpha_i z_i^n$ as $g(n)+h(n)$, where $g(n)=\sum \alpha_i z_i^n$ for all i such that $|z_i|<1$, leads to a decomposition $A(\mathbf{N})=C_0(\mathbf{N})\oplus A(\mathbf{Z})$, where the first summand is an ideal and the second a subalgebra. Thus $\mathrm{ap}(\mathbf{N})$ can be realized as the maximal ideal space of $C_0(\mathbf{N})\oplus A(\mathbf{Z})$. The homomorphisms which are non-zero on $C_0(\mathbf{N})$ correspond to evaluation at points of $\mathbf{N}$ and the remaining homomorphisms are induced by elements of ap $(\mathbf{Z})$ acting on $A(\mathbf{Z})$. Accordingly $\mathrm{ap}(\mathbf{N})$ is realized as $\mathrm{ap}(\mathbf{Z})\cup\mathbf{N}$ with the Gelfand topology, the usual multiplication on the component sets and the linking formula

$$n+y=\tilde{n}+y \quad (n\in\mathbf{N},\, y\in\mathrm{ap}(\mathbf{Z})),$$

where $\tilde{n}$ is obtained by injecting n in $\mathbf{Z}$ and then applying $k_{\mathbf{Z}}$. Observe that the copy of $\mathbf{N}$ appearing in the formula for $\mathrm{ap}(\mathbf{N})$ is indeed the canonical image of $\mathbf{N}$ in its almost periodic compactification—thus we have already adopted the convenient abuse of notation that the embedding map $\mathbf{N}\to\mathrm{ap}(\mathbf{N})$ is to be written as $n\mapsto n$, i.e. $k_{\mathbf{N}}(n)=n$.

The above shows, in particular, that each $n\in\mathbf{N}$ is an isolated point of $\mathrm{ap}(\mathbf{N})$ and makes possible a quick proof that $\mathrm{ap}(\mathbf{N}\times H)$ is isomorphic to $\mathrm{ap}(\mathbf{N})\times H$ for every compact abelian group H. In fact let $\phi\colon\mathbf{N}\times H\to S$ be a continuous homomorphism for some compact commutative semigroup S. Let ψ be the restriction of ϕ to $\mathbf{N}\times\{e\}$, $\tilde{\psi}$ the induced map $\tilde{\psi}\colon \mathrm{ap}\,(\mathbf{N})\times\{e\}\to S$. Now define $\tilde{\phi}$ as follows:

$$\tilde{\phi}(x+1,t)=\tilde{\psi}(x,e)\phi(1,t) \quad (x\in\mathrm{ap}(\mathbf{N})) \quad \tilde{\phi}(1,t)=\phi(1,t).$$

Now consider $\mu\in\mathbf{B}'$, with the associated countable subgroup D of $\mathbf{T}$. Let $\Gamma(\mu)$ have the same significance as before. There is an injection $\alpha\colon D\to\Gamma(\mu)$ of the discrete group D in the compact group $\Gamma(\mu)$ defined by $\alpha(d)(\phi)=\phi(d)$, for all ϕ in $\Gamma(\mu)$ and hence a commutative diagram

$$\begin{tikzcd} D \arrow[r, "\alpha"] \arrow[d, "k_D"'] & \Gamma(\mu)^{\wedge} \\ ap(D) \arrow[ur, "\tilde{\alpha}"'] & \end{tikzcd}$$

We are now ready to state and prove

(6.2) THEOREM. *Let μ be a member of* $\mathbf{B}'$ *and let D be the countable subgroup of* $\mathbf{T}$ *generated by the (d_n) appearing in the definition of μ. Further let $\Gamma(\mu)$ be the discrete group comprising those characters ϕ of D for which $\prod_{n=1}^{\infty}\frac{1}{2}|1+\phi(d_n)|$ converges. Then*

(i) $\sum(N(\mu))$ *is the closure in* $ap(\mathrm{N})\times\Gamma(\mu)^{\wedge}$ *of the semigroup*

$$\bigcup_{n=1}^{\infty}\left\{\left(n,\ \sum_{i=1}^{m}\varepsilon_i\alpha(d_i)\right): 0\leqslant\varepsilon_i\leqslant n \quad (i=1,2,\ldots,m),\quad m=1,2,3\ldots\right\}$$

(ii) $\Sigma(A_c(\mu))=\mathrm{ap}\,(N)\times\Gamma(\mu)^{\wedge}$

(iii) $\Sigma(A(\mu))=\mathrm{ap}\,(D)\cup(\mathrm{ap}\,(N)\times\Gamma(\mu)^{\wedge}$, *where the topology is that of the disjoint union, and the multiplication is that of the disjoint union together with the linking formula*

$$x+(y,z)=(y,\alpha(x)+z)\quad (x\in\mathrm{ap}(D),\ y\in\mathrm{ap}\,(N),\ z\in\Gamma(\mu)^{\wedge})$$

Proof. Let G be the set of all elements of $\Delta(M(\mathbf{T}))_{\mu}$ which have absolute value equal to the constant function 1. Note that G inherits from $\Delta(M(\mathbf{T}))$ a multiplication under which it becomes a group. Moreover G contains as a subgroup a copy of $\mathbf{T}$, viz. those constant functions of unit absolute value. As a consequence of (4.10) we have that $G/\mathbf{T}$ is isomorphic with $\Gamma(\mu)$. In other words we have the exact sequence

$$0\longrightarrow\mathbf{T}\xrightarrow{i}G\xrightarrow{\gamma}\Gamma(\mu)\longrightarrow 0.$$

where i is the map taking a member θ of $\mathbf{T}$ to the constant function with value $\exp(2\pi i\theta)$. Since $\mathbf{T}$ is divisible the exact sequence splits, i.e. there are homomorphisms $\pi: G\to\mathbf{T}$ and $\chi:\Gamma(\mu)\to G$ such that $\pi\circ i$ and $\gamma\circ\chi$ are the identity maps on $\mathbf{T}$ and $\Gamma(\mu)$ respectively. This leaves considerable freedom in the choice of χ and π, although a choice of either of these determines the other. It will be possible at a later stage of the argument to indicate how this choice is to be made, for the moment we assume that χ has been fixed. Naturally, there is now an isomorphism $\tau: G\to\mathbf{T}\times\Gamma(\mu)$ defined by $\tau(\phi)=(\pi(\phi),\gamma(\phi))$. Now topologize G by giving $\mathbf{T}$ the usual topology, $\Gamma(\mu)$ the discrete topology and by demanding that τ is a homeomorphism. The dual $G^{\wedge}$ of the LCA group G has a compact open subgroup isomorphic with $\Gamma(\mu)^\frown$. In fact the dual homeomorphism $\tau^*: \mathbf{Z}\oplus\Gamma(\mu)^\frown \to G^\frown$ is such that $\tau^*((0)\oplus\Gamma(\mu)^\frown)$ is the annihilator of $\tau^{-1}(\mathbf{T}\oplus 0))$.

It is not difficult to see that the topology of G is finer than topology induced on G as a subset of $\Delta(M(\mathbf{T}))_\mu$ in the $\sigma(L^\infty(\mu), L^1(\mu))$-topology. Moreover the results of § 4 allow us to consider G as a subgroup of the maximal ideals of $A(\mu)$. Thus any positive measure ν in $A(\mu)$ gives rise to a continuous positive definite function $\hat{\nu}|_G$ on G. By Bochner's theorem, there is a positive measure $\theta(\nu)$ on $G^\frown$ such that the Fourier-Stieltjes transform of $\theta(\nu)$ coincides with $\nu^\frown$ on G. Since convolution on $\mathbf{T}$ and on $G^\frown$ both correspond to pointwise multiplication of transforms on G it is possible to extend the map θ defined in this way to a positive norm-decreasing algebra homomorphism $\theta: A(\mu)\to M(G^\frown)$. In fact we have the intermediate result.

(6.3) LEMMA. *θ is an isometric CM-morphism.*

Proof. The fact that every continuous character of $\mathbf{T}$ induces a complex homomorphism χ of $M(\mathbf{T})$ with $|\chi_\mu|\equiv 1$, guarantees an injection $\xi: \mathbf{T}^\frown \to G$. Dualising we have a homomorphism $\xi^*: G^\frown \to \mathbf{T}$ with dense image. Hence, if $P=\sum_{r=1}^n a_r\gamma_r$, $\gamma_r\in T^\frown$, $a_r\in\mathbf{C}$, is a trigonometric polynomial on $\mathbf{T}$, the corresponding polynomial, $\xi(P)=\sum_{r=1}^n a_r\xi(\gamma_r)$, on $G^\frown$ has supremum norm $\|P\circ\xi^*\|_\infty=\|P\|_\infty$. Now a standard application of Eberlein's criterion shows that θ is norm-preserving.

It remains only to prove that if $\omega\ll\theta(\nu)$ then ω is in the image of θ. An approximation argument using the linear and isometric properties of θ reduces this to checking that $\phi\cdot\theta(\nu)\in\theta(A(\mu))$ whenever $\phi\in G(=G^{\frown\frown})$ and $\nu\in A(\mu)$. Recall however that elements of G may be regarded as belonging to $\Delta(A(\mu))$. We may therefore define $\lambda=\phi_\nu\cdot\nu$. Then for all ψ in G,

$$\theta(\lambda)^\frown(\psi)=\lambda^\frown(\psi)=\hat{\nu}(\phi\psi)=(\theta\nu)^\frown(\phi\psi)=(\phi\cdot(\theta\nu))^\frown(\psi),$$

and this proves the lemma.

The homemorphism τ^* induces a CM-isomorphism between $M(G^\frown)$ and $M(\mathbf{Z}\oplus\Gamma(\mu)^\frown)$, and composing this map with θ we obtain an isometric CM-morphism $\Theta: A(\mu)\to M(\mathbf{Z}\oplus\Gamma(\mu)^\frown)$. We continue the proof of (6.2) by obtaining some properties of θ. The first of these is ndependent of the unspecified choice in the definition of τ, but τ will be defined explicitly in the course of the proof to give the second.

(6.4) LEMMA (i) *For all d in D,* $\Theta(\delta(d))=\delta(0,\alpha(d))$

(ii) *Let*

$$A_n=\mathrm{cl}\left\{(0,t): t\in\Gamma(\mu), t=\sum_{i=1}^m \varepsilon_i(d_i),\quad 0\leqslant\varepsilon_i\leqslant n\quad (0\leqslant i\leqslant m)\; m=1,2,3\ldots\right\}.$$

Then $\operatorname{supp}\theta(\mu^n)=(n,0)+A_n$

Proof. (i) The map γ: $S(\mu)\to D^\wedge$, incorporated in the definition of τ, induces a homomorphism β: $G\to D^\wedge$ which is continuous when $D^\wedge$ is topologized as the dual of the discrete group D; also $\beta(G)=\Gamma(\mu)$. The dual map β^*: $D\to G^\wedge$ is also given explicitly by

$$(\phi, \beta^*(d)) = \delta(d)^\wedge(\chi),$$

where $\phi\in G$ and $\chi\in\Delta(M(\mathbf{T}))$ satisfies $\chi_\mu=\phi$. Since γ maps all constant functions to the identity character, it is clear that $\beta^*(D)$ is contained in the annihilator of $\tau^{-1}(\mathbf{T}\oplus(0))$ and a simple check shows that the map $(\tau^*)^{-1}\circ\beta^*$: $D\to\mathbf{Z}\oplus\Gamma(\mu)^\wedge$ is given by $d\mapsto(0, \alpha(d))$.

The equality $\theta(\delta(d))=\delta(\beta^*(d))$ is immediate from a consideration of Fourier transforms, so we obtain $\Theta(\delta(d))=\delta(0,\ \alpha(d))$ as required. (ii) Since $A_n+A_1=A_{n+1}$ and $\operatorname{supp}\Theta(\mu^n)+\operatorname{supp}\Theta(\mu)=\operatorname{supp}\Theta(\mu^{n+1})$, it will suffice to prove this part for $n=1$. In view of part (i) the measures $\Theta(\mu_{0,m})$ are discrete probability measures on $(0)\oplus\Gamma(\mu)^\wedge$ and hence have a weak $*$ limit point ν, say, which by compactness of $\Gamma(\mu)^\wedge$ is itself a probability measure. It is elementary that

$$\operatorname{supp}(\nu) = A_1.$$

Fix a net $(\Theta(\mu_{0,m(\alpha)}))$ converging to ν. We check that $(\Theta(\mu_{r,m(\alpha)}))$ converges for each positive integer r. Note that $\Theta(\mu_{r,n(\alpha)})$ is supported on

$$B_r=\operatorname{cl}\left\{(0,t):t\in\Gamma(\mu)^\wedge, t=\sum_{i=r+1}^{m}\varepsilon_i\alpha(d_i),\ \varepsilon_i=0,1\ (r+1\leqslant i\leqslant m),\quad m=1,2,3,\ldots\right\}$$

and that
$$B_r=B_{r+1}\cup((0,\alpha(d_r))+B_{r+1}).$$

This union is disjoint, for

$$\xi^*\sigma(\tau^*)^{-1}(B_{r+1})=\left\{\sum_{i=r+1}^{\infty}\varepsilon_i d_i:\ \varepsilon_i=0,1\right\}$$

and
$$\xi^*\circ(\tau^*)^{-1}((0,\alpha(d_r))+B_{r+1})=d_r+\xi^*\circ(\tau^*)^{-1}(B_{r+1}),$$

and these two subsets of the circle are disjoint. Accordingly we can find a continuous function $f_r\in C((0)\oplus\Gamma(\mu)^\wedge)$ such that $f_r\equiv 1$ on B_{r+1} and $f_r\equiv 0$ on $(0,\alpha(d_r))+B_{r+1}$.

Now we can make the inductive assumption that $(\Theta(\mu_{r,m(\alpha)}))$ converges to a measure ν_r on $(0)\oplus\Gamma(\mu)^\wedge$ for some $r\geqslant 0$. Then $(f_r\cdot\Theta(\mu_{r,m(\alpha)}))$ also converges. However

$$f_r\cdot\Theta(\mu_{r,m(\alpha)})=\tfrac{1}{2}(f_r\cdot\Theta(\mu_{r+1,m(\alpha)})+f_r\cdot\delta(0,\alpha(d_{r+1}))*\Theta(\mu_{r+1,m(\alpha)}))=\tfrac{1}{2}\Theta(\mu_{r+1,m(\alpha)}).$$

Thus $(\Theta(\mu_{r+1,m(\alpha)}))$ converges, and by induction $\Theta(\mu_{r,m(\alpha)})$ converges for all $r=0,1,2,\ldots$ This implies that, for all $\phi\in\Gamma(\mu)$,

$$(\mu_{r,m(\alpha)})^\wedge(\phi)=\prod_{i=r+1}^{m(\alpha)}\tfrac{1}{2}(1+\phi(d_i))\ \text{ converges.}$$

But recall from the proof of (4.7) that, if $\prod_{i=r+1}^{m(\alpha)}\frac{1}{2}(1+\phi(d_i))$ converges and $\phi(d_i)\neq-1$ for $i>r$, then

$$f_{m(\alpha)}(\phi) = \sum_{d \in D_{m(\alpha)}} \phi(d)\, c_{m(\alpha)}(d)$$

converges to, say, $\boldsymbol{\chi}(\phi) \in G$ with $\boldsymbol{\gamma}(\boldsymbol{\chi}(\phi)) = \phi$.

Now we can write any $\psi \in G$ as $a(\psi)\chi(\boldsymbol{\gamma}(\psi))$, where $a(\psi) \in \mathbf{T}$ and we define $\tau(\psi) = (a(\psi), \boldsymbol{\chi}(\boldsymbol{\gamma}(\psi)))$. The equations $f_n(\phi) f_n(\phi') = f_n(\phi\phi')$, valid for $n \in \mathbf{N}$, $\phi, \phi' \in \Gamma(\mu)$, ensure that $\boldsymbol{\chi}$ is a homomorphism. Of course, π and τ are now explicitly determined.

We have also, for all ψ in G,

$$\begin{aligned}(\delta(1,0) \times \nu)^{\wedge}(\tau(\psi)) &= a(\psi)\, \nu^{\wedge}(\boldsymbol{\chi}(\boldsymbol{\gamma}(\psi))) \\ &= \int a(\psi)\, \boldsymbol{\chi}(\boldsymbol{\gamma}(\psi))\, d\mu \\ &= \int \psi d\mu = (\theta, \mu)^{\wedge}(\psi) = (\Theta(\mu))^{\wedge}(\tau(\psi)),\end{aligned}$$

and this completes the proof.

Observe that modulo the choice of net, the definition of τ is canonical but one can prove that different choices of nets give rise to different isomorphisms τ. On the other hand there is no technical need for introducing τ. It would be possible to start the proof of (6.2) by defining Θ directly on atoms as in (i) of (6.4). After fixing a suitable net, one could extend the definition of Θ to measures of the form $\lim \mu_{r,m}$ in such a way as to ensure (ii) of 6.4. The algebra $A(\mu)$ qua commutative Banach algebra is generated by the measures already specified, subject to certain obvious relations (e.g. $\mu_{0,m} = (\frac{1}{2}\delta(0) + \frac{1}{2}\delta(d_1)) \ast \mu_{1,m}$). The independent power property shows that this is the only sense in which $A(\mu)$ fails to be "free" and makes possible a constructive extension of Θ to $A(\mu)$. The remaining steps of the proof of (6.2), which we now give, are similar in these two approaches.

Let us first verify assertion (ii) of (6.2). Lemma (6.4) proves that, for every measure $\nu \in A_c(\mu)$, $\Theta(\nu)$ is supported on $\mathbf{N} \oplus \Gamma(\mu)$. The canonical injection $k_{\mathbf{N}}: \mathbf{N} \to \mathrm{ap}(\mathbf{N})$ induces a CM-morphism from $M(\mathbf{N} \oplus \Gamma(\mu)^{\wedge})$ to $M(\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge})$. Composing this map with Θ we obtain a CM-morphism Λ: $A_c(\mu) \to M(\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge})$. Since semicharacters of ap (N) separate points the same is true of $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)$. Furthermore $\alpha(D)$ is dense in $\Gamma(\mu)^{\wedge}$, so that the union of the supports of the measures $\Lambda(\delta(d) \ast \mu^n)$ $(d \in D, n = 1, 2, \ldots)$ is dense in $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)$. Thus $\Lambda(A_c(\mu))$ is weak $\ast$-dense in $M(\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge})$.

To prove (ii), therefore, it remains only to show that the non-zero complex homomorphisms of $A_c(\mu)$ correspond to evaluation at a semicharacter of $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge}$. First consider $\chi \in \Delta(A_c(\mu))$ such that $|\chi_\mu| \equiv 1$. By (4.4), χ_μ belongs to $S(\mu)$ and hence to G. Thus $\tau(\chi_\mu)$ can be regarded as a character of $\mathbf{Z} \oplus \Gamma(\mu)^{\wedge}$. By restriction to $\mathbf{N} \oplus \Gamma(\mu)^{\wedge}$ and subsequent transference to $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge}$ via $k_{\mathbf{N}}$ this yields a semicharacter ϕ of $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge}$. Previous definitions give

$$\int \tau(\chi_\mu)\, d\Theta(\nu) = \int \chi_\mu d\theta(\nu) = \int \chi_\nu d\nu,$$

for all $\nu \in A_c(\mu)$. Thus, for such ν,

$$\int \phi\, d\Lambda(\nu) = \nu^\wedge(\chi).$$

Now given arbitrary $\chi \in \Delta(A_c(\mu))$, we decompose $\chi = \chi_1\chi_2$ where $|\chi_1|_\mu = 1$ (μ a.e.) and $(\chi_2)_\mu = a(\mu$ a.e.), where a is a constant satisfying $0 < a \leqslant 1$. We define ϕ_1 corresponding to χ_1 as above, and define ϕ_2 by first setting

$$\psi(n, x) = a^n \quad (n = 1, 2, 3 \ldots;\ x \in \Gamma(\mu)^\wedge)$$

and using the semicharacter ψ of $\mathbf{N} \oplus \Gamma(\mu)^\wedge$ to induce ϕ_2 in $(\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^\wedge)^\wedge$. Now set $\phi = \phi_1 \cdot \phi_2$.

Any measure $\nu \in A_c(\mu)$ can be decomposed as a norm convergent sum, $\nu = \sum_{n=1}^\infty \nu_n$, where each ν_n is a sum of translates (by members of D) of measures absolutely continuous with respect to μ^n. Since $(\chi_2)_{\mu^n} = a^n$ (μ^n a.e.) and $\chi_{\delta(d)}(d) = 1$ for all $d \in D$, we have

$$\nu^\wedge(\chi) = \sum_{n=1}^\infty a^n \nu_n{}^\wedge(\chi_1).$$

Now $\Lambda(\nu_n)$ is supported on $\{n\} \times \Gamma(\mu)^\wedge$, so that

$$\int \phi d\Lambda(\nu_n) = a^n \int \phi_1 d\Lambda(\nu_n) = a^n \nu_n{}^\wedge(\chi_1).$$

Since Λ is an isometric L-homorphism we have

$$\int \phi d\Lambda(\nu) = \sum_{n=1}^\infty \int \phi d\Lambda(\nu_n) = \sum_{n=1}^\infty a^n \nu_n{}^\wedge(\chi_1) = \nu^\wedge(\chi).$$

As every semicharacter of $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)$ clearly gives rise to an element of $\Delta(A_c(\mu))$ this completes the proof that $\Sigma(A_c(\mu))$ can be identified with $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^\wedge$.

The proof of (6.2) (i) is analogous. In this case the supports of the measures $\Theta(\nu)$ for $\nu \in N(\mu)$ are contained in the closure of $\bigcup_{n=1}^\infty (n, 0) + A_n$ in $\mathbf{Z} \oplus \Gamma(\mu)^\wedge$, and this is easily transferred to a closed subsemigroup of $\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^\wedge$. The rest of the proof of (i) is now obvious.

Let us write $\Sigma = \mathrm{ap}(D) \cup (\mathrm{ap}(\mathbf{N}) \times \Gamma(\mu)^\wedge)$ with the topology and multiplication described in (6.2) (iii). Every $\nu \in A(\mu)$ can be decomposed in the form $\nu = \nu' + \nu''$, where $\nu' \in A_c(\mu)$ and ν'' is the discrete part of ν. Then $\Lambda'\colon A(\mu) \to M(\Sigma)$ is defined by

$$\Lambda'(\nu) = \Lambda(\nu') + \Phi(\nu'') \quad (\nu \in A(\mu))$$

where Φ is the canonical map from $A_d(\mu)$ to $M_d(\mathrm{ap}\,(D))$ regarded as a subalgebra of $M(\Sigma)$. Λ' is evidently an isometric L-homomorphism and the extra ingredient in the verification that Λ' is multiplicative corresponds to the fact that the diagram defining $\tilde{\alpha}$: $\operatorname{ap}(D) \to \Gamma(\mu)$ commutes. It is, therefore, easy to see that Λ' is a CM-morphism with weak $*$ dense image.

Any element ψ of $\Sigma^{\wedge}$ which is non-zero on $\operatorname{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge}$ is of the form (a, ϕ) with $a \in C$, $0 < |a| \leqslant 1$, $\phi \in \Gamma(\mu)$. Then, for all $n \in \mathbf{N}$, $x \in \operatorname{ap}(D)$, $t \in \Gamma(\mu)^{\wedge}$,

$$a^n \phi(\alpha(x) + t) = \psi(x + (n, t)) = \psi(x)\psi(n, t) = \psi(x) a^n \phi(t),$$

which gives
$$\psi(x) = \phi \cdot \tilde{\alpha}(x), \quad \text{for all } x \in \operatorname{ap}(D).$$

It follows from this that $\int \psi d\Phi(\nu'') = \int \varphi d\nu''$, for all $\nu \in A(\mu)$, and hence that

$$\int \psi d\Lambda'(\nu) = \int \chi d\nu, \quad (\nu \in A(\mu)),$$

where $\chi \in \Delta(A(\mu))$ is determined by the property that

$$\chi_\mu(s) = a\boldsymbol{\chi}(\phi)\,(s) \quad (\mu \text{ a.e. } s).$$

Any other $\psi \in \Sigma^{\wedge}$ is zero on $\operatorname{ap}(\mathbf{N}) \times \Gamma(\mu)^{\wedge}$ and is induced by some $\gamma \in D^{\wedge}$ on $\operatorname{ap}(D)$. In this case, writing χ for the generalized character which is zero on $A_c(\mu)$ and induced by γ on $A_d(\mu)$, we have

$$\int \chi_\nu d\nu = \int \gamma d\nu'' = \int \psi d\Lambda'(\nu), \quad (\nu \in A(\mu)).$$

Since the elements of $\Sigma^{\wedge}$ evidently separate points of Σ we have indeed found the structure semigroup of $A(\mu)$. This completes the proof of the theorem.

Observe that in the case where μ is coarse $\Gamma(\mu) \approx \mathbf{T}^{\wedge}$ and so $\Gamma(\mu) \approx \mathbf{T}$. In this case a canonical decomposition $G \cong \mathbf{T} \oplus \mathbf{Z}$ is self-evident. In particular there is no question of choosing a net in the proof of (6.4) since the sequence $\theta(\mu_{0,n})$ already converges. On the other hand if μ is fine then inevitably $\theta(\mu_{0,n})$ does *not* converge so that the definition of τ genuinely involves choice.

The first part of the next result indicates that in certain cases the structure semigroup is, indeed, a sharper tool for distinguishing CM-algebras than the maximal ideal space (even regarded as a topological semigroup.) On the other hand, the second part gives a perhaps surprising demonstration that the study of generalized characters can be more effective in this regard.

6.5 COROLLARY. (i) *There exist (infinitely many pairs of) L-subalgebras N, N' of $M(\mathbf{T})$ such that $\Delta(N)$, $\Delta(N')$ are isomorphic as topological semigroups but $\Sigma(N)$ and $\Sigma(N')$ are not isomorphic as semigroups.* (ii) *There exists a collection of (infinitely many) L-subalgebras of $M(\mathbf{T})$, any pair N, N' of which fail to be isomorphic as CM-algebras but are such that $\Sigma(N)$, $\Sigma(N')$ are isomorphic as topological semigroups.*

Proof. (i) We simply take any coarse μ in $\mathbf{B}'$ and let $N = N(\mu)$, $N' = A_c(\mu)$. The verification reduces to proving that there is no semigroup isomorphism $\sigma\colon \Sigma_1 \to \Sigma_2$, where $\Sigma_2 = \mathrm{ap}\,(\mathbf{N}) \times \mathbf{T}$, and Σ_1 is the closure in Σ_2 of the semigroup

$$S = \bigcup_{n=1}^{\infty}\left\{\left(n, \sum_{i=1}^{m} \varepsilon_i d_i\right): 0 \leqslant \varepsilon_i \leqslant n \quad (i = 1, 2, \ldots, m), \quad m = 1, 2, 3 \ldots\right\} = \bigcup_{n=1}^{\infty} (n, 0) + \operatorname{supp} \mu^n.$$

To see this note that $\operatorname{supp} \mu$ is a proper closed subset of $\mathbf{T}$ and hence that there exist positive integers q, p with $p < q$ and $p/q \in \operatorname{supp} \mu$. We see then that the equation $qx = (q, 0)$ (where, of course, qx denotes $x + x + \ldots + x$ q times) has strictly less than q distinct solutions x in Σ_1. Writing $\sigma(1, 0) = (y, t) \in \mathrm{ap}(\mathbf{N}) \times \mathbf{T}$ we note that $\{(y, t + (r/q)): r = 0, 1, \ldots q-1\}$ are q distinct solutions of the equation $qz = \sigma(q, 0)$, for z in Σ_2. This contradicts surjectivity of σ, and gives the required assertion.

(ii) Since $\Sigma(A_c(\mu)) \approx \mathrm{ap}(\mathbf{N}) \times \mathbf{T}$ for every coarse μ in $\mathbf{B}'$, it will suffice to find infinitely many μ's such that the maximal ideal spaces $\Delta(A_c(\mu))$ are non-isomorphic as topological semigroups. Of course, the maximal ideal spaces will be algebraically isomorphic since they are all isomomphic to the semigroup of continuous semicharacters of $\Sigma(A_c(\mu))$. The difference between two of these maximal ideal spaces will be in their Gelfand topologies.

Note that, by (4.4), $\Delta(A_c(\mu))$ is isomorphic to $S(\mu)$ which for a coarse measure can be visualised as a countable union of punctured discs; formally,

$$S(\mu) = \{a\gamma\colon 0 < |a| \leqslant 1; \gamma \in T^{\wedge}\}.$$

We shall prove in a moment that if $S(\mu)$ is topologically isomorphic to $S(\mu')$, and $(\gamma_n) \subset \mathbf{T}^{\wedge}$ converges in $S(\mu)$, then (γ_n) cannot tend to zero in the $\sigma(L^\infty(\mu'), L^1(\mu'))$ – topology. Using this fact it is not difficult to produce infinitely many coarse measures $\mu_r (r = 0, 1, 2, \ldots)$ for which $\Delta(A_c(\mu_r))$ are all different (as topological semigroups). The measures μ_r are defined by

$$\mu_r = \mathop{\ast}_{n=1}^{\infty} \tfrac{1}{2}(\delta(0) + \delta(2^{-r}3^{-n})),$$

$(r = 0, 1, 2, \ldots)$. We make use of (2.1) to prove the convergence of a sequence (γ_n) of continuous characters in $S(\mu)$. To determine whether a sequence (γ_n) tends to 0 in the $\sigma(L^\infty(\mu)$, $L^1(\mu))$, we apply the following elementary result from [3]:

(6.6) LEMMA. *Let μ be a positive measure on* $\mathbf{T}$, *and suppose that $\hat{\mu}(n_i + p) \to 0$ as $i \to \infty$ for all $p \in \mathbf{Z}$. Then* $\exp 2\pi i n_i t \to 0$, $\sigma(L^\infty(\mu), L^1(\mu))$.

New consider the sequence $(2^k 3^n)_n$. It is clear that $\exp 2\pi i 2^k 3^n d \to 1$ for all $d \in D(\mu_r)$ provided $r \leqslant k$. Moreover in this case

$$|\hat{\mu}_r(2^k 3^n)| = \prod_{m=1}^{\infty} |\cos \pi\, 2^{k-r} 3^{-m}| > 0$$

for all $n>0$. By (2.1), there is, for each $r\leqslant k$, some subsequence of (2^k3^n) which converges in $S(\mu_r)$. On the other hand, when $r=k+1$,

$$|\mu_r(2^k3^n+p)|<\left|\cos\pi\left(\frac{1}{2}+\frac{p}{2^k\,3^n}\right)\right|\to 0 \quad \text{as } n\to\infty,$$

for every integer p. Therefore

$$\exp 2\pi i 2^k 3^n t\to 0,\ \sigma(L^\infty(\mu_{k+1}),\, L^1(\mu_{k+1})) \text{ as } n\to\infty.$$

Thus the proof of the pairwise non-isomorphism of the topological semigroups $\Delta(A_c(\mu_r))$ $(r=0, 1, 2, \dots)$ will be completed when we have checked our earlier assertion on the behaviour of sequences of continuous characters in isomorphic $S(\mu)$ and $S(\mu')$.

Let i: $S(\mu)\to S(\mu')$ be a topological semigroup isomorphism, and let $I(\mu)$, $I(\mu')$ be the subgroups of invertible elements in $S(\mu)$, $S(\mu')$ respectively. Clearly i induces an isomorphism (also denoted by i) between $I(\mu)$ and $I(\mu')$. It is evident that

$$I(\mu)=\{a\gamma\colon |a|=1,\ \gamma\in\mathbf{T}^\wedge\}$$

and because μ is coarse the proof of (3.2) implies that $I(\mu)$ with the Gelfand topology is isomorphic to $\mathbf{T}\times\mathbf{Z}$. Thus i must map the connected component of $I(\mu)$ to that of $I(\mu')$, the connected component being, in both cases, the group of constants of absolute value 1. Now because the only automorphisms of $\mathbf{Z}$ (which is isomorphic to $I(\mu)$ factored by its connected component), are the identity and $n\mapsto -n$, it follows easily that either $i(\gamma)=b(\gamma)\gamma$ where $|b(\gamma)|=1$ for all $\gamma\in T^\wedge$ or $i(\gamma)=b(\gamma)\gamma^{-1}$ for all $\gamma\in T$. In either case, if (γ_n) converges in $S(\mu)$, then $(i(\gamma_n))$ converges in $S(\mu')$, and so some subsequence (γ_{n_k}) tends to 0 in the $\sigma(L^\infty(\mu), L^1(\mu))$-topology. This completes the proof.

It is possible to obtain trivial examples of the phenomenon described in (ii) by considering the measure algebras $L^1(G)$ where G is an LCA group. For these, it is known (see [19]) that $\Delta(L^1(G))\approx G^\wedge$ with the dual topology, and $\Sigma(L^1(G))\approx bG$—the Bohr compactification of G. Thus to obtain a version of (ii) it is sufficient to produce infinitely many non-isomorphic LCA groups with the same Bohr compactification. Dualising, this is equivalent to finding infinitely many essentially different locally compact group topologies on a fixed group. A simple example of this can be described as follows. Let G be the algebraic group $\prod_{n=1}^\infty \mathbf{Z}(p_n)$ where (p_n) is an enumeration of the primes. For any fixed subset C of $\mathbf{N}$, we assign a topology τ_C to G by making $\prod_{n\in C} Z(p_n)$ a compact open subgroup of (G, τ_C). It is not difficult to see that unless C and C' differ by only finitely many integers, then (G, τ_C) and $(G, \tau_{C'})$ are non-isomorphic.

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