POISSON FORMULA AND COMPOUND DIFFUSION ASSOCIATED TO AN OVERDETERMINED ELLIPTIC SYSTEM ON THE SIEGEL HALFPLANE OF RANK TWO

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Introduction

Let $z=(z_1, z_2, z_3)\in\mathbf{C}^3$; we write $z_j=x_j+iy_j$ $(1\leqslant j\leqslant 3)$; we also use polar coordinates (ϱ, θ) in the (y_2, y_3)-plane when convenient. $H=\{z\,|\,y_1-\varrho>0\}$ is a tube domain over a circular cone; by a linear change of coordinates it is equivalent to the Siegel upper halfplane

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of two-by-two matrices. With the Bergman metric $ds^2 = \partial\bar\partial \log (y_1^2 - \varrho^2)$, H is a Hermitian symmetric space. We denote by Δ_1 the Laplacian associated with the metric.

It is a special case of a theorem of Furstenberg [2] that every bounded solution of $\Delta_1 F = 0$ has a Poisson integral representation on the "maximal boundary" which in our case can be identified with a compactification of $\mathbf{R}^3 \times \mathbf{T}^1$. The values $f(x, \theta)$ of the boundary function can be recovered from F by taking limits along paths in H whose x-coordinates are fixed and whose imaginary parts tend to the vertex of the cone while becoming tangent to the generator of the cone having the coordinate θ [6].

The Bergman-Šilov boundary of H is $\mathbf{R}^3$ (or more precisely, a compactification of $\mathbf{R}^3$). As first shown by C. C. Moore, it coincides with one of the non-maximal Furstenberg-Satake boundaries. There is a Poisson kernel on $\mathbf{R}^3$, it can be gotten from the previous one by integrating out the θ-variable, which still has the property of reproducing all bounded holomorphic functions on H. All functions representable by this more special Poisson integral are annihilated by further second order differential operators besides Δ_1.

This last observation, in another case, was first made by Hua [4]. In general, for symmetric domains of tube type, E. M. Stein, J. A. Wolf and the first named author showed several years ago that there are always k independent such operators, where k is the dimension of the isotropy group (unpublished). Stein posed the question if these operators characterize the class of the more special Poisson integrals.

In the present paper, in the case of H, we take one elliptic operator Δ_2 independent of Δ_1, which annihilates all the special Poisson integrals, and we prove that every bounded solution of the system

$$\Delta_1 F = 0; \quad \Delta_2 F = 0,$$

can be represented as a special Poisson integral on $\mathbf{R}^3$.

The method we use can be described as follows. We introduce a linear combination Δ_1' of Δ_1 and Δ_2 which is still elliptic and consider the diffusion processes $z_\omega^{(1)}(t)$, $z_\omega^{(2)}(t)$ associated to Δ_1' and Δ_2, respectively. The sample paths of both processes tend towards the vertex of the cone. At the same time Δ_2 tends rapidly towards the boundary $\{\varrho = y\}$, while Δ_1' tends away from it, towards the axis. With the aid of two hypersurfaces in H we will introduce a sequence of stopping times and define a *compound process* $z_\omega(t)$ which between consecutive stopping times will alternately be governed by Δ_1' and Δ_2. For a solution of the system $\Delta_1 F = \Delta_2 F = 0$, $F(z_\omega(t))$ will then be a martingale.

Properly choosing the two hypersurfaces we can arrange that, as $t \to \infty$,

(i) the x_j-coordinates of $z_\omega(t)$ tend to definite limits a.s.

(ii) y_1 and $y_1 - \varrho/y_1$ tend to 0 a.s.

(iii) θ oscillates indefinitely a.s.

Using these properties we prove the main result as follows. By a regularization argument we may assume that the Furstenberg boundary function $f(x, \theta)$ of F is continuous. Now $f(x, \theta)$ must be independent of θ, or else the martingale convergence theorem would lead to a contradiction to the Fatou-type theorems that allow to recover f from F. This is enough to show that F is a Poisson integral on the Bergman-Šilov boundary.

Our method of *compound processes* can, of course, be applied much more generally, to any overdetermined elliptic system, and more specifically to boundary problems for analytic functions of several complex variables. It could be also generalized, the shift between the two elliptic operators being realized by a Poisson law with density depending on the position. But in all the cases the effectiveness of the method depends on delicate numerical estimates which are worked out in our case in a very specific situation.

For many useful discussion on the subject of this paper we would like to express our gratutude to Professor Stanley Sawyer.

I. Preliminaries

1.1. Definitions and notations

Given $z=(z_1, z_2, z_3)\in \mathbf{C}^3$, we write $z_j=x_j+iy_j$ $(1\leqslant j\leqslant 3)$. We consider the domain

$$H=\{z\in \mathbf{C}^3 \,|\, y_1>0,\ y_1^2-y_2^2-y_3^2>0\},$$

which is a symmetric generalized halfplane. Its Poisson kernel with respect to the Bergman-Šilov boundary has the form [5]

$$P(z,u)=c\frac{A(-i(z-\bar{z}))^{3/2}}{|A(-i(z-u))|^3}, \tag{1.1.1}$$

where $z\in H$, $u\in \mathbf{R}^3$, c is a constant and $A(z)=z_1^2-z_2^2-z_3^2$.

Introducing the new coordinates

$$\begin{aligned} w_1 &= 2^{-\frac{1}{2}}(z_1+z_2),\\ w_2 &= 2^{-\frac{1}{2}}(z_1-z_2),\\ w_3 &= z_3, \end{aligned}$$

and writing

$$W=\begin{pmatrix} 2^{1/2}w_1, & w_3\\ w_3, & 2^{1/2}w_2 \end{pmatrix},$$

it becomes apparent that H is just the Siegel upper halfplane $\{\operatorname{Im} W \succ 0\}$.

If we write $V = \operatorname{Im} W$ and

$$\partial_W = \begin{pmatrix} 2^{1/2}\dfrac{\partial}{\partial W_1}, & \dfrac{\partial}{\partial W_3} \\ \dfrac{\partial}{\partial W_3}, & 2^{1/2}\dfrac{\partial}{\partial W_2} \end{pmatrix}$$

then the entries of the formal product matrix $\bar{\partial}_W \cdot V \cdot \partial_W$ give four second order differential operators, each of which annihilate $P(z, u)$ for all fixed $u \in \mathbf{R}^3$. This can be seen by imitating the argument in [4, p. 117] or by direct checking.

A linear combination (with non-constant coefficients) of these operators is the Laplace-Beltrami operator with respect to the Bergman metric which, rewritten in terms of the z_j-coordinates, has the form

$$\Delta_1 = \frac{2}{3}\left[A(y)\left(-\partial_1\bar{\partial}_1 + \sum_{j=2}^{3}\partial_j\bar{\partial}_j\right) + 2\sum_{k=1}^{3}\sum_{l=1}^{3} y_k y_l \partial_k \bar{\partial}_l\right]$$

(we wrote ∂_j for $\partial/\partial z_j$). Another combination of the operators, in fact a multiple of $\operatorname{tr} \bar{\partial}_W \cdot V \cdot \partial_W$, becomes in terms of the z_j-coordinates

$$\Delta_2 = 4y_1^2 \sum_{k=1}^{3} \partial_k \bar{\partial}_k + 4\sum_{j=2}^{3} y_1 y_j (\partial_1 \bar{\partial}_j + \bar{\partial}_1 \partial_j).$$

This operator is also elliptic, and it is invariant under real translations and under real rotations in the (z_2, z_3)-plane.

Our objects of study are the bounded solutions of the system

$$\Delta_1 F = 0,$$
$$\Delta_2 F = 0.$$

It will turn out that such an F can always be written as the Poisson integral with the aid of (1.1.1) of a bounded measurable function on $\mathbf{R}^3$.

We will also use the operator

$$\Delta_1' = 3\Delta_1 - \frac{1}{2}\frac{y_1 - \varrho}{y_1}\Delta_2,$$

which is easily seen to be (degenerate) elliptic. ϱ here is defined by

$$\varrho = \varrho(z) = (y_2^2 + y_3^2)^{1/2}.$$

We also define

$$\theta = \theta(z) = \cos^{-1}\frac{y_2}{\varrho},$$

and two other important functions by

$$\xi(z) = -\log y_1,$$

$$\chi(z) = -\log \frac{y_1 - \varrho}{y_1}.$$

Either one of the triples (y_1, y_2, y_3), (y_1, ϱ, θ) or (ξ, χ, θ) can be used to describe the imaginary part of a point $z \in H$; the last one is in many ways the most natural.

Next we choose a function $\Lambda \in C^2(\mathbf{R})$ having the following properties:

(i) Λ is monotone increasing,

(ii) $\Lambda \geqslant 2$,

(iii) $|\Lambda'|, |\Lambda''| < 1/16$,

(iv) $\Lambda(t) = \log t$ for large values of t.

It is obvious that such a Λ exists. Now we define the function ϕ on H by

$$\phi(z) = \Lambda(\xi(z)) = \Lambda(-\log y_1),$$

and the subdomains $H^{(i)}$ $(1 \leqslant i \leqslant 2)$ by

$$H^{(1)} = \{z \in H \mid \chi(z) > \phi(z)\},$$

$$H^{(2)} = \{z \in H \mid \chi(z) < \phi(z) + 1\}.$$

1.2. Recall of some results

Suppose that

$$\Delta = \frac{1}{2} \sum_{jk} e_{jk} D_j D_k + \sum_j f_j D_j$$

is an elliptic operator on a domain in $\mathbf{R}^n$, and let $u_\omega(t)$ be the process governed by Δ in the sense of [8, p. 90]. We denote by ∇ the "gradient" associated to Δ, i.e. for any F we write

$$\|\nabla F\|^2 = \sum_{jk} e_{jk} (D_j F)(D_k F).$$

We note here the obvious but useful formulas

$$\|\nabla \psi \circ F\|^2 = (\psi')^2 \cdot \|\nabla F\|^2, \tag{1.2.1}$$

$$\Delta(\psi \circ F) = \tfrac{1}{2} \psi'' \cdot \|\nabla F\|^2 + \psi' \cdot \Delta F. \tag{1.2.2}$$

An application of Ito's Lemma [8, pp. 32, 44] to the stochastic differential $dF(u_\omega(t))$ gives

$$F(u_\omega(t)) = F(u_\omega(0)) + \int_0^t (\Delta F)(u_\omega(s))\, ds + \int_0^t \|\nabla F\| (u_\omega(s))\, db_\omega(s), \tag{1.2.3}$$

where $b_\omega(s)$ is an ordinary Brownian motion. (1.2.3) is valid for all stopping times t not greater than the lifetime of the process.

Suppose now that $u_\omega(t)$ has infinite lifetime, but the one-dimensional process $F(u_\omega(t))$ is being stopped by an absorbing barrier at $F(u_\omega(0))+h$ $(h>0)$. Let τ be the lifetime of this process, i.e.

$$\tau = \min\{t>0 \mid F(u_\omega(t)) - F(u_\omega(0)) \geqslant h\}.$$

We can study τ with the aid of the following considerations, used also in [7].

On each sample path we define the intrinsic time t^* by (cf. also [3])

$$t^* = \int_0^t \|\nabla F\|^2(u_\omega(s))\,ds. \tag{1.2.4}$$

We write $u_\omega^*(t^*) = u_\omega(t)$, i.e. u_ω^* is the process u_ω reparametrized by the intrinsic time. Then

$$F(u_\omega^*(t^*)) = F(u_\omega(0)) + \int_0^{t^*} \frac{\Delta F}{\|\nabla F\|^2}(u_\omega^*(s))\,ds + b_{\omega_1}(t^*), \tag{1.2.5}$$

where b_{ω_1} is again ordinary Brownian motion and $\omega \to \omega_1$ is a probability-preserving map.

If we have a uniform estimate

$$0 < k \leqslant \|\nabla F\|^2, \tag{1.2.6}$$

then, by (1.2.4),

$$k\tau \leqslant \tau^*.$$

If we also have a uniform estimate

$$\beta \leqslant \frac{\Delta F}{\|\nabla F\|^2}, \tag{1.2.7}$$

then, denoting by $Y_\omega(t)$ the process on $\mathbf{R}$ governed by

$$\tfrac{1}{2}D^2 + \beta D$$

and started at 0, we have

$$Y_\omega(t^*) \leqslant F(u_\omega^*(t^*))$$

for all t^*. If we denote by τ_0 the absorption time at h of $Y_\omega(t)$, i.e.

$$\tau_0 = \min\{t>0 \mid Y_\omega(t) \geqslant h\},$$

then it follows that

$$\tau^* \leqslant \tau_0.$$

So (1.2.6) and (1.2.7) together imply

$$\tau \leqslant \frac{\tau_0}{k}. \tag{1.2.8}$$

If we have the reverse inequalities in both (1.2.6) and (1.2.7), all the other inequalities also turn around, and instead of (1.2.8) we get the reverse estimate

$$\tau \geqslant \frac{\tau_0}{k}.$$

About τ_0 we have rather precise information. By the methods indicated in [1, vol. 2, pp. 450–453] it is easy to check that, for $\beta < 0$,

$$P[\tau_0 < \infty] = \frac{-1}{2\beta} e^{2\beta h}. \tag{1.2.9}$$

For $\beta > 0$,

$$E[e^{-\lambda\tau_0}] = \exp\left(-h[(\beta^2+2\lambda)^{1/2}-\beta]\right) \tag{1.2.10}$$

for all $\lambda < -(\beta^2/2)$. (In particular, $\tau_0 < \infty$, a.s.)

At one point we will also need the two-sided absorption time τ_{00} of $Y_\omega(t)$, i.e.

$$\tau_{00} = \min\{t > 0 \,\big|\, |Y_\omega(t)| \geqslant h\}.$$

If $\beta \neq 0$, we have

$$E[e^{-\lambda\tau_{00}}] = \frac{\cosh \beta h}{\cosh \beta h\left(1 + 2\dfrac{\lambda}{\beta^2}\right)^{1/2}}. \tag{1.2.11}$$

For small λ both (1.2.10) and (1.2.11) are majorized by $e^{-c\lambda}$ with some positive constant c.

1.3. Some numerical estimates

In this section we collect some results that will be used repeatedly in the sequel. We denote by ∇_1', ∇_2 the gradients associated to Δ_1', Δ_2 in the sense of section 1.2. The other definitions are in section 1.1.

LEMMA 1.3.1. *At every point of $H^{(1)}$ we have*

$$\frac{3}{2} < \|\nabla_1'\xi\|^2 < 2, \qquad \frac{\Delta_1'\xi}{\|\nabla_1'\xi\|^2} = \frac{1}{2},$$

$$\frac{3}{2} < \|\nabla_1'\chi\|^2 < 2, \qquad -\frac{1}{4} < \frac{\Delta_1'\chi}{\|\nabla_1'\chi\|^2} < -\frac{1}{6}.$$

Proof. Direct computation gives

$$\|\nabla_1'(\xi)\|^2(z) = \frac{\varrho}{y_1^2}(y_1 + \varrho).$$

Since $\varrho < y_1$ on all H, this is majorized by 2. Since $\phi = \Lambda \circ \xi > 2$ by definition of Λ and ϕ, on $H^{(1)}$ we have $\chi(z)>2$, which means $(y_1-\varrho)/y_1<e^{-2}$. This implies the lower estimate of $\|\nabla_1'\xi\|^2$. The proof of the other assertions is similarly based on direct computations.

LEMMA 1.3.2. *On $H^{(2)}$ we have*

$$\|\nabla_2\xi\|^2=2, \quad \frac{\Delta_2\xi}{\|\nabla_2\xi\|^2}=\frac{1}{2},$$

$$2<\|\nabla_2\chi\|^2<4e^{\chi}, \quad \frac{3}{4}<\frac{\Delta_2\chi}{\|\nabla_2\chi\|^2}.$$

On $\Gamma=\{z\in H\,|\,\varrho(z)>y_1/2\}$ also

$$\frac{\Delta_2\chi}{\|\nabla_2\chi\|^2}<\frac{7}{6}.$$

Proof. One proceeds by computing precise expressions and then using elementary estimates. For example,

$$\|\nabla_2\chi\|^2(z)=2\frac{y_1+\varrho}{y_1-\varrho}<4\frac{y_1}{y_1-\varrho}=4e^{\chi(z)}.$$

LEMMA 1.3.3. *Let $\bar{\chi}=\chi-\phi$ and $\bar{\bar{\chi}}=\chi-\log\phi$. On $H^{(1)}$, $\bar{\chi}$ and $\bar{\bar{\chi}}$ satisfy inequalities of the same type as χ with constants having the same sign as in Lemma 1.3.1.*

On $H^{(2)}$ we have

$$1<\|\nabla_2\bar{\chi}\|^2<5e^{\chi}, \quad \frac{9}{16}<\frac{\Delta_2\bar{\chi}}{\|\nabla_2\bar{\chi}\|^2},$$

and, on $\Gamma=\{z\in H|\varrho(z)>y_1/2\}$,

$$\frac{\Delta_2\bar{\chi}}{\|\nabla_2\bar{\chi}\|^2}<2.$$

$\bar{\bar{\chi}}$ also satisfies the same inequalities.

Proof. Since $\phi=\Lambda\circ\xi$, the formulas (1.2.1) and (1.2.2) together with property (iii) of Λ and Lemma 1.3.2 give

$$\|\nabla_2\phi\|^2=(\Lambda')^2\|\nabla_2\xi\|^2<\tfrac{1}{100},$$

$$|\Delta_2\phi|^2\leqslant|\Lambda''|\cdot\tfrac{1}{2}\|\nabla_2\xi\|^2+|\Lambda'|\cdot|\Delta\xi|<\tfrac{1}{8}.$$

The same estimates hold also for $\log\phi$, since the derivatives of $\log\Lambda$ have the same bounds as those of Λ (since $\Lambda\geqslant 2$ everywhere). It follows that

$$|\Delta_2\bar{\chi}-\Delta_2\chi|<\tfrac{1}{8},$$

$$\Big|\|\nabla_2\bar{\chi}\|-\|\nabla_2\chi\|\Big|\leqslant\tfrac{1}{10},$$

the same inequalities being true also for $\bar{\chi}$ in place of $\bar{\chi}$. Comparing these with the inequalities in Lemma 1.3.2 the assertions about $\Delta_2\bar{\chi}$, $\|\nabla_2\bar{\chi}\|$, $\Delta_2\bar{\chi}$, $\|\nabla_2\bar{\chi}\|$ follow by an elementary calculation.

The proofs of the assertions about $\Delta_1'\bar{\chi}$, etc., are exactly analogous.

LEMMA 1.3.4. *At all points* $z \in H$,

$$\Delta_1' x_j = \Delta_2 x_j = 0,$$
$$\|\nabla_1' x_j\|^2, \ \|\nabla_2 x_j\|^2 \leqslant 2y_1^2,$$

for all $1 \leqslant j \leqslant 3$. *Furthermore,*

$$\Delta_1'\theta = \Delta_2\theta = 0, \quad \|\nabla_2\theta\|^2 > 2.$$

Proof. Direct computation.

II. The compound process $z_\omega(t)$

2.1. Construction of $z_\omega(t)$

We denote by $\partial H^{(i)}$ $(1 \leqslant i \leqslant 2)$ the boundary of $H^{(i)}$ *relative to* H. So $\partial H^{(1)} \subset H^{(2)}$ and $\partial H^{(2)} \subset H^{(1)}$; they are the hypersurfaces defined by $\bar{\chi} = \chi - \varphi = 0$, resp. $\bar{\chi} = 1$.

We consider the processes $z_\omega^{(i)}(t)$ $(1 \leqslant i \leqslant 2)$ defined on $H^{(i)}$ and governed by the operators Δ_1', resp. Δ_2. We denote their lifetimes by $\tau^{(1)}$ resp. $\tau^{(2)}$.

For $z^0 \in H^{(i)}$ $(1 \leqslant i \leqslant 2)$ we denote by $\Omega^{(i)}(z^0)$ the sample space of paths $z_\omega^{(i)}(t)$ starting at z^0. The corresponding expectation we denote by $E_{z^0}^{(i)}$, or simply E_{z^0} where this can not cause confusion.

The following lemma shows that $z_\omega^{(i)}(t)$ actually ends by being absorbed at $\partial H^{(i)}$, and that $\tau^{(i)}$ is finite $(1 \leqslant i \leqslant 2)$. These facts are needed for the construction of our compound process. Statement (ii) of the lemma is a byproduct of the proof of (i), and will be used in section 3.2.

LEMMA 2.1.1. (i) $\tau^{(i)} < +\infty$ *and* $z_\omega^{(i)}(\tau^{(i)}) \in \partial H^{(i)}$ *a.s.* $(1 \leqslant i \leqslant 2)$.

(ii) *There exists a constant* $c > 0$ *such that, for sufficiently small* $|\lambda|$,

$$E_{z^0}[\exp(\lambda\tau^{(1)})] < e^{c\lambda} \tag{2.1.1}$$

uniformly for $z^0 \in \partial H^{(2)}$, *and, for sufficiently small* $\lambda > 0$,

$$E_{z^0}[\exp(\lambda\tau^{(2)})] < e^{c\lambda} \tag{2.1.2}$$

uniformly for $z^0 \in \partial H^{(1)}$. *The left-hand sides of* (2.1.1) *and* (2.1.2) *are finite for every* z^0 *in* $H^{(1)}$ *resp.* $H^{(2)}$.

Proof. First we show that $\tau^{(i)}$ equals the lifetime of the process $\bar{\chi}(z_\omega^{(i)}(t))$. Since ξ, θ, $\bar{\chi}$, x_j $(1 \leqslant j \leqslant 3)$ can be used as a coordinate system, it will be sufficient for this to show that the projections of $z_\omega^{(i)}(t)$ onto the other coordinates, i.e. the one-dimensional processes $\xi(z_\omega^{(i)}(t))$, ... have infinite lifetime a.s. (see [10]).

The comparison method of section 1.2 and the estimates in Lemmas 1.3.1 and 1.3.2 imply at once that $\xi(z_\omega^{(i)}(t))$ has infinite lifetime. One also sees from (1.2.5) that

$$\xi(z_\omega^{(i)*}(t^*)) = \xi(z^0) + \frac{t^*}{2} + b_{\omega_1}(t^*),$$

and

$$\tfrac{3}{2}t < t^* \leqslant 2t.$$

Using that $|b_{\omega_1}(t^*)| < (t^*)^{\varepsilon + \frac{1}{2}}$ for large t^*, a.s., this implies

$$y_1(z_\omega^{(i)}(t)) \leqslant y_1(z^0) \exp\left(-\frac{t}{2}\right), \tag{2.1.3}$$

for sufficiently large t, a.s.

To show that the lifetime of $x_j(z_\omega^{(i)}(t))$ is infinite we use the same method. By Lemma 1.3.4 the intrinsic time t^* is majorized by

$$t^* < 2\int_0^t y_1(z_\omega^{(i)}(s))\,ds,$$

and reparametrized by t^* the process is a pure Brownian motion. (2.1.3) shows that $t^* < +\infty$ for all $t > 0$, a.s. From this the infiniteness of the lifetime follows.

It is obvious that there can be no limitation on the lifetime of $\theta(z_\omega^{(i)}(t))$, so we have shown that $\tau^{(i)}$ is a.s. equal to the lifetime of $\chi(z_\omega^{(i)}(t))$, which also shows that $z_\omega^{(i)}(\tau^{(i)}) \in H^{(i)}$ $(1 \leqslant i \leqslant 2)$.

By definition of $H^{(1)}$ and $H^{(2)}$ the lifetime of $\bar{\chi}(z_\omega^{(i)}(t))$ is the time of its absorption by 0 resp. 1. $z^0 \in H^{(1)}$ means that $\bar{\chi}(z^0) > 0$; $\tau^{(1)}$ can now be estimated by applying the method of section 1.2 to $-\bar{\chi}$. From (1.2.10) and from Lemma 1.3.3 it follows that $\tau^{(1)}$ is finite. If we know in addition that $z^0 \in \partial H^{(2)}$, then $\bar{\chi}(z^0) = 1$ and (1.2.10) even gives the part concerning $\tau^{(1)}$ of statement (*ii*). We proceed exactly analogously with $\tau^{(2)}$, finishing the proof of the Lemma.

Now we begin the construction of $z_\omega(t)$. We take disjoint copies $\tilde{H}^{(1)}$ and $\tilde{H}^{(2)}$ of $H^{(1)}$, $H^{(2)}$, and set $\tilde{H} = \tilde{H}^{(1)} \cup \tilde{H}^{(2)}$. This is a two-sheeted domain with a natural projection onto H. If $z \in H$, $\tilde{z}$ will always denote a point in $\tilde{H}$ whose projection is z. Every function F on H lifts naturally to $\tilde{H}$; we do not use a different symbol for the lift of F.

Suppose first that $\tilde{z}^0 \in \tilde{H}^{(1)}$. The elements of the sample space $\Omega(\tilde{z}^0)$ of the new process will be sequences

$$\omega = (\omega_1^{(1)}, \omega_1^{(2)}, \omega_2^{(1)}, \omega_2^{(2)}, \omega_3^{(1)}, \dots),$$

such that $\omega_1^{(1)} \in \Omega^{(1)}(z^0)$ and then, inductively,

$$\omega_j^{(2)} \in \Omega^{(2)}(z_{\omega_j^{(1)}}(\tau^{(1)})),$$

$$\omega_{j+1}^{(1)} \in \Omega^{(1)}(z_{\omega_j^{(2)}}(\tau^{(2)})).$$

We will use the notation

$$T_j = T_j(\omega) = \sum_{i=1}^{j} (\tau^{(1)}(\omega_i^{(1)}) + \tau^{(2)}(\omega_i^{(2)})),$$

$$T_{j+1}^{(1)} = T_j + \tau^{(1)}(\omega_{j+1}^{(1)}),$$

and also, for brevity, $\tau_j^{(i)}$ for $\tau^{(i)}(\omega_j^{(i)})$.

For $\omega \in \Omega(\tilde{z}^0)$ we define the sample path $z_\omega(t)$ inductively by

$$z_\omega(t) = \begin{cases} \tilde{z}_{\omega_{j+1}^{(1)}}(t - T_j) \in \tilde{H}^{(1)} & \text{if} \quad T_j \leqslant t < T_{j+1}^{(1)} \\ \tilde{z}_{\omega_{j+1}^{(2)}}(t - T_{j+1}^{(1)}) \in \tilde{H}^{(2)} & \text{if} \quad T_{j+1}^{(1)} \leqslant t < T_{j+1} \end{cases}.$$

The probability measure $P_{\tilde{z}^0}(d\omega)$ on $\Omega(\tilde{z}^0)$ will be given, for functions φ depending only on finitely many components of ω, by

$$\int \varphi(\omega) P_{\tilde{z}^0}(d\omega) = \int_{\Omega^{(1)}(z^0)} P(d\omega_1^{(1)}) \int_{\Omega^{(2)}(z_{\omega_1^{(1)}}(\tau^{(1)}))} P(d\omega_1^{(2)}) \dots \varphi(\omega),$$

and by extension for more general φ.

If $\tilde{z}^0 \in \tilde{H}^{(2)}$ the elements of $\Omega(\tilde{z}^0)$ will be sequences starting with $\omega_1^{(2)} \in \Omega^{(2)}(z^0)$. In this case we set $\tau_1^{(1)} = \tau^{(1)}(\omega_1^{(1)}) = 0$ by definition; we define T_j, $T_j^{(1)}$, $z_\omega(t)$ by the same formulas as before.

2.2. Basic properties of $z_\omega(t)$

$z_\omega(t)$ is now a process on $\tilde{H}$; $\dots < T_j < T_{j+1}^{(1)} < T_{j+1} < \dots$ are stopping times for it. At the moment it is defined only up to some explosion time, but it will follow from section 3.2 that its lifetime is actually infinite.

LEMMA 2.2.1. *The formula* (1.2.3) *remains valid for the process* $z_\omega(t)$, *i.e. for any* $F \in C^2(H)$ *and any stopping time* t *we have*

$$F(z_\omega(t)) = F(z^0) + \int_0^t (\Delta F)(z_\omega(s))\, ds + \int_0^t \|\nabla F\|(z_\omega(s))\, db_\omega(s), \tag{2.2.1}$$

where b_ω *is a Brownian motion and where* ΔF *(resp.* $\|\nabla F\|$*) means* $\Delta_1' F$ *or* $\Delta_2 F$ *(resp.* $\|\nabla_1' F\|$ *or* $\|\nabla_2 F\|$*) depending on whether* $z_\omega(s)$ *is on* $\tilde{H}^{(1)}$ *or on* $\tilde{H}^{(2)}$.

Proof. We carry out the proof only for the case where $\tilde{z}^0 \in \tilde{H}^{(1)}$; the case of $\tilde{H}^{(2)}$ requires only minor modifications.

For each $z \in H^{(i)}$ $(1 \leqslant i \leqslant 2)$ we have a Brownian motion $b_{\omega^{(i)}}(t)$ defined on $\Omega^{(i)}(\tilde{z})$. With the aid of these, retaining the notations of section 2.1, we define $b_\omega(t)$ for $\omega \in \Omega(z^0)$ inductively by

$$b_\omega(t) = \begin{cases} b_\omega(T_j) + b_{\omega^{(1)}_{j+1}}(t - T_j) & \text{if} \quad T_j \leqslant t < T^{(1)}_{j+1} \\ b_\omega(T^{(1)}_{j+1}) + b_{\omega^{(2)}_{j+1}}(t - T^{(1)}_{j+1}) & \text{if} \quad T^{(1)}_{j+1} \leqslant t < T_{j+1}. \end{cases}$$

One can check then that $b_\omega(t)$ is a Brownian motion.

Now (1.2.3) gives

$$F(z_\omega(T^{(1)}_{j+1})) - F(z_\omega(T_j)) = \int_0^{\tau^{(1)}} (\Delta_1' F)(z^{(1)}_{\omega^{(1)}_{j+1}}(s))\, ds + \int_0^{\tau^{(1)}} \|\nabla_1' F\| (z^{(1)}_{\omega^{(1)}_{j+1}}(s))\, db_{\omega^{(1)}_{j+1}}(s). \tag{2.2.2}$$

Using the definitions of $z_\omega(t)$ and $b_\omega(t)$, the right-hand side of this equality can be written as

$$\int_{T_j}^{T^{(1)}_{j+1}} (\Delta F)(z_\omega(s))\, ds + \int_{T_j}^{T^{(1)}_{j+1}} \|\nabla F\| (z_\omega(s))\, db_\omega(s). \tag{2.2.3}$$

An analogous formula holds also for the intervals $[T^{(1)}_{j+1}, T_{j+1}]$. Summation over j of these formulas proves (2.2.1) for the cases where t is equal to one of the T_j's or $T^{(1)}_j$'s.

To extend this to the case of an arbitrary t, we introduce the stopping times $T_j \wedge t$, $T^{(1)}_j \wedge t$. Using these instead of T_j, $T^{(1)}_j$, (2.2.2) and (2.2.3) remain valid: This is trivial if $t \notin [T_j, T^{(1)}_{j+1}]$, and follows by general properties of stopping times if $t \in [T_j, T^{(1)}_{j+1}]$. Again we have analogous formulas for the intervals $[T^{(1)}_{j+1}, T_{j+1}]$, and summation over j, as before, proves the general case of the lemma.

COROLLARY 1. *If F is a function on H such that $\Delta_1 F = \Delta_2 F = 0$, then $F(z_\omega(t))$ is a martingale with continuous sample paths.*

COROLLARY 2. *The method of time changes and comparison equations described in section 1.2 is applicable to $z_\omega(t)$.*

COROLLARY 3. *If $F \in C^2(H)$ and if there exist constants, c, c' such that*

$$\Delta F \geqslant c > 0, \quad \|\nabla F\| \leqslant c' < \infty$$

on H (meaning $\Delta_1' F \geqslant c$ on $H^{(1)}$, $\Delta_2 F \geqslant c$ on $H^{(2)}$, etc.), then, for any ε, $\delta > 0$ there exists a constant β such that, for all $\tilde{z}^0 \in \tilde{H}$.

$$P_{\tilde{z}^0}[F(z_\omega(t)) \geqslant F(z^0) + (c - \varepsilon)t - \beta,\ \forall t > 0] > 1 - \delta.$$

Proof. We apply the martingale inequality of McKean [8, p. 25]

$$P\left[\operatorname*{Max}_{t>0}\left(\int_0^t e\,db - \frac{\alpha}{2}\int_0^t e^2 ds > \beta\right)\right] < e^{-\alpha\beta}, \tag{2.2.4}$$

with $e = -\|\nabla F\|(z_\omega(s))$, α such that $\alpha c'^2 < 2\varepsilon$ and β such that $e^{-\alpha\beta} < \delta$.

III. Asymptotic behaviour of the sample paths

3.1. Behaviour of $y_1(z_\omega(t))$ and $x_j(z_\omega(t))$

LEMMA 3.1.1. (i) *For any* ε, $\delta > 0$, *there exist positive constants* c_1, c_2 *such that, for all* $\tilde{z}^0 \in \tilde{H}$,

$$P_{\tilde{z}^0}[c_1 y_1(z^0) e^{-(1+\varepsilon)t} < y_1(z_\omega(t)) < c_2 y_1(z^0) \exp\left(-(\tfrac{3}{4} - \varepsilon)t\right),\ \forall t > 0] > 1 - \delta.$$

(ii) *Let* $i \leqslant j \leqslant 3$. *For any* η, $\delta > 0$, *there exists* $\lambda > 0$ *such that, if* $y_1(z^0) < \lambda$, *then*

$$P_{\tilde{z}^0}[\,|x_j(z_\omega(t)) - x_j(z^0)| < \eta,\ \forall t > 0] > 1 - \delta.$$

Proof. (i) follows by applying Corollary 3 of Lemma 2.2.1 to the function $\xi(z) = -\log y_1$ and using the estimates of Lemmas 1.3.1 and 1.3.2.

To prove (ii) we note that, by Lemmas 2.2.1 and 1.3.4,

$$x_j(z_\omega(t)) - x_j(z^0) = \int_0^t \|\nabla x_j\|(z_\omega(s))\,db_\omega(s).$$

So the martingale inequality (2.2.4) gives

$$P_{\tilde{z}^0}\left[x_j(z_\omega(t)) - x_j(z^0) < \frac{\alpha}{2}\int_0^t \|\nabla x_j\|^2(z_\omega(s))\,ds + \frac{\eta}{2},\ \forall t > 0\right] > 1 - e^{-(\alpha\eta)/2} > 1 - \frac{\delta}{4}, \tag{3.1.1}$$

with appropriate choice of α.

By Lemma 1.3.4 and by part (i) of this lemma used with $\delta/4$ instead of δ, we have

$$P_{\tilde{z}^0}\left[\int_0^t \|\nabla x_j\|^2(z_\omega(s))\,ds < c_2^2(y_1(z^0))^2\left(\frac{3}{2} - 2\varepsilon\right)^{-1},\ \forall t > 0\right] > 1 - \frac{\delta}{4}. \tag{3.1.2}$$

Together with (3.1.1) this implies

$$P_{\tilde{z}^0}[x_j(z_\omega(t)) - x_j(z^0) < \eta] > 1 - \frac{\delta}{2},$$

whenever

$$y_1(z^0)^2 < \lambda^2 = \frac{\eta}{\alpha c_2^2}\left(\frac{3}{2} - 2\varepsilon\right).$$

Applying the same argument to the function $-x_j$ instead of x_j finishes the proof.

Remark. With some extra work the exponent in the upper estimate in part (i) can be improved to $-(1-\varepsilon)t$. For this one has to use some information about the behaviour of $\chi(z_\omega(t))$ to get a finer estimate of $\|\nabla_2\xi\|^2$ than the crude uniform estimate of Lemma 1.3.1.

The same remark also applies to the following lemma, which gives a somewhat different kind of information about $y_1(z_\omega(t))$.

LEMMA 3.1.2. *For any $\tilde{z}^0 \in \tilde{H}$, $a>0$, and any $\varepsilon>0$,*

$$\sum_{j>0} P_{\tilde{z}^0}[\operatorname{Max}_{t\geq aj} y_1(z_\omega(t)) > y_1(z^0)\exp(-(\tfrac{3}{4}-\varepsilon)aj)] < \infty,$$

$$\sum_{j>0} P_{\tilde{z}^0}[\min_{t\geq aj} y_1(z_\omega(t)) < y_1(z^0)\, e^{-(1+\varepsilon)aj}] < \infty.$$

Proof. The j'th term of the first sum is obviously majorized by p_j+q_j, where

$$p_j = P_{\tilde{z}^0}\left[\xi(z_\omega(aj)) - \xi(z^0) < \left(\frac{3}{4}-\frac{\varepsilon}{2}\right)aj\right],$$

$$q_j = P_{\tilde{z}^0}\left[\xi(z_\omega(aj)) - \xi(z^0) \geq \frac{\varepsilon}{2}aj,\ \min_{t\geq aj}\xi(z_\omega(t)) - \xi(z^0) < \left(\frac{3}{4}-\frac{\varepsilon}{2}\right)aj\right].$$

Applying the time change of section 1.2 we have, by Lemmas 1.3.1, 1.3.2,

$$\xi(z^*_\omega(t^*)) - \xi(z^0) = \tfrac{1}{2}t^* + b_{\omega_1}(t^*), \quad \tfrac{2}{3}t < t^* \leq 2t.$$

Using these, we estimate p_j as follows:

$$p_j < P_{\tilde{z}^0}\left[\frac{3}{4}aj + b_{\omega_1}((aj)^*) < \left(\frac{3}{4}-\frac{\varepsilon}{2}\right)aj\right] \leq P\left[\min_{0\leq t\leq 2aj} b_{\omega_1}(t) < -\frac{\varepsilon}{2}aj\right]$$

$$= 2P\left[b_{\omega_1}(2aj) < -\frac{\varepsilon}{2}aj\right] \leq \frac{4}{(\pi aj)^{1/2}}\exp\left(-\frac{\varepsilon^2 a}{16}j\right),$$

where the last two lines follow from elementary properties of the Brownian motion ([3, p. 7] and [8, p. 4]).

To estimate q_j, we observe that it is just the probability of the finiteness of the lifetime of the process $\xi(z_\omega(t))-\xi(z^0)$ starting at some value larger than $(\varepsilon/2)aj$ and absorbed at $(\frac{3}{4}-\frac{1}{2}\varepsilon)aj$. So (1.2.9) applies with $h>(\frac{3}{4}-\varepsilon)aj$ and $\beta=-\frac{1}{2}$, giving

$$q_j < \exp(-(\tfrac{3}{4}-\varepsilon)aj).$$

So Σp_j and Σq_j are both majorized by convergent geometric series, proving the first assertion of the lemma. The proof of the second one is analogous.

COROLLARY. *For sufficiently large t, a.s. we have*

$$y_1(z^0)\,e^{-(1+\varepsilon)t} < y_1(z_\omega(t)) < y_1(z^0)\exp(-(\tfrac{3}{4}-\varepsilon)t).$$

Of course, this corollary can also be proved directly very easily by the time change method.

3.2. Estimate of T_j

LEMMA 3.2.1. *There exist positive constants c' and c'' such that, for all $\tilde{z}^0 \in \tilde{H}$,*

$$\sum_{j\geq 1} P_{\tilde{z}^0}[T_j > c''j] < \infty, \quad \sum_{j\geq 1} P_{\tilde{z}^0}[T_j < c'j] < \infty.$$

Proof. By Lemma 2.1.1 (ii) we can pick a number $\lambda > 0$ such that, with some $\gamma > 0$,

$$E_z^{(i)}[\exp(\lambda\tau^{(i)} - \gamma)] < \exp\left(-\frac{\gamma}{2}\right), \tag{3.2.1}$$

uniformly for $z \in \partial H^{(i')}$, where either $i=1$ and $i'=2$ or $i=2$ and $i'=1$.

Now let $\tilde{z}^0 \in \tilde{H}$. Let $B_j^{(1)}$ be the sigma-field of the "past" $t \leq T_j^{(1)}$ for $z_\omega(t)$, and $E^{B_j^{(1)}}$ the corresponding conditional expectation. By the uniform estimate (3.2.1) we have

$$\begin{aligned} E_{\tilde{z}^0}[\exp(\lambda T_j - 2j\gamma)] &= E_{\tilde{z}^0}[\exp(\lambda T_j^{(1)} - (2j-1)\gamma) E^{B_j^{(1)}}[\exp(\tau_j^{(2)} - \gamma)]] \\ &\leq e^{-\gamma/2} E_{\tilde{z}^0}[\exp(\lambda T_j^{(1)} - (2j-1)\gamma)]. \end{aligned}$$

Repeating this argument now with the sigma-field B_{j-1} belonging to $t \leq T_{j-1}$ and then using induction, we get

$$E_{\tilde{z}^0}[e^{\lambda T_j - 2j\gamma}] \leq \begin{cases} \exp\left(-\left(j-\frac{1}{2}\right)\gamma\right) E_{\tilde{z}^0}\left[\exp\left(\lambda\tau_1^{(1)} - \frac{\gamma}{2}\right)\right] \\ e^{-(j-1)\gamma} E_{\tilde{z}^0}\left[\exp\left(\gamma\tau_1^{(2)} - \frac{\gamma}{2}\right)\right] \end{cases},$$

depending on whether $\tilde{z}^0$ is in $\tilde{H}^{(1)}$ or $\tilde{H}^{(2)}$. In either case we have an inequality

$$E_{\tilde{z}^0}[e^{\lambda T_j - 2j\gamma}] \leq Ce^{-j\gamma},$$

with a constant C depending only on $\tilde{z}^0$.

Now the Markov-Chebyshev inequality gives

$$P_{\tilde{z}^0}\left[T_j > \frac{2\gamma}{\lambda} j\right] = P_{\tilde{z}^0}[e^{\lambda T_j - 2j\gamma} > 1] \leq Ce^{-j\gamma},$$

and the first statement of the Lemma follows, with $c'' = (2\gamma)/\lambda$.

To prove the second statement, again using Lemma 2.1.1 (ii) we pick $\lambda > 0$ and have, with some $\gamma > 0$,

$$E_z^{(1)}[e^{\gamma - \lambda\tau^{(1)}}] < e^{-\gamma},$$

uniformly for $z \in \partial H^{(2)}$. For $\tau^{(2)}$ we use the trivial estimate

$$E_z[e^{-\lambda\tau^{(2)}}] < 1.$$

The argument used in proving the first statement now gives

$$E_{\tilde{z}^0}[e^{j\gamma-\lambda T_j}] < Ce^{-j\gamma},$$

with C depending only on z^0. Setting $c' = \gamma/\lambda$ the Markov-Chebyshev inequality gives

$$P_{\tilde{z}^0}[T_j < c'j] = P_{\tilde{z}^0}[e^{j\gamma-\lambda T_j} > 1] \leqslant Ce^{-j\gamma},$$

finishing the proof.

COROLLARY. *Given $\tilde{z}^0 \in \tilde{H}$, for sufficiently large j we have a.s.*

$$c'j \leqslant T_j \leqslant c''j.$$

We note that the same results obviously hold for $T_j^{(1)}$ in place of T_j.

3.3. Behaviour of $\chi(z_\omega(t))$

The next lemma shows that the imaginary parts of the sample paths $z_\omega(t)$ stay away from the axis of the cone; as $t\to\infty$ they tend towards the vertex by becoming tangential to the boundary. Recall that $\bar{\bar{\chi}}$ was defined as $\chi - \log\phi$.

LEMMA 3.3.1. *For every $\tilde{z}^0 \in \tilde{H}$,*

$$\sum_{j\geqslant 1} P_{\tilde{z}^0}[\min_{T_j\leqslant t\leqslant T_{j+1}} \bar{\bar{\chi}}(z_\omega(t)) \leqslant 0] < \infty.$$

Proof. We use the abbreviated notation

$$\varphi = \tfrac{8}{9}\phi^{9/8}e^{-(9/8)\phi},$$

and show first that, for $z\in\partial H^{(1)}$,

$$P_z^{(2)}[\min_{t>0} \bar{\bar{\chi}}(z_\omega^{(2)}(t)) \leqslant 0] < \varphi(z). \tag{3.3.1}$$

In fact, this is the probability of the finiteness of the lifetime of $\bar{\bar{\chi}}(z_\omega^{(2)}(t))$ starting at $\bar{\bar{\chi}}(z) = \phi(z) - \log\phi(z)$ (since $z\in\partial^{(1)}$) and absorbed at 0. (3,3.1) is immediate from the method of section 1.2, in particular (1.2.9), and from Lemma 1.3.3.

Let A_j denote the event the Lemma is about, i.e.

$$A_j = \left\{\min_{T_j\leqslant t\leqslant T_{j+1}} \bar{\bar{\chi}}(z_\omega(t)) < 0\right\}.$$

Note that for $T_j < t < T_{j+1}^{(1)}$ we have $z_\omega(t)\in\tilde{H}^{(1)}$, hence $\bar{\bar{\chi}}(z_\omega(t)) > 0$. By (3.3.1) we have an estimate of a conditional probability,

$$P_{\tilde{z}^0}[A_j \mid z_\omega(T_{j+1}^{(1)}) = z] \leqslant \varphi(z).$$

So it follows that

$$P_{\tilde{z}^0}[A_j] \leqslant E_{\tilde{z}^0}[\varphi(z_\omega(T_{j+1}^{(1)}))]. \tag{3.3.2}$$

We proceed to estimate this as follows.

Using the events

$$C_j = \{T^{(1)}_{j+1} < c'j\},$$

$$D_j = \left\{\operatorname*{Max}_{t > c'j} y_1(z_\omega(t)) > \exp\left(-\frac{c'}{2}j\right)\right\},$$

we have from (3.3.2)

$$P_{\tilde z^0}[A_j] \leqslant \|\varphi\|_\infty (P_{\tilde z^0}[C_j] + P_{\tilde z^0}[D_j]) + E_{\tilde z^0}[\varphi(z_\omega(T^{(1)}_{j+1}))\mathbf{1}_{C_j^c \cap D_j^c}]. \tag{3.3.3}$$

On $C_j^c \cap D_j^c$ we have

$$y_1(z_\omega(T^{(1)}_{j+1})) \leqslant \exp\left(-\frac{c'}{2}j\right).$$

By definition of φ and ϕ (section 1.1) we have, for small $y_1 = y_1(z)$,

$$\varphi(z) = \frac{8}{9}\left(\frac{\log(-\log y_1)}{-\log y_1}\right)^{9/8}.$$

So, for large enough j, we have on $C_j^c \cap D_j^c$

$$\varphi(z_\omega(T^{(1)}_{j+1})) \leqslant \frac{8}{9}\left(\frac{\log \frac{c'}{2}j}{\frac{c'}{2}j}\right)^{9/8}.$$

Using this in (3.3.3) we have, for sufficiently large j, with some positive constants M_1, M_2,

$$P_{\tilde z^0}[A_j] \leqslant M_1(P_{\tilde z^0}[C_j] + P_{\tilde z^0}[D_j]) + M_2\left(\frac{1}{j}\right)^{17/16}.$$

By Lemmas 3.1.2 and 3.2.1 this finishes the proof.

COROLLARY. *For any $\tilde z^0 \in \tilde H$, we have*

$$\frac{y_1 - \varrho}{y_1}(z_\omega(t)) < \frac{1}{\log \frac{t}{2}}$$

for all sufficiently large t, a.s.

Proof. The Lemma immediately implies $\bar{\tilde\chi}(z_\omega(t)) \geqslant 0$ for large t, a.s. which means

$$\frac{y_1 - \varrho}{y_1}(z_\omega(t)) \leqslant e^{-\log \phi(z_\omega(t))}$$

for large t, a.s. From the definition of ϕ and from the Corollary of Lemma 3.1.2 the statement follows.

3.4. Behaviour of $\theta(z_\omega(t))$

The goal of this section is to prove Lemma 3.4.4. For this an estimate of $\tau^{(2)}$ from below is needed; this is given in Lemma 3.4.2.

LEMMA 3.4.1. *There exist $\mu>0$ and $c_1>0$ such that, for all $z\in\partial H^{(1)}$ with $y_1(z)<\mu$,*

$$E_z^{(2)}\left[\exp\left(-\int_0^{\tau^{(2)}}\exp\left(\phi(z_\omega^{(2)}(t))-\phi(z)-1\right)dt\right)\right]\leqslant\exp\left(-c_1e^{-\phi(z)}\right).$$

Proof. We apply the time change method of section 1.2 to $\bar{\chi}(z_\omega^{(2)}(t))$. We have

$$t^*=\int_0^t\|\nabla_2\bar{\chi}\|^2(z_\omega^{(2)}(s))\,ds, \tag{3.4.1}$$

and

$$\bar{\chi}(z_\omega^{(2)}(t))=b_{\omega_1}(t^*)+\int_0^{t^*}\frac{\Delta_2\bar{\chi}}{\|\nabla_2\bar{\chi}\|^2}(z_\omega^{(2)*}(s))\,ds. \tag{3.4.2}$$

Let $\Gamma=\{z\in H\,|\,2\varrho(z)>y_1(z)\}$, and let σ be the last-leaving time

$$\sigma=\sup\{0<t\leqslant\tau^{(2)}\,|\,z_\omega(t)\notin\Gamma\}$$

or $\sigma=0$ if this set is empty.

From (3.4.2) and from Lemma 1.3.3,

$$b_{\omega_1}(\tau^{(2)*})-b_{\omega_1}(\sigma^*)+2(\tau^{(2)*}-\sigma^*)=\bar{\chi}(z_\omega(\tau^{(2)}))-\bar{\chi}(z_\omega(\sigma)).$$

The right-hand side here is at least 1, since $\bar{\chi}$ is 1 on $\partial H^{(2)}$, is 0 on $\partial H^{(1)}$ (this corresponds to the case $\sigma=0$), and is negative outside of Γ. It follows that if we introduce the process

$$Y_{\omega_1}(t)=b_{\omega_1}(t)+2t$$

then we have

$$2\max_{0\leqslant t\leqslant\tau^{(2)*}}|Y_{\omega_1}(t)|\geqslant Y_{\omega_1}(\tau^{(2)*})-Y_{\omega_1}(\sigma^*)\geqslant 1.$$

Writing

$$\tau^Y=\min\left\{t>0\,\middle|\,|Y_{\omega_1}(t)|\geqslant\frac{1}{2}\right\},$$

it follows that we have $\tau^{(2)*}\geqslant\tau^Y$. $Y_{\omega_1}(t)$ is governed by the operator $\frac{1}{2}D^2+2D$; using (1.2.11) and Lemma 1.3.3 we therefore get the estimate

$$E_z[e^{-\lambda\tau^{(2)*}}]\leqslant E[e^{-\lambda\tau^Y}]\leqslant e^{-c\lambda}, \tag{3.4.3}$$

with some constant $c>0$, for all sufficiently small $\lambda>0$.

(3.4.1) and Lemma 1.3.3 give

$$\tau^{(2)*}\leqslant 5\int_0^{\tau^{(2)}}e^{\phi(z_\omega(s))+1}\,ds,$$

using this and setting $\lambda = \frac{1}{8}e^{-\phi(z)-2}$ in (3.4.3) gives the inequality of the lemma. By the definition of $\phi(z)$, the smallness of $y_1(z)$ guarantees that λ is sufficiently small.

Recall the definition $\xi(z) = -\log y_1(z)$.

LEMMA 3.4.2. *There exist $\nu>0$ and $c>0$ such that, for all $z\in\partial H^{(1)}$ with $y_1(z)<\nu$,*

$$E_z^{(2)}[e^{-\tau^{(2)}}] \leqslant \exp\left(-\frac{c}{\xi(z)}\right).$$

Proof. Define $T=\min\{t>0 \mid \phi(z_\omega(t)) \geqslant \phi(z)+1\}$. Then

$$E_z^{(2)}[e^{-\tau^{(2)}}] = E_z^{(2)}[1_{\{\tau^{(2)}\leqslant T\}}\, e^{-\tau^{(2)}}] + E_z^{(2)}[1_{\{T<\tau^{(2)}\}}\, e^{-\tau^{(2)}}] = I_1 + I_2. \tag{3.4.4}$$

Lemma 3.4.1 and the definition of T immediately give

$$I_1 \leqslant \exp(-c_1 e^{-\phi(z)}) = \exp\left(-\frac{c_1}{\xi(z)}\right), \tag{3.4.5}$$

with the last equality holding only for sufficiently small $y_1(z)$ (i.e. large $\xi(z)$, i.e. large $\phi(z)$).

For I_2 we use the elementary estimate, valid for any constant R,

$$I_2 \leqslant P[T<\tau^{(2)}] \leqslant P[T\leqslant R] + P[R<\tau^{(2)}] = I_3 + I_4 \tag{3.4.6}$$

and we set $R=\xi(z)^{\frac{1}{2}}$.

For small $y_1(z)$, by definition of ϕ, $T=\min\{t>0 \mid \xi(z_\omega^{(2)}(t)) \geqslant e\cdot\xi(z)\}$. So Lemma 1.3.2 and (1.2.10) give

$$E_z[e^{-\lambda T}] < e^{-(e-1)\xi(z)\lambda^{1/2}},$$

for sufficiently large λ, in particular for $\lambda=\xi(z)$. Now, by the Markov-Chebyshev inequality,

$$I_3 = P[e^{-\lambda T} > e^{-\lambda R}] \leqslant e^{-(e-2)\xi(z)^{3/2}}. \tag{3.4.7}$$

For I_4 we choose a sufficiently small $\lambda>0$, use the Markov-Chebyshev inequality and then (2.1.2) to obtain

$$I_4 \leqslant E_z[e^{\lambda\tau^{(2)}}]e^{-\lambda R} \leqslant Ce^{-\lambda\xi(z)^{1/2}}, \tag{3.4.8}$$

with some positive constant C.

It follows that for large enough $\xi(z)$ (i.e. small $y_1(z)$), (3.4.4) is majorized by $\exp(-\frac{1}{2}c_1/\xi(z))$, finishing the proof.

LEMMA 3.4.3. *For any $\tilde{z}^0\in\tilde{H}$,*

$$\sum_{k>0} \tau_k^{(2)} = +\infty \quad a.s.$$

Proof. Let $h(z)=c/\xi(z)$ if $y_1(z)<\nu$ and 0 otherwise. With this notation Lemma 3.4.2 can be rewritten as

$$E_z^{(2)}[e^{-(\tau^{(2)}-h(z))}]\leqslant 1, \tag{3.4.9}$$

for all $z\in\partial H^{(1)}$.

Given $\tilde{z}^0\in\tilde{H}$, now write

$$a_k=E_{\tilde{z}^0}\left[\exp\left(-\sum_{j=1}^{k}(\tau_j^{(2)}-h(z_\omega(T_j^{(1)})))\right)\right].$$

Letting $B_j^{(1)}$, as in section 3.2, be the sigma field of $z_\omega(t)$, $t\leqslant T_j^{(1)}$, using the corresponding conditional expectation and (3.4.9), we have, for $k\geqslant 2$,

$$a_k=E_{\tilde{z}^0}\left[\exp\left(-\sum_{j=1}^{k-1}(\tau_j^{(2)}-h(z_\omega(T_j^{(1)})))\right)E^{B_k^{(1)}}\left[\exp\left(-(\tau_k^{(2)}-h(z_\omega(T_k^{(1)})))\right)\right]\right]\leqslant a_{k-1}.$$

Hence, by induction, $a_k\leqslant a_1$ for all $k\geqslant 1$. a_1 depends only on $\tilde{z}^0$.

Now note that

$$\sum_{j\geqslant 1}h(z_\omega(T_j^{(1)}))=+\infty \quad \text{a. s.} \tag{3.4.10}$$

In fact, by the Corollaries to Lemmas 3.2.1 and 3.2.2 we have, with some positive constants C', C''

$$e^{-C''j}<y_1(z_\omega(T_j^{(1)}))<e^{-C'j},$$

for sufficiently large j, a.s. These inequalities show that, for large j, $y_1(z_\omega(T_j^{(1)}))<\mu,\nu$ (the constants in Lemmas 3.4.1, 3.4.2) a.s., and then $h(z_\omega(T_j^{(1)}))>c/(C''j)$. This proves (3.4.10).

To finish the proof of the lemma write $H_M=\{\sum_{k\geqslant 1}\tau_k^{(2)}\leqslant M\}$; we must show $P_{\tilde{z}}[H_M]=0$ for all $M>0$.

Suppose that, for some $M>0$, $P_{\tilde{z}^0}[H_M]>0$. Then, for a.a. $\omega\in H_M$,

$$-\sum_{k=1}^{n}(\tau_k^{(2)}-h(z_\omega(T_k^{(1)})))\to+\infty,$$

as $n\to\infty$. Hence, by Fatou's lemma $a_k\to\infty$ as $k\to\infty$, which is a contradiction.

LEMMA 3.4.4. *For any $\tilde{z}^0\in\tilde{H}$, $\theta(z_\omega(t))$ a.s. assumes all values infinitely often as $t\to\infty$.*

Proof. $\varrho(z_\omega(t))\neq 0$ for all $t>0$, a.s. since the set $\{\varrho=0\}$ has codimension two in H. Therefore, as in [8, p. 108] we can lift $z_\omega(t)$ to a covering domain, i.e. we can look at $\theta(z_\omega(t))$ as an $\mathbf{R}$-valued process and we can apply Lemma 2.2.1 to it. Using also Lemma 1.3.4 and the time change method, we have

$$\theta(z_\omega(t))-\theta(z^0)=b_{\omega_1}(t^*),$$

$$t^*=\int_0^t\|\nabla\theta\|^2(z_\omega(s))\,ds.$$

It is well-known that $b_{\omega_1}(t^*)$ assumes all values infinitely often as $t^* \to \infty$. Therefore it suffices to prove that if $t \to \infty$ then $t^* \to \infty$, a.s.

By an estimate in Lemma 1.3.4 we have

$$t^* \geqslant \sum_{j=1}^{j(t)} \int_{T_j^{(1)}}^{T_j} \|\Delta_2 \theta\|^2 (z_\omega(s))\, ds \geqslant 2 \sum_{j=1}^{j(t)} \tau_j^{(2)},$$

where $j(t) = \text{Max}\,\{j \mid T_j \leqslant t\}$. By section 3.2, $j(t) \to \infty$ a.s. if $t \to \infty$. By Lemma 3.4.3 it follows that $t^* \to \infty$, a.s.

Remark. One easily computes $\Delta_1 \theta = 0$ and

$$\|\nabla_1 \theta\|^2 = \frac{y_1^2 - \varrho^2}{6\varrho^2},$$

from which one can see that, for the process $z_\omega^0(t)$ governed by Δ_1, $\theta(z_\omega^0(t))$ has a limit as $t \to \infty$. This corresponds to the fact that, for a bounded solution of the single equation $\Delta_1 F = 0$, the "boundary function" will be a function of x *and* θ.

IV. The Poisson representation

4.1. Some versions of Fatou's theorem

We prove first a lemma about general symmetric spaces; it is only a slight variation of a Fatou-type theorem proved in [6] After that we will specialize everything to the case of harmonic functions on H and state explicitly the particular consequence of the first lemma that will play a crucial role in the final section.

LEMMA 4.1.1. *Let all notations be as in* [6, § 4]. *Let* $E = \phi$. *Let* $f \in L^\infty(G/B)$ *and let* F *be its Poisson integral. Let* $Q \subset G$ *be compact, let* $\dot{Q}$ *be the image of* Q *in* G/B *under the natural map, and assume that* f *is continuous at all points of* $\dot{Q}$. *Then, for all* C *as in* [6, *Definition* 4.1] *and all* $\varepsilon > 0$, *there exists* $T \in \mathfrak{a}$ *such that* $x \in g \cdot \mathcal{A}_C^T(\dot{e})$ *implies* $|F(x) - f(\dot{g})| < \varepsilon$ *uniformly for all* $g \in Q$.

Proof. We may assume that $\dot{Q}$ is contained in the orbit of $\dot{e}$ under $\bar{N}$, since finitely many translates of this orbit cover G/B.

Next we note that Lemma 4.1 and its Corollary 1 in [6] remain true, with their original proof, for g' varying in any given compact subset of G.

These remarks, together with Lemma 4.2 imply that to prove our statement it suffices to add the following remark to the proof of Theorem 4.1 of [6]:

If $f_1 \in L^\infty(\bar{N})$ and is continuous at all points of a compact subset Q_1, then for all

$\varepsilon>0$ and all U, V, there exists $T\in\mathfrak{a}$ such that $x\in\Gamma^T_{U,V}(\bar{n}_0)$ implies $|F(x)-f_1(\bar{n}_0)|<\varepsilon$ uniformly for all $\bar{n}_0\in Q_1$.

That this is true can be seen by following the original proof up to its last two lines, and then noting that f_1 is relatively left-uniformly continuous on Q_1, and therefore the required T can be chosen uniformly for all $\bar{n}_0$ in Q_1.

Note. This lemma remains true for the case of arbitrary E, but we stated it only for the case $E=\phi$ which we need.

Now we describe the explicit meaning of Lemma 4.3.1 for the case of H. Our assertions can easily be checked with the aid of some computations in [9, pp. 87–91] and of the indications in [5, § 5].

$o=(i, 0, 0)$ will be the base point, K its isotropy group in the identity component G of the group of all holomorphic automorphisms of H. A is a vector group, its Lie algebra $\mathfrak{a}$ can be identified with $\mathbf{R}^2$. If $a\in A$ and $\log a=(t_1, t_2)\in\mathbf{R}^2=\mathfrak{a}$, then a acts on H by

$$\begin{cases} z_1\mapsto e^{-(t_1+t_2)}(z_1\cosh t_2+z_2\sinh t_2)\\ z_2\mapsto e^{-(t_1+t_2)}(z_1\sinh t_2+z_2\cosh t_2)\,.\\ z_3\mapsto e^{-(t_1+t_2)}z_3\end{cases}\tag{4.1.1}$$

The positive Weyl chamber $\mathfrak{a}^+$ can be chosen to be the positive quadrant of $\mathbf{R}^2$, so $\log a>T=(T_1, T_2)$ means $t_1>T_1$ and $t_2>T_2$.

Let E be the singleton set consisting of the simple root $(t_1, t_2)\mapsto t_2$. Then $G/B(E)$ is isomorphic as a G-space with the Bergman-Šilov boundary of the canonical bounded realization of H.

$\bar{N}(E)$ is isomorphic with $\mathbf{R}^3$. For $u\in\mathbf{R}^3$ we write the corresponding element as $\bar{n}(u)$; it acts on H by $z\mapsto z+u$. The orbit of $\dot{e}$ in $G/B(E)$ is dense open, and $\bar{N}(E)$ is simply transi tive on it. It has a natural identification with the subset $\{z\in\mathbf{C}^3|\operatorname{Im} z=0\}$ of the boundary of H (called the Bergman-Šilov boundary of H in [5]).

K^E is isomorphic with $\mathbf{T}^1$. Its generic element, denoted $k(\varphi)$, acts on H by

$$\begin{cases} z_1\mapsto z_1\\ z_2\mapsto z_2\cos\varphi-z_3\sin\varphi.\\ z_3\mapsto z_2\sin\varphi+z_3\cos\varphi\end{cases}\tag{4.1.2}$$

$\bar{N}(E)K^E\cong\mathbf{R}^3\cdot\mathbf{T}^1$ is a semidirect product. The orbit of $\dot{e}$ in G/B (in the "maximal boundary") is open dense and $\bar{N}(E)K^E$ is simply transitive on it. We shall identify this orbit with $\mathbf{R}^3\times\mathbf{T}^1$, writing for $\bar{n}(u)k(\varphi)\cdot\dot{e}$ simply (u, φ). If a bounded function f is given on $\mathbf{R}^3\times\mathbf{T}^1$, its Poisson integral is well defined, since this set is open dense in G/B.

Now to obtain the needed consequences of Lemma 4.1.1, let f be continuous and bounded on $\mathbf{R}^3\times\mathbf{T}^1$ let F be its Poisson integral and let $Q\subset\mathbf{R}^3\times\mathbf{T}^1$ be compact. Then for all

$\varepsilon > 0$ and all C there exists $T = (T_1, T_2)$ such that $z \in \bar{n}(u)\,k(\varphi)\,\mathcal{A}_C^T(e)$ implies $|F(z) - f(u, \varphi)| < \varepsilon$ for all $(u, \varphi) \in Q$.

It would not be hard to see what the exact shape of $\mathcal{A}_C^T(\dot{e})$ is, and so get a precise geometric description of "admissible convergence" to the maximal boundary, but for our purposes we can be satisfied with less. It is clear that, for any C, $\mathcal{A}_C^T(\dot{e}) \supset \{a \cdot o \mid a \in A, \log a > T\}$. By (4.1.1) the latter set in turn contains $\{i(y_1, y_2, 0) \mid y_1, (y_1 - y_2)/y_1 < \delta\}$ if $\delta \leqslant e^{-T_1}, e^{-T_2}$ (we suppose, as we may, that $T_1, T_2 > 0$). Now $\bar{n}(u)\,k(\varphi)\,\mathcal{A}_C^T(\dot{e}) \supset A^\delta(u, \varphi)$, where

$$A^\delta(u, \varphi) = \left\{ z \in H \,\middle|\, x(z) = u,\ \theta(z) = \varphi,\ y_1(z) < \delta,\ \frac{y_1 - \varrho}{y_1}(z) < \delta \right\}.$$

So we have proved the following weak but sufficient consequence of Lemma 4.1.1:

LEMMA 4.1.2. *Let f be continuous and bounded on $\mathbf{R}^3 \times \mathbf{T}^1$ and let F be its Poisson integral. Let $Q \subset \mathbf{R}^3 \times \mathbf{T}^1$ be compact. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that $z \in A^\delta(u, \varphi)$ implies*

$$|F(z) - f(u, \varphi)| < \varepsilon,$$

uniformly for all $(u, \varphi) \in Q$.

4.2. The main result

THEOREM. *A function F on H can be written as the Poisson integral of a bounded measurable function on the Bergman-Šilov boundary if and only if it is bounded and satisfies the equations $\Delta_1 F = 0$, $\Delta_2 F = 0$.*

Proof. The "only if" part is a simple direct checking, it is essentially contained in the discussion in section 1.1.

To prove the "if" part we first note that, by Furstenberg's theorem [2], F bounded and $\Delta_1 F = 0$ implies that F is the Poisson integral of a bounded measurable function on the maximal boundary G/B. Let $f = f(u, \varphi)$ be the restriction of this function to $\mathbf{R}^3 \times \mathbf{T}^1$ which we regard, as in section 4.1, as an open dense subset of G/B.

We claim that it suffices to show that $f(u, \varphi)$ is independent of φ. Indeed, in this case f lifted to (a dense open subset of) G is right-invariant under K^E, so it induces a function on a dense open subset of $G/B(E)$, of which F is still the Poisson integral (cf. [6, § 1]).

We shall now first prove the independence of $f(u, \varphi)$ of φ in the special case where f is continuous, and then reduce the general case to this special one.

Suppose that f is continuous and $f(u_0, \varphi_1) \neq f(u_0, \varphi_2)$ for some $u_0 \in \mathbf{R}^3$; $\varphi_1, \varphi_2 \in \mathbf{T}^1$. Then we have

$$|f(u_0, \varphi_1) - f(u_0, \varphi_2)| = 5\varepsilon,$$

with $\varepsilon > 0$. By uniform continuity we can choose a compact neighborhood U of u_0 in $\mathbf{R}^3$ such that

$$|f(u, \varphi_i) - f(u_0, \varphi_i)| < \varepsilon,$$

for all $u \in U$, $1 \leqslant i \leqslant 2$. By Lemma 4.1.2 there exists $\delta > 0$ such that if $z \in A^\delta(u, \varphi_i)$, then

$$|F(z) - f(u, \varphi_i)| < \varepsilon,$$

for all $u \in U$, $1 \leqslant i \leqslant 2$. It follows that, writing

$$A^\delta(U, \varphi_i) = \bigcup_{u \in U} A^\delta(u, \varphi_i) \quad (1 \leqslant i \leqslant 2),$$

we have

$$|F(z_1) - F(z_2)| > \varepsilon, \tag{4.2.1}$$

whenever $z_i \in A^\delta(U, \varphi_i)$ $(1 \leqslant i \leqslant 2)$.

Now choose $\tilde{z}^0 \in \tilde{H}$ with $x(\tilde{z}^0) = u$ and with $y_1(\tilde{z}^0)$ sufficiently small so that Lemma 3.1.1 (ii) ensures that

$$P_{\tilde{z}^0}[x(z_\omega(t)) \in U, \quad \forall t > 0] > \tfrac{1}{2}.$$

By the Corollaries to Lemmas 3.1.2 and 3.3.1, and by Lemma 3.4.4, as $t \to \infty$, $z_\omega(t)$ meets both $A^\delta(U, \varphi_1)$ and $A^\delta(U, \varphi_2)$ infinitely often with probability at least $\frac{1}{2}$. But, by Corollary 1 to Lemma 2.2.1, $F(z_\omega(t))$ is a martingale, and now the martingale convergence theorem gives a contradiction to (4.2.1).

Now we drop the hypothesis that f is continuous. As discussed in section 4.1, f can be regarded as a function on the group $\bar{N}(E)K^E$ which we now briefly denote by L. The Poisson integral can be written as a Haar integral on L with the aid of some kernel $\mathcal{P}_z$,

$$F(z) = \int_L \mathcal{P}_z(l) f(l)\, dl. \tag{4.2.2}$$

Since the Poisson integral is a G-equivariant map from G/B to H, and since the action of L on G/B is the same as its action on itself by left translation, we have, for all $h \in L$,

$$F(h^{-1}z) = \int_L \mathcal{P}_z(l) f(h^{-1}l)\, dl. \tag{4.2.3}$$

Now let $\{\alpha_n\}$ be a continuous approximate identity on L, and let $f_n = \alpha_n * f$, i.e.

$$f_n(l) = \int_L f(h^{-1}l)\, \alpha_n(h)\, dh.$$

Let F_n be the Poisson integral of f_n. Then, by Fubini's theorem and by (4.2.3),

$$F_n(z) = \int_L F(h^{-1}z)\, \alpha_n(h)\, dh. \tag{4.2.4}$$

As one sees directly, the operator Δ_2 commutes with L in the sense that $\Delta_2(\Psi \circ l) = (\Delta_2 \Psi) \circ l$ for all $l \in L$ and all functions Ψ. Since $\Delta_2 F = 0$, (4.2.4) implies therefore that $\Delta_2 F_n = 0$. Since F_n is the Poisson integral of f_n which is bounded and continuous, we are in the situation discussed before. It follows that, writing f_n again as a function on $\mathbf{R}^3 \times \mathbf{T}$, $f_n(u, \varphi)$ is independent of φ. Since $\lim f_n = f$, the same is true about $f(u, \varphi)$, finishing the proof of the theorem.

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