TWO NEW INTERPOLATION METHODS BASED ON THE DUALITY MAP

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0. Introduction

In his classical paper [11] (1927) Marcel Riesz proved a theorem on linear operators mapping L_p spaces on one measure space onto L_q spaces on another measure space. In the case when the underlying measure spaces are *finite sets* it can be stated as follows. *Let T be a linear operator mapping functions on one finite set onto functions on another finite set (in other words: an $n \times m$ matrix) and denote by M_{pq} the norm of T considered as an operator $T: L_p \to L_q$ where $p, q \in [1, \infty]$, $p \leqslant q$. Then* $\log M_{pq}$ *is a convex function of the pair* $(1/p, 1/q)$. Several years later his student Olof Thorin [14] (compare also [15]) found a very nice proof based on function theory (three line theorem of Doetch). It works in the complex case only but removes the restriction $p \leqslant q$. Accordingly the theorem is now known as the Riesz–Thorin theorem. It has become a standard tool in many branches of analysis and it has been generalized in many directions (see e.g. [17], chap. 12). The current text-books always give Thorin's proof and Riesz's original proof has fallen into oblivion. The purpose of this paper is to reinterpret Riesz's proof in the light of the theory of *interpolation spaces*.

To show how this is done, we shall first sketch Riesz's proof. Putting $M_0 = M_{p_0, q_0}$, $M_1 = M_{p_1, q_1}$ and $M = M_{p,q}$, where $1/p = 1/p_\theta = (1-\theta)/p_0 + \theta/p_1$ and analogously for q, it suffices to show that $M \leqslant M_0^{1-\theta} M_1^\theta$ for some $\theta \in (0, 1)$. Riesz does this by choosing $a \in L_p$ and $\beta \in L_{q'}$ $(1/q + 1/q' = 1)$ with unit norms such that $M = \langle Ta, \beta \rangle$ and combines this choice with suitable Hölder inequalities. (Since we are presently dealing with *finite dimensional spaces*, the question of existence of a and β does not cause any difficulty. Elements of dual spaces we usually denote by Greek letters.) The details can be arranged as follows. By Lagrange multipliers, say, we find $Ta = M \operatorname{grad} \|\beta\|_{q'}$ and $T^t\beta = M \operatorname{grad} \|a\|_p$ so that in particular $\|Ta\|_q = \|T^t\beta\|_{p'} = M$, where $\|a\|_p = \|a\|_{L_p}$ etc. Introducing the *duality maps* $D_{p'} = D_{L_{p'}}: L_{p'} \to L_p$ and $D_q = D_{L_q}: L_q \to L_{q'}$, where

$$D_{p'}\alpha = \tfrac{1}{2}\,\mathrm{grad}\,\|\alpha\|_{p'}^2\,(= \|\alpha\|_{p'}^{2-p'}|\alpha|^{p'-1}\,\mathrm{sgn}\,\alpha)$$

and analogously for $D_q b$, we can rewrite this as

$$Ma = D_{p'}T^t\beta, \quad M\beta = D_q Ta. \tag{0.1}$$

The Hölder inequalities in question can be stated as

$$\left.\begin{aligned} &\|D_{p'}\alpha\|_{p_0} \leqslant \|\alpha\|_{p_1'}^{\theta/(1-\theta)}\|\alpha\|_{p'}^{(1-2\theta)/(1-\theta)}, && \theta \leqslant \frac{1}{p}, \\ &\|D_q b\|_{q_1'} \leqslant \|b\|_{q_1}^{(1-\theta)/\theta}\|b\|_q^{(2\theta-1)/\theta}, && \frac{1}{q} \leqslant \theta. \end{aligned}\right\} \tag{0.2}$$

A straightforward verification reveals that if $p_0 \leqslant q_0$, $p_1 \leqslant q_1$ then there exists indeed a θ such that $1/q \leqslant \theta \leqslant 1/p$ so that both inequalities (0.2) are applicable. Take now $\alpha = T^t\beta$, $b = Ta$ in (0.2) and use (0.1). The result is the two inequalities

$$M^{1-\theta}\|a\|_{p_0}^{1-\theta} \leqslant M_1^\theta\|\beta\|_{q_1'}^\theta M^{2\theta-1},$$
$$M^\theta\|\beta\|_{q_1'}^\theta \leqslant M_0^{1-\theta}\|a\|_{p_0}^{1-\theta}M^{1-2\theta}.$$

Forming their product, the desired inequality $M \leqslant M_0^{1-\theta}M_1^\theta$ follows.

The idea of this paper is now to use inequalities like (0.2) to associate with each Banach couple $\bar{A} = (A_0, A_1)$ spaces $R_\theta^*(\bar{A})$ and $R_\theta(\bar{A})$. In this way one gets two new interpolation methods which we jointly refer to as *the Riesz method*. Since we now want to deal with *infinite dimensional spaces* certain technical difficulties arise (connected with the existence of a and β in the above proof), which we have not entirely overcome. Under suitable additional assumptions we can, however, prove that if $\bar{A}$ and $\bar{B}$ are two Banach couples and $T: \bar{A} \to \bar{B}$ a bounded linear map then T: $R_\theta^*(\bar{A}) \to R_\theta(\bar{B})$. In particular if we here take $\bar{A} = (L_{p_0}, L_{p_1})$, $\bar{B} = (L_{q_0}, L_{q_1})$ (the underlying measure spaces are now general) we have $L_p \supset R_\theta^*(\bar{A})$ and $R_\theta(\bar{B}) \subset L_q$ so formally we get back Riesz's theorem.

Rather surprisingly, the Riesz method in turn is related to a new method (*the method of quadratic means*) recently discovered by Pusz–Woronowicz [10], and by Uhlmann [16], in the context of Quantum Theory, and studied in some detail in an unfortunately unpublished preprint by Simon [12], and by Grahame Bennett [1]. On the other hand the latter [2] has also been able to extend the Riesz–Thorin theorem to the case when $p < 0$ (with a convenient interpretation). A closer examination quickly reveals that indeed Riesz's proof too yields a result of the Bennett type.

Our investigation is organized as follows.

In section 1 we define the method R_θ and develop some of its very simplest properties.

In section 2 we do the same for the dual method R_θ^*.

In section 3 we put together the results of section 1 and section 2 and establish a general interpolation theorem (the one just described) which might be thought of as an abstract version of Riesz's theorem. (For the proof we need a slight extension of a theorem of Lindenstrauss [7] which we have deferred to an appendix.)

In section 4 we give some concrete illustrations of the previous developments. As we have already seen we get of course back Riesz's theorem but it is perhaps again a surprise that the same proof lends itself to a derivation of a version of Marcinkiewicz's interpolation theorem [8]. Another two cases which we can cover are interpolation with change of measure (Stein–Weiss [13]) and finally interpolation between a space and its dual (Girardeau [4]).

In section 5 we quickly develop the theory of the method of quadratic means, mainly following [12] and [1], and make a comparison with the Riesz method.

In section 6 we make an attempt to merge Bennett's point of view (interpolation between a space and "the dual of a space which does not exist" [2]) with the theory of K- and J-spaces (see [3]), introducing new functionals called H and I. (In another appendix we outline also an abstract formulation of Bennett's theorem [2].)

Finally, section 7 is devoted to various comments on the previous discussion.

What we do in the two appendices we have just told.

It is assumed that the reader has some familiarity with the theory of interpolation spaces. Regarding terminology we have tried as far as practicable to follow [3]. For the reader's benefit we recall here briefly some of the basic definitions.

By a Banach couple $\bar{A} = (A_0, A_1)$ we mean two Banach spaces which are continuously imbedded in some Hausdorff topological vector space (usually unspecified). We put $\Delta = \Delta(\bar{A}) = A_0 \cap A_1$ (intersection) and $\Sigma = \Sigma(\bar{A}) = A_0 + A_1$ (hull). In Δ and Σ respectively we have the following two one-parameter families of norms (with $t \in (0, \infty)$):

$$J(t, a) = \max(\|a\|_{A_0}, t\|a\|_{A_1}), \quad a \in \Delta$$

and

$$K(t, a) = \inf_{a = a_0 + a_1} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad a \in \Sigma.$$

With the aid of J we can define the J-spaces $\bar{A}_{\theta q; J}$ and with the aid of K the K-spaces $\bar{A}_{\theta q; K}$. In particular by definition $a \in \bar{A}_{\theta q; K}$ if and only if $a \in \Sigma$ and

$$\left(\int_0^\infty (t^{-\theta} K(t, a))^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Here $\theta \in (0, 1)$, $q \in (0, \infty]$. These are examples of interpolation spaces. One can show that they actually coincide up to (quasi-)norm (equivalence theorem) so we can drop the final subscripts writing simply $\bar{A}_{\theta q}$. Besides the "real" spaces $\bar{A}_{\theta q}$ we have also the "complex" spaces $[\bar{A}]_\theta$. For more details see [3].

Some auxiliary notation (which we have already made use of above): If p_0, p_1 are real numbers $\neq 0$ and $\theta \in (0, 1)$ we define p_θ by the formula $1/p_\theta = (1-\theta)/p_0 + \theta/p_1$. This notation will appear in section 4 and section 5 chiefly. We likewise define p' (the conjugate exponent) by $1/p + 1/p' = 1$. T^t denotes the transpose of the operator T, A' the dual of the space A.

Acknowledgement. I would like to thank Barry Simon and Grahame Bennett for interesting correspondence, and for sending me their unpublished manuscripts. Without their interference this paper would never have been written. I also express my gratitude to the editors for suggesting several improvements of the original version, including a change of title.

1. The spaces R_θ

Consider a Banach couple $\bar{A} = (A_0, A_1)$. For simplicity I assume that $\Delta = \Delta(\bar{A})$ is dense in both A_0 and A_1 so that one can speak of the dual couple $\bar{A}' = (A_0', A_1')$. Consider pairs (Y, v) where Y is a Banach space and $v: \Delta \to Y$ a continuous linear map. Let D_Y be the duality map associated with Y, i.e. D_Y is a set-valued function defined on Y taking as values subsets of Y' such that $\eta \in D_Y y$ if and only if

$$\langle \eta, y \rangle = \|y\|_Y^2 = \|\eta\|_{Y'}^2.$$

It is clear that $D_Y y$ is convex and non-empty. (Moreover we have

$$\langle \eta, f \rangle = \frac{d}{d\lambda}\left(\tfrac{1}{2}\|y + \lambda f\|_Y^2\right)\Big|_{\lambda=0}, \quad \text{i.e.} \quad \eta = \tfrac{1}{2}\operatorname{grad} \|y\|_Y^2,$$

if the norm in Y is differentiable so that D_Y is single-valued in the latter case. We then write abusively $\eta = D_Y y$ when $\eta \in D_Y y$.) I say that (Y, v) satisfies the condition (R_θ), where $\theta \in (0, 1)$, if for every $a \in \Delta$ holds $v^t(D_Y v(a)) \subset A_1'$ and

$$\|v^t(\eta)\|_{A_1'} \leq \|a\|_{A_0}^{(1-\theta)/\theta} \|v(a)\|_Y^{(2\theta-1)/\theta} \quad \text{if } \eta \in D_Y v(a). \tag{1.1}$$

(Again if the norm in Y is differentiable (1.1) simply means that

$$\|v^t(D_Y v(a))\|_{A_1'} \leq \|a\|_{A_0}^{(1-\theta)/\theta} \|v(a)\|_Y^{(2\theta-1)/\theta}.$$

In praxis v is often an injection and Δ a dense subspace of Y.) Now define a norm denoted by $\|a\|_{R_\theta(\bar{A})}$, or simply $\|a\|_{R_\theta}$, by setting

$$\|a\|_{R_\theta} = \sup_{(Y,v)} \|v(a)\|_Y \tag{1.2}$$

where (Y, v) runs through all pairs (Y, v) satisfying the condition (R_θ). The completion of Δ in the norm $\|a\|_{R_\theta}$ I denote by $R_\theta(\bar{A})$, or simply R_θ.

We begin by comparing $R_\theta(\bar{A})$ with the K- and the J-method.

PROPOSITION 1.1. *We have the inequalities*

$$\|a\|_{R_\theta} \leqslant t^{-\theta} J(t, a) \quad \textit{if } a \in \Delta, \tag{1.3}$$

$$K(t, a) \leqslant t^\theta \|a\|_{R_\theta} \quad \textit{if } a \in R_\theta. \tag{1.4}$$

It follows that $\bar{A}_{\theta 1} \subset R_\theta \subset \bar{A}_{\theta\infty}$.

Proof. Since

$$\|v(a)\|_Y^2 = \langle \eta, v(a) \rangle = \langle v^t(\eta), a \rangle \leqslant \|v^t(\eta)\|_{A_1'} \|a\|_{A_1} \quad \text{if } a \in A_1,\ \eta \in D_Y v(a)$$

(1.1) gives

$$\|v(a)\|_Y^2 \leqslant \|a\|_{A_0}^{(1-\theta)/\theta} \|a\|_{A_1} \|v(a)\|_Y^{(2\theta-1)/\theta}$$

or

$$\|v(a)\|_Y \leqslant \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^\theta$$

(provided $v(a) \neq 0$; if $v(a)=0$ there is nothing to prove). Therefore by (1.2)

$$\|a\|_{R_\theta} \leqslant \|a\|_{A_0}^{1-\theta} \|a\|_{A_1}^\theta \leqslant t^{-\theta} J(t, a)$$

proving (1.3).

(1.4): Take $Y = \mathbf{R}$ and $v(a) = \langle \alpha, a \rangle$ where $\alpha \in \Delta(\bar{A}')$. Then, identifying Y' and $\mathbf{R}$, $\eta \in D_Y y$ if and only if $\eta = y$ and $v^t(\eta) = \eta\alpha$. Thus (1.1) means that

$$|\langle \alpha, a \rangle| \, \|\alpha\|_{A_1'} \leqslant \|a\|_{A_0}^{(1-\theta)/\theta} |\langle \alpha, a \rangle|^{(2\theta-1)/\theta}$$

or

$$\|\alpha\|_{A_1'}^\theta |\langle \alpha, a \rangle|^{1-\theta} \leqslant \|a\|_{A_0}^{1-\theta}.$$

This clearly implies

$$\|\alpha\|_{A_0'}^{1-\theta} \|\alpha\|_{A_1'}^\theta \leqslant 1.$$

On the other hand the latter inequality is fulfilled if $t^\theta J(t^{-1}, \alpha) \leqslant 1$. Since K and J are dual norms this gives (1.4). $\square$

Next we come to the interpolation property.

PROPOSITION 1.2. *R_θ is an interpolation space of exponent θ.*

Proof. Given two Banach couples $\bar{A}$ and $\bar{B}$ and a bounded linear operator $T: \bar{A}\to\bar{B}$ we have to show that $T: R_\theta(\bar{A})\to R_\theta(\bar{B})$ with $M\leqslant M_0^{1-\theta}M_1^\theta$. (Here $M_0=\|T\|_{A_0,B_0}$, etc.) Let (Y, v) be any pair satisfying the condition (R_θ) with respect to $\bar{B}$, i.e.

$$\|v^t(\eta)\|_{B_1'}\leqslant\|b\|_{B_0}^{(1-\theta)/\theta}\|v(b)\|_Y^{(2\theta-1)/\theta}\quad\text{if } \eta\in D_Y v(b).$$

We apply this with b replaced by Ta

$$\|v^t(\eta)\|_{B_1'}\leqslant M_0^{(1-\theta)/\theta}\|a\|_{A_0}^{(1-\theta)/\theta}\|v(Ta)\|_Y^{(2\theta-1)/\theta}\quad\text{if } \eta\in D_Y v(Ta).$$

On the other hand

$$\|T^t v^t(\eta)\|_{A_1'}\leqslant M_1\|v^t(\eta)\|_{B_1'}.$$

With no loss of generality we may assume that $M_0^{1-\theta}M_1^\theta\leqslant 1$. It then follows that $(Y, v\circ T)$ satisfies the condition (R_θ) with respect to $\bar{A}$. Thus we have by (1.2)

$$\|v(Ta)\|_Y\leqslant\|a\|_{R_\theta(\bar{A})}.$$

But since (Y, v) satisfies the condition (R_θ) this inequality combined with (1.2) once more gives

$$\|Ta\|_{R_\theta(\bar{B})}\leqslant\|a\|_{R_\theta(\bar{A})}\quad\text{or}\quad M\leqslant 1. \qquad\square$$

2. The spaces R_θ^*

Let again A be a Banach couple, with $\Delta=\Delta(\bar{A})$ dense in both A_0 and A_1, and consider now pairs (X, u) where X is a *reflexive* Banach space and $u: X\to\Sigma=\Sigma(\bar{A})$ a continuous linear map. I say that (X, u) satisfies the condition (R_θ^*), where $\theta\in(0, 1)$, if for every $\alpha\in(\Sigma(\bar{A}))'\approx\Delta(\bar{A}')$ holds $u(D_{X'}u^t(\alpha))\subset A_0$ and

$$\|u(x)\|_{A_0}\leqslant\|\alpha\|_{A_1'}^{\theta/(1-\theta)}\|u^t(\alpha)\|_{X'}^{(1-2\theta)/(1-\theta)}\quad\text{if } x\in D_{X'}u^t(\alpha). \tag{2.1}$$

Now let $\|a\|_{R_\theta^*(\bar{A})}$ or simply $\|a\|_{R_\theta^*}$ be the greatest norm on $\Delta(\bar{A})$ which is $\leqslant$ all *pseudo-norms* of the type

$$\sup_{0\neq\alpha\in\Delta(\bar{A}')}\frac{|\langle a, \alpha\rangle|}{\|u^t(\alpha)\|_{X'}}$$

where (X, u) is a pair satisfying the condition (R_θ^*), i.e. we have

$$\|a\|_{R_\theta^*}=\inf\sum_{i\in I}\sup_{0\neq\alpha\in\Delta(\bar{A}')}\frac{|\langle a_i, \alpha\rangle|}{\|u_i^t(\alpha)\|_{X_i'}} \tag{2.2}$$

where the infimum is taken over all families $(a_i)_{i\in I}$ and $(X_i, u_i)_{i\in I}$ such that $a=\sum a_i$ and (X_i, u_i) satisfies the condition (R_θ^*). The completion of $\Delta(\bar{A})$ in the norm $\|a\|_{R_\theta^*}$ I denote by $R_\theta^*(\bar{A})$ or simply R_θ^*.

Parallel to Proposition 1.1 we now have:

PROPOSITION 2.1. *We have the inequalities*

$$\|a\|_{R_\theta^*} \leqslant t^{-\theta} J(t, a) \quad \text{if } a \in \Delta, \tag{2.3}$$

$$K(t, a) \leqslant t^{\theta} \|a\|_{R_\theta^*} \quad \text{if } a \in R_\theta^*. \tag{2.4}$$

It follows that $\bar{A}_{\theta 1} \subset R_\theta^* \subset \bar{A}_{\theta\infty}$.

Proof. (2.4): We have the formula

$$K(t, a) = \sup_{0 \neq \alpha \in \Delta(\bar{A}')} \frac{|\langle a, \alpha\rangle|}{J\left(\frac{1}{t}, \alpha\right)}.$$

Now repetition of the argument of the proof of (1.3) gives

$$\|u^t(\alpha)\|_{X'} \leqslant t^{\theta} J\left(\frac{1}{t}, \alpha\right).$$

if (X, u) satisfies the condition (R_θ). Therefore

$$t^{-\theta} K(t, a) \leqslant \sup \frac{|\langle a, \alpha\rangle|}{\|u^t(\alpha)\|_{X'}}$$

from which (2.4) follows from the definition of $\|a\|_{R_\theta^*}$.

(2.3): Now we take $X=\mathbf{R}$ and $u(x)=xa$ with $a\in\Delta$. An argument similar to the one of the proof of (1.4) shows that if $t^{-\theta}J(t, a)\leqslant 1$ then (X, u) satisfies the condition (R_θ^*). Since obviously $u^t(\alpha)=\langle\beta, a\rangle$ this gives in the said hypothesis $\|a\|_{R_\theta^*}\leqslant 1$. □

We further have corresponding to 1.2.

PROPOSITION 2.2. R_θ^* *is an interpolation space of exponent* θ.

Proof. By a routine argument similar to the one of the proof of Proposition 1.2. □

3. The main result

The purpose of this section is to establish the following result, which thus may be conceived as an abstract generalization of M. Riesz's theorem. It is the main result of this paper.

THEOREM 3.1. *Let $\bar{A}$ and $\bar{B}$ be two Banach couples and $T\colon \bar{A}\to\bar{B}$ a bounded linear map. Assume that:* (*) *there exists a sequence (T_n) of bounded linear maps $T_n\colon \bar{A}\to\bar{B}$ such that $T_n\colon \Sigma(\bar{A})\to\Delta(\bar{B})$ which approximates T in the sense that*

$$\lim_{n\to\infty}\|T_n a - Ta\|_{\Delta(\bar{B})}=0 \quad \textit{if } a\in\Delta(\bar{A}) \tag{3.1}$$

and moreover holds

$$\overline{\lim_{n\to\infty}}\,\|T_n\|_{A_0,B_0}=\|T\|_{A_0,B_0}, \tag{3.2}$$

$$\overline{\lim_{n\to\infty}}\,\|T_n\|_{A_1,B_1}=\|T\|_{A_1,B_1}.$$

Then holds $T\colon R^*_\theta(\bar{A})\to R_\theta(\bar{B})$ together with the convexity $M\leqslant M_0^{1-\theta}M_1^\theta$ (where again $M_0=\|T\|_{A_0,B_0}$ etc.).

Proof. It suffices to prove the theorem under the additional hypothesis that $T\colon \Sigma(\bar{A})\to\Delta(\bar{B})$. For if the theorem is true in that special case we can apply it to each of the operators T_n. The convexity inequality together with (3.2) then gives for any $\varepsilon>0$ and large n

$$\|T_n a\|_{R_\theta(\bar{B})}\leqslant (M_0^{1-\theta}M_1^\theta+\varepsilon)\,\|a\|_{R^*_\theta(\bar{A})}.$$

Again using (3.1) and (1.3) this plainly yields

$$\|Ta\|_{R_\theta(\bar{B})}\leqslant (M_0^{1-\theta}M_1^\theta+\varepsilon)\,\|a\|_{R^*_\theta(\bar{A})}$$

which implies $M\leqslant M_0^{1-\theta}M_1^\theta$.

Assume thus from now on that $T\colon \Sigma(\bar{A})\to\Delta(\bar{B})$.

Let (X, u) be any pair satisfying the condition (R^*_θ) with respect to $\bar{A}$ and similarly let (Y, v) be any pair satisfying the condition (R_θ) with respect to $\bar{B}$.

We wish to show that

$$\|Sx\|_Y\leqslant M_0^{1-\theta}M_1^\theta\|x\|_X \quad \text{for } x\in X, \text{ with } S=v\circ T\circ u. \tag{3.3}$$

Indeed assume that (3.3) holds true. Obviously we can here replace

$$\|x\|_X=\sup\frac{|\langle x,\xi\rangle|}{\|\xi\|_{X'}}$$

by

$$\sup\frac{|\langle x, u^t(\alpha)\rangle|}{\|u^t(\alpha)\|_{X'}}=\sup\frac{|\langle u(x),\alpha\rangle|}{\|u^t(\alpha)\|_{X'}}.$$

Thus with $a=u(x)$ (3.3) gives

$$\|v(Ta)\|_Y\leqslant M_0^{1-\theta}M_1^\theta\sup\frac{|\langle a,\alpha\rangle|}{\|u^t(\alpha)\|_{X'}}.$$

(We notice that in view of Hahn-Banach if the right hand side is $<\infty$ then a certainly can be represented in this form.) In view of the definition of the norms $\|b\|_{R_\theta(\bar{B})}$ and $\|a\|_{R_\theta^*(\bar{A})}$ this clearly gives $M \leqslant M_0^{1-\theta} M_1^\theta$.

Replace now A_0 by $A_0+\varepsilon^{-1}A_1$, A_1 by $\varepsilon^{-1}A_0+A_1$, B_0 by $B_0 \cap \varepsilon B_1$, B_1 by $\varepsilon B_0 \cap B_1$. We are then in a situation when $A_0=A_1=\Sigma(\bar{A})$, $B_0=B_1=\Delta(\bar{B})$, algebraically. It suffices to prove (3.3) in the latter case. Indeed (X, u) still satisfies the condition (R_θ^*) with respect to $(A_0+\varepsilon^{-1}A_1, \varepsilon^{-1}A_0+A_1)$ and similarly (Y, v) satisfies the condition (R_θ) with respect to $(B_0 \cap \varepsilon B_1, \varepsilon B_0 \cap B_1)$. Moreover we have

$$\|T\|_{A_0+\varepsilon^{-1}A_1,\, B_0\cap\varepsilon B_1} \leqslant \max\left(\|T\|_{A_0,B_0}, \varepsilon\|T\|_{A_1,B_0}, \varepsilon\|T\|_{A_0,B_1}, \varepsilon^2\|T\|_{A_1,B_1}\right)$$

and a similar estimate with $\|T\|_{\varepsilon^{-1}A_0+A_1,\, \varepsilon B_0\cap B_1}$. We can therefore afterwards safely pass to the limit $\varepsilon \to 0$.

We can therefore from now on assume that $A_0=A_1=\Sigma(\bar{A})=:A$, $B_0=B_1=\Delta(\bar{B})=:B$, algebraically.

By the extension of Lindenstrauss's theorem given in Appendix I we can assume that S attains its norm. We can find $x \in X$ and $\eta \in Y'$ with $\|x\|_X=\|\eta\|_{Y'}=1$ such that $N=\langle \eta, Sx\rangle = \langle S^t\eta, x\rangle$ with $N=\|S\|_{X,Y}$. We then must have $N\eta \in D_Y Sx$, $Nx \in D_X S^t\eta$. Also $\|Sx\|_Y=\|S^t\eta\|_{X'}=N$. From (1.1)—applied to the couple $\bar{B}$—now follows

$$N\|v^t(\eta)\|_{B_1'} \leqslant \|T(u(x))\|_{B_0}^{(1-\theta)/\theta}\|Sx\|_Y^{(2\theta-1)/\theta} \leqslant M_0^{(1-\theta)/\theta}\|u(x)\|_{A_0}^{(1-\theta)/\theta}N^{(2\theta-1)/\theta}$$

and from (2.1)

$$N\|u(x)\|_{A_0} \leqslant \|T^t(v^t(\eta))\|_{A_1'}^{(1-\theta)/\theta}\|S^t\eta\|_{Y'}^{(1-2\theta)/\theta} \leqslant M_1^{\theta/(1-\theta)}\|v^t(\eta)\|_{B_1'}^{\theta/(1-\theta)}N^{(1-2\theta)/(1-\theta)}.$$

We raise the first inequality to power θ and the second one to power $1-\theta$ and multiply together. Then results $N \leqslant M_0^{1-\theta}M_1^\theta$ which is the same as (3.3). □

From Theorem 3.1 follows at once

COROLLARY 3.1. *Let $\bar{A}=\bar{B}$ and assume that (*) is fulfilled with $T=id_{\bar{A}}$. Then holds $R_\theta^*(\bar{A}) \subset R_\theta(\bar{A})$.* □

4. Illustrations

In this section we will find what the spaces R_θ and R_θ^* are in several concrete cases.

1° L_p spaces. This case has essentially already been discussed in the introduction. Consider the couple (L_{p_0}, L_{p_1}). We want to compare $R_\theta(L_{p_0}, L_{p_1})$ and L_p where $p=p_\theta$, $\theta \in (0, 1)$. To this end we take $Y=L_p$ and $v=id$. Then as is well-known

$$D_{L_p}y = |y|^{p-1}\operatorname{sgn} y/\|y\|_{L_p}^{p-2}, \quad y \in L_p.$$

Thus (1.1) becomes

$$\| |y|^{p-1} \|_{L_{p_1'}} \leqslant \|y\|_{L_{p_0}}^{(1-\theta)/\theta} \|y\|_{L_p}^{(2\theta-1)/\theta+p-2}$$

or

$$\|y\|_{L_{p_1'(p-1)}} \leqslant \|y\|_{L_{p_0}}^{(1-\theta)/(p-1)\theta} \|y\|_{L_p}^{(\theta p-1)/(p-1)\theta}$$

which is Hölder's inequality if the exponents are in $(0, 1)$, i.e. we get the condition $\theta \geqslant 1/p$. Thus we conclude that

$$R_\theta(L_{p_0}, L_{p_1}) \subset L_p \quad \text{if } p = p_\theta, \quad \theta \geqslant \frac{1}{p}. \tag{4.1}$$

In the same way utilizing (2.1), with $X = L_p$, $u = id$, we find

$$L_p \subset R_\theta^*(L_{p_0}, L_{p_1}) \quad \text{if } p = p_\theta, \quad \theta \leqslant \frac{1}{p}. \tag{4.2}$$

If we now apply Theorem 3.1 to the couples (L_{p_0}, L_{p_1}) and (L_{q_0}, L_{q_1})—assumption (*) is certainly fulfilled—and use (4.1) and (4.2) we get a proof of M. Riesz' theorem; in fact this is essentially the original proof (compare the introduction). Notice also that (4.1) and (4.2) put together yield

$$L_p = R_\theta(L_{p_0}, L_{p_1}) = R_\theta^*(L_{p_0}, L_{p_1}) \quad \text{if } p = p_\theta = 1/\theta. \tag{4.3}$$

2° L_{pq} (Lorentz)-spaces. Exactly the same calculations can be made for the couple $(L_{p_0 q_0}, L_{p_1 q_1})$. Of course we won't get the constant 1 in the inclusions corresponding to (4.1) and (4.2), because the constant in the "Lorentz-Hölder" inequality is not 1 either. Thus Riesz' original proof can indeed be used to prove (a version of) Marcinkiewicz theorem [8] too.

3° Interpolation between a space and its dual. Consider the couple (Z, Z') where Z is any *reflexive* Banach space imbedded as a dense subspace of a Hilbert space H so that H can be imbedded into Z'; we thus have the situation $Z \subset H \subset Z'$. (Notice the special case when Z too is a Hilbert space!) We take $Y = H$ and $v = id$. Now (1.1) with $\theta = \frac{1}{2}$ is just $\|a\|_Z \leqslant \|a\|_Z$, thus trivially fulfilled. In the same way taking $X = H$ and $u = id$ we see that (2.1) too is trivially fulfilled. We conclude that

$$H = R_{1/2}(Z, Z') = R_{1/2}^*(Z, Z'), \tag{4.4}$$

which in particular generalizes (4.3) with $\theta = 1/p = \frac{1}{2}$. Obviously (4.4) corresponds to the interpolation theorem of Girardeau [4]. (Girardeau considers general locally convex spaces.)

4° Interpolation with change of measure. Let $L_p(h)$ denote L_p with the original measure $d\mu$ replaced by $h d\mu$, h a positive μ-measurable weight function; i.e. the norm in $L_p(h)$ is

$$\|a\|_{L_p(h)} = \left(\int |a(u)|^p h(u)\, d\mu(u)\right)^{1/p}.$$

Thus we have

$$D_{L_p(h)} a = |a|^{p-1} \operatorname{sgn} a h^p / \|a\|^{p-2}_{L_p(h)}$$

for the duality

$$\langle \alpha, a\rangle = \int \alpha(u)\, a(u)\, d\mu(u).$$

Notice also that

$$(L_p(h))' = L_{p'}(h') \quad \text{with } h' = h^{1/(p-1)} = h^{p'-1}.$$

Now consider the couple $(L_{p_0}(h_0), L_{p_1}(h_1))$ and take first $Y = L_p(h)$ and $v = id$, with θ as in 1°. Then (1.1) becomes

$$\||a|^{p-1} h_1\|_{L_{p_1'}(h_1')} \leqslant \|a\|^{(1-\theta)/\theta}_{L_p(h_0)} \|a\|^{(p\theta-1)/\theta}_{L_p(h)}.$$

This is Hölder's inequality if also

$$h = h_0^{p(1-\theta)/p_0} h_1^{p\theta/p_1}.$$

Thus we conclude that

$$R_\theta(L_{p_0}(h_0), L_{p_1}(h_1)) \subset L_p(h).$$

In the same way using (2.1) we obtain an inclusion in the opposite sense. We can use this to prove the interpolation theorem of Stein-Weiss [13]. We also find

$$L_p(h) = R_\theta(L_{p_0}(h_0), L_{p_1}(h_1)) = R^*_\theta(L_{p_0}(h_0), L_{p_1}(h_1)) \tag{4.5}$$

if $p = p_\theta = 1/\theta$ which thus generalizes (4.3).

5. On QM and QM^*

As already told in the introduction this section mainly reproduces results drawn from unpublished work by Simon [12] and Bennett [1].

Let us return to the situation of section 1. ($\bar{A}$ is thus a Banach couple etc.) We now modify the definition of the condition (R_θ) in the sense that we restrict Y to be a Hilbert space, with scalar product $(y|y')_Y$ or simply $(y|y')$, and we further specialize to $\theta = \frac{1}{2}$. We then get the inequality

$$\|v^t v(a)\|_{A_1'} \leqslant \|a\|_{A_0}. \tag{5.1}$$

Equivalently

$$|(v(a)|v(a'))| \leqslant \|a\|_{A_0} \|a'\|_{A_1} \tag{5.1'}$$

where we of course identify Y and Y'. If thus (5.1) holds for any $a \in \Delta$, or (5.1′) for any a, $a' \in \Delta$, we say that (Y, v) satisfies the condition (QM). We now imitate the definition of the Riesz method and get the space $QM(\bar{A})$ or simply QM.

By a dual procedure (the one of section 2) we get also spaces $QM^*(\bar{A})$ or QM^*.

Let us now investigate to what extent the properties of R_θ extend to the spaces QM.

Proposition 1.2 can be generalized so that we have notably $\bar{A}_{\frac{1}{2},1} \subset QM \subset \bar{A}_{\frac{1}{2},\infty}$. Here the second inclusion follows also from the observation that obviously $QM \supset R_{\frac{1}{2}}$. (If a pair (Y, v) satisfies condition (QM) then it satisfies a fortiori the condition $(R_{\frac{1}{2}})$.) However as to the first inclusion we have a much stronger result, namely

$$\bar{A}_{\frac{1}{2},2} \cap [\bar{A}]_{\frac{1}{2}} \subset QM. \tag{5.2}$$

(Notice that $\bar{A}_{\theta 1} \subset \bar{A}_{\theta p}$ and $\bar{A}_{\theta 1} \subset [\bar{A}]_\theta$; see [3], p. 44 and p. 102. $[\bar{A}]_\theta$ are the complex spaces.) This depends on multi-linear interpolation. To fix the ideas let us here consider the real case only. (It is the complex case that is treated in [12].) Consider the bilinear mapping L: $(a, a') \to (v(a) \mid v(a'))$. Then we have

$$L: A_0 \times A_1 \to \mathbf{R},$$

$$L: A_1 \times A_0 \to \mathbf{R}.$$

Interpolation now yields (see [3], p. 76, excercise 5).

$$L: (A_0, A_1)_{\theta p} \times (A_1, A_0)_{\theta q} \to \mathbf{R}, \quad \frac{1}{p} + \frac{1}{q} \geqslant 1. \tag{5.3}$$

Noticing the symmetry $(A_0, A_1)_{\theta p} = (A_1, A_0)_{1-\theta, p}$ and taking $\theta = \frac{1}{2}$, $p = q = 2$ we get from (5.3)

$$|L(a, a')| \leqslant C \|a\|_{\bar{A}_{\frac{1}{2},2}} \|a'\|_{\bar{A}_{\frac{1}{2},2}}$$

which (take $a = a'$ and maximize) gives

$$\|a\|_{QM} \leqslant C \|a\|_{\bar{A}_{\frac{1}{2},2}}.$$

This proves the part of (5.2) we were interested in. We can also prove a result on reiteration, viz.

$$QM(\bar{A}_{\theta p}, \bar{A}_{1-\theta, q}) \subset QM(\bar{A}). \tag{5.4}$$

Just use (5.3) with general parameters. Since $[\bar{A}]_\theta \subset \bar{A}_{\theta\infty}$ we have in particular

$$QM([\bar{A}]_\theta, [\bar{A}]_{1-\theta}) \subset QM(\bar{A})$$

which is the case considered in [12]. Finally it is clear that Proposition 1.2 too can be generalized: QM is effectively an interpolation space.

The spaces QM^* have analogous properties. Also the proof of Theorem 3.1 gives the following generalization of Corollary 3.1: $QM^* \subset QM$, in the same assumptions on $\bar{A}$.

We next consider QM in some concrete cases:

Examples. 1° L_p spaces. We have $(L_{p_0}, L_{p_1})_{\frac{1}{2},2} = L_p$ and $[L_{p_0}, L_{p_1}]_{\frac{1}{2}} = L_p$ with $p = p_{\frac{1}{2}}$. Thus (5.2) gives at once $L_{p,\max(2,p)} \subset QM(L_{p_0}, L_{p_1})$. A closer examination reveals that we can here replaces $\subset$ by $=$. For $2 \leq p < \infty$ this was done in [12]. This is very easy. The idea is to use for a given the scalar product

$$(y|y') = \|a\|^{2-p} \int y(t)\overline{y'(t)}\,|a(t)|^{p-2}\,d\mu(t) \tag{5.5}$$

with Hölder's inequality. We leave the details to the reader. The case $1 < p < 2$ was treated by Bennett [1] with a quite different technique. We offer here a proof more along the lines of [12], though valid only when p_0 and $p_1 > 1$. It is no essential restriction to take the measure space to be $\mathbf{R}_+^*$ (with $d\mu(t) = t^{-1}dt$). Now we use instead of (5.5) the scalar product

$$(y|y') = \int_0^\infty t^{2(1/p-1)} \int_0^t y(u)\,du \int_0^t \overline{y'(v)}\,dv\,\frac{dt}{t} \tag{5.6}$$

Since everything is rearrangement invariant we can assume that a is nonincreasing. We then have $\|a\|_{L_{p2}} \leq C\|a\|_Y$ which will imply

$$\|a\|_{L_{p2}} \leq C\|a\|_{QM(L_{p_0}, L_{p_1})}$$

if we can also show that (5.1′) is fulfilled (possibly with a constant.) To this end we first prove, using a technique which ought to be familiar to the readers of [3], that

$$|(y|y')| \leq C\|y\|_{L_{p_0,1}}\|y'\|_{L_{p_1,1}}.$$

We then again apply (5.3) with general θ and $p = q = 2$.

2° Hilbert spaces. If A_0 and A_1 are both Hilbert spaces we know that $\bar{A}_{\frac{1}{2},2}$ and $[\bar{A}]_{\frac{1}{2}}$ too are Hilbert spaces—actually they coincide up to equivalence of norm. It is easy to see that the scalar product on this space fulfills condition (QM). In view of (5.2) we thus get in this case $QM = \bar{A}_{\frac{1}{2},2} = [\bar{A}]_{\frac{1}{2}}$.

6. The functionals I and H

As was mentionned in the introduction Grahame Bennett [2] has extended Riesz–Thorin to "the case $p<0$". The precise meaning of this is the following. If T is the linear operator to be interpolated then the condition $T: L^p \to L^q$ with p or $q<0$ is interpreted as $T^t: L^{q'} \to L^{p'}$ provided this is a meaningful statement which is the case if p' and q' are both >0. (Also the author of [2] restricts himself to *discrete* measures.) In this way Riesz–Thorin gets generalized to the region $\hat{\mathcal{R}}$ which is the image of the usual "positive" quadrant $\mathcal{R}$ of the $(1/p, 1/q)$-plane under the map $(1/p, 1/q) \to (1/p, 1/q)$. The proof depends on the three line theorem in conjunction with a certain factorization theorem due to Maurey [9]. In [2] a short direct proof of the special case of Maurey's theorem needed is reproduced.

It is not difficult to formalize Bennett's argument [2], although it seems difficult to imagine a non-trivial situation other than the L_p-case where one has such a factorization. For the reader's benefit a brief sketch of this will be given in Appendix II.

Here we try instead to establish a connection with the J- and K-spaces.

Consider thus the following situation. There is given a quasi-Banach space A_0 and another one A_1' "which is the dual of a space which need not exist". In order not to enter into too many technicalities let us only treat the *finite dimensional* case: I.e. A_0 is a finite dimensional vector space V equipped with a quasi-norm and A_1' its (algebraic) dual V^* equipped with a quasi-norm.

We now define the I-functional $I(t, a)$ by the formula

$$I(t, a) = \sup \|v(a)\|_Y.$$

Here the sup goes over all pairs (Y, v)—with Y and v as in section 1—subject to the following restriction.

$$\|a\|_{A_0} \leqslant \|v(a)\|_Y,\ \eta \in D_Y v(a) \ \Rightarrow\ \|v^t(\eta)\|_{A_1'} \leqslant t\|v(a)\|_Y. \tag{6.1}$$

We can then define I-spaces $\bar{A}_{\theta q;I}$ in the obvious way (imitating the construction of the J-spaces $\bar{A}_{\theta q;J}$, [3], p. 42).

Let us see what this gives in the "classical" case, i.e. A_1' is effectively the dual of a space A_1. (Since we are dealing with the finite dimensional case this means that A_1 is our V equipped with a quasi-norm, in general different from the one defining A_0.) We have $\langle \eta, y\rangle = \|y\|^2 = \|\eta\|^2$ (see section 1). This gives

$$\|v(a)\|_Y^2 = \langle \eta, v(a)\rangle = \langle v^t(\eta), a\rangle \leqslant \|v^t(\eta)\|_{A_1'} \|a\|_{A_1} \leqslant t\|v(a)\|_Y \|a\|_{A_1} \leqslant \|v(a)\|_Y J(t, a)$$

or

$$\|v(a)\|_Y \leqslant J(t, a) \quad \text{if } \|a\|_{A_0} \leqslant \|v(a)\|_Y.$$

Thus $I(t, a) \leqslant J(t, a)$. But $(A_0 \cap tA_1, id)$ obviously satisfies (6.1). Therefore we have in fact equality. *In the classical case the I-functional coincides with the J-functional. In particular the I-spaces are the same as the J-spaces.*

Returning to the general case we notice that (6.1) is intimately related to (1.1). In fact it is a kind of limiting case of the latter. We see also that condition (R_θ) implies that $(Y, t^\theta v)$ fulfills (5.1). This gives

$$t^\theta \|v(a)\|_Y \leqslant I(t, a)$$

which again implies

$$\|a\|_{R_\theta(\bar{A})} \leqslant \frac{I(t, a)}{t^\theta}.$$

This again shows, with the same proof as for J-spaces ([3], p. 44), that we have the imbedding $\bar{A}_{\theta 1; I} \subset R_\theta(\bar{A})$.

In a dual manner we can define the H-functional. In the classical case we have again $K(t, a) = H(t, a)$ and in the general case the inclusion $R_\theta^*(\bar{A}) \subset \bar{A}_{\theta\infty; H}$. Maybe one also has the estimate

$$H(t, a) \leqslant \min\left(1, \frac{t}{s}\right) I(s, a)$$

(analogous to the one for K and J; [3], p. 42) which would imply the imbedding $\bar{A}_{\theta p; I} \subset \bar{A}_{\theta p; H}$ (half of the usual equivalence theorem; [3], p. 44). Since we are in the finite dimensional case in this context means just that we have a "universal" estimate for the norms.

7. Comments

1° I do not like at all the *ad hoc* restriction made in section 2 that X should be reflexive. Perhaps one should instead take X (as well as Y) finite dimensional. This would also avoid the use of Lindenstrauss' theorem [5]. On the other hand this complicates the treatment of the examples (section 4).

2° More generally, when dealing with the Riesz method or the method of quadratic means, one could—in the spirit of Bennett's paper [2]; see section 6 of the present paper—allow quasi-Banach spaces. I.e. as in section 6 we take one quasi-Banach space A_0 and another one A_1' which need not be a dual space. In particular in the L_p case Riesz' proof thus permits to extend his interpolation theorem to the whole region $(p \leqslant q)$.

3° The spaces QM can also be defined when *both* A_0 and A_1 are quasi-Banach spaces. (No need for duals, *provided* one uses (5.1′) rather than (5.1)!) As a generalization of (5.2) one can now prove $\bar{A}_{\frac{1}{2}, r} \subset QM(\bar{A})$ where $r \leqslant 1$ is a number depending on the moduli of concavity (in the sense of Rolewicz; see [5], p. 165) of the spaces involved. On the other hand we have in general *no* inclusion in the opposite sense. Indeed we may even have $\|a\|_{QM(\bar{A})}=0$ for all $a \in \Delta$; $\|a\|_{QM}$ need therefore not be a norm, although it is always a semi-norm. Take $A_0=L_{p_0}$, $A_1=L_{p_1}$ with $0<p_0, p_1<1$ and the theorem of Day (to the effect that $L_p'=0$ if $0<p<1$; see [5], p. 161–162).

4° Inequality (1.1) can be generalized in the following direction:

$$\|v^t(\eta)\|^{\lambda}_{A_0'}\|v^t(\eta)\|^{\mu}_{A_1'} \leqslant \|a\|^{k}_{A_0}\|a\|^{m}_{A_1}\|v(a)\|^{1-k-m}_{Y} \tag{7.1}$$

where λ, μ, k, m are $\geqslant 0$ with $\lambda+\mu=1$. In this way A_0 and A_1 get treated in a symmetric way. However this excludes the generalization of type 2° above.

5° I have no single example with $R_\theta(\bar{A}) \neq R^*_\theta(\bar{A})$.

6° Perhaps one can prove under not too restrictive assumptions on $\bar{A}$ a duality theorem, viz. $(R_\theta(\bar{A}))' \approx R^*_\theta(\bar{A}')$.

7° The construction leading to the Riesz method and the method of quadratic means can be formalized as follows. Let us imagine that one has for each Banach couple $\bar{A}$ a family $\mathfrak{Y}(A)$ or simply $\mathfrak{Y}$ of pairs (Y, v) where Y is a Banach space and $v: \Delta(\bar{A}) \to Y$ a continuous linear map which depends functorially on $\bar{A}$ in the following sense. If $T: \bar{A} \to \bar{B}$ has norm(s) $\leqslant 1$ and if (Y, v) belongs to $\mathfrak{Y}(\bar{B})$ then $(Y, v \circ T)$ belongs to $\mathfrak{Y}(\bar{A})$. Then we get an interpolation space $S_{\mathfrak{Y}}=S_{\mathfrak{Y}}(\bar{A})$ by taking the completion in the norm

$$\|a\|_{S_{\mathfrak{Y}}}=\sup_{(Y,v)\in\mathfrak{Y}}\|v(a)\|_Y, \quad a \in \Delta(\bar{A}).$$

As yet another example of this general construction let us mention the case when $\mathfrak{Y}(\bar{A})$ is the family of pairs (Y, v) such that

$$\|v(a)\| \leqslant \|a\|^{1-\theta}_{A_0}\|a\|^{\theta}_{A_1}, \quad a \in \Delta(\bar{A})$$

with a fixed number $\theta \in (0, 1)$. Indeed in this case as is readily seen $S_{\mathfrak{Y}}(\bar{A})=\bar{A}_{\theta 1}$. As is well-known it is this inequality which has been the point of departure of much of the early work of S. G. Krein and his associates; see e.g. [6].

Appendix I. On operators which attain their norm

Let X, Y, A, B be Banach spaces and $u: X \to A$ and $v: B \to Y$ bounded linear operators. With every bounded linear operator $T: A \to B$ we can then associate the bounded linear operator $S: X \to Y$ defined by $S=v \circ T \circ u$. Here is the relevant commutative diagram:

$$\begin{tikzcd} A \arrow[r, "T"] & B \arrow[d, "v"] \\ X \arrow[u, "u"] \arrow[r, "S"] & Y \end{tikzcd}$$

Assume that X is reflexive. Then we have the following result which is an immediate extension of a theorem by Lindenstrauss [7] (the case $X=A$, $Y=B$, $u=id_X$, $v=id_Y$):

THEOREM. *Every bounded linear operator $T: A\to B$ can be approximated (in $L(A, B)$) by bounded linear operators $\hat{T}: A\to B$ such that the corresponding operator $\hat{S}: X\to Y$ attains its norm (in $L(X, Y)$).*

For the reader's benefit we reproduce here the essentials of the

Proof. It suffices to find for every $\varepsilon>0$ an operator $\hat{T}$ such that (i) $\|T-\hat{T}\|_{A,B}<\varepsilon$ and (ii) $|\eta_j(Sx_k)|>\|S\|_{X,Y}-\varepsilon_j$ for $k\geqslant j$ where (ε_j) is a sequence of positive numbers tending to 0 and (x_j) and (η_j) sequences in X and Y' respectively with $\|x_j\|_X=\|\eta_j\|_{Y'}=1$. Indeed since X is reflexive (ii) entails that $\hat{S}$ attains its norm ([7], Lemma 1). Again to fulfill (ii) it suffices to find a sequence of operators (T_j) with $\hat{T}=\lim T_j$ such that (ii') $|\eta_j(S_jx_k)|>\|S_j\|_{X,Y}-\varepsilon_j$ for $k\geqslant j$. We define T_j recursively by

$$T_{j+1}a = T_ja+\varepsilon_j\eta_j(v(T_ja))\,T_j(u(x_j)),\quad a\in A \tag{1}$$

with $T_1=T$ and set $\hat{T}=\lim_{j\to\infty}T_j$ (if the limit exists). This is indeed the case under the restriction $\|x_j\|_X=\|\eta_j\|_{Y'}=1$ provided ε_j decreases sufficiently fast and then obviously (i) too can be made to be true. (1) gives

$$S_{j+1}x = S_jx+\varepsilon_j\eta_j(S_jx)S_jx_j,\quad x\in X. \tag{2}$$

Therefore we get by the triangle inequality taking $x=x_k$

$$\varepsilon_j|\eta_j(S_jx_k)|\;\|S_j\|+\|S_j\|\geqslant\|S_{j+1}a_k\|\geqslant\|S_ka_k\|-\|S_k-S_{j+1}\|\geqslant\|S_{j+1}\|-\tfrac{1}{2}\varepsilon_j^2,\quad j>k, \tag{3}$$

provided S_k is made to converge sufficiently fast, and also $\|S_k\|\geqslant\|S_j\|$ if $j>k$. Now select x_j and η_j such that $\eta_j(S_jx_j)=\|S_jx_j\|$, which is possible in view of Hahn-Banach, and $\|S_jx_j\|$ sufficiently close to $\|S_j\|$. Then by taking $x=x_j$ we get

$$\|S_{j+1}\|\geqslant\|S_{j+1}x_j\|=\|S_jx_j\|\,(1+\varepsilon_j\|S_jx_j\|)\geqslant\|S_j\|+\varepsilon_j\|S_j\|-\tfrac{1}{2}\varepsilon_j^2 \tag{4}$$

which in particular entails $\|S_{j+1}\| \geqslant \|S_j\|$. (3) and (4) together give

$$\varepsilon_j|\eta_j(S_j x_k)|\,\|S_j\| + \|S_j\| \geqslant \|S_j\| + \varepsilon_j\|S_j\| - \varepsilon_j^2$$

which again gives (ii′). □

Appendix II. Abstract version of Bennett's theorem

Let A_0 and A_1' be as in section 6. Let B_0 and B_1' be another two spaces in a similar relation (i.e. B_1 is a finite dimensional vector space W equipped with a quasi-norm and B_1' is the dual W^* equipped with a quasi-norm). Let T be a linear operator with $T: A_0 \to B_0$ and $T^t: B_1' \to A_1'$. We wish to establish a result of the type $T: A \to B$ along with the inequality $M \leqslant M_0^{1-\theta} M_1^{\theta}$, where

$$M_0 = \|T\|_{A_0,B_0}, \quad M_1 = \|T\|_{A_1,B_1} = \|T^t\|_{B_1',A_1'}, \quad M = \|T\|_{A,B}.$$

Here A is V equipped with a quasi-norm (other than the one for A_0) and B is W equipped with a quasi-norm (other than the one for B_0).

We assume that 1° there exists for each $b \in W$ a linear $\omega: B \to B_0$ such that

$$\|b\|_B = \|\omega(b)\|_{B_0}, \quad \|\omega^{1/\theta}\|_{B_0,B_1} \leqslant 1$$

and that 2° there exist linear operators $w: A_1 \to A$ and $S: A \to B_1$ such that

$$T = S \circ w, \quad \|S\|_{A,B_1} \|w^{\theta/(1-\theta)}\|_{A,A_0}^{(1-\theta)/\theta} \leqslant M_1,$$

with some interpretation of the fractional powers of ω and w. (In the concrete case of L_p spaces 1° is just Hölder's inequality while as 2° is Maurey's theorem [9].) We consider the function

$$F(z) = \|\omega^{z/\theta} S w^{(1-z)/(1-\theta)} a\|_{B_0}.$$

We also assume that the three line theorem applies to F. Obviously we have

$$F(\theta) = \|Ta\|_B,$$
$$F(0+it) \leqslant \|Tw^{\theta/(1-\theta)} a\|_{B_0} \leqslant M_0 \|w^{\theta/(1-\theta)}\|_{A,A_0} \|a\|_A,$$
$$F(1+it) \leqslant \|\omega^{1/\theta} Sa\|_{B_0} = \|S\|_{A,B_1} \|a\|_A$$

so the three line theorem gives (use 2° once more!)

$$\|Ta\|_B \leqslant M_0^{1-\theta} M_1^{\theta} \|a\|_A$$

thus establishing our goal.

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Received June 6, 1978