

# THE TOPOLOGY OF SPACES OF RATIONAL FUNCTIONS

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## § 1. Introduction

A rational function  $f$  of the form

$$f(z) = \frac{p(z)}{q(z)} = \frac{z^n + a_1 z^{n-1} + \dots + a_n}{z^n + b_1 z^{n-1} + \dots + b_n}, \quad (1)$$

where  $a_i$  and  $b_i$  are complex numbers, defines a continuous map of degree  $n$  from the Riemann sphere  $S^2 = \mathbb{C} \cup \infty$  to itself. If the coefficients  $(a_1, \dots, a_n; b_1, \dots, b_n)$  vary continuously in  $\mathbb{C}^{2n}$  the map  $f$  varies continuously providing the polynomials  $p$  and  $q$  have no root in common; but the topological degree of the map  $f$  jumps when a root of  $p$  moves into coincidence with a root of  $q$ .

Let  $F_n^*$  denote the open set of  $\mathbb{C}^{2n}$  consisting of pairs of monic polynomials  $(p, q)$  of degree  $n$  with no common root.  $F_n^*$  is the complement of an algebraic hypersurface, the “resultant locus”, in  $\mathbb{C}^{2n}$ . On the other hand it can be identified with a subspace of the space  $M_n^*$  of maps  $S^2 \rightarrow S^2$  which take  $\infty$  to 1 and have degree  $n$ . In this paper I shall prove that when  $n$  is large the  $2n$ -dimensional complex variety  $F_n^*$  is a good approximation to the homotopy type of the space  $M_n^*$ , or, more precisely

**PROPOSITION (1.1).** *The inclusion  $F_n^* \rightarrow M_n^*$  is a homotopy equivalence up to dimension  $n$ .*

Equivalently one can consider the space  $F_n$  of rational functions of the form

$$\frac{a_0 z^n + \dots + a_n}{b_0 z^n + \dots + b_n},$$

where again the numerator and denominator have no common factor, and  $a_0$  and  $b_0$  are not both zero. This space is the complement of a hypersurface in  $\mathbb{P}^{2n+1}$ . It can be regarded as a subspace of the space  $M_n$  of all maps  $S^2 \rightarrow S^2$  of degree  $n$ . Proposition (1.1) implies at once

PROPOSITION (1.1'). *The inclusion  $F_n \rightarrow M_n$  is a homotopy equivalence up to dimension  $n$ .*

The preceding results can be generalised in two directions. First we can consider maps  $S^2 \rightarrow \mathbf{P}^m$ , where  $\mathbf{P}^m$  is  $m$ -dimensional complex projective space. If  $M_n^*(S^2; \mathbf{P}^m)$  denotes the space of base-point-preserving continuous maps  $S^2 \rightarrow \mathbf{P}^m$  of degree  $n$ , and  $F_n^*(S^2; \mathbf{P}^m)$  denotes those of the form

$$z \mapsto (p_0(z), p_1(z), \dots, p_m(z)),$$

where  $p_0, \dots, p_m$  are monic polynomials of degree  $n$ , then we have

PROPOSITION (1.2). *The inclusion*

$$F_n^*(S^2; \mathbf{P}^m) \rightarrow M_n^*(S^2; \mathbf{P}^m)$$

*is a homotopy equivalence up to dimension  $n(2m-1)$ .*

Again there is a version of this without base-points.

Secondly we can consider rational functions on a compact Riemann surface  $X$  of genus  $g$ . If  $F_n(X; \mathbf{P}^m)$  is the space of rational algebraic maps of degree  $n$ , and  $M_n(X; \mathbf{P}^m)$  is the corresponding space of continuous maps, we have

PROPOSITION (1.3). *If  $g > 0$  the inclusion*

$$F_n(X; \mathbf{P}^m) \rightarrow M_n(X; \mathbf{P}^m)$$

*is a homology equivalence up to dimension  $(n-2g)(2m-1)$ .*

This result even extends to the case when the Riemann surface  $X$  has singularities. It seems likely that the homology equivalence is actually a homotopy equivalence up to the same dimension.

I am confident that the methods of this paper suffice to treat the case of rational maps from a Riemann surface to a class of algebraic varieties which at least includes Grassmannians and flag manifolds. I think there is even some hope that analogous results hold when the dimension of the domain of the maps is greater than one, but the present methods do not apply to that case.

One reason for expecting that Proposition (1.1') might be true was pointed out to me by M. F. Atiyah. It is known [6] that when  $S^2$  is given its usual Riemannian structure the "energy" function

$$E(f) = \frac{1}{2} \int_{S^2} \|Df(x)\|^2 dx$$

on the space  $M_n^m$  of smooth maps  $S^2 \rightarrow S^2$  of degree  $n$  has no critical points apart from the rational maps  $F_n$ , on which it attains its absolute minimum. (Critical points of  $E$  are called

harmonic maps.) An extrapolation of Morse theory might lead one to hope that  $F_n$  is a deformation retract of  $M_n^{sm}$ . Proposition (1.1') shows that this is false, but that nevertheless it becomes approximately true as  $n \rightarrow \infty$ . Unfortunately there does not seem to be a simple physical interpretation of the "energy"  $E$ , so one's intuition is unreliable as a guide. (A discussion of a closely related question of infinite dimensional Morse theory can be found in [1a].)

The Morse theory point of view applies also to maps  $X \rightarrow S^2$  when  $X$  is a compact Riemann surface of genus  $g$ , for again (cf. [6]) the only harmonic maps of degree  $n$  are the rational maps, providing  $n > g$ . The case of  $\mathbf{P}^m$  when  $m > 1$  is more problematical.

Partial results about the space  $F_n^*$  have been obtained by J. D. S. Jones, and his work stimulated me to prove (1.1). In particular he proved that  $\pi_1(F_n^*) \cong \mathbf{Z}$ , a result which I shall use in my argument. (I give his proof in § 6.) But my attention was first drawn to the space by R. Brockett, who was interested in it because of its role in control theory. He was concerned with the real part  ${}^{\mathbf{R}}F_n^*$  of  $F_n^*$ , i.e. the functions of the form (1) with  $a_i$  and  $b_i$  real. These functions belong to the space  ${}^{\mathbf{R}}M_n^*$  of base-point-preserving maps  $S^2 \rightarrow S^2$  of degree  $n$  which commute with complex conjugation, and they induce maps of the real axis  $S^1 = \mathbf{R} \cup \infty \rightarrow S^1$ . Brockett [2] showed that  ${}^{\mathbf{R}}F_n^*$  consists of  $n+1$  connected components  ${}^{\mathbf{R}}F_{n,r}^*$ , distinguished by the degree  $r$  of the restriction to  $S^1$ , which is congruent to  $n$  modulo 2 and varies from  $-n$  to  $n$ . I shall reprove that in § 7 of this paper, and at the same time shall prove

**PROPOSITION (1.4).** *The inclusion  ${}^{\mathbf{R}}F_{n,r}^* \rightarrow {}^{\mathbf{R}}M_{n,r}^*$  is a homotopy equivalence up to dimension  $\frac{1}{2}(n - |r|)$ .*

Here  ${}^{\mathbf{R}}M_{n,r}^*$  is the space of base-point-preserving maps  $S^2 \rightarrow S^2$  of degree  $n$  which commute with complex conjugation and have degree  $r$  when restricted to the real axis  $S^1$ .

In view of the theorem of [8] the two Propositions (1.1) and (1.4) are together equivalent to the assertion that  $F_n^* \rightarrow M_n^*$  becomes an equivariant homotopy equivalence (with respect to complex conjugation) as  $n \rightarrow \infty$ .

A rational function of the form (1) is determined by its sets  $\xi$  and  $\eta$  of zeros and poles. These are sets with multiplicities: one should think of them as elements of the free abelian monoid  $A(\mathbf{C})$  on the set  $\mathbf{C}$ . I shall refer to them as (positive) *divisors* in  $\mathbf{C}$ . If  $\xi = n_1 z_1 + \dots + n_k z_k$  is a divisor, with  $n_i \in \mathbf{Z}$  and  $z_i \in \mathbf{C}$ , I shall write  $\deg(\xi)$  for the degree, or cardinality,  $n_1 + \dots + n_k$ . Thus  $F_n^*$  can be identified with the space  $Q_n$  of pairs of disjoint divisors  $(\xi, \eta)$  of degree  $n$  in  $\mathbf{C}$ .

It is known [15] that there is a relation between the space  $C_n$  of unordered  $n$ -tuples of *distinct* points of  $\mathbf{C}$  and the space of maps  $M_n^*$ . In fact there is a map  $E: C_n \rightarrow M_n^*$  which

is a homology equivalence up to dimension  $[n/2]$ . This map is not a homotopy equivalence in any range: indeed  $\pi_k C_n = 0$  when  $k > 1$ , while the fundamental group of  $C_n$  is by definition the braid group  $Br_n$  on  $n$  strands, and that of  $M_n^*$  is  $\mathbf{Z}$ . One construction of  $E: C_n \rightarrow M_n^*$  is to assign to an  $n$ -tuple  $\xi = \{z_1, \dots, z_n\}$  the map  $f_\xi$  given by

$$f_\xi(z) = 1 + \sum \frac{1}{z - z_i} = 1 + \frac{\psi'_\xi(z)}{\psi_\xi(z)},$$

where  $\psi_\xi(z) = \prod (z - z_i)$ . Thus  $E$  factorizes through  $Q_n$ , taking  $\xi \in C_n$  to  $(\xi, \eta) \in Q_n$ , where  $\eta$  is the set of zeros of  $\psi_\xi + \psi'_\xi$ . But of course  $E$  does not extend from  $C_n$  to the contractible space  $A_n(\mathbf{C})$ .

Up to homotopy one can think of  $Q_n$  as a subspace of  $Q_{n+1}$ . To do so, replace  $Q_n$  by the homeomorphic space  $Q_n^\dagger$  of disjoint pairs of divisors  $(\xi, \eta)$  which are contained in the region  $\{\operatorname{Re}(z) < n\}$ . Then  $Q_n^\dagger$  can be embedded in  $Q_{n+1}^\dagger$  by multiplying by  $(z - x_n)/(z - y_n)$ , where  $x_n$  and  $y_n$  are distinct points satisfying  $n \leq \operatorname{Re}(x_n)$ ,  $\operatorname{Re}(y_n) < n + 1$ . Doing this suggests introducing a stabilized space  $\hat{Q}$ . Consider all disjoint pairs  $(\xi, \eta)$  of formal infinite positive divisors in  $\mathbf{C}$ , and let  $\hat{Q}$  consist of those pairs  $(\xi, \eta)$  which "almost coincide" with  $(\xi_0, \eta_0)$ , where  $\xi_0 = \sum_{k \geq 0} x_k$ ,  $\eta_0 = \sum_{k \geq 0} y_k$ , and "almost coincide" means that  $\xi - \xi_0$  and  $\eta - \eta_0$  are finite, but not necessarily positive, divisors. The space  $\hat{Q}$  has  $\mathbf{Z} \times \mathbf{Z}$  as its set of connected components. A point  $(\xi, \eta)$  belongs to the  $(m, n)$ -component if  $\deg(\xi - \xi_0) = m$  and  $\deg(\eta - \eta_0) = n$ . The  $(0, 0)$ -component is precisely  $\bigcup_{n \geq 0} Q_n^\dagger$ , and the other components are homeomorphic to it.

The idea of our method is to deduce Proposition (1.1) from the following apparently rather different result.

**PROPOSITION (1.5).** *There is a homotopy equivalence*

$$\hat{Q} \rightarrow \Omega^2(\mathbf{P} \vee \mathbf{P}),$$

where  $\mathbf{P}$  is infinite-dimensional complex projective space, and  $\Omega^2$  denotes the second loop-space.

This is closely related to Proposition (1.1), because the connected components of the space  $\Omega^2 S^2$  and  $\Omega^2(\mathbf{P} \vee \mathbf{P})$  are homotopy equivalent. The equivalence is induced by a map  $q: S^2 \rightarrow \mathbf{P} \vee \mathbf{P}$  which makes  $S^2$ , up to homotopy, a bundle over  $\mathbf{P} \vee \mathbf{P}$  whose fibre is a circle.

The map of Proposition (1.5) can be understood geometrically as follows. Recall that the infinite symmetric product of  $S^2 = \mathbf{C} \cup \infty$ , with  $\infty$  as base-point, can be identified with  $\mathbf{P}$ . Given a point  $(\xi, \eta) \in F_n^*$  we assign to it a map  $f_{\xi, \eta}: \mathbf{C} \rightarrow \mathbf{P} \vee \mathbf{P}$  obtained by scanning  $\mathbf{C}$  with a microscope of large magnification and very small field of vision. When one centres the microscope at a point  $x \in \mathbf{C}$  one sees either nothing or else an enlarged fragment of  $\xi$  or of  $\eta$ , but not of both. An enlarged fragment of  $\xi$  or of  $\eta$  can be thought of as an element

of the infinite symmetric product of  $\mathbf{C} \cup \infty$ , i.e. as a point of  $P$ . One defines  $f_{\xi, \eta}(x) \in \mathbf{P} \vee \mathbf{P}$  by assigning it to the left- or right-hand copy according as it belongs to  $\xi$  or to  $\eta$ . If  $x$  is far from the origin one sees nothing, so  $f_{\xi, \eta}$  defines an element of  $\Omega^2(\mathbf{P} \vee \mathbf{P})$ .

The reason that  $\Omega^2(\mathbf{P} \vee \mathbf{P})$  arises in the proof rather than  $\Omega^2 S^2$  is that the rational functions are really a fibre bundle over the space of pairs of divisors  $(\xi, \eta)$  with fibre  $\mathbf{C}^\times$ —though that is obscured by normalizing the functions as 1 at  $\infty$ . Locally a rational function has a value belonging to  $\mathbf{C} \cup \infty = S^2$ , and locally a configuration  $(\xi, \eta)$  has a germ belonging to  $\mathbf{P} \vee \mathbf{P}$ . In fact we shall see (Proposition (4.8)) that if  $F(U)$  is the space of non-zero meromorphic functions on the open unit disk  $U$  then the map  $F(U) \rightarrow F(U)/\mathbf{C}^\times$  is homotopically just the above map  $S^2 \rightarrow \mathbf{P} \vee \mathbf{P}$ .

The proof of (1.5) follows the lines of [11], but the details are somewhat different, and some new points arise. One must beware, of course, of thinking of a pair of divisors as a configuration of particles and antiparticles: here the two kinds cannot annihilate each other.

To treat maps  $S^2 \rightarrow \mathbf{P}^m$  is no harder. The base-point-preserving holomorphic maps of degree  $n$  are given by  $(m+1)$ -tuples of monic polynomials of degree  $n$ , and hence by  $(m+1)$ -tuples of divisors  $(\xi_0, \xi_1, \dots, \xi_m)$  of degree  $n$  such that  $\xi_0 \cap \xi_1 \cap \dots \cap \xi_m = \emptyset$ . They form a space  $Q_n^{(m)}$ . Given  $(\xi_0, \dots, \xi_m) \in Q_n^{(m)}$  one can define a map  $X \rightarrow W_{m+1} \mathbf{P}$  by “scanning”, where  $W_{m+1} \mathbf{P}$  is the subspace of the  $(m+1)$ -fold product  $\mathbf{P} \times \dots \times \mathbf{P} = \prod^{m+1} \mathbf{P}$  consisting of  $(p_0, \dots, p_m)$  such that at least one  $p_i$  is the base-point in  $\mathbf{P}$ . ( $W_{m+1} \mathbf{P}$  is sometimes called the  $(m+1)$ -fold “fat wedge” of  $\mathbf{P}$ .) Generalizing (1.5) we have

**PROPOSITION (1.6).** *There is a homotopy equivalence*

$$Q^{(m)} \rightarrow \Omega^2(W_{m+1} \mathbf{P}).$$

Here  $Q^{(m)} = \bigcup_{n \geq 0} Q_n^{(m)}$ .

The connected components of  $\Omega^2(W_{m+1} \mathbf{P})$  are homotopically the same as those of  $\Omega^2 \mathbf{P}^m$ , for we shall see that there is a map  $q: \mathbf{P}^m \rightarrow W_{m+1} \mathbf{P}$  whose homotopic fibre is a torus  $\mathbf{T}^m$ . (A better way to express this is as follows. The projective transformations of  $\mathbf{P}^m$  which leave fixed the vertices of the simplex of reference form the group  $(\mathbf{C}^\times)^m$ . The homotopical orbit space of the action of  $(\mathbf{C}^\times)^m$  on  $\mathbf{P}^m$  is  $W_{m+1} \mathbf{P}$ .)

The proof of (1.3) is considerably harder than that of (1.1), as rational functions can no longer be identified with pairs of divisors. A pair of divisors  $(\xi, \eta)$  on  $X$  comes from a rational function only if  $\xi$  and  $\eta$  have the same image in the Jacobian variety of  $X$ . In the case of Riemann surfaces with singularities the generalized Jacobians of Rosenlicht [14], [16] must be used.

The plan of the work is as follows.

§ 2 describes the maps  $g: S^2 \rightarrow \mathbf{P} \vee \mathbf{P}$  and  $q: \mathbf{P}^m \rightarrow W_{m+1}\mathbf{P}$ .

§ 3 gives the proofs of (1.3) and (1.5). This is very simple and geometrical, but it uses the result that  $\pi_1(F_n^*) \cong \mathbf{Z}$ .

§ 4 treats the corresponding result for Riemann surfaces  $X$  of higher genus. This involves analysing the relation between rational functions and pairs of divisors.

§ 5 discusses the stability of the homology of the spaces  $F_n^*$  and  $F_n(X; \mathbf{P}^m)$  as  $n$  increases. (This completes the homological part of the proofs of (1.1) and (1.4).) The method here was introduced by Arnol'd [1] in connection with the braid groups, but I have simplified it so as to show that it applies to a considerable variety of situations. As an example of this there is an appendix to § 5 proving the homological stability of the configuration spaces of a general manifold, obtaining a more precise result than that of [11]. § 5 also contains a splitting theorem for the homology of  $F_n^*$ .

§ 6 completes the proof of (1.1) by studying the action of the fundamental group of  $F_n^*$  on its higher homotopy groups. This amounts to finding the monodromy of the "resultant" map  $F_n^* \rightarrow \mathbf{C}^\times$ , which is given by a weighted homogeneous polynomial. I have included here Jones's proof that  $\pi_1(F_n^*) \cong \mathbf{Z}$ .

§ 7 treats the case of real coefficients, and proves (1.4).

§ 8 extends the results to Riemann surfaces with singularities.

In §§ 3, 4 and 5 it makes very little difference whether one is considering maps into  $S^2$  or into  $\mathbf{P}^m$ . As the case of  $S^2$  is simpler and clearer I have treated it first, and then indicated the changes needed for  $\mathbf{P}^m$  at the end of each section.

In writing the paper I have been much helped by discussions with M. F. Atiyah, R. Brockett and J. D. S. Jones, and I am very grateful to them.

*Note.* To say that a map  $f: X \rightarrow Y$  is a homology or homotopy equivalence up to dimension  $n$  is intended to mean that the relative homology  $H_i(Y, X; \mathbf{Z})$  or homotopy  $\pi_i(Y, X)$  vanishes when  $i \leq n$ , or equivalently that  $H_i(X) \rightarrow H_i(Y)$  or  $\pi_i(X) \rightarrow \pi_i(Y)$  is bijective for  $i < n$  and surjective when  $i = n$ . This is the case, for example, if  $Y$  is a  $CW$  complex and  $X$  is a subcomplex with the same  $n$ -skeleton as  $Y$ .

## § 2. The fibration sequence

$$\mathbf{T}^m \rightarrow \mathbf{P}^m \xrightarrow{q} W_{m+1}\mathbf{P} \rightarrow \prod^m \mathbf{P}.$$

When  $m=1$  one can see these fibrations very directly. Consider the standard circle-bundle  $S^\infty \rightarrow \mathbf{P}$  whose total space is the infinite-dimensional sphere, which is contractible. If one attaches together two copies of this along a fibre one obtains a circle-bundle on

$\mathbf{P} \vee \mathbf{P}$  whose total space consists of two contractible pieces intersecting in a circle. This gives

$$\mathbf{T} \rightarrow S^2 \rightarrow \mathbf{P} \vee \mathbf{P}.$$

Now consider the  $S^2$ -bundle on  $\mathbf{P}$  associated to the principal  $\mathbf{T}$ -bundle  $S^\infty \rightarrow \mathbf{P}$  by the action of  $\mathbf{T}$  on  $S^2$  which rotates it about its poles. The total space is made up of two disk-bundles (corresponding to the hemispheres of  $S^2$ ) intersecting in a circle-bundle. The disk-bundles are each homotopy equivalent to  $\mathbf{P}$ , and their intersection is  $S^\infty$ , which is contractible. This gives

$$S^2 \rightarrow \mathbf{P} \vee \mathbf{P} \rightarrow \mathbf{P}.$$

As  $\mathbf{P}$  is the classifying-space of the group  $\mathbf{T}$  this asserts that the homotopical orbit-space  $S^2//\mathbf{T} = S^2 \times_{\mathbf{T}} E\mathbf{T}$  of the action of  $\mathbf{T}$  on  $S^2$  is  $\mathbf{P} \vee \mathbf{P}$ . Thinking of  $S^2$  as  $\mathbf{P}^1$ , and observing that the action of  $\mathbf{T}$  extends to an action of  $\mathbf{C}^\times$ , we can say equivalently that  $\mathbf{P}^1//\mathbf{C}^\times \simeq \mathbf{P} \vee \mathbf{P}$ . Of course  $\mathbf{C}^\times$  is precisely the group of projective transformations of  $\mathbf{P}^1$  leaving fixed 0 and  $\infty$ .

Passing to the general case, observe that the group of projective transformations of  $\mathbf{P}^m$  which leave fixed the vertices of the simplex of reference is  $G = (\mathbf{C}^\times)^m$ . The classifying space  $BG$  can be identified with the  $m$ -fold product  $\mathbf{P} \times \dots \times \mathbf{P} = \prod^m \mathbf{P}$ . Let  $U_i$  ( $0 \leq i \leq m$ ) be the part of  $\mathbf{P}^m$  where the  $i$ th homogeneous coordinate is non-zero. Then  $\{U_i\}$  is an open covering of  $\mathbf{P}^m$  by  $G$ -invariant open sets. The bundle  $E$  on  $BG$  with fibre  $\mathbf{P}^m$  is therefore covered by  $m+1$  open subbundles  $E_i$  whose fibres are the  $U_i$ . Because  $U_i$  is contractible  $E_i \simeq \prod^m \mathbf{P}$ . Now  $U_{i_0} \cap \dots \cap U_{i_k} \simeq \mathbf{T}^k$ , so  $E_{i_0} \cap \dots \cap E_{i_k} \simeq \prod^{m-k} \mathbf{P}$ . Inspecting the inclusion maps shows that  $E \simeq W_{m+1}\mathbf{P}$ .

To describe the map  $q: \mathbf{P}^m \rightarrow W_{m+1}\mathbf{P}$  explicitly it is best to replace  $W_{m+1}\mathbf{P}$  by the equivalent open subspace  $W'_{m+1}\mathbf{P}$  of  $\prod^{m+1} \mathbf{P}$  consisting of all  $(p_0, \dots, p_m)$  such that the first coordinate of at least one  $p_i$  is non-zero. (Here  $\mathbf{P}$  is thought of as the projective space of  $\mathbf{C}^\infty$ .) Let  $A: \mathbf{P}^m \rightarrow \mathbf{P}^m$  be the map which permutes the homogeneous coordinates cyclically, i.e.

$$A(\lambda_0, \dots, \lambda_m) = (\lambda_1, \lambda_2, \dots, \lambda_m, \lambda_0).$$

Then the desired map  $q: \mathbf{P}^m \rightarrow W_{m+1}\mathbf{P}$  is given by  $p \mapsto (p, Ap, A^2p, \dots, A^m p)$ , where  $\mathbf{P}^m$  is regarded as a subspace of  $\mathbf{P}$ .

### § 3. The homotopy equivalence $\hat{Q} \rightarrow \Omega^2(\mathbf{P} \vee \mathbf{P})$

For any space  $X$  let  $A(X)$  denote the free abelian monoid generated by the points of  $X$ . If  $Y$  is a closed subspace of  $X$  let  $A(X, Y)$  denote the quotient monoid of  $A(X)$  by the relation which identifies the points of  $Y$  with zero. Thus if  $Y$  is non-empty  $A(X, Y)$  is the

infinite symmetric product of  $X/Y$ . By the theorem of Dold & Thom [5] one knows that in this case  $\pi_k(A(X, Y)) \cong H_k(X, Y)$  if  $X$  is connected.

The space  $A(S^2, \infty)$  can be identified with the infinite-dimensional complex projective space  $\mathbf{P}$  formed from the vector space  $\mathbb{C}[z]$  of polynomials by mapping the divisor  $\sum n_i z_i$ , where  $z_i = (u_i: v_i) \in \mathbf{P}^1 = S^2$ , to the polynomial  $\prod (u_i - v_i z)$ .

Let  $Q(X, Y)$  denote the subspace of  $A(X, Y) \times A(X, Y)$  consisting of pairs of divisors with disjoint support. (The support of  $\sum n_i z_i$  means the set of  $z_i$  in  $X - Y$  such that  $n_i > 0$ .) The space  $Q(X, Y)$  depends only on  $X/Y$ . The space  $F_n^*$  we wish to study is the connected component of  $Q(\mathbb{C})$  consisting of pairs  $(\xi, \eta)$  with  $\deg(\xi) = \deg(\eta) = n$ . I shall write  $Q_{m,n}$  for the component with  $\deg(\xi) = m$  and  $\deg(\eta) = n$ , and  $Q_n$  for  $Q_{n,n}$ . The case  $m \neq n$  is not very interesting, as  $Q_{m,n} \cong Q_{n,n} \times \mathbb{C}^{m-n}$  if  $m \geq n$ . (For if  $m > n$  a rational function  $p/q$ , where  $p$  and  $q$  are monic polynomials of degrees  $m$  and  $n$ , can be written canonically as  $h + r/q$ , where  $h$  and  $r$  are monic of degrees  $m - n$  and  $n$ .)

**PROPOSITION (3.1).**  $\mathbf{P} \vee \mathbf{P} \xrightarrow{\cong} Q(S^2, \infty)$ .

*Proof.* The map is the inclusion of the axes in

$$Q(S^2, \infty) \subset A(S^2, \infty) \times A(S^2, \infty) = \mathbf{P} \times \mathbf{P}.$$

Let  $Q_\varepsilon$  be the open set of  $Q(S^2, \infty)$  consisting of pairs  $(\xi, \eta)$  such that either  $\xi$  or  $\eta$  is disjoint from the closed disk of radius  $\varepsilon > 0$  about the origin. Radial expansion defines a deformation retraction of  $Q_\varepsilon$  into  $\mathbf{P} \vee \mathbf{P}$ , so  $Q_\varepsilon \simeq \mathbf{P} \vee \mathbf{P}$ . But  $Q(S^2, \infty)$  is the union of the  $Q_\varepsilon$  for  $\varepsilon > 0$ , so  $Q(S^2, \infty) \simeq \mathbf{P} \vee \mathbf{P}$  also.

Now let  $X$  be the closed square  $[0, 1] \times [0, 1]$  in  $\mathbb{R}^2 = \mathbb{C}$ , and let  $Y = [0, 1] \times \{0, 1\}$  be a pair of opposite edges of  $X$ .

**PROPOSITION (3.2).**  $Q(X, Y) \simeq \Omega(\mathbf{P} \vee \mathbf{P})$ .

*Proof.* Let  $R$  be the rectangle  $[-1, 2] \times [0, 1]$ , and let  $\partial R$  be its boundary. Then  $R = X_0 \cup X \cup X_1$ , where  $X_0 = [-1, 0] \times [0, 1]$  and  $X_1 = [1, 2] \times [0, 1]$ . Consider the quotient map  $\pi: Q(R, \partial R) \rightarrow Q(R, \partial R \cup X)$ . Each fibre of  $\pi$  is homeomorphic to  $Q(\partial R \cup X, \partial R) = Q(X, Y)$ . On the other hand  $Q(R, \partial R \cup X) = Q(X_0, \partial X_0) \times Q(X_1, \partial X_1)$ . If we assume for the moment that  $\pi$  is a quasifibration then Proposition (3.2) follows at once. In fact  $Q(R, \partial R)$  and  $Q(X_0, \partial X_0)$  and  $Q(X_1, \partial X_1)$  can each be identified with  $Q(S^2, \infty)$ , and then the maps  $Q(R, \partial R) \rightarrow Q(X_i, \partial X_i)$  are homotopic to the identity. But for any space  $Q$  the homotopic fibre of the diagonal map  $Q \rightarrow Q \times Q$  is the loop-space  $\Omega Q$ , for  $Q \rightarrow Q \times Q$  is equivalent to the restriction  $\text{Map}([0, 1]; Q) \rightarrow \text{Map}(\{0, 1\}; Q)$ . Thus in the present case  $Q(X, Y) \simeq \Omega Q(S^2, \infty) \simeq \Omega(\mathbf{P} \vee \mathbf{P})$ , providing we prove



LEMMA (3.3).  $\pi: Q(R, \partial R) \rightarrow Q(R, \partial R \cup X)$  is a quasifibration.

Postponing the proof of the lemma, we complete the proof of Proposition (1.5) by repeating the previous device.

Let  $S$  denote the pair of opposite edges  $\{-1, 2\} \times [0, 1]$  of  $R$ . Consider this time the quotient map

$$\pi': Q(R, S) \rightarrow Q(R, S \cup X).$$

Each fibre of  $\pi'$  is homeomorphic to  $Q(X)$ . On the other hand  $Q(R, S \cup X) = Q(X_0, Y_0) \times Q(X_1, Y_1)$ , where  $Y_i = S \cup (X \cap X_i)$ . By Proposition (3.2) we know that  $Q(R, S) \simeq Q(X_i, Y_i) \simeq \Omega(\mathbf{P} \vee \mathbf{P})$ , so if  $\pi'$  were a quasifibration we could conclude that  $Q(X) \simeq \Omega^2(\mathbf{P} \vee \mathbf{P})$ . Of course  $Q(X) \simeq Q(\mathbf{C})$ , so this would be too much: we need to stabilize the space  $Q(X)$ .

First we replace  $R$  by the half-open rectangle  $R' = [-1, 2] \times (0, 1)$ . We write  $S' = S \cap R'$ ,  $X' = X \cap R'$ , etc. Then we choose disjoint sequences  $\xi_0$  and  $\eta_0$  in  $R'$  tending, say, to  $(\frac{1}{2}, 0) \in \partial X$ . Let  $\hat{Q}(R, S)$  denote the stabilized version of  $Q(R, S)$  consisting of pairs  $(\xi, \eta)$  of formal infinite divisors which almost coincide (in the sense described earlier) with  $(\xi_0, \eta_0)$ . We still have a projection  $\pi': \hat{Q}(R', S') \rightarrow Q(R', S' \cup X')$ ; but now each fibre is homeomorphic to  $\hat{Q}(X')$ , which is essentially the same as the space  $\hat{Q}$  of (1.5). So the proof of (1.5) will be complete when we have proved

LEMMA (3.4).  $\pi': \hat{Q}(R', S') \rightarrow Q(R', S' \cup X')$  is a quasifibration.

The proof of this uses Jones's result that  $\pi_1(F_n^*) \cong \mathbf{Z}$ .

*Remark.* The preceding argument has been formulated in a way that does not make it obvious that the map  $F \rightarrow \Omega^2(\mathbf{P} \vee \mathbf{P})$  obtained is essentially the composite  $F_n^* \subset \Omega^2 S^2 \rightarrow \Omega^2(\mathbf{P} \vee \mathbf{P})$  mentioned in the introduction. It is not hard to verify this, but we shall leave the question for the moment, as it will be discussed fully in the next section (cf. Propositions (4.7) and (4.8)).

It remains to prove the lemmas.

*Proof of Lemma (3.3).* We use a technique devised by Dold & Thom [5] to prove the corresponding assertions for the infinite symmetric product, and used again in [10] and [11]. The following argument is so close to those in [11] that I shall give it fairly briefly.

Let us write  $R_0 = \partial R \cup X$ . We filter the base-space  $B = Q(R, R_0)$  by an increasing family of closed subspaces  $\{B_{p,q}\}$ .  $B_{p,q}$  consists of those pairs  $(\xi, \eta)$  such that

$$\deg(\xi \cap (R - R_0)) \leq p \quad \text{and} \quad \deg(\eta \cap (R - R_0)) \leq q.$$

Over  $B'_{p,q} = B_{p,q} - (B_{p-1,q} \cup B_{p,q-1})$  the total space  $\pi^{-1}(B'_{p,q})$  is the product  $B'_{p,q} \times Q(X, Y)$ , and hence a fibration. Using (3.3) of [11] (cf. [5] (§ 2) and [10] (p. 62)) what

needs to be checked to show that  $\pi$  is a quasifibration is that the map  $Q(X, Y) \rightarrow Q(X, Y)$  which takes  $(\xi, \eta)$  to  $(\xi + P, \eta')$ , where  $P \in \partial X - Y$ , and  $\eta'$  is  $\eta$  slightly shrunk so that it lies in  $(\varepsilon, 1 - \varepsilon) \times [0, 1]$  and avoids  $P$ , is a homotopy equivalence. This is true because  $P$  can be moved continuously to a point of  $Y$ , where it becomes zero.

*Proof of Lemma (3.4).* Similarly here it is enough to show that the map  $\hat{Q}(X') \rightarrow \hat{Q}(X')$  again defined by  $(\xi, \eta) \mapsto (\xi + P, \eta')$  is a homotopy equivalence. This is however not homotopic to the identity: on the set  $\mathbf{Z} \times \mathbf{Z}$  of connected components it is  $(m, n) \mapsto (m + 1, n)$ . But if one thinks of  $\hat{Q}(X')$  as  $\mathbf{Z} \times \mathbf{Z} \times \varinjlim Q_{m,n}$  its restriction to  $Q_{m,n}$  is homotopic to the stabilization map  $Q_{m,n} \rightarrow Q_{m+1,n}$ . This implies that it is a homology equivalence. It does not by itself imply that it is a homotopy equivalence (cf. the example of the shift map of  $B\mathfrak{S}_\infty$  discussed in [12]). But in the present case we know that  $\pi_1(Q_{m,n})$  is abelian, and that  $\pi_1(Q_{m,n}) \rightarrow \pi_1(Q_{m+1,n})$  is an isomorphism. This allows us to conclude that  $\hat{Q}(X') \rightarrow \hat{Q}(X')$  is a homology equivalence with any twisted coefficients, and hence a homotopy equivalence.

To conclude this section let us observe that the preceding arguments apply without any change at all to prove Proposition (1.6) also. One has only to replace pairs of disjoint divisors by  $(m + 1)$ -tuples of divisors  $(\xi_0, \dots, \xi_m)$  such that  $\xi_0 \cap \dots \cap \xi_m = \emptyset$ , and  $\mathbf{P} \vee \mathbf{P}$  by  $W_{m+1}\mathbf{P}$ . The case  $m > 1$  is actually easier, as then the spaces  $Q_n^{(m)}(\mathbf{C})$  and  $M_n^*(S^2; \mathbf{P}^m)$  are both simply connected, the first because it is obtained by removing a variety of complex codimension  $m$  from  $\mathbf{C}^{mn}$ , and the second because its fundamental group is  $\pi_3(\mathbf{P}^m)$ .

#### § 4. Riemann surfaces of higher genus

As in the preceding section we shall begin by discussing maps into  $S^2$ .

Let  $X$  be a compact Riemann surface of genus  $g$ . The rational (i.e. meromorphic) functions on  $X$  form an infinite-dimensional field  $K_X$ . For a fixed positive divisor  $\eta = \sum n_i z_i$  of degree  $n$  on  $X$  the rational functions which have poles of order at most  $n_i$  at  $z_i$  form a complex vector space  $K_\eta$  in  $K_X$  which (by the Riemann–Roch theorem) has dimension  $n - g + 1$  providing  $n \geq 2g - 1$ . As  $\eta$  varies over the  $n$ -fold symmetric product  $A_n(X)$ , which is a compact non-singular variety of dimension  $n$ , one obtains all the rational functions of degree  $\leq n$ . They form an  $(n - g + 1)$ -dimensional algebraic vector bundle on  $A_n(X)$ . The functions with exactly  $n$  poles form a Zariski open set  $F_n(X) = F_n(X; \mathbf{P}^1)$  which is the complement of a hypersurface in this vector bundle.  $F_n(X)$  is the space of all holomorphic maps  $X \rightarrow S^2$  of degree  $n$ . If  $M_n(X)$  is the space of all continuous maps  $X \rightarrow S^2$  of degree  $n$  we have

**PROPOSITION (4.1).** *The inclusion  $F_n(X) \rightarrow M_n(X)$  is a homology equivalence up to dimension  $n - 2g$ .*

Let us choose a base-point  $x_0$  in  $X$ , and let  $F_n^*(X)$  (resp.  $M_n^*(X)$ ) denote the rational (resp. continuous) functions  $f$  such that  $f(x_0)=1$ . Proposition (4.1) is equivalent to the assertion that  $F_n^*(X) \rightarrow M_n^*(X)$  is a homology equivalence up to dimension  $n-2g$ , as one sees from the Wang sequences associated to the fibrations  $F_n(X) \rightarrow S^2$  and  $M_n(X) \rightarrow S^2$  with fibres  $F_n^*(X)$  and  $M_n^*(X)$ .

A meromorphic function on  $X$  is determined up to a multiplicative constant by its divisors  $(\xi, \eta)$  of zeros and poles. Let  $X' = X - x_0$ , and as before let  $Q_n(X')$  denote the space of disjoint pairs  $(\xi, \eta)$  of divisors of degree  $n$ . The manifold  $X'$  is parallelizable. Let us choose a definite parallelization. Then, just as when  $X = S^2$ , a configuration  $(\xi, \eta)$  looks locally like a point of  $\mathbf{P} \vee \mathbf{P}$ , and there is a map  $S: Q_n(X') \rightarrow \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P})$  defined by "scanning". ( $\text{Map}_n^*$  denotes the space of base-point-preserving maps which have degree  $n$  on to each copy of  $\mathbf{P}$ . To see that scanning a configuration  $\xi$  of degree  $n$  gives a map  $X \rightarrow \mathbf{P}$  of degree  $n$  it suffices by continuity to consider the case where the points of  $\xi$  are distinct. But then the map factorizes  $X \rightarrow S^2 \vee \dots \vee S^2 \rightarrow \mathbf{P}$ , where  $X \rightarrow S^2 \vee \dots \vee S^2$  collapses the complement of a neighbourhood of  $\xi$ , and each  $S^2 \rightarrow \mathbf{P}$  is the standard inclusion of  $\mathbf{P}^1$  in  $\mathbf{P}$ .) It will turn out that this map is a homology equivalence up to dimension  $n-2g+1$ ; but before seeing that we need to construct a stabilized space like that of Proposition (1.3).

Let  $\{U_k\}$  be a contracting sequence of closed disk-like neighbourhoods of  $x_0$  in  $X$ , with intersection  $x_0$ . Write  $X_k = X - U_k$ . Then  $Q(X_k)$  is homeomorphic to  $Q(X')$ . Embed  $Q(X_k)$  in  $Q(X_{k+1})$  by  $(\xi, \eta) \mapsto (\xi + x_k, \eta + y_k)$ , where  $x_k$  and  $y_k$  belong to  $U_k - U_{k+1}$ . The union of the  $Q(X_k)$  is  $\hat{Q}(X')$ , and can be thought of as the space of pairs of disjoint infinite divisors which almost coincide with a standard pair  $(\xi_0, \eta_0)$ . The scanning process defines a map  $S: Q(X_k) \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$  for each  $k$ , and there are homotopy commutative diagrams

$$\begin{array}{ccc} Q(X_k) & \xrightarrow{S} & \text{Map}^*(X; \mathbf{P} \vee \mathbf{P}) \\ \downarrow & & \downarrow \alpha_k \\ Q(X_{k+1}) & \xrightarrow{S} & \text{Map}^*(X; \mathbf{P} \vee \mathbf{P}), \end{array}$$

where  $\alpha_k$  is a homotopy equivalence which increases the bidegree by  $(1, 1)$ . We can obtain from them a map  $S: \hat{Q}(X') \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$ . (Strictly speaking, it is unique up to phantom homotopy; but the non-uniqueness does not matter.) In analogy with (1.5) we have

**PROPOSITION (4.2).**  $S: \hat{Q}(X') \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$  is a homotopy equivalence.

Postponing the proof of (4.2), let us pass on to discuss the relation between the space  $Q_n(X')$  of pairs of divisors and the space  $F_n^*(X)$  of holomorphic maps  $X \rightarrow S^2$ .

An arbitrary pair  $(\xi, \eta)$  will not usually be the zeros and poles of a rational function

on  $X$ . Let us recall that there is a complex torus  $J$ , of complex dimension  $g$ , associated to  $X$ , called its Jacobian variety, and a holomorphic map  $j: X \rightarrow J$  taking  $x_0$  to 0. The map  $j$  extends to a homomorphism  $j: A(X, x_0) \rightarrow J$ , where  $A(X, x_0)$  is the infinite symmetric product of  $X$  with  $x_0$  as base-point.

It may be worth recalling the definition of the Jacobian. Let  $V$  be the  $g$ -dimensional complex vector space of holomorphic 1-forms on  $X$ . An embedding  $H_1(X; \mathbf{Z}) \rightarrow V^*$  is defined by the pairing  $H_1(X) \times V \rightarrow \mathbf{C}$  given by

$$(\gamma, \alpha) \mapsto \int_{\gamma} \alpha.$$

Its image is a lattice  $L \cong \mathbf{Z}^{2g}$ . One defines  $J = V^*/L$ . To define  $j: X \rightarrow J$  choose for each  $x$  in  $X$  a path  $\gamma_x$  from  $x_0$  to  $x$ . Then  $j(x)$  is the linear form  $\alpha \mapsto \int_{\gamma_x} \alpha$  in  $V^*$ , which is well-defined modulo  $L$ .

The following facts are well-known.

**PROPOSITION (4.3).** (a) *A pair  $(\xi, \eta)$  in  $Q_n(X)$  arises from a rational function on  $X$  if and only if  $j(\xi) = j(\eta)$ .*

(b) *The map  $j: A_n(X) \rightarrow J$  is a smooth fibre bundle with fibre the projective space  $\mathbf{P}^{n-g}$  if  $n \geq 2g - 1$ .*

(c) *The map  $j: A_n(X') \rightarrow J$  is a smooth fibre bundle with fibre the affine space  $\mathbf{C}^{n-g}$  if  $n \geq 2g$ .*

In fact (a) is Abel's theorem, (b) is proved in [9], and (c) follows from (b) because  $A_n(X')$  is the complement of  $A_{n-1}(X)$  in  $A_n(X)$ .

It follows from assertion (a) that the rational functions  $F_n^*(X)$  are precisely the fibre at 0 of the map  $j: Q_n(X') \rightarrow J$  given by  $(\xi, \eta) \mapsto j(\xi) - j(\eta)$ . We know something about  $Q_n(X')$  from Proposition (4.2), and we know the homotopy type of the Jacobian  $J$ . To be able to draw a conclusion about the homotopy type of  $F_n^*(X)$  we need to know that the map  $Q_n(X') \rightarrow J$  resembles a fibration to some extent. It is certainly not a fibration: for example if  $X$  is a torus (i.e.  $g = 1$ ) we can identify  $X$  with  $J$  so that  $j$  is the identity. Then  $F_1^*(X)$  is empty, for an elliptic function cannot have just one pole. But every other fibre of  $Q_n(X') \rightarrow J$  is isomorphic to  $X' - \{\text{point}\}$ . (These fibres are the spaces of theta functions with a given automorphy factor, i.e. the spaces of meromorphic sections of the line bundle corresponding to the point of the Jacobian.) In fact it is easy to see that the fibres  $Q_n(X') \rightarrow J$  never become homologically equivalent to each other even for large  $n$ . In view of this we have to generalise the concept of homology fibration introduced in [12].

**Definition (4.4).** A map  $p: E \rightarrow B$  is a *homology fibration up to dimension  $m$*  if each  $b \in B$  has arbitrarily small contractible neighbourhoods  $U$  such that the inclusion  $p^{-1}(b') \rightarrow p^{-1}(U)$  is a homology equivalence up to dimension  $m$  for each  $b'$  in  $U$ .

The essential property of such homology fibrations is that the fibre  $p^{-1}(b)$  at any point  $b$  of the base is homology equivalent to the homotopic fibre at  $b$  up to dimension  $m$ . In fact the proof of Proposition 5 of [12], which asserts this when  $B$  is paracompact and locally contractible, applies without change when the words “up to dimension  $m$ ” are inserted. (See the note at the end of this section.)

In the next section I shall prove

PROPOSITION (4.5). *The map  $j: Q_n(X') \rightarrow J$  is a homology fibration up to dimension  $n - 2g$ .*

Assuming (4.5) and also (4.2) I shall now complete the proof of (4.1).

For some integer  $N$  choose a rational function  $f$  on  $X$  which has a pole of order  $N$  at  $x_0$ , and no other poles. Define  $\zeta_A = f^{-1}(A)$  for any  $A$  in  $\mathbb{C}$ . This is a divisor of degree  $N$  in  $X'$ , and its image in  $J$  is independent of  $A$ . Then define a new stabilization map  $i: Q(X') \rightarrow Q(X')$  by

$$(\xi, \eta) \mapsto (\xi + \zeta_A, \eta + \zeta_{-A})$$

where  $A = 1 + \sup \{|f(x)| : x \in \xi \cup \eta\}$ .

We have a commutative diagram

$$\begin{array}{ccccccc} Q_n(X') & \xrightarrow{i} & Q_{n+N}(X') & \xrightarrow{i} & Q_{n+2N}(X') & \xrightarrow{i} & \dots \\ \downarrow j & & \downarrow j & & \downarrow j & & \\ J & \longrightarrow & J & \longrightarrow & J & \longrightarrow & \dots \end{array}$$

where the maps in the bottom now are the identity. Each map  $j$  is a homology fibration up to dimension  $n - 2g$  at least, and by the stability theorem (5.2) which will be proved in the next section each map  $i$  induces a homology equivalence  $j^{-1}(\alpha) \rightarrow j^{-1}(\alpha)$  of the fibres at any point  $\alpha$  of  $J$  up to dimension  $n - 2g$ . So by Proposition 3 of [12] the map of telescopes induced by the diagram is a homology fibration up to dimension  $n - 2g$ . On the other hand homotopically the map of telescopes is a map  $\hat{j}: \hat{Q}_n(X') \rightarrow J$ , where  $\hat{Q}_n(X')$  is one connected component of  $\hat{Q}(X')$ . Now we need

LEMMA (4.6). *There is a homotopy commutative diagram*

$$\begin{array}{ccc} \hat{Q}_n(X') & \xrightarrow{S} & \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P}) \\ \downarrow \hat{j} & & \downarrow \\ J & \xrightarrow{D} & \text{Map}_0^*(X; \mathbf{P}), \end{array}$$

in which the map  $S$  is that of (4.2), the map  $D$  is an equivalence, and the vertical map on the right is subtraction  $\mathbf{P} \vee \mathbf{P} \subset \mathbf{P} \times \mathbf{P} \rightarrow \mathbf{P}$  in the  $H$ -space  $\mathbf{P}$ .

This lemma identifies the homotopic fibre of  $j$  with that of the right-hand map, which is the desired space  $\text{Map}_n^*(X; S^2)$  in view of the fibration sequence

$$S^2 \rightarrow \mathbf{P} \vee \mathbf{P} \rightarrow \mathbf{P}.$$

The proof of (4.1) would now be complete but that we have not shown that the map  $F_n^*(X) \rightarrow \text{Map}_n^*(X; S^2)$  which arises is the natural inclusion. So we must prove in addition:

LEMMA (4.7). *There is a homotopy-commutative diagram*

$$\begin{array}{ccc} F_n^*(X) & \longrightarrow & \text{Map}_n^*(X; S^2) \\ \downarrow & & \downarrow q \\ Q_n(X') & \xrightarrow{S} & \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P}), \end{array}$$

in which the top map is the inclusion.

This really does complete the proof of (4.1), as the homotopy classes of maps  $F_n^*(X) \rightarrow \text{Map}_n^*(X; S^2)$  which make the diagram commute differ among each other only by the action of the group  $\pi_1 \text{Map}_n^*(X; \mathbf{P}) = [X; S^1] = H^1(X; \mathbf{Z})$  on  $\text{Map}_n^*(X; S^2)$ , and so if one is an equivalence up to a certain dimension then the others are too. (The indeterminacy group  $[X; S^1]$  appears here because the scanning map was defined by choosing a parallelization of  $X'$ , and the possible choices of that are an orbit of the group  $[X; S^1]$ .)

Now we shall prove Lemmas (4.6) and (4.7), and then return to the proof of (4.2).

*Proof of (4.6).* First observe that the space  $\text{Map}_0^*(X; \mathbf{P})$  is, up to homotopy at least, a connected abelian group with homotopy groups  $\pi_i = [S^i X; \mathbf{P}] = \mathbb{H}^{2-i}(X; \mathbf{Z})$ . Thus homotopically it is a torus with fundamental group  $H^1(X; \mathbf{Z})$ . On the other hand the Jacobian  $J$  is a torus with fundamental group  $H_1(X; \mathbf{Z})$ . These are canonically identified by Poincaré duality, and this defines the map  $D$  in the diagram.

As to the commutativity, it is enough to consider the diagram

$$\begin{array}{ccc} Q_n(X') & \xrightarrow{S} & \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P}) \\ \downarrow j & & \downarrow \\ J & \xrightarrow{D} & \text{Map}_0^*(X; \mathbf{P}). \end{array}$$

This is the restriction of a diagram

$$\begin{array}{ccc} A_n(X') \times A_n(X') & \xrightarrow{S \times S} & \text{Map}_n^*(X; \mathbf{P} \times \mathbf{P}) \\ \downarrow j \times (-j) & & \downarrow \\ J & \xrightarrow{D} & \text{Map}_0^*(X; \mathbf{P}), \end{array}$$

in which the top map is defined by scanning zeros and poles separately, and the right-hand vertical map is subtraction in the  $H$ -space  $P$ . So it suffices to show that

$$\begin{array}{ccc} A_n(X') & \xrightarrow{S} & \text{Map}_n^*(X; \mathbf{P}) \\ & \searrow j & \nearrow D \\ & & J \end{array}$$

commutes up to homotopy. That is the fundamental assertion that “scanning” defines the  $S$ -duality of  $X$  with itself. As  $j$ ,  $D$ , and  $S$  are all  $H$ -maps it is enough to see that

$$\begin{array}{ccc} X' & \xrightarrow{S} & \text{Map}_1^*(X; \mathbf{P}) \\ & \searrow j & \nearrow D \\ & & J \end{array}$$

commutes. It suffices to consider the effect of the maps on  $H_1(X')$  and I shall leave that to the reader.

*Proof of (4.7).* For any open Riemann surface  $Y$  let  $F(Y)$  denote the space of non-zero meromorphic functions on  $Y$ . This is the space of all holomorphic maps  $Y \rightarrow S^2$  with the two constant functions 0 and  $\infty$  omitted. We give it the compact-open topology (which is quite different from the topology it acquires as the multiplicative group of an infinite dimensional field).

Let  $U$  be the open unit disk in  $\mathbf{C}$ . Because the punctured Riemann surface  $X'$  is parallelizable one can define a map  $\sigma: X' \rightarrow \text{Emb}(U; X')$  from  $X'$  into the space of holomorphic embeddings of  $U$  in  $X'$ : for any  $x$  in  $X'$ ,  $\sigma(x)$  is an embedding with centre  $x$  and small radius. As  $Y \mapsto F(Y)$  is a continuous contravariant functor with respect to open embeddings the map  $\sigma$  defines by adjunction a map

$$S_F: F(X') \rightarrow \text{Map}(X'; F(U)).$$

Now the scanning map  $S: F(X') \rightarrow \text{Map}(X'; \mathbf{P} \vee \mathbf{P})$  is obtained by composing  $S_F$  with a standard map  $\pi: F(U) \rightarrow \mathbf{P} \vee \mathbf{P}$ . On the other hand the embedding  $F(X') \subset \text{Map}(X'; S^2)$  is the composite of  $S_F$  with evaluation at the origin  $\varepsilon: F(U) \rightarrow S^2$ . So it is enough to prove

**PROPOSITION (4.8).** *There is a homotopy commutative diagram*

$$\begin{array}{ccc} F(U) & \xrightarrow{\varepsilon} & S^2 \\ \downarrow & & \downarrow q \\ F(U)/\mathbf{C}^\times & \xrightarrow{\pi} & \mathbf{P} \vee \mathbf{P} \end{array}$$

*in which the rows are homotopy equivalences and the map  $q$  is that described in § 2.*

*Proof.* The map  $\varepsilon$  is a homotopy equivalence because the space of all holomorphic maps  $U \rightarrow S^2$  is obviously equivalent to  $S^2$  (for  $U$  is holomorphically contractible), and deleting two points from an infinite dimensional manifold does not change its homotopy type.

The map  $\pi: F(U)/\mathbb{C}^\times \rightarrow \mathbf{P} \vee \mathbf{P}$  is the composite of the map  $\text{div}: F(U)/\mathbb{C}^\times \rightarrow Q(U, U - \frac{1}{2}U)$ , which assigns to a function its divisors of zeros and poles, with the equivalences  $\mathbf{P} \vee \mathbf{P} \rightarrow Q(S^2, \infty) \rightarrow Q(U, U - \frac{1}{2}U)$  described in § 3. To see that  $\text{div}$  is an equivalence we observe that

- (i)  $\mathbb{C}^\times$  acts freely and principally on  $F(U)$ ,
- (ii)  $F(U) = F_0 \cup F_\infty$ , where  $F_\lambda$  is the open set  $\{f \in F(U): f(0) \neq \lambda\}$ ,
- (iii)  $F_0$  and  $F_\infty$  are contractible, and  $\varepsilon: F_0 \cap F_\infty \cong \mathbb{C}^\times$ ,
- (iv)  $Q(U, U - \frac{1}{2}U) = Q_0 \cup Q_\infty$  correspondingly, where  $Q_0$  and  $Q_\infty$  consist of the pairs  $(\xi, \eta)$  such that  $0 \in \xi$  and  $0 \in \eta$  respectively, and
- (v)  $Q_0$  and  $Q_\infty$  are each equivalent to  $\mathbf{P}$ , and their intersection is contractible.

To see that the diagram commutes up to homotopy, observe that the evaluation map  $\varepsilon$  has a homotopy inverse  $\psi: S^2 \rightarrow F(U)$  given by

$$\psi(\zeta) = (z + \zeta)/(1 + \zeta z).$$

The composite  $S^2 \rightarrow F(U) \xrightarrow{\text{div}} Q(U, U - \frac{1}{2}U)$  is exactly  $S^2 \xrightarrow{a} \mathbf{P} \vee \mathbf{P} \rightarrow Q(U, U - \frac{1}{2}U)$ .

*Proof of Proposition (4.2).* The open manifold  $X'$  can be formed by taking an open disk  $Y$  and attaching to it  $2g$  handles  $A_1, \dots, A_{2g}$  each homeomorphic to  $[0, 1] \times (0, 1)$ . (See Figure 1.) (The boundary  $\partial A_i = \{0, 1\} \times (0, 1)$  of each  $A_i$  is identified with part of the boundary of  $Y$ .)

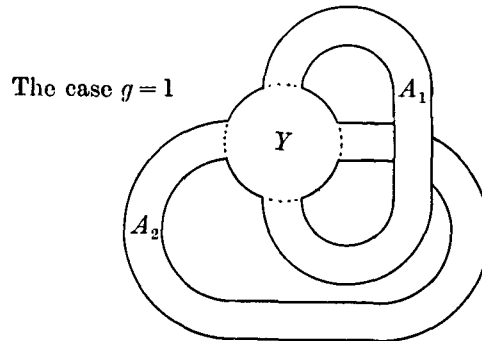


Fig. 1.



Consider the map  $\pi: Q(X') \rightarrow Q(X', \bar{Y})$ , where  $\bar{Y}$  denotes the closure of  $Y$  in  $X'$ . Each fibre is homeomorphic to  $Q(\bar{Y})$ , which we know approximates to  $\Omega^2(\mathbf{P} \vee \mathbf{P})$ . On the other hand  $Q(X', \bar{Y})$  is the same as  $\prod Q(A_i, \partial A_i)$ , and we know that  $Q(A_i, \partial A_i) \simeq \Omega(\mathbf{P} \vee \mathbf{P})$  from (3.2). The map  $\pi$  will not be a quasifibration, so we stabilize  $Q(X')$  by replacing it by the space  $\hat{Q}(X')$  of pairs  $(\xi, \eta)$  of infinite divisors which almost coincide with a fixed pair  $(\xi_0, \eta_0)$  in  $Y$  which tend to the boundary of  $Y$  but remain distant from the handles  $A_i$ . Then, just as in Lemma (3.4), we shall have a quasifibration with fibre  $\hat{Q}(\bar{Y})$ . So we have a fibration sequence

$$\hat{Q}(Y) \rightarrow \hat{Q}(X') \rightarrow \prod Q(A_i, \partial A_i) \dots \quad (\hat{Q})$$

(For the inclusion  $\hat{Q}(Y) \subset \hat{Q}(\bar{Y})$  is a homotopy equivalence).

We should like to compare  $(\hat{Q})$  with a corresponding sequence of mapping spaces. The union  $\coprod A_i$  is a closed subset of  $X'$ , so its one-point compactification  $(\coprod A_i)^+ = \vee A_i^+$  is a closed subset of  $(X')^+ = X$ , and there is a cofibration sequence  $\vee A_i^+ \rightarrow X \rightarrow Y^+$ . This leads to a fibration sequence of mapping spaces

$$\text{Map}^*(Y^+; \mathbf{P} \vee \mathbf{P}) \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P}) \rightarrow \prod \text{Map}^*(A_i^+; \mathbf{P} \vee \mathbf{P}) \dots \quad (M)$$

The sequence

$$Q(Y) \rightarrow Q(X') \rightarrow \prod Q(A_i, \partial A_i) \dots \quad (Q)$$

almost maps into  $(M)$ . To get a good map one must make some slight adjustments. Choose a complete metric on  $X'$ . Then replace  $Q(X')$  by its subspace consisting of pairs of divisors  $(\xi, \eta)$  which are separated by at least  $\varepsilon > 0$ . Do the same for  $Q(A_i, \partial A_i)$  and  $Q(Y)$ , and further restrict the divisors in  $Q(Y)$  to be separated by at least  $\varepsilon$  from  $A_i$ . Finally, replace  $\text{Map}^*(A_i^+; \mathbf{P} \vee \mathbf{P})$  by  $\text{Map}^*(\check{A}_i^+; \mathbf{P} \vee \mathbf{P})$ , where  $\check{A}_i$  is the closed subset of  $A_i$  consisting of points distant at least  $\varepsilon$  from  $\partial A_i$ . None of these changes affects the homotopy type; but now scanning with a disk of radius  $\varepsilon$  defines a map from the sequence  $(Q)$  to the sequence  $(M)$ . We still need to stabilize this map. But the stabilization affects the configurations and maps only in a region well separated from the handles  $A_i$ , so the construction described before the statement of (4.2) can be carried out to obtain the desired map  $(\hat{Q}) \rightarrow (M)$ . We know from (3.3) and (3.4) that the maps of base and fibre are homotopy equivalences (observing that  $A_i^+ \simeq S^1$  and  $Y^+ \simeq S^2$ ), so we can deduce that  $\hat{Q}(X') \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$  is a homotopy equivalence.

### Maps into $\mathbf{P}^m$

As was the case in § 3 very few changes are needed to pass from  $\mathbf{P}^1$  to  $\mathbf{P}^m$ . No change at all is needed to show that there is a homotopy equivalence

$$Q^{(m)}(X') \rightarrow \text{Map}^*(X; W_{m+1}\mathbf{P}).$$

A collection of divisors  $(\xi_0, \dots, \xi_m)$  in  $X'$  with empty intersection corresponds to a base-point-preserving map  $f: X \rightarrow \mathbf{P}^m$  if and only if all the  $\xi_i$  have the same image in the Jacobian  $J$ . In one direction this is obvious: when  $f$  is given the  $\xi_i$  are the inverse images of hyperplanes of  $\mathbf{P}^m$ , and so are linearly equivalent divisors. On the other hand if the  $\xi_i$  are equivalent there is a line-bundle  $L$  on  $X$  with sections  $s_i$  such that  $\xi_i$  is the set of zeros of  $s_i$ . One can suppose that  $s_0(x_0) = \dots = s_m(x_0) \neq 0$ . Then

$$x \mapsto (s_0(x), \dots, s_m(x))$$

is the desired map  $X \rightarrow \mathbf{P}^m$ . (Although the  $s_i(x)$  lie in  $L_x$  their ratios are in  $\mathbf{C}$ .)

Thus  $F_n^*(X; \mathbf{P}^m)$  is the fibre at 0 of a map  $j: Q_n^{(m)}(X') \rightarrow J^m$  which takes  $(\xi_0, \dots, \xi_m)$  to  $(j(\xi_0) - j(\xi_1), j(\xi_1) - j(\xi_2), \dots)$ . It turns out that this is a homology fibration up to dimension  $(n - 2g)(2m - 1)$ .

Apart from this only the generalizations of (4.7) and (4.8) deserve comment. In the proof of (4.7) one replaces  $F(Y)$  by  $F^{(m)}(Y)$ , the space of holomorphic maps  $Y \rightarrow \mathbf{P}^m$  whose image is not contained in one of the coordinate hyperplanes. Evaluation at the origin still defines a homotopy equivalence

$$\varepsilon: F^{(m)}(U) \rightarrow \mathbf{P}^m,$$

for again the space of all holomorphic maps  $U \rightarrow \mathbf{P}^m$  is equivalent to  $\mathbf{P}^m$ , and the part deleted from it has infinite codimension. A homotopy inverse to  $\varepsilon$  is given by

$$p \mapsto \{z \mapsto p + zAp + z^2A^2p + \dots + z^m A^m p\}$$

in the notation of § 2. The  $(m + 1)$ -tuple of divisors associated to the last map is  $(\xi_0, \dots, \xi_m)$ , where  $\xi_i$  is the set of zeros of the polynomial

$$p_i + zp_{i+1} + \dots + z^m p_{i-1}.$$

The only common zeros of these polynomials are roots of unity, so  $(\xi_0, \dots, \xi_m)$  defines a point of  $Q^{(m)}(U, U - \frac{1}{2}U)$ . When one identifies  $W_{m+1}\mathbf{P}$  with  $Q^{(m)}(U, V)$  as in § 3 the composite map

$$\mathbf{P}^m \rightarrow F^{(m)}(U) \rightarrow Q^{(m)}(U, U - \frac{1}{2}U)$$

is exactly the map  $q: \mathbf{P}^m \rightarrow W_{m+1}\mathbf{P}$  described in § 2.

*Note.* D. B. A. Epstein has pointed out to me that the proofs of Propositions 5 and 6 of [12] are not quite adequate. I used the following result:

Let  $S$  be a partially ordered set, and  $\{U_\alpha\}_{\alpha \in S}$  a collection of open sets of a space  $X$  indexed by  $S$ , such that

- (i)  $\alpha \leq \beta \Rightarrow U_\alpha \subset U_\beta$ , and
- (ii) if  $x \in U_\alpha \cap U_\beta$  then  $x \in U_\gamma$ , where  $\gamma \leq \alpha$  and  $\gamma \leq \beta$ . Then there is a simplicial space  $Y$  with

$$Y_k = \coprod_{\alpha_0 \leq \dots \leq \alpha_p} U_{\alpha_0},$$

and a spectral sequence with  $E_{pq}^1 = H_q(Y_p)$  which converges to  $H_*(X)$ .

If sheaf cohomology is used instead of singular homology then this is classical. To obtain it for singular homology one must observe that by [15a] there is a spectral sequence with  $E_{pq}^1 = H_q(Y_p)$  which converges to  $H_*(|Y|)$ , and by the appendix to [15b] the natural map  $|Y| \rightarrow X$  induces an isomorphism  $H_*(|Y|) \rightarrow H_*(X)$ . The hypothesis of paracompactness in the two propositions is thus quite unnecessary.

### § 5. Stabilization

As in the last section let  $X$  be a compact Riemann surface of genus  $g$  with a base-point  $x_0$ , and let  $X' = X - x_0$ .  $X'$  will be fixed throughout this section, so I shall write  $A_n$  for the  $n$ -fold symmetric product  $A_n(X')$ ,  $Q_n$  for  $Q_n(X')$ , and so on.

The discussion of stabilization is best carried out using cohomology with compact supports. Recall that if  $Y$  is a locally compact space one defines  $H_{\text{cpt}}^i(Y)$  as the reduced cohomology  $\tilde{H}^i(Y^+)$  of the one-point compactification  $Y^+$  of  $Y$  (with  $\infty$  as base-point). If  $Y$  is an orientable open manifold of dimension  $N$  the Poincaré duality theorem asserts that  $H_{\text{cpt}}^i(Y) \cong H_{N-i}(Y)$ .

We have seen that there is a stabilization map  $Q_n \rightarrow Q_{n+1}$  which takes  $(\xi, \eta)$  to  $(\xi + x, \eta + y)$ , where  $x$  and  $y$  are distinct points of  $X'$  near  $x_0$ . (One can suppose that  $(\xi, \eta)$  in  $Q_n$  is constrained to lie outside a small neighbourhood of  $x_0$  containing  $x$  and  $y$ .) Let  $V_x$  and  $V_y$  be very small disjoint disks in  $X'$  with centres  $x$  and  $y$ . The closed embedding  $Q_n \rightarrow Q_{n+1}$  extends to an open embedding  $Q_n \times V_x \times V_y \rightarrow Q_{n+1}$  given by  $((\xi, \eta), x', y') \mapsto (\xi + x', \eta + y')$ . Now  $Q_n$  is an open manifold of dimension  $4n$ , so  $H_i(Q_n) \cong H_{\text{cpt}}^{4n-i}(Q_n)$ . The stabilization  $H_i(Q_n) \rightarrow H_i(Q_{n+1})$  corresponds to the map  $H_{\text{cpt}}^i(Q_n) \rightarrow H_{\text{cpt}}^{i+4}(Q_{n+1})$  which is the composite of the suspension isomorphism  $H_{\text{cpt}}^i(Q_n) \rightarrow H_{\text{cpt}}^{i+4}(Q_n \times V_x \times V_y)$  and the natural map induced by the embedding  $Q_n \times V_x \times V_y \rightarrow Q_{n+1}$ . (The functor  $H_{\text{cpt}}^i$  is covariant for open embeddings because one-point compactification is contravariant for open embeddings.) Now we can prove

**PROPOSITION (5.1).** *The stabilization map  $Q_n \rightarrow Q_{n+1}$  is a homology equivalence up to dimension  $n - a$ , where  $a = 0$  if  $g = 0$  and  $a = 2g - 1$  if  $g > 0$ .*

*Proof.* Let  $P_n = A_n \times A_n$  be the space of pairs of divisors  $(\xi, \eta)$  of degree  $n$ , not necessarily disjoint.  $P_n$  is filtered by closed subspaces

$$P_n = P_{n,0} \supset P_{n,1} \supset \dots \supset P_{n,n} = A_n,$$

where  $P_{n,k} = \{(\xi, \eta) : \deg(\xi \cap \eta) \geq k\}$ .

Clearly  $P_{n,k} - P_{n,k+1} \cong Q_{n-k} \times A_k$ .

The open embedding  $Q_n \times V_x \times V_y \rightarrow Q_{n+1}$  extends to an open embedding  $P_n \times V_x \times V_y \rightarrow P_{n+1}$ , which takes  $P_{n,k} \times V_x \times V_y$  into  $P_{n+1,k}$  for each  $k$ .

Proceeding inductively, let us assume

(\*)<sub>n</sub>: if  $m < n$  then  $H_i(Q_m) \rightarrow H_i(Q_{m+1})$  is bijective when  $i < m - a$ , and surjective when  $i = m - a$ .

The statement (\*)<sub>1</sub> is certainly true. By Poincaré duality (\*)<sub>n</sub> is equivalent to

(\*)'<sub>n</sub>: if  $m < n$  then  $H_{\text{cpt}}^j(Q_m) \rightarrow H_{\text{cpt}}^{j+4}(Q_{m+1})$  is bijective when  $j > 3m + a$ , and surjective when  $j = 3m + a$ .

I shall abbreviate the last conclusion to "... is stable for  $j \geq 3m + a$ ".

From (\*)'<sub>n</sub> we deduce

(†)<sub>n</sub>: if  $k > 0$  then  $H_{\text{cpt}}^j(P_{n,k}) \rightarrow H_{\text{cpt}}^{j+4}(P_{n+1,k})$  is stable for  $j \geq 3n - k + a$ .

This is proved by downwards induction on  $k$ . It is true when  $k = n$  because then  $j \geq 2n = \dim(P_{n,n})$  and  $j + 4 \geq 2n + 4 = \dim(P_{n+1,n})$ . One passes from  $k + 1$  to  $k$  by applying the 5-lemma to the diagram

$$\begin{array}{ccccccccc} H_{\text{cpt}}^{j-1}(P_{n,k+1}) & \longrightarrow & H_{\text{cpt}}^j(Q_{n-k} \times A_k) & \longrightarrow & H_{\text{cpt}}^j(P_{n,k}) & \longrightarrow & H_{\text{cpt}}^j(P_{n,k+1}) & \longrightarrow & H_{\text{cpt}}^{j+1}(Q_{n-k} \times A_k) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\text{cpt}}^{j+3}(P_{n+1,k+1}) & \longrightarrow & H_{\text{cpt}}^{j+4}(Q_{n-k+1} \times A_k) & \longrightarrow & H_{\text{cpt}}^{j+4}(P_{n+1,k}) & \longrightarrow & H_{\text{cpt}}^{j+4}(P_{n+1,k+1}) & \longrightarrow & H_{\text{cpt}}^{j+5}(Q_{n-k+1} \times A_k) \end{array}$$

which arises from the pair  $(P_{n,k}, P_{n,k+1})$  because  $P_{n,k} - P_{n,k+1} = Q_{n-k} \times A_k$ . (Because  $\dim(A_k) = 2k$  the hypothesis (\*)'<sub>n</sub> implies that  $H_{\text{cpt}}^j(Q_m \times A_k) \rightarrow H_{\text{cpt}}^{j+4}(Q_{m+1} \times A_k)$  is stable when  $j \geq 3m + 2k + a$ ).

Now consider the pair  $(P_n, P_{n,1})$ . There is a diagram

$$\begin{array}{ccccccccc} H_{\text{cpt}}^{j-1}(P_n) & \longrightarrow & H_{\text{cpt}}^{j-1}(P_{n,1}) & \longrightarrow & H_{\text{cpt}}^j(Q_n) & \longrightarrow & H_{\text{cpt}}^j(P_n) & \longrightarrow & H_{\text{cpt}}^j(P_{n,1}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H_{\text{cpt}}^{j+3}(P_{n+1}) & \longrightarrow & H_{\text{cpt}}^{j+3}(P_{n+1,1}) & \longrightarrow & H_{\text{cpt}}^{j+4}(Q_{n+1}) & \longrightarrow & H_{\text{cpt}}^{j+4}(P_{n+1}) & \longrightarrow & H_{\text{cpt}}^{j+4}(P_{n+1,1}). \end{array}$$

But  $P_n = A_n \times A_n$ , and when  $n > a$  we know that  $A_n$  is a fibre bundle over the torus  $J$  with fibre  $\mathbb{C}^{n-a}$ . So by the Thom isomorphism theorem  $H_{\text{cpt}}^{j-1}(P_n) \cong H_{\text{cpt}}^{j-1-4n+4a}(J \times J) \cong H_{\text{cpt}}^{j+3}(P_{n+1})$ . Thus (†)<sub>n</sub>  $\Rightarrow$  (\*)<sub>n+1</sub> if  $n > a$ . But if  $n \leq a$  then (\*)<sub>n+1</sub> is trivially true. So altogether we have shown (\*)<sub>n</sub>  $\Rightarrow$  (\*)<sub>n+1</sub>, and the proof of (4.1) is complete.

*Proof of Proposition (4.5).* It turns out that exactly the same argument suffices to prove Proposition (4.5), asserting that the map  $j: Q_n(X') \rightarrow J$  is a homology fibration up to dimension  $n - 2g$ .

The map  $j$  extends to  $j: P_n \rightarrow J$ . If  $\sigma$  is any subset of  $J$  let  $P_n^\sigma, P_{n,k}^\sigma, Q_n^\sigma$  denote the parts of  $P_n, P_{n,k}, Q_n$  lying above  $\sigma$ . Proposition (4.5) amounts to the assertion that if  $\alpha \in J$  and  $\sigma$  is a small contractible open neighbourhood of  $\alpha$  in  $J$  then the inclusion  $Q_n^\alpha \subset Q_n^\sigma$  is a homology equivalence up to dimension  $n - 2g$ . For any subset  $\sigma$  we have

$$P_n^\sigma = P_{n,0}^\sigma \supset P_{n,1}^\sigma \supset \dots \supset P_{n,n}^\sigma,$$

and  $P_{n,k}^\sigma - P_{n,k+1}^\sigma = Q_{n-k}^\sigma \times A_k$ .

We make the inductive hypothesis

(\*)<sub>n</sub>: if  $m < n$  then  $H_i(Q_m^\alpha) \rightarrow H_i(Q_m^\sigma)$  is stable when  $i \leq m - 2g$ , or equivalently, since  $Q_m^\alpha$  and  $Q_m^\sigma$  are manifolds of dimensions  $4m - 2g$  and  $4m$  respectively,

(\*)'<sub>n</sub>: if  $m < n$  then  $H_{\text{cpt}}^j(Q_m^\alpha) \rightarrow H_{\text{cpt}}^{j+2g}(Q_m^\sigma)$  is stable when  $j \geq 3m$ . As before we deduce first

(†)<sub>n</sub>: if  $k > 0$  then  $H_{\text{cpt}}^j(P_{n,k}^\alpha) \rightarrow H_{\text{cpt}}^{j+2g}(P_{n,k}^\sigma)$  is stable when  $j \geq 3n - k$ ,

and then show that (†)<sub>n</sub> implies (\*)<sub>n+1</sub>. Of course the essential input for the argument is the fact already pointed out that  $j: P_n \rightarrow J$  is a fibre bundle when  $n \geq 2g$ . The difference between the stability ranges in (5.1) and (4.5) arises at the bottom of the induction: in the former case (\*)<sub>n</sub> is trivially true for  $n \leq 2g - 1$ , and in the latter case only for  $n \leq 2g$ .

By this point it will be clear that the proof of (5.1) works equally well for  $Q_n^\sigma$  when  $\sigma$  is an arbitrary subset of the Jacobian, providing we use the stabilization map  $i$  described in § 4 which has the property that  $ji = j$ .

**PROPOSITION (5.2).** *The stabilization map  $i: Q_n^\sigma \rightarrow Q_{n+N}^\sigma$  is a homology equivalence up to dimension  $n - 2g$  for any subset  $\sigma$  of  $J$ . In particular  $i$  induces a homology equivalence  $F_n^*(X) \rightarrow F_{n+N}^*(X)$  up to dimension  $n - 2g$ .*

Before leaving the subject of stabilization it may be worth describing briefly another approach to the question, which, though it does not lead to an explicit stability dimension, proves more in another respect. This is the method used by Dold [4] for the symmetric groups, and applied also in [11]. We shall consider only the case of the rational functions  $F_n^*$  on the Riemann sphere.

**PROPOSITION (5.3).** *The embedding  $F_{n-1}^* \rightarrow F_n^*$  makes  $H_*(F_{n-1}^*)$  a direct summand in  $H_*(F_n^*)$ . More precisely, there is a sequence of graded groups  $\{K_m\}$  and a canonical isomorphism*

$$H_*(F_n^*) \cong K_0 \oplus K_1 \oplus \dots \oplus K_n$$

*compatible with the embeddings  $F_m^* \rightarrow F_n^*$  for  $m \leq n$ .*

*Proof.* Let  $Q_{m,n}$  denote the space of pairs of disjoint divisors  $(\xi, \eta)$  in  $\mathbb{C}$  such that  $\deg(\xi) = m$  and  $\deg(\eta) = n$ . If  $p \leq m$  there is a "many-valued map" from  $Q_{m,n}$  to  $Q_{p,n}$  which assigns to  $(\xi, \eta)$  the collection of  $\binom{m}{p}$  pairs  $(\xi', \eta)$ , where  $\xi'$  runs through the collection of subdivisors of  $\xi$  with degree  $p$ . Such a many-valued map induces (cf. [11, p. 103]) a transfer  $r_{m,p}: H_*(Q_{m,n}) \rightarrow H_*(Q_{p,n})$ , which clearly satisfies the identity

$$r_{m+1,p} i_m = r_{m,p} + i_{p-1} r_{m,p-1},$$

where  $i_m: H_*(Q_{m,n}) \rightarrow H_*(Q_{m+1,n})$  is the natural map. It follows from [4] (Lemma 2) that  $H_*(Q_{m,n})$  can be decomposed as the sum of  $m+1$  pieces, of which the first  $p+1$  form the image of  $H_*(Q_{p,n})$ . One can similarly decompose  $H_*(Q_{m,n})$  into  $n+1$  pieces by breaking up the second divisor in the pair. The two decompositions commute, giving a decomposition of the form  $H_*(Q_{m,n}) = \bigoplus_{p \leq m, q \leq n} K_{p,q}$ . But  $K_{p,q} = 0$  if  $p \neq q$ , for we saw in § 3 that  $Q_{p,q} \simeq Q_{p,p}$  if  $p \leq q$ .

Proposition (5.3) implies that  $H_*(F_n^*)$  is independent of  $n$  when  $n$  is large, as we know that  $\varinjlim_n H_*(F_n^*)$  is finitely generated.

The same argument gives a splitting of  $H_*(Q_n(X'))$  for any punctured Riemann surface  $X'$ ; but it does not seem obvious that there is an induced splitting of  $H_*(F_n^*(X))$ .

*The case of  $Q_n^{(m)}(X')$ .* No essential changes are needed in any of the preceding discussion to treat  $(m+1)$ -tuples instead of pairs of divisors. But then in the fundamental filtration

$$P_n = P_{n,0} \supset P_{n,1} \supset P_{n,n} = A_n,$$

where  $P_{n,k} = \{(\xi_0, \dots, \xi_m): \deg(\xi_0 \cap \dots \cap \xi_m) \geq k\}$ , each layer has complex codimension  $m$  in its predecessor. This leads to a higher stabilization dimension. We have

**PROPOSITION (5.1').** *The stabilization map  $Q_n^{(m)} \rightarrow Q_{n+1}^{(m)}$  is a homology equivalence up to dimension  $(n-a)(2m-1)$ , where  $a=0$  if  $g=0$ , and  $a=2g-1$  if  $g>0$ .*

Similarly the generalization of (4.5) is

**PROPOSITION (4.5').** *The map*

$$j: Q_n^{(m)}(X') \rightarrow J^m$$

*defined by*

$$j(\xi_0, \dots, \xi_m) = (j(\xi_0) - j(\xi_1), \dots, j(\xi_{m-1}) - j(\xi_m))$$

*is a homology fibration up to dimension  $(n-2g)(2m-1)$ .*

## § 6. The action of $\pi_1(F_n^*)$

We now return to the rational functions on the Riemann sphere. From what has preceded we know that the inclusion of the rational functions  $F_n^*$  in the space of maps  $M_n^*$

is a homology equivalence up to dimension  $n$ . J. D. S. Jones has proved that  $\pi_1(F_n^*) \cong \mathbf{Z}$  (I shall recall his proof below), so we know that  $F_n^* \subset M_n^*$  induces an isomorphism of fundamental groups as well. We should like to show that it is actually a homotopy equivalence up to dimension  $n$ . To do so we must show that the space  $F_n^*$  is simple up to dimension  $n$ , i.e. that  $\pi_1 = \pi_1(F_n^*)$  acts trivially on  $\pi_k(F_n^*)$  for  $k < n$ . (The space  $M_n^*$  is simple because it is homotopy equivalent to  $M_0^*$ , which is an  $H$ -space.) It suffices to show that  $\pi_1$  acts trivially on  $H_k(\tilde{F}_n^*)$  for  $k < n$ , where  $\tilde{F}_n^*$  is the universal covering space of  $F_n^*$ ; and in fact [7] it is even enough to show that  $\pi_1$  acts nilpotently on  $H_k(\tilde{F}_n^*)$ , i.e. that  $H_k(\tilde{F}_n^*)$  has a  $\pi_1$ -stable filtration such that  $\pi_1$  acts trivially on the associated graded module. To prove this we consider the map  $R: F_n^* \rightarrow \mathbb{C}^\times$  which assigns to a rational function  $f = p/q$  the “resultant”  $R_{p,q}$  of the polynomials  $p$  and  $q$ .

Recall that if  $p(z) = z^n + a_1 z^{n-1} + \dots + a_n$  and  $q(z) = z^n + b_1 z^{n-1} + \dots + b_n$  are two monic polynomials with roots  $\alpha_1, \dots, \alpha_n$  and  $\beta_1, \dots, \beta_n$  their resultant  $R_{p,q}$  is  $\prod_{i,j} (\alpha_i - \beta_j)$ , a polynomial in  $a_1, \dots, a_n$  and  $b_1, \dots, b_n$  which is homogeneous of weight  $n^2$  when  $a_k$  and  $b_k$  are assigned weight  $k$ .  $R_{p,q}$  vanishes if and only if  $p$  and  $q$  have a common root, so  $F_n^*$  is precisely the space of pairs of polynomials  $(p, q)$  such that  $R_{p,q} \neq 0$ . The homogeneity means that if  $\mathbb{C}^\times$  acts on  $F_n^*$  by acting on the roots of the polynomials, i.e. by

$$\lambda \cdot (a_1, \dots, a_n; b_1, \dots, b_n) = (\lambda a_1, \dots, \lambda^n a_n; \lambda b_1, \dots, \lambda^n b_n),$$

then  $R_{\lambda \cdot (p,q)} = \lambda^{n^2} R_{(p,q)}$ . This implies

**PROPOSITION (6.1).** (a)  $R: F_n^* \rightarrow \mathbb{C}^\times$  is a fibre bundle with non-singular fibres and structural group  $\{\zeta \in \mathbb{C}: \zeta^{n^2} = 1\}$ .

(b) The monodromy  $T: R^{-1}(1) \rightarrow R^{-1}(1)$ , i.e. the action of the generator of the structural group, is given by

$$(a_1, \dots, a_n; b_1, \dots, b_n) \rightarrow (\zeta a_1, \dots, \zeta^n a_n; \zeta b_1, \dots, \zeta^n b_n),$$

where  $\zeta$  is a primitive root of unity of order  $n^2$ .

Because  $R$  is a fibration and (as we shall see) induces an isomorphism of fundamental groups the fibre  $R^{-1}(1)$  can be identified up to homotopy with the universal cover  $\tilde{F}_n^*$ , and the action of  $\pi_1$  is given by the monodromy.

**PROPOSITION (6.2).** The monodromy  $T_*: H_k(R^{-1}(1)) \rightarrow H_k(R^{-1}(1))$  is “nilpotent” when  $k < n$ , in the sense that it preserves a filtration of  $H_*(R^{-1}(1))$  and acts as the identity on the associated graded module.

**COROLLARY (6.3).**  $F_n^*$  is a nilpotent space up to dimension  $n$ .

*Proof of (6.2).* Because the fibre  $Y = R^{-1}(1)$  is non-singular one can identify  $H_i(Y)$  with  $H_{\text{cpt}}^{4n-2-i}(Y)$ . Let us filter  $Y$  by closed subspaces:

$$Y = Y_n \supset Y_{n-1} \supset \dots \supset Y_1 \supset Y_0 = \emptyset,$$

where  $Y_m$  consists of the rational functions where the denominator has at most  $m$  distinct zeros. This filtration is preserved by the monodromy. Because of the exact triangles

$$\dots \rightarrow H_{\text{cpt}}^k(Y_m - Y_{m-1}) \rightarrow H_{\text{cpt}}^k(Y_m) \rightarrow H_{\text{cpt}}^k(Y_{m-1}) \rightarrow \dots$$

it will be enough to show that the monodromy acts nilpotently on  $H_{\text{cpt}}^k(Y_m - Y_{m-1})$  for each  $m$  when  $k < 3n - 2$ .

The space  $Y_n - Y_{n-1}$  is fibred over  $C_n$ , the space of distinct  $n$ -tuples in  $\mathbb{C}$ , with fibre  $(\mathbb{C}^\times)^{n-1}$ . For if  $p/q \in Y_n - Y_{n-1}$ , and  $q$  has roots  $\beta_1, \dots, \beta_n$ , then  $p$  is completely determined by  $p(\beta_1), \dots, p(\beta_n)$  in  $\mathbb{C}^\times$ , which are arbitrary except that their product is  $R_{p,q} = 1$ . In general  $Y_m - Y_{m-1}$  has one connected component for each partition of  $n$  into  $m$  pieces, and the component corresponding to  $n = k_1 + \dots + k_m$  is fibred over  $\check{C}_k$ , the space of distinct  $m$ -tuples in  $\mathbb{C}$  which are unordered but are labelled with the multiplicities  $k_1, \dots, k_m$ . The fibre is  $G_k$ , the kernel of the homomorphism  $(\mathbb{C}^\times)^m \rightarrow \mathbb{C}^\times$  given by  $(\xi_1, \dots, \xi_m) \mapsto \xi_1^{k_1} \xi_2^{k_2} \dots \xi_m^{k_m}$ . For if  $p/q \in Y_m - Y_{m-1}$  and  $q(z) = \prod (z - \beta_i)^{k_i}$  then  $p$  is determined by giving

$$p(\beta_1), p'(\beta_1), \dots, p^{(k_1-1)}(\beta_1); \dots; p(\beta_m), p'(\beta_m), \dots, p^{(k_m-1)}(\beta_m),$$

which are arbitrary except for the constraint  $R_{p,q} = \prod p(\beta_i)^{k_i} = 1$ .

Thus  $Y_n - Y_{n-1}$  is the quotient of  $Z_n = (\mathbb{C}^\times)^{n-1} \times \check{C}_n$ , where  $\check{C}_n$  is the space of ordered distinct  $n$ -tuples in  $\mathbb{C}$ , by a free action of the symmetric group  $\mathfrak{S}_n$ . (Think of  $(\mathbb{C}^\times)^{n-1}$  as  $\{(\xi_1, \dots, \xi_n) \in (\mathbb{C}^\times)^n : \xi_1 \xi_2 \dots \xi_n = 1\}$ .) The monodromy  $T$  is induced by

$$\check{T}: (\xi_1, \dots, \xi_n; z_1, \dots, z_n) \mapsto (\zeta^n \xi_1, \dots, \zeta^n \xi_n; \zeta z_1, \dots, \zeta z_n)$$

on  $Z_n$  (where  $\zeta^n = 1$ ), which commutes with the action of  $\mathfrak{S}_n$ . Now  $\check{T}$  acts as the identity on  $H_{\text{cpt}}^*(Z_n)$ —in fact it is homotopic to the identity, though not equivariantly with respect to  $\mathfrak{S}_n$ —so one can conclude from the spectral sequence

$$H^*(\mathcal{S}_n; H_{\text{cpt}}^*(Z_n)) \Rightarrow H_{\text{cpt}}^*(Y_n - Y_{n-1})$$

that  $T$  acts nilpotently on  $H_{\text{cpt}}^*(Y_n - Y_{n-1})$ .

The case of  $Y_m - Y_{m-1}$  for  $m < n$  is similar. One considers the connected components separately. For the component corresponding to the partition  $\mathbf{k} = \{k_1, \dots, k_m\} = \{1^{a_1} a_2^{a_2} \dots\}$  the group  $\mathfrak{S}_n$  is replaced by  $\mathfrak{S}_{a_1} \times \mathfrak{S}_{a_2} \times \dots \subset \mathfrak{S}_m$ . The action of  $T$  comes from the map  $\check{T}: G_{\mathbf{k}} \times \check{C}_{\mathbf{k}} \rightarrow G_{\mathbf{k}} \times \check{C}_{\mathbf{k}}$  defined by  $\check{T}(\xi_1, \dots, \xi_m; \beta_1, \dots, \beta_m) = (\zeta^n \xi_1, \dots, \zeta^n \xi_m; \zeta \beta_1, \dots, \zeta \beta_m)$ , where  $\zeta^n = 1$ . This will be homotopic to the identity providing  $(\zeta^n, \dots, \zeta^n)$  is in the identity component of  $G_{\mathbf{k}}$ . But  $G_{\mathbf{k}}$  is connected—in fact is isomorphic to  $(\mathbb{C}^\times)^{m-1}$ —unless the integers  $\{k_1, \dots, k_m\}$



have a common factor. There can be no common factor unless  $k_i \geq 2$  for all  $i$ , in which case  $m \leq n/2$ . But the complex dimension of  $Y_m$  is  $m + n - 1$ , so  $H_{\text{cpt}}^k(Y_m - Y_{m-1}) = 0$  if  $k > 2m + 2n - 2$ , and if  $m \leq n/2$  there is nothing to prove.

The preceding proof was suggested by Jones's proof that  $\pi_1(F_n^*) \simeq \mathbf{Z}$ . For completeness, and because it fits in naturally here, I shall give his argument.

**PROPOSITION (6.4).**  *$\pi_1(F_n^*) \cong \mathbf{Z}$ , and is generated by the loop which moves one zero of a rational function once around one pole.*

*Proof.* Let  $U$  be the part of  $F_n^*$  consisting of all rational functions  $p/q$  such that  $q$  has  $n$  distinct roots. Because the complement of  $U$  in  $F_n^*$  is of codimension 2 (it is an algebraic hypersurface) we know that  $\pi_1(U) \rightarrow \pi_1(F_n^*)$  is surjective. But from the preceding discussion we know that  $U$  is fibred over  $C_n$  with fibre  $(\mathbf{C}^\times)^n$ ; and the fibration has a cross-section. Hence  $\pi_1(U)$  is the semidirect product  $Br_n \times \mathbf{Z}^n$ , where  $Br_n = \pi_1(C_n)$  is the  $n$ th braid group, which acts on  $\mathbf{Z}^n$  via its homomorphism  $Br_n \rightarrow \mathfrak{S}_n$ . Let  $V$  be the part of  $U$  consisting of all  $p/q$  such that the roots of  $p$  are in the upper half-plane and those of  $q$  are in the lower half-plane. Then  $\pi_1(V) = Br_n$ . On the other hand the inclusion  $V \rightarrow F_n^*$  is evidently homotopic to a constant. Hence  $\pi_1(F_n^*)$  is a quotient of  $Br_n \times \mathbf{Z}^n$  by a normal subgroup which contains  $Br_n$ . The biggest such quotient is  $\mathbf{Z}$ . But we know that  $\pi_1(F_n^*)$  cannot be smaller than  $\mathbf{Z}$  because of the map  $R: F_n^* \rightarrow \mathbf{C}^\times$ .

### § 7. The real case

Let  ${}^{\mathbf{R}}F_n^*$  denote the rational functions of the form

$$\frac{z^n + a_1 z^{n-1} + \dots + a_n}{z^n + b_1 z^{n-1} + \dots + b_n}$$

with  $a_i$  and  $b_i$  real. These define maps  $\mathbf{C}U_\infty \rightarrow \mathbf{C}U_\infty$  which are equivariant with respect to complex conjugation and preserve  $\mathbf{R}U_\infty$ . We shall see that the degree of the restriction to  $\mathbf{R}U_\infty = S^1$  is congruent to  $n$  modulo 2 and lies between  $-n$  and  $n$ , and that the space  ${}^{\mathbf{R}}F_n^*$  has  $n + 1$  connected components  ${}^{\mathbf{R}}F_{n,r}^*$ , indexed by this degree. This was proved by R. Brockett [2]. More precisely one has

**PROPOSITION (7.1).**  *${}^{\mathbf{R}}F_{n,r}^*$  is homeomorphic to the space of complex rational functions  $F_{p,q}$ , where  $p + q = n$  and  $p - q = r$ .*

Here  $F_{p,q}$  is the space of functions of the form  $f/g$ , where  $f$  and  $h$  are monic complex polynomials of degrees  $p$  and  $q$  respectively.

*Proof.* If  $f \in {}^{\mathbf{R}}F_n^*$  let  $\xi = f^{-1}(i)$ , a divisor in  $\mathbf{C}$ . The function  $f$  is completely determined by  $\xi$ : in fact if  $\varphi$  is the monic polynomial with  $\xi$  as roots then

$$f = \frac{\operatorname{Re}(\varphi) + \operatorname{Im}(\varphi)}{\operatorname{Re}(\varphi) - \operatorname{Im}(\varphi)}.$$

The divisor  $\xi$  is disjoint from  $\bar{\xi} = f^{-1}(-i)$ , and hence from the real axis, but is subject to no other constraints. Write  $\xi = \xi_+ + \xi_-$ , where  $\xi_+$  is in the upper half-plane  $H_+$  and  $\xi_-$  is in the lower half-plane. Then  $(\xi_+, \bar{\xi}_-)$  determines  $\xi$ , and is an element of  $Q(H_+)$ , which is homeomorphic to  $Q(\mathbf{C})$ . The degree of  $f|_{S^1}$  is the winding number of

$$z \mapsto \frac{f(z) - i}{f(z) + i}$$

about the origin when  $z$  runs around the boundary of the upper half-plane, i.e. it is the difference between the number of zeros and the number of poles of  $(f-i)/(f+i)$  in  $H_+$ , i.e. it is  $r = \deg(\xi_+) - \deg(\xi_-) = p - q$  if  $(\xi_+, \bar{\xi}_-)$  belongs to  $Q_{p,q}(H_+)$ .

Now let  ${}^{\mathbf{R}}M_{n,r}^*$  denote the space of equivariant maps  $S^2 \rightarrow S^2$  which have degree  $n$ , take  $\infty$  to 1, and have degree  $r$  on  $S^1$ .

**PROPOSITION (7.2).** *The inclusion  ${}^{\mathbf{R}}F_{n,r}^* \subset {}^{\mathbf{R}}M_{n,r}^*$  is a homotopy equivalence up to dimension  $\frac{1}{2}(n - |r|)$ .*

*Proof.* An equivariant map  $S^2 \rightarrow S^2$  is determined by its restriction to the closed upper half-plane  $\bar{H}_+$ . Because the space of based maps  $S^1 \rightarrow S^1$  of degree  $r$  is contractible, and  $S^1$  is contractible in  $S^2$ , the space of maps  $(\bar{H}_+, S^1) \rightarrow (S^2, S^1)$  with degree  $r$  on  $S^1$  is homotopy equivalent to  $\Omega^2 S^2$ . Thus, using (7.1), one knows that  ${}^{\mathbf{R}}F_{n,r}^*$  and  ${}^{\mathbf{R}}M_{n,r}^*$  have the same homotopy type up to dimension  $\min(p, q) = \frac{1}{2}(n - |r|)$ . To see that the inclusion actually induces this equivalence, consider the homotopy commutative diagram

$$\begin{array}{ccc} {}^{\mathbf{R}}F_{n,r}^* & \longrightarrow & {}^{\mathbf{R}}M_{n,r}^* \\ \downarrow & & \downarrow \\ Q_{p,q}(H_+) & \longrightarrow & M_r \end{array}$$

where  $M_r$  is the space of maps  $\bar{H}_+ \rightarrow S^2$  which take  $S^1$  into  $S^2 - \{i, -i\}$  with degree  $r$ . The left-hand vertical map is that of (6.1). The right-hand vertical map is restriction, and is a homotopy equivalence on to a connected component. The bottom map takes  $(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q)$  to  $f$ , where

$$\frac{f(z) - i}{f(z) + i} = \frac{(z - \alpha_1) \dots (z - \alpha_p)}{(z - \beta_1) \dots (z - \beta_q)}.$$

If  $p=q$  then  $r=0$ , and this is an equivalence with one component of  $M_r \simeq \Omega^2 S^2$  by Proposition (1.1). If  $p \neq q$  one reduces to the previous case by multiplying by  $z^{q-p}$ . That completes the proof.

**§ 8. Riemann surfaces with singularities**

If  $X$  is a compact Riemann surface with singularities (i.e. an algebraic curve with singularities) one must distinguish between the holomorphic maps  $X \rightarrow S^2$  and the field  $K_X$  of rational functions on  $X$ . For example, let  $X$  be the plane cubic curve with a double-point at the origin whose equation is  $y^2 = x^2(x+1)$ . Then  $t = y/x$  belongs to  $K_X$  (indeed  $K_X = \mathbb{C}(t)$ , for  $x = t^2 - 1$  and  $y = t(t^2 - 1)$ ), but  $t$  does not define a map  $X \rightarrow S^2$  because it tends to the two distinct values  $\pm 1$  as one approaches the origin along the two branches of the curve. In fact in general  $K_X$  consists precisely of the holomorphic maps  $\tilde{X} \rightarrow S^2$ , where  $\tilde{X}$  is the desingularized curve of which  $X$  is a quotient. I shall write  $F_n(X)$  for the holomorphic maps  $X \rightarrow S^2$  of degree  $n$ .

PROPOSITION (8.1). *If  $X$  is a compact Riemann surface with singularities the inclusion  $F_n(X) \subset \text{Map}_n(X; S^2)$  is a homology equivalence up to dimension  $n - 2(\pi - k + 1)$ , where  $\pi$  is the numerical genus of  $X$  (cf. [16] p. 73) and  $k$  is the number of singular points.*

The numerical genus of  $X$  is the dimension of  $H^1(X; O)$ , where  $O$  is the sheaf of holomorphic functions on  $X$ .

I think it is worth discussing explicitly the two most obvious particular cases—the cubic curve with a double point and the cubic curve with a cusp—before passing to the general proof.

(a) *The cubic curve  $X$  with a double-point at  $x_0$*

$X$  can be obtained from the Riemann sphere  $S^2$  by identifying two distinct points, say  $0$  and  $\infty$ . Let  $p: S^2 \rightarrow X$  be the quotient map. Then  $F_n(X) = \{f \in F_n(S^2): f(0) = f(\infty)\}$ , and  $\text{Map}_n(X; S^2) = \{f \in \text{Map}_n(S^2; S^2): f(0) = f(\infty)\}$ . Clearly it is enough to show that  $F_n^*(X) \subset \text{Map}_n^*(X; S^2)$  is an equivalence up to dimension  $n - 2$ , where  $*$  indicates the maps taking  $x_0$  to  $1$ .

The elements  $f$  of  $F_n^*(X)$  correspond to pairs of divisors  $(\xi, \eta)$  in  $X' = X - x_0 = S^2 - \{0, \infty\} = \mathbb{C}^\times$ . But conversely if  $\xi = \{\alpha_1, \dots, \alpha_n\}$  and  $\eta = \{\beta_1, \dots, \beta_n\}$  are in  $\mathbb{C}^\times$  then the corresponding rational function

$$f(z) = \frac{(z - \alpha_1) \dots (z - \alpha_n)}{(z - \beta_1) \dots (z - \beta_n)}$$

such that  $f(\infty) = 1$  satisfies  $f(0) = 1$  if and only if  $\alpha_1 \alpha_2 \dots \alpha_n = \beta_1 \beta_2 \dots \beta_n$ . Thus if  $j: Q_n(\mathbb{C}^\times) \rightarrow \mathbb{C}^\times$  is defined by  $(\xi, \eta) \mapsto (\alpha_1 \alpha_2 \dots \alpha_n) (\beta_1 \beta_2 \dots \beta_n)^{-1}$  then  $F_n^*(X)$  is the fibre of  $j$  at  $1$  in  $\mathbb{C}^\times$ . In this case the group  $\mathbb{C}^\times$  is the “generalised Jacobian” of  $X$ .

On the other hand there is a homotopy equivalence  $\hat{Q}(X') \rightarrow \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$ . This is not quite a particular case of (4.2), for the open surface  $X'$  now has two ends rather than one, but the argument of (4.2) applies without change. (In fact  $X'$  is obtained from an open disk by attaching one handle.)

It is easy to see that there is a homotopy commutative diagram

$$\begin{array}{ccc} Q_n(X') & \longrightarrow & \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P}) \\ j \downarrow & & \downarrow \\ \mathbb{C}^\times & \longrightarrow & \text{Map}_0^*(X; \mathbf{P}), \end{array}$$

in which the bottom map is an equivalence: notice that homotopically  $X$  is  $S^2 \vee S^1$ . So to complete the proof of (8.1) in this case we need to show that  $j$  is a homology fibration up to dimension  $n-2$ , and that  $Q_n(X') \rightarrow \hat{Q}_n(X')$  is a homology equivalence up to the same dimension.

(b) *The cubic curve  $X$  with a cusp at  $x_0$*

This case is easier than the preceding one. Again the desingularization of  $X$  is  $S^2$ , but now the quotient map  $S^2 \rightarrow X$  is a homeomorphism. Let us suppose it takes  $\infty$  to  $x_0$ . Then  $F_n(X) = \{f \in F_n(S^2): f'(\infty) = 0\}$ . Again it is enough to show that  $F_n^*(X) \subset \text{Map}_n^*(X; S^2)$  is an equivalence up to dimension  $n-2$ . Now  $X' = \mathbb{C}$ , and  $F_n^*(X) \subset Q_n(\mathbb{C})$ . On the other hand if  $\xi = \{\alpha_1, \dots, \alpha_n\}$  and  $\eta = \{\beta_1, \dots, \beta_n\}$  are divisors in  $\mathbb{C}$  then the corresponding rational function  $f$  satisfies  $f'(\infty) = 0$  if and only if  $\alpha_1 + \dots + \alpha_n = \beta_1 + \dots + \beta_n$ . In this case the generalised Jacobian is  $\mathbb{C}$ , and  $j: Q_n(X') \rightarrow \mathbb{C}$  is

$$(\xi, \eta) \mapsto \alpha_1 + \dots + \alpha_n - \beta_1 - \dots - \beta_n.$$

But now  $\hat{Q}_n(X') \simeq \text{Map}_n^*(X; S^2)$  because  $X$  is homeomorphic to  $S^2$ , and we know already that  $Q_n(X') \subset \hat{Q}_n(X')$  is an equivalence up to dimension  $n$ . So we have only to show that  $j: Q_n(X') \rightarrow \mathbb{C}$  is a homology fibration up to dimension  $n-2$ . Its fibres are hyperplane sections of the complement of the resultant locus in  $\mathbb{C}^{2n}$ .

From the two examples (a) and (b) it is clear how to treat the next case.

(c) *A curve  $X$  with just one singular point  $x_0$*

There is a generalised Jacobian  $J$  which is the quotient of the dual of the  $\pi$ -dimensional complex vector space of regular differentials on  $X$  by the lattice  $H_1(X'; \mathbf{Z})$  (cf. [16] p. 108).  $F_n^*(X)$  is the fibre of  $j: Q_n(X') \rightarrow J$  at identity. On the other hand  $\hat{Q}(X') \simeq \text{Map}^*(X; \mathbf{P} \vee \mathbf{P})$  by the argument of (4.2).

The space  $\text{Map}_0^*(X; \mathbf{P})$  is homotopically an abelian group, and  $\pi_i \text{Map}_0^*(X; \mathbf{P}) \simeq H^{2-i}(X; \mathbf{P}) = H_{\text{cpt}}^{2-i}(X')$ . Thus it is a torus with fundamental group  $H_{\text{cpt}}^1(X')$ , which is

canonically isomorphic to  $H_1(X')$  by Poincaré duality. So there is a homotopy equivalence  $D: J \rightarrow \text{Map}_0^*(X; \mathbf{P})$ , and, just as in § 4 a homotopy commutative diagram

$$\begin{array}{ccc} Q_n(X') & \longrightarrow & \text{Map}_n^*(X; \mathbf{P} \vee \mathbf{P}) \\ j \downarrow & & \downarrow \\ J & \xrightarrow{D} & \text{Map}_0^*(X; \mathbf{P}). \end{array}$$

The whole proof will proceed just as in § 4 if one proves the two stability theorems:

- (i)  $Q_n(X') \rightarrow \tilde{Q}_n(X')$  is a homology equivalence up to dimension  $n - 2\pi$ , and
- (ii)  $j: Q_n(X') \rightarrow J$  is a homology fibration up to dimension  $n - 2\pi$ .

These results in turn are proved as in § 5. The only new ingredient needed is the fact that the  $n$ -fold symmetric product  $A_n(X')$  is a fibre bundle over  $J$  with fibre  $\mathbb{C}^{n-\pi}$ , providing  $n \geq 2\pi$ . This follows essentially from the Riemann–Roch theorem for singular curves ([16] (p. 80)) in the same way that (4.3) (c) follows from the usual Riemann–Roch theorem, for again  $A_n(X') = A_n(X) - A_{n-1}(X)$ .

(d) *The general case*

It turns out that this does not now need anything new. Let  $S$  be the set of singular points of  $X$ , and define  $\check{X} = X/S$ , a curve with just one singular point which has numerical genus  $\pi - k + 1$ . We shall prove

**PROPOSITION (8.2).** *The map  $e: F_n(X) \rightarrow (S^2)^S$  which assigns to a function its value on the set  $S$  is a homology fibration up to dimension  $n' = n - 2(\pi - k + 1)$ , where  $k = \text{card}(S)$ .*

This implies the desired result (8.1), because the fibre of  $e$  at  $(1, 1, \dots, 1)$  is  $F_n^*(\check{X})$ , which coincides up to dimension  $n'$  with  $\text{Map}_n^*(\check{X}; S^2)$  by the result of case (c); while  $\text{Map}_n^*(\check{X}; S^2)$  is the fibre of the evaluation map  $\text{Map}_n(X; S^2) \rightarrow (S^2)^S$ .

*Proof of (8.2).* Let  $F_n^\dagger(X)$  denote the holomorphic maps  $X \rightarrow S^2$  which have neither a zero nor a pole in  $S$ . It is enough to prove that the restriction

$$e: F_n^\dagger(X) \rightarrow (\mathbb{C}^\times)^S$$

is a homology fibration up to dimension  $n'$ , for the translates of  $(\mathbb{C}^\times)^S$  by  $SL_2(\mathbb{C})$  form an open covering of  $(S^2)^S$ .

Let  $\check{J}$  be the generalized Jacobian of  $\check{X}$ . If  $X' = X - S$  the fibre of  $j: Q_n(X') \rightarrow J$  is the space  $F_n^*(\check{X})$ . Let  $p: \check{X} \rightarrow X$  be the desingularization of  $X$ , and  $\check{S} = p^{-1}(S)$ . There is an exact sequence of groups

$$0 \rightarrow K \rightarrow \check{J} \rightarrow J \rightarrow 0,$$

where  $J$  is the Jacobian of  $X$ . The space  $j^{-1}(K)$  can be identified with  $F_n^\dagger(\check{X})/\mathbb{C}^\times$ , where  $F_n^\dagger(\check{X})$  denotes the rational functions on  $\check{X}$  which have neither zeros nor poles on  $\check{S}$ , and

the composite map  $F_n^\dagger(\tilde{X}) \rightarrow j^{-1}(K) \rightarrow K$  measures the obstruction to such a function's belonging to  $F_n^*(\tilde{X})$ . From [16] we know that  $K$  is a quotient-group of

$$\left(\prod_{s \in \tilde{S}} K_s\right)/\mathbb{C}^\times,$$

where  $K_s$  is the group of  $k_s$ -jets of maps  $\tilde{X} \rightarrow \mathbb{C}^\times$  for some integers  $k_s$ . But the space  $F_n^\dagger(X) \subset F_n^\dagger(\tilde{X})$  is defined by a condition on the same jets. In fact there is an embedding  $(\mathbb{C}^\times)^S/\mathbb{C}^\times \rightarrow K$  such that  $F_n^\dagger(X)/\mathbb{C}^\times$  is precisely  $j^{-1}((\mathbb{C}^\times)^S/\mathbb{C}^\times)$ . Obviously it is enough to show that  $j: F_n^\dagger(X)/\mathbb{C}^\times \rightarrow (\mathbb{C}^\times)^S/\mathbb{C}^\times$  is a homology fibration up to dimension  $n'$ . But this map is just a restriction of  $j: Q_n(X') \rightarrow \mathcal{J}$ , for which the property is known; and it follows from Propositions 5 and 6 of [12] that a homology fibration whose base is a manifold remains one when it is restricted to a submanifold of the base.

*Remark.* It is worth emphasizing that the proof of (8.2) did not use the fact that  $S$  was the set of singular points of  $X$ . It holds, for example, if  $X$  is non-singular and  $S$  is an arbitrary finite subset of  $X$ . This is an interesting strengthening of the classical lemma on the independence of valuations.

### Appendix to § 5. The homological stability of configuration spaces

The argument used in § 5 was invented by Arnol'd [1] to prove the stability of the homology of the braid groups. It applies more generally to the configuration spaces  $C_n(M)$  of unordered  $n$ -tuples of distinct points of an arbitrary open manifold  $M$ , as I shall now explain. But Arnol'd's argument in [1] is more complicated than the following version, and seems to involve considerably more ingredients. As I understand it, the essential point is to reduce the problem for configuration spaces to the analogous one for the symmetric products  $A_n(M)$ . The latter problem was trivial in Arnol'd's case, as  $A_n(\mathbb{C}) \cong \mathbb{C}^n$ . But in any case  $A_n(M)$  depends only on the (proper) homotopy type of  $M$ , so it is more accessible than  $C_n(M)$ .

We define a map  $C_n(M) \rightarrow C_{n+1}(M)$  by adding a point to the configuration in a standard way "near infinity". (Up to homotopy there is one such map for each end of  $M$ . Of course we are assuming  $M$  is connected.)

**PROPOSITION (A.1).**  $C_n(M) \rightarrow C_{n+1}(M)$  is a homology equivalence up to dimension  $d_n = [n/2]$ .

From now on I shall write  $C_n$  for  $C_n(M)$ . The argument depends on Poincaré duality:  $C_n$  is a manifold, but it is orientable only if  $M$  is orientable and even-dimensional (or

$\dim(M) = 1$ ). Proposition (A.1) is true in all cases, but the proof is much more straightforward when the  $C_n$  are orientable, so I shall assume that at first, and shall indicate at the end how to treat the general case.

The map  $C_n \rightarrow C_{n+1}$  extends to an open embedding  $\mathbf{R}^q \times C_n \rightarrow C_{n+1}$ , where  $q$  is the dimension of  $M$ . By duality (A.1) is equivalent to

(\*)<sub>n</sub>:  $\mathbf{R}^q \times C_n \rightarrow C_{n+1}$  is a compact cohomology equivalence above dimension  $(n+1)q - d_n$ .

(This means of course that the map induces an isomorphism of  $H_{cpt}^i$  when  $i > (n+1)q - d_n$ , and a surjection when  $i = (n+1)q - d_n$ .)

Arnol'd introduces the following filtration of the  $n$ -fold symmetric product  $A_n$  of  $M$ . Any  $\xi \in A_n$  can be written uniquely in the form  $\xi = 2\eta + \zeta$ , where the points in  $\zeta$  all have multiplicity 1. Then

$$A_n = A_{n,0} \supset A_{n,1} \supset A_{n,2} \supset \dots,$$

where  $A_{n,k}$  consists of the divisors  $\xi = 2\eta + \zeta$  with  $\deg(\eta) \geq k$ . Of course  $A_{n,k} = \emptyset$  if  $2k > n$ .

We have  $A_{n,k} - A_{n,k+1} = A_k \times C_{n-2k}$ .

The exact sequences corresponding to the cofibrations

$$(\mathbf{R}^q \times A_{n,k+1})^+ \rightarrow (\mathbf{R}^q \times A_{n,k})^+ \rightarrow (\mathbf{R}^q \times A_k \times C_{n-2k})^+$$

and

$$A_{n+1,k+1}^+ \rightarrow A_{n+1,k}^+ \rightarrow (A_k \times C_{n-2k+1})^+$$

show, by downwards induction on  $k$ , that if (\*)<sub>m</sub> holds for  $m < n$  then

(†)<sub>n,k</sub>:  $\mathbf{R}^q \times A_{n,k} \rightarrow A_{n+1,k}$  is a compact cohomology equivalence above dimension  $qk + q(n-2k+1) - d_{n-2k} = (n+1)q - d_n$

holds providing  $k > 0$ .

But in view of the diagram

$$\begin{array}{ccccc} (\mathbf{R}^q \times A_n)^+ & \longrightarrow & (\mathbf{R}^q \times C_n)^+ & \longrightarrow & S(\mathbf{R}^q \times A_{n,1})^+ \\ \uparrow & & \uparrow & & \uparrow \\ A_{n+1}^+ & \longrightarrow & C_{n+1}^+ & \longrightarrow & S(A_{n+1,1}^+) \end{array}$$

we find (†)<sub>n,1</sub>  $\Rightarrow$  (\*)<sub>n</sub> if the result corresponding to (A.1) holds for the symmetric products, i.e. if we prove

**PROPOSITION (A.2).**  $\mathbf{R}^q \times A_n \rightarrow A_{n+1}$  is a compact cohomology equivalence above dimension  $(n+1)q - d_n$ .

*Proof.* Here the assertion involves only the homotopy-type of  $M^+$ , for  $A_n(M)^+ = A_n(M^+)/A_{n-1}(M^+)$ . There is clearly no loss of generality in assuming that  $M$  is the interior of a compact manifold with boundary. In that case  $M^+$  can be obtained by attaching a

$q$ -cell to a compact space  $M_0^+$  of dimension  $q-1$ . ( $M_0$  is a closed set of  $M$ .) We shall introduce a new filtration of  $A_n = A_n(M)$ :

$$A_n = A_n^0 \supset A_n^1 \supset A_n^2 \supset \dots \supset A_n^{n+1} = \emptyset,$$

where  $A_n^k = \{\xi \in A_n : \deg(\xi \cap M_0) \geq k\}$ .

Then  $A_n^k - A_n^{k+1} = A_k(M_0) \times A_{n-k}(U)$ , where  $U = M - M_0 \cong \mathbf{R}^q$ .

The stabilization map  $\mathbf{R}^q \times A_n \rightarrow A_{n+1}$  takes  $\mathbf{R}^q \times A_n^k$  into  $A_{n+1}^k$ . If we assume

(\*)  $\mathbf{R}^q \times A_m(U) \rightarrow A_{m+1}(U)$  is a compact cohomology equivalence above dimension  $(m+1)q - d_m$ , for all  $m$

then  $\mathbf{R}^q \times (A_n^k - A_n^{k+1}) \rightarrow A_{n+1}^k - A_{n+1}^{k+1}$  will be a compact cohomology equivalence above dimension

$$k(q-1) + (n-k+1)q - d_{n-k} = (n+1)q - k - d_{n-k} \leq (n+1)q - d_n.$$

This will be true for all  $k$ , so the desired result (A.2) follows at once. It remains to justify the assumption (\*), i.e. to prove the following particular case of (A.2):

**PROPOSITION (A.3).**  $\mathbf{R}^q \times A_n(\mathbf{R}^q) \rightarrow A_{n+1}(\mathbf{R}^q)$  is a compact cohomology equivalence above dimension  $(n+1)q - d_n$ , when  $q$  is even.

*Proof.* I do not know a "geometrical" proof of this, but as the cohomology  $H_{\text{cpt}}^*(A_n(\mathbf{R}^q))$  has been calculated by Nakaoka [13] the result can be checked directly. Nakaoka's theorem seems to me very beautiful, so I shall explain it briefly.

We write  $A_n = A_n(\mathbf{R}^q)$ . Notice that  $A_n = A_n(S^q) - A_{n-1}(S^q)$ . Now the infinite symmetric product  $A_\infty(S^q)$  is an Eilenberg-MacLane space  $K(\mathbf{Z}, q)$ . Its filtration by the  $A_n(S^q)$  splits it homologically (by the argument given at the end of § 4), i.e.

$$H^*(K(\mathbf{Z}, q)) \cong \bigoplus_{n \geq 0} H_{\text{cpt}}^*(A_n)$$

canonically as rings. The multiplication on the right is given by the transfer  $H_{\text{cpt}}^*(A_n \times A_m) \rightarrow H_{\text{cpt}}^*(A_{n+m})$ .

The map of (A.3) can be identified with the multiplication  $H_{\text{cpt}}^m(A_n) \rightarrow H_{\text{cpt}}^{m+q}(A_{n+1})$  by the generator of  $H_{\text{cpt}}^q(A_1)$  in this ring. It is enough to prove the assertion with coefficients in every prime field  $\mathbf{F}_p$ . Let us take the case  $p=2$ . Then it is well-known that  $H^*(K(\mathbf{Z}, q))$  is a polynomial algebra on certain generators  $Sq^i Sq^j \dots Sq^{i_k} \varepsilon_q$ , where  $\varepsilon_q$  is the fundamental class, and

$$i_j < i_{j+1} + i_{j+2} + \dots + i_k + q$$

for each  $j$ . Nakaoka proves that this generator lies in  $H_{\text{cpt}}^*(A_n)$  where  $n = 2^k$ . In particular  $\varepsilon_q$  is the generator of  $H_{\text{cpt}}^*(A_1)$ , and is a polynomial generator. Furthermore the dimension of any polynomial generator lying in  $H_{\text{cpt}}^*(A_n)$ , if  $n = 2^k$ , is at most

$$q + (q-1) + 2(q-1) + \dots + 2^{k-1}(q-1) = n(q-1) + 1.$$



To prove (A.3) we must show that any element of  $H_{\text{cpt}}^m(A_{n+1})$  is divisible by  $\varepsilon_q$  if  $m \geq (n+1)q - d_n$ . But if a monomial in  $H_{\text{cpt}}^*(A_{n+1})$  is a product of generators coming from  $A_{n_1}, A_{n_2}, \dots, A_{n_r}$ , where  $n_1 + n_2 + \dots + n_r = n+1$ , then its dimension is at most

$$m = \sum (n_i(q-1) + 1) = (n+1)q - (n+1-r).$$

If  $m \geq (n+1)q - d_n$  then  $r \geq n+1 - d_n$ . But if  $\varepsilon_q$  is not a factor then all  $n_i$  are  $\geq 2$ , so  $r \leq [(n+1)/2]$ . Then  $d_n \geq n+1 - [(n+1)/2]$ , a contradiction.

The argument for the other primes is exactly similar.

The proof is now complete when  $M$  is even-dimensional and orientable. If it is even-dimensional but not orientable one does not need to make much change. Poincaré duality tells one that  $H_i(C_n) \cong H_{\text{cpt}}^{nq-i}(C_n; O_n)$ , where  $O_n$  is the orientation bundle of  $C_n$ . But  $O_n$  extends to a bundle on  $A_n$ , and if  $m \leq n$  the bundle  $O_m$  on  $A_m \subset A_n$  is the restriction of  $O_n$ . So one can carry through the whole preceding argument using twisted coefficients.

If  $M$  is odd-dimensional, however, the orientation bundle of  $C_n$  does not extend to  $A_n$ , and one needs a new device. It suffices to consider the cohomology with coefficients in  $\mathbb{F}_p$  where  $p$  is odd, for no questions of orientability or even-dimensionality arise if  $p=2$ . We let  $\tilde{A}_n$  denote the quotient of the product  $M^n$  by the alternating group on  $n$  letters, and let  $\tilde{C}_n$  denote the double covering of  $C_n$  contained in  $\tilde{A}_n$ . The spaces  $\tilde{A}_n$  and  $\tilde{C}_n$  have obvious involutions, and Poincaré duality asserts that  $H_*(C_n) \cong H_{\text{cpt}}^*(\tilde{C}_n)_{\text{odd}}$ , where the subscript "odd" indicates the  $-1$  eigenspace of the involution. We write  $\tilde{A}_{n,1} = \tilde{A}_n - \tilde{C}_n$ . There is an exact triangle

$$\dots \longrightarrow H_{\text{cpt}}^*(\tilde{C}_n)_{\text{odd}} \longrightarrow H_{\text{cpt}}^*(\tilde{A}_n)_{\text{odd}} \longrightarrow H_{\text{cpt}}^*(\tilde{A}_{n,1})_{\text{odd}} \longrightarrow \dots$$

But  $\tilde{A}_{n,1} = A_{n,1}$ , so the involution acts trivially on it, and  $H_{\text{cpt}}^*(\tilde{C}_n)_{\text{odd}} \cong H_{\text{cpt}}^*(\tilde{A}_n)_{\text{odd}}$ .

We reduce the stability of  $H_{\text{cpt}}^*(\tilde{A}_n(M))_{\text{odd}}$  to that of  $H_{\text{cpt}}^*(\tilde{A}_n(\mathbb{R}^q))_{\text{odd}}$  by writing  $M = M_0 \cup U$  as above. It is not hard to see that, in obvious notation,

$$H_{\text{cpt}}^*(\tilde{A}_n^k - \tilde{A}_n^{k+1})_{\text{odd}} \cong H_{\text{cpt}}^*(\tilde{A}_k(M_0))_{\text{odd}} \otimes H^*(\tilde{A}_{n-k}(U))_{\text{odd}}.$$

It remains to prove the analogue of (A.3). It turns out that  $\bigoplus_{n \geq 0} H_{\text{cpt}}^*(\tilde{A}_n(\mathbb{R}^q))_{\text{odd}}$ , although it can no longer be interpreted as the cohomology of a space, forms a ring very similar in structure to  $\bigoplus_{n \geq 0} H_{\text{cpt}}^*(A_n(\mathbb{R}^q))$  when  $q$  is even. In particular the generator  $\varepsilon_q$  of  $H_{\text{cpt}}^*(\mathbb{R}^q) = H_{\text{cpt}}^*(A_1(\mathbb{R}^q))_{\text{odd}}$  is, although odd-dimensional, a polynomial generator, and the other generators are obtained by applying "twisted" Steenrod operations to  $\varepsilon_q$ . So the argument runs as before. Of course in this case it actually determines the homology of the configuration spaces completely:

PROPOSITION (A.4). *If  $q$  is odd then the Pontrjagin ring  $\bigoplus_{n \geq 0} H_*(C_n(\mathbb{R}^q); \mathbb{F}_p)$  is the free anticommutative algebra on a family of generators  $\{Q_J e_0\}$ , where the multi-index  $J = (j_1, \dots, j_k)$  satisfies*

$$\begin{aligned} j_i &\equiv 0 \text{ or } -1 \text{ modulo } 2p-2, \\ j_i &\leq p j_{i+1}, \\ j_1 &> (p-1)(j_2 + \dots + j_k), \\ j_k &\leq (p-1)(n-1). \end{aligned}$$

The element  $Q_J e_0$  belongs to  $H_{|J|}(C_{p^k})$ , where  $|J| = j_1 + \dots + j_k$ . This result is well-known to experts: cf. [3].

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