SUBDOMINANT SOLUTIONS OF THE DIFFERENTIAL EQUATION

$$y'' - \lambda^2(x-a_1)(x-a_2)\dots(x-a_m)\,y = 0$$

BY

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I. Introduction

1. The solutions f_k and f_{-k}

In order to state the main results of this paper, we shall start with the differential equation

$$d^2y/dz^2 - P(z, c_1, c_2, \dots, c_m)\,y = 0, \tag{1.1}$$

where $c_1, c_2, \dots, c_m$ are complex parameters and

$$P(z, c) = (z - c_1)(z - c_2)\dots(z - c_m). \tag{1.2}$$

Let

$$\left\{\prod_{j=1}^{m}(1 - c_j/z)\right\}^{\frac{1}{2}} = 1 + \sum_{h=1}^{\infty} b_h(c)\,z^{-h}, \tag{1.3}$$

where $b_h(c)$ are homogeneous polynomials of $c_1, \dots, c_m$ of degree h respectively, and let us put

$$\mathcal{A}_m(z, c) = \begin{cases} z^{\frac{1}{2}m}\left\{1 + \sum\limits_{h=1}^{\frac{1}{2}(m+1)} b_h(c)\,z^{-h}\right\} & (m = \text{odd}), \\ z^{\frac{1}{2}m}\left\{1 + \sum\limits_{h=1}^{\frac{1}{2}m} b_h(c)\,z^{-h}\right\} + \dfrac{b_{1+\frac{1}{2}m}(c)}{1+z} & (m = \text{even}), \end{cases} \tag{1.4}$$

where

$$z^r = \exp\{r[\log|z| + i\arg z]\} \tag{1.5}$$

for any constant r. Previously, P. F. Hsieh and Y. Sibuya [7] constructed a unique solution

$$y = \mathcal{Y}_m(z, c) \tag{1.6}$$

of the differential equation (1.1) such that

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(i) *$\mathcal{Y}_m$ is an entire function of $(z, c_1, \ldots, c_m)$;*

(ii) *$\mathcal{Y}_m$ and $d\mathcal{Y}_m/dz$ admit respectively the asymptotic representations*

$$\left.\begin{aligned} \mathcal{Y}_m &= z^{-\frac{1}{4}m}\{1+O(z^{-\frac{1}{2}})\} \exp\left\{-\int_0^z \mathcal{A}_m(t,c)\,dt\right\}, \\ d\mathcal{Y}_m/dz &= z^{\frac{1}{4}m}\{-1+O(z^{-\frac{1}{2}})\} \exp\left\{-\int_0^z \mathcal{A}_m(t,c)\,dt\right\} \end{aligned}\right\} \tag{1.7}$$

uniformly on each compact set in the $(c_1, \ldots, c_m)$ space as z tends to infinity in any sector of the form

$$|\arg z| \leqslant \frac{3\pi}{(m+2)} - \delta_0, \tag{1.8}$$

where the path of integration should be taken in the sector (1.8) *and δ_0 is an arbitrary positive constant.*

The solution $\mathcal{Y}_m$ tends to zero as z tends to infinity in the sector $|\arg z| < \pi/(m+2)$. We call $\mathcal{Y}_m$ a subdominant solution of the differential equation in the sector $|\arg z| < \pi/(m+2)$.

Now let λ be a complex parameter, and $a_1, \ldots, a_m$ fixed real constants such that

$$-\infty < a_m < a_{m-1} < \ldots < a_2 < a_1 < +\infty. \tag{1.9}$$

Then let us put

$$\omega = \exp\,(2\pi i/(m+2)), \tag{1.10}$$

$$\varrho = \lambda^{2/(m+2)}, \qquad \arg \varrho = \frac{2}{m+2} \arg \lambda, \tag{1.11}$$

$$k = \begin{cases} 0, 1, 2, \ldots, (m+1)/2 & (m = \text{odd}), \\ 0, 1, 2, \ldots, m/2,\ 1 + m/2 & (m = \text{even}), \end{cases} \tag{1.12}$$

and

$$\left.\begin{aligned} f_k(x,\lambda) &= \mathcal{Y}_m(\varrho\omega^k x, \varrho\omega^k a_1, \varrho\omega^k a_2, \ldots, \varrho\omega^k a_m), \\ f_{-k}(x,\lambda) &= \mathcal{Y}_m(\varrho\omega^{-k} x, \varrho\omega^{-k} a). \end{aligned}\right\} \tag{1.13}$$

It is easily verified that *the functions $y = f_k(x,\lambda)$ and $y = f_{-k}(x,\lambda)$ are solutions of the differential equation*

(E) $$y'' - \lambda^2 P(x)\, y = 0,$$

where $'' = d^2/dx^2$ and

$$P(x) = P(x,a) = (x-a_1)(x-a_2)\ldots(x-a_m). \tag{1.14}$$

Furthermore, from the properties (i) and (ii) of the function $\mathcal{Y}_m$ we can derive the following properties of f_k and f_{-k}:

(i) *The functions* f_k *and* f_{-k} *are entire functions of* (x, ϱ);

(ii) f_k, f'_k, f_{-k} *and* f'_{-k} *admit the asymptotic representations*

$$\left.\begin{aligned} f_k &= (\varrho\omega^k)^{-\frac{1}{4}m}\, x^{-\frac{1}{4}m}\,\{1+O(x^{-\frac{1}{2}})\}\ \exp\left\{-\varrho\omega^k\int_0^x A_m(\varrho\omega^k t, \varrho\omega^k a)\,dt\right\},\\ f'_k &= (\varrho\omega^k)^{-\frac{1}{4}m}\,\lambda x^{\frac{1}{4}m}\,\{(-1)^{k+1}+O(x^{-\frac{1}{2}})\}\ \exp\left\{-\varrho\omega^k\int_0^x A_m(\varrho\omega^k t, \varrho\omega^k a)\,dt\right\},\end{aligned}\right\} \quad (1.15\text{-k})$$

uniformly on each compact set of the ϱ*-plane as* x *tends to infinity in the sector*

$$\left|\arg \varrho + \arg x + \frac{2\pi k}{(m+2)}\right| \leqslant \frac{3\pi}{(m+2)} - \delta_0, \quad (1.16\text{-k})$$

and

$$\left.\begin{aligned} f_{-k} &= (\varrho\omega^{-k})^{-\frac{1}{4}m}\, x^{-\frac{1}{4}m}\,\{1+O(x^{-\frac{1}{2}})\}\ \exp\left\{-\varrho\omega^{-k}\int_0^x A_m(\varrho\omega^{-k} t, \varrho\omega^{-k} a)\,dt\right\},\\ f'_{-k} &= (\varrho\omega^{-k})^{-\frac{1}{4}m}\,\lambda x^{\frac{1}{4}m}\,\{(-1)^{k+1}+O(x^{-\frac{1}{2}})\}\ \exp\left\{-\varrho\omega^{-k}\int_0^x A_m(\varrho\omega^{-k} t, \varrho\omega^{-k} a)\,dt\right\},\end{aligned}\right\} \quad (1.15\text{-(-k)})$$

uniformly on each compact set of the ϱ*-plane as* x *tends to infinity in the sector*

$$\left|\arg \varrho + \arg x - \frac{2\pi k}{(m+2)}\right| \leqslant \frac{3\pi}{(m+2)} - \delta_0. \quad (1.16\text{-(-k)})$$

The paths of integrations in the right-hand members of (1.15) should be taken in the sectors (1.16) respectively.

Let S_k be the sector in the x-plane which is defined by

$$\left|\arg x + \frac{2\pi k}{(m+2)}\right| < \frac{\pi}{(m+2)} \quad (1.17\text{-k})$$

and let S_{-k} be the sector in the x-plane which is defined by

$$\left|\arg x - \frac{2\pi k}{(m+2)}\right| < \frac{\pi}{(m+2)}. \quad (1.17\text{-(-k)})$$

If $\varrho>0$, f_k tends to zero as x tends to infinity in S_k, while f_{-k} tends to zero as x tends to infinity in S_{-k}. Hence we call f_k and f_{-k} subdominant solutions in S_k and S_{-k} respectively.

We shall denote by I_h the intervals

$$a_{h+1} < x < a_h \quad (h = 1, 2, \dots, m-1) \tag{1.18}$$

on the real axis of the x-plane respectively. The intervals $-\infty < x < a_m$ and $a_1 < x < +\infty$ will be denoted by I_m and I_0 respectively. Furthermore, I_{m+1} denotes the empty set. Then *one of our main concerns in this paper will be to evaluate* f_k, f'_k, f_{-k} *and* f'_{-k} *when* $x \in I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$ *as* λ *tends to* ∞ *in a sector* $|\arg \lambda| \leq \theta_0$, *where*

$$k = \begin{cases} 1, 2, \dots, (m+1)/2 & (m = odd), \\ 1, 2, \dots, m/2 & (m = even). \end{cases} \tag{1.19}$$

By using these results we shall also compute *large positive eigenvalues* λ *of the boundary-value problem*

$$\text{(P)} \qquad y'' - \lambda^2 P(x)\, y = 0, \qquad \int_{-\infty}^{+\infty} |y(x)|^2\, dx < +\infty,$$

when m *is an even integer.*

2. The associated Riccati equation

We denote by Ω_0 the domain obtained from the complex x-plane by deleting the line-segments $a_{2j} < x < a_{2j-1}$ on the real axis, where

$$j = \begin{cases} 1, 2, \dots, (m+1)/2 & (m = \text{odd}), \\ 1, 2, \dots, m/2 & (m = \text{even}) \end{cases} \tag{2.1}$$

and $a_{m+1} = -\infty$. Let $\mathcal{A}(x)$ be a branch of $P(x)^{\frac{1}{2}}$ such that

$$\mathcal{A}(x) > 0 \tag{2.2}$$

for large positive values of x. It is easily verified that $\mathcal{A}(x)$ is single-valued and holomorphic in the interior of Ω_0.

By the transformation $w = y^{-1} dy/dx$ we derive the Riccati equation $dw/dx + w^2 = \lambda^2 P(x)$ from the equation (E) of Section 1. In the domain Ω_0 we can write this Riccati equation in the form

$$w' + w^2 = (\lambda \mathcal{A}(x))^2. \tag{2.3}$$

In order to evaluate the quantities f_k, f_{-k}, f'_k and f'_{-k} when $x \in I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$ as λ tends to ∞, we shall study the Riccati equation (2.3).

First of all, we shall prove the following lemma.

LEMMA 1. *The Riccati equation* (2.3) *has two formal solutions*

$$w=\lambda A(x)-\tfrac{1}{4}R(x)+p(x,\lambda) \tag{2.4}$$

and

$$w=-\lambda A(x)-\tfrac{1}{4}R(x)+p(x,-\lambda), \tag{2.5}$$

where

$$R(x)=P'(x)/P(x), \tag{2.6}$$

$$p(x,\lambda)=\sum_{n=1}^{\infty}(\lambda A(x))^{-n}p_n(x), \tag{2.7}$$

$$p_n(x)=Q_n(x)P(x)^{-n-1}, \tag{2.8}$$

and the quantity Q_n is a polynomial in x of degree not greater than $(m-1)(n+1)$.

In fact, if we define p_n by

$$p_1(x)=\tfrac{1}{2}\{\tfrac{1}{4}R'(x)-\tfrac{1}{16}R(x)^2\} \tag{2.9}$$

and

$$p_n(x)=\tfrac{1}{2}\{\tfrac{1}{2}nR(x)p_{n-1}(x)-p'_{n-1}(x)-\sum_{j+h=n-1}p_j(x)p_h(x)\}\quad(n>1), \tag{2.10}$$

the two formal series (2.4) and (2.5) satisfy the equation (2.3), since $A'(x)=\tfrac{1}{2}A(x)R(x)$.

Secondly, we shall prove the following lemma.

LEMMA 2. *For each fixed point ξ in $I_{2k}\cup I_{2k-1}\cup I_{2(k-1)}$, where*

$$k=\begin{cases}1,2,\ldots,(m+1)/2 & (m=odd),\\ 1,2,\ldots,m/2 & (m=even),\end{cases} \tag{2.11}$$

there is a curve

$$C_{k,1}(\xi):\quad x=z_k(s;\xi)\quad(0\leqslant s<+\infty) \tag{2.12}$$

and a positive constant $\theta_0(\xi)$ such that

(i) *$z_k(s;\xi)$ is continuous in s for $0\leqslant s<+\infty$;*

(ii) *$\dot{z}_k(s;\xi)$ is bounded and piecewise continuous in s for $0\leqslant s<+\infty$, where denotes d/ds;*

(iii) *$z_k(0;\xi)=\xi$;*

(iv) *$\lim_{s\to+\infty}z_k(s;\xi)=\infty$, $\lim_{s\to+\infty}\arg z_k(s;\xi)=2\pi k/(m+2)$;*

(v) *$\mathrm{Im}\, z_k(s;\xi)>0$ for $0<s<+\infty$;*

(vi) *the quantity $(-1)^k\,\mathrm{Re}\left[e^{i\theta}\int_0^t A(z_k(s;\xi))\dot{z}_k(s;\xi)\,ds\right]$ is nondecreasing for $0\leqslant t<+\infty$ if $|\theta|\leqslant\theta_0(\xi)$.*

This lemma will be proved in Sections 5, 6 of Chapter II. In the proof we shall use the idea that was suggested in the paper of M. A. Evgrafov and M. V. Fedorjuk [3; p. 9]. We shall use some properties of the trajectories of the autonomous system

$$dx/dt = i\mathcal{A}(\bar{x}), \tag{2.13}$$

where t is a real independent variable and $^{-}$ denotes the complex conjugate. If $x = x(t)$ is a trajectory of (2.13), then we have

$$\frac{d}{dt}\int_0^t \mathcal{A}(x(s))\,\dot{x}(s)\,ds = \mathcal{A}(x(t))\,\dot{x}(t) = i\mathcal{A}(x(t))\,\mathcal{A}(\overline{x(t)})$$
$$= i\mathcal{A}(x(t))\,\overline{\mathcal{A}(x(t)}) = i\,|\mathcal{A}(x(t))|^2.$$

Hence the real part of the holomorphic function

$$\int^x \mathcal{A}(\tau)\,d\tau$$

is constant along the trajectory $x = x(t)$ of (2.13). This is the reason why we shall study the system (2.13).

As a corollary of Lemma 2, we obtain the following lemma.

LEMMA 3. *Put*

$$\zeta_k(s;\xi) = \overline{z_k(s;\xi)} \quad (0 \leqslant s < +\infty). \tag{2.14}$$

Then the curve $\mathcal{C}_{k,2}(\xi):\quad x = \zeta_k(s;\xi)$

satisfies the conditions

(i) $\zeta_k(s;\xi)$ *is continuous in* s *for* $0 \leqslant s < +\infty$;

(ii) $\dot{\zeta}_k(s;\xi)$ *is bounded and piecewise continuous in* s *for* $0 \leqslant s < +\infty$;

(iii) $\zeta_k(0;\xi) = \xi$;

(iv) $\lim_{s\to+\infty} \zeta_k(s;\xi) = \infty$, $\lim_{s\to+\infty} \arg \zeta_k(s;\xi) = -2\pi k/(m+2)$;

(v) $\operatorname{Im} \zeta_k(s;\xi) < 0$ *for* $0 < s < +\infty$;

(vi) *the quantity* $(-1)^k \operatorname{Re}\left[e^{i\theta}\int_0^t \mathcal{A}(\zeta_k(s;\xi))\,\dot{\zeta}_k(s;\xi)\,ds\right]$ *is nondecreasing for* $0 \leqslant t < +\infty$ *if* $|\theta| \leqslant \theta_0(\xi)$.

Since ξ is real, the conditions (i), (ii), (iii), (iv) and (v) of Lemma 3 can be easily derived from the corresponding conditions of Lemma 2. In order to derive the condition (vi) of Lemma 3 from that of Lemma 2, we use the identity $\mathcal{A}(\bar{x}) = \overline{\mathcal{A}(x)}$.

Finally we shall prove the following lemma.

LEMMA 4. *For each ξ in $I_{2k}\cup I_{2k-1}\cup I_{2(k-1)}$, where*

$$k=\begin{cases}1,2,\ldots,(m+1)/2 & (m=odd),\\ 1,2,\ldots,m/2 & (m=even),\end{cases}\tag{2.15}$$

there exists a function $w_{k,1}(s,\lambda;\xi)$ such that

(i) *$w_{k,1}$ is defined and continuous in (s,λ) and holomorphic in λ for*

$$|\lambda|\geqslant M_0(\xi),\quad |\arg\lambda|\leqslant\theta_0(\xi),\quad 0\leqslant s<+\infty,\tag{2.16}$$

where M_0 is a sufficiently large positive constant depending on ξ;

(ii) *$\dot{w}_{k,1}$ is piecewise continuous in s for $0\leqslant s<+\infty$, where $\dot{}$ denotes d/ds;*

(iii) *$w_{k,1}$ satisfies the differential equation*

$$\dot{w}_{k,1}=\dot{z}_k(s;\xi)\{[\lambda\mathcal{A}(z_k(s;\xi))]^2-w_{k,1}^2\}\tag{2.17}$$

for (2.16) except at the points of discontinuity of $\dot{w}_{k,1}$ in s;

(iv) *$w_{k,1}$ satisfies the inequalities*

$$\Big|w_{k,1}(s,\lambda;\xi)-(-1)^{k+1}\lambda\mathcal{A}(z_k)+\tfrac{1}{4}R(z_k)-\sum_{n=1}^{N}[(-1)^{k+1}\lambda\mathcal{A}(z_k)]^{-n}p_n(z_k)\Big|$$
$$\leqslant K_N(\xi)|\lambda\mathcal{A}(z_k)|^{-N-1}\quad(N=1,2,\ldots)\tag{2.18}$$

for (2.16), where $z_k=z_k(s;\xi)$ and K_N are positive constants depending on ξ.

This lemma will be proved in Section 7 of Chapter II. From Lemma 4 we can derive the following lemma.

LEMMA 5. *Put*

$$w_{k,2}(s,\lambda;\xi)=\overline{w_{k,1}(s,\bar{\lambda};\xi)}.\tag{2.19}$$

Then the function $w_{k,2}$ satisfies the conditions

(i) *$w_{k,2}$ is defined and continuous in (s,λ) and holomorphic in λ for (2.16);*

(ii) *$\dot{w}_{k,2}$ is piecewise continuous in s for $0\leqslant s<+\infty$;*

(iii) *$w_{k,2}$ satisfies the differential equation*

$$\dot{w}_{k,2}=\dot{\zeta}_k(s;\xi)\{[\lambda\mathcal{A}(\zeta_k(s;\xi))]^2-w_{k,2}^2\}\tag{2.20}$$

for (2.16) except at the points of discontinuity of $\dot{w}_{k,2}$ in s;

(iv) $w_{k,2}$ *satisfies the inequalities*

$$|w_{k,2}(s,\lambda;\xi)-(-1)^{k+1}\lambda\mathcal{A}(\zeta_k)+\tfrac{1}{4}R(\zeta_k)-\sum_{n=1}^{N}[(-1)^{k+1}\lambda\mathcal{A}(\zeta_k)]^{-n}p_n(\zeta_k)|$$
$$\leqslant K_N(\xi)\,|\lambda\mathcal{A}(\zeta_k)|^{-N-1}\quad (N=1,2,\dots) \tag{2.21}$$

for (2.16), *where* $\zeta_k=\zeta_k(s;\xi)$.

It is easily seen that the quantities $\dot{w}_{k,1}$ and $\dot{w}_{k,2}$ have a discontinuity in s only at the points of discontinuity of $\dot{z}_k$ and $\dot{\zeta}_k$ respectively. The differential equations (2.17) and (2.20) are derived from the Riccati equation (2.3) by the changes of the variable $x=z_k(s;\xi)$ and $x=\zeta_k(s;\xi)$ respectively. The inequalities (2.18) and (2.21) give asymptotic expansions of $w_{k,1}$ and $w_{k,2}$ as $\lambda\mathcal{A}(z_k)$ and $\lambda\mathcal{A}(\zeta_k)$ tend to infinity for each fixed ξ. We have

$$w_{k,1}(s,\lambda;\xi)\cong(-1)^{k+1}\lambda\mathcal{A}(z_k)-\tfrac{1}{4}R(z_k)+\sum_{n=1}^{\infty}[(-1)^{k+1}\lambda\mathcal{A}(z_k)]^{-n}p_n(z_k) \tag{2.22}$$

as $\lambda\mathcal{A}(z_k)$ tends to infinity, and

$$w_{k,2}(s,\lambda;\xi)\cong(-1)^{k+1}\lambda\mathcal{A}(\zeta_k)-\tfrac{1}{4}R(\zeta_k)+\sum_{n=1}^{\infty}[(-1)^{k+1}\lambda\mathcal{A}(\zeta_k)]^{-n}p_n(\zeta_k) \tag{2.23}$$

as $\lambda\mathcal{A}(\zeta_k)$ tends to infinity. The estimates (2.18) and (2.21) depend on ξ and they are not uniform over any interval which contains the points a_{2k-1} or a_{2k}. The quantities $\lambda\mathcal{A}(z_k)$ and $\lambda\mathcal{A}(\zeta_k)$ tend to infinity either if λ tends to infinity or if s tends to infinity.

3. Main results

Let us denote by $\mathcal{B}(x)$ a branch of $P(x)^{\frac{1}{4}}$ such that

$$\mathcal{B}(x)>0 \tag{3.1}$$

for large positive values of x. For real τ, we shall put

$$\mathcal{A}(\tau-)=\lim_{t\to 0+}\mathcal{A}(\tau-it),\qquad \mathcal{A}(\tau+)=\lim_{t\to 0+}\mathcal{A}(\tau+it) \tag{3.2}$$

and
$$\mathcal{B}(\tau-)=\lim_{t\to 0+}\mathcal{B}(\tau-it),\qquad \mathcal{B}(\tau+)=\lim_{t\to 0+}\mathcal{B}(\tau+it). \tag{3.3}$$

Let
$$Q_{k,1}(s,\lambda;\xi)=w_{k,1}(s,\lambda;\xi)-(-1)^{k+1}\lambda\mathcal{A}(z_k)+\tfrac{1}{4}R(z_k) \tag{3.4}$$

and
$$Q_{k,2}(s,\lambda;\xi)=w_{k,2}(s,\lambda;\xi)-(-1)^{k+1}\lambda\mathcal{A}(\zeta_k)+\tfrac{1}{4}R(\zeta_k), \tag{3.5}$$

where $z_k=z_k(s;\xi)$ and $\zeta_k=\zeta_k(s;\xi)$. Finally, put

$$\alpha_k = (-1)^k b_{1+m/2}(a). \tag{3.6}$$

We remark that the function $\mathcal{B}(x)$ is single-valued and holomorphic in the x-plane cut along the interval $-\infty < x < a_1$ on the real axis. In (3.3), we use this holomorphic function $\mathcal{B}(x)$.

Our main result of this paper is the following theorem:

THEOREM 1. *For each fixed point ξ in $I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$, where*

$$k = \begin{cases} 1, 2, \ldots, (m+1)/2 & (m = odd), \\ 1, 2, \ldots, m/2 & (m = even), \end{cases} \tag{2.11}$$

we have

$$\left.\begin{aligned} f_k(\xi,\lambda) &= C_k(\lambda)\,\mathcal{B}(\xi-)^{-1} \exp\left[(-1)^{k+1}\lambda\int_0^{\xi} \mathcal{A}(\tau-)\,d\tau - \int_0^{+\infty} Q_{k,2}(s,\lambda;\xi)\,\zeta_k(s;\xi)\,ds\right] \\ f_{-k}(\xi,\lambda) &= C_{-k}(\lambda)\,\mathcal{B}(\xi+)^{-1} \exp\left[(-1)^{k+1}\lambda\int_0^{\xi} \mathcal{A}(\tau+)\,d\tau - \int_0^{+\infty} Q_{k,1}(s,\lambda;\xi)\,\dot z_k(s;\xi)\,ds\right], \\ f_k'(\xi,\lambda) &= \{(-1)^{k+1}\lambda\mathcal{A}(\xi-) - \tfrac{1}{4}R(\xi) + Q_{k,2}(0,\lambda;\xi)\}\,f_k(\xi,\lambda), \\ f_{-k}'(\xi,\lambda) &= \{(-1)^{k+1}\lambda\mathcal{A}(\xi+) - \tfrac{1}{4}R(\xi) + Q_{k,1}(0,\lambda;\xi)\}\,f_{-k}(\xi,\lambda), \end{aligned}\right\} \tag{3.7}$$

where

$$C_k(\lambda) = \begin{cases} (\varrho\omega^k)^{-\frac{1}{4}m} \exp\left[(-1)^{k+1}\lambda\displaystyle\int_0^{+\infty}\{A_m(\tau,a) - \mathcal{A}(\tau-)\}\,d\tau\right] & (m = odd) \\ (\varrho\omega^k)^{-\frac{1}{4}m-\lambda\alpha_k} \exp\left[(-1)^{k+1}\lambda\displaystyle\int_0^{+\infty}\{A_m(\tau,a) - \mathcal{A}(\tau-)\}\,d\tau\right] & (m = even), \end{cases} \tag{3.8}$$

$$C_{-k}(\lambda) = \begin{cases} (\varrho\omega^{-k})^{-\frac{1}{4}m} \exp\left[(-1)^{k+1}\lambda\displaystyle\int_0^{+\infty}\{A_m(\tau,a) - \mathcal{A}(\tau+)\}\,d\tau\right] & (m = odd) \\ (\varrho\omega^{-k})^{-\frac{1}{4}m-\lambda\alpha_k} \exp\left[(-1)^{k+1}\lambda\displaystyle\int_0^{+\infty}\{A_m(\tau,a) - \mathcal{A}(\tau+)\}\,d\tau\right] & (m = even), \end{cases} \tag{3.9}$$

and s and τ are real variables, and

$$|\lambda| \geqslant M_0(\xi), \qquad |\arg\lambda| \leqslant \theta_0(\xi). \tag{3.10}$$

Remark. In order to derive asymptotic representations for f_k, f_{-k}, f_k' and f_{-k}' as λ tends to infinity in the domain (3.10), use the definitions (3.4) and (3.5) for $Q_{k,1}$ and $Q_{k,2}$ respectively, and use the asymptotic expansions (2.22) and (2.23) for $w_{k,1}$ and $w_{k,2}$ respectively. It was already mentioned that these expansions are not uniform over any interval which contains a_{2k-1} or a_{2k}. Hence the corresponding asymptotic representations for f_k, f_{-k}, f'_k and f'_{-k} are also not uniform over any interval containing a_{2k} or a_{2k-1}.

This theorem will be proved in Section 8 of Chapter II.

4. Computation of eigenvalues of Problem (*P*)

Notice that, since $\omega^{m+2}=1$, the two solutions $f_{1+m/2}$ and $f_{-1-m/2}$ are identical. Furthermore, the solution f_0 is subdominant along the positive real axis, while the solution $f_{1+m/2}$ is subdominant along the negative real axis. Therefore, eigenvalues of Problem (P), when m is even, are zeros of the Wronskian of f_0 and $f_{1+m/2}$:

$$W(\lambda)=\begin{vmatrix} f_0(x,\lambda) & f_{1+m/2}(x,\lambda) \\ f'_0(x,\lambda) & f'_{1+m/2}(x,\lambda) \end{vmatrix}. \tag{4.1}$$

We may compute large positive zeros of $W(\lambda)$ by using asymptotic evaluations of $f_0, f'_0, f_{1+m/2}$ and $f'_{1+m/2}$ as λ tends to $+\infty$. However, a straightforward application of the theory of asymptotic solutions of ordinary differential equations with respect to parameters would yield asymptotic evaluations of $f_0, f'_0, f_{1+m/2}$ and $f'_{1+m/2}$ in different x-intervals that are disjoint. Hence such results cannot be used for the computation of zeros of $W(\lambda)$. This difficulty is due to the fact that the coefficient $P(x)$ of (E) has zeros $a_1, \dots, a_m$, These zeros of the coefficient $P(x)$ of (E) are called transition points of (E). For a general discussion and the history of the study of the problem of transition points see R. E. Langer [9], J. Heading [6] and W. Wasow [11].

In order to overcome such a difficulty, we need a more elaborate analysis. While in case $m=2$ the problem is reduced to the study of the parabolic cylinder function, there is no such well-known special function that is helpful in case $m>2$. For general discussion of the problem of two transition points see A. Erdélyi, M. Kennedy, and J. L. McGregor [2], R. E. Langer [10], and N. D. Kazarinoff [8]. In this paper, the function $\mathcal{Y}_m$ plays a role similar to that of the parabolic cylinder function in case $m=2$. Actually the functions f_k and f_{-k} were derived directly from the function $\mathcal{Y}_m$. We shall explain briefly how to evaluate $W(\lambda)$ asymptotically as λ tends to $+\infty$. The proof will be given in detail later on in Chapter III. Let us put

$$\Delta_k(\lambda)=\begin{vmatrix} f_k & f_{-k} \\ f'_k & f'_{-k} \end{vmatrix} \quad (k=1,2,\dots,m/2), \tag{4.2}$$

$$D_{k,1}(\lambda)=\begin{vmatrix} f_k & f_{k+1} \\ f'_k & f'_{k+1} \end{vmatrix} \quad (k=0,1,\dots,m/2), \tag{4.3}$$

$$D_{k,2}(\lambda) = \begin{vmatrix} f_{-k} & f_{-k-1} \\ f'_{-k} & f'_{-k-1} \end{vmatrix} \quad (k = 0, 1, \dots, m/2), \tag{4.4}$$

$$\Delta_{k,1}(\lambda) = \begin{vmatrix} f_{-k} & f_{k+1} \\ f'_{-k} & f'_{k+1} \end{vmatrix} \quad (k = 1, 2, \dots, -1 + m/2), \tag{4.5}$$

and

$$\Delta_{k,2}(\lambda) = \begin{vmatrix} f_{k} & f_{-k-1} \\ f'_{k} & f'_{-k-1} \end{vmatrix} \quad (k = 1, 2, \dots, -1 + m/2). \tag{4.6}$$

First of all by using Theorem 1, we shall evaluate $\Delta_k(\lambda)$ at a fixed point of the interval I_{2k-1} for large positive values of λ. This will show that f_k and f_{-k} are linearly independent if λ is positive and large. Let $\Phi_k(x, \lambda)$ be the matrix of these two independent solutions, i.e.

$$\Phi_k(x, \lambda) = \begin{pmatrix} f_k(x, \lambda) & f_{-k}(x, \lambda) \\ f'_k(x, \lambda) & f'_{-k}(x, \lambda) \end{pmatrix}. \tag{4.7}$$

Then

$$\Phi_k(x, \lambda)^{-1} = \Delta_k(\lambda)^{-1} \begin{pmatrix} f'_{-k} & -f_{-k} \\ -f'_k & f_k \end{pmatrix}. \tag{4.8}$$

Hence if we put

$$H_k(\lambda) = \begin{pmatrix} \Delta_{k,2}(\lambda) & D_{k,2}(\lambda) \\ -D_{k,1}(\lambda) & -\Delta_{k,1}(\lambda) \end{pmatrix} \quad (k = 1, \dots, -1 + m/2), \tag{4.9}$$

we get

$$\Phi_k(x, \lambda) = \Delta_{k+1}(\lambda)^{-1} \Phi_{k+1}(x, \lambda) H_k(\lambda), \tag{4.10}$$

and

$$\Phi_1(x, \lambda) = \{\Delta_2(\lambda) \Delta_3(\lambda) \dots \Delta_{m/2}(\lambda)\}^{-1} \Phi_{m/2}(x, \lambda) H(\lambda), \tag{4.11}$$

where

$$H(\lambda) = H_{-1+m/2}(\lambda) \dots H_2(\lambda) H_1(\lambda). \tag{4.12}$$

On the other hand, we have

$$\begin{pmatrix} f_0(x, \lambda) \\ f'_0(x, \lambda) \end{pmatrix} = \Delta_1(\lambda)^{-1} \Phi_1(x, \lambda) \begin{pmatrix} D_{0,2}(\lambda) \\ -D_{0,1}(\lambda) \end{pmatrix} \tag{4.13}$$

and

$$\begin{pmatrix} f_{1+m/2}(x, \lambda) \\ f'_{1+m/2}(x, \lambda) \end{pmatrix} = \Delta_{m/2}(\lambda)^{-1} \Phi_{m/2}(x, \lambda) \begin{pmatrix} -D_{\frac{1}{2}m,2}(\lambda) \\ D_{\frac{1}{2}m,1}(\lambda) \end{pmatrix}. \tag{4.14}$$

Hence we get

$$W(\lambda) = \{\Delta_1(\lambda) \Delta_2(\lambda) \dots \Delta_{\frac{1}{2}m}(\lambda)\}^{-1} [D_{\frac{1}{2}m,1}(\lambda), D_{\frac{1}{2}m,2}(\lambda)] H(\lambda) \begin{pmatrix} D_{0,2}(\lambda) \\ -D_{0,1}(\lambda) \end{pmatrix}. \tag{4.15}$$

Therefore, large positive eigenvalues of Problem (P) are zeros of

$$[D_{\frac{1}{2}m,1}(\lambda), D_{\frac{1}{2}m,2}(\lambda)]\, H(\lambda) \begin{pmatrix} D_{0,2}(\lambda) \\ -D_{0,1}(\lambda) \end{pmatrix}. \tag{4.16}$$

We shall evaluate the quantities $D_{k,1}$ and $D_{k,2}$ at $x=\infty$ by using asymptotic representations (1.15) of f_k, f_{-k} and their derivatives. On the other hand, by using Theorem 1, we shall evaluate the quantities $\Delta_{k,1}$ and $\Delta_{k,2}$ at a fixed point of I_{2k}. In this manner, we shall prove the following theorem.

THEOREM 2. *Let*

$$\lambda_k(\eta) = \frac{(\eta+\frac{1}{2})\pi}{\int_{a_{2k}}^{a_{2k-1}} \sqrt{|P(\tau)|}\, d\tau} + \sum_{n=1}^{\infty} \mathcal{E}_{k,n}\eta^{-n} \quad (k=1,2,\dots,m/2)$$

be formal power series in η^{-1} with constant coefficients $\mathcal{E}_{k,n}$ which satisfy the formal equations

$$\lambda_k \int_{a_{2k}}^{a_{2k-1}} \sqrt{|P(\tau)|}\, d\tau = (\eta+\tfrac{1}{2})\pi - (i/2)\sum_{n=1}^{\infty} \lambda_k^{-2n+1} \int_{\gamma_k} [(-1)^{k+1} A(x)]^{-2n+1} p_{2n-1}(x)\, dx \tag{4.17}$$

$$(k=1,2,\dots,m/2),$$

where γ_k is a circle which encircles only a_{2k} and a_{2k-1}, and the integration must be taken in the counterclockwise sense. Then there is a positive integer ν_0 such that we can denote almost all positive eigenvalues of the boundary-value problem (P) by $\lambda_{k,\nu}$ ($k=1,2,\dots,m/2$; $\nu=1+\nu_0, 2+\nu_0,\dots$) so that we have

$$\lambda_{k,\nu} = \frac{(\nu+\frac{1}{2})\pi}{\int_{a_{2k}}^{a_{2k-1}} \sqrt{|P(\tau)|}\, d\tau} + \sum_{n=1}^{N} \mathcal{E}_{k,n}\nu^{-n} + O(\nu^{-N-1})$$

as ν tends to $+\infty$, where N may be any positive integer.

Remark. Previously M. V. Fedorjuk [4] claimed that the eigenvalues $\lambda_{k\nu}$ are determined asymptotically as ν tends to $+\infty$ by the formal equations

$$\lambda_{k,\nu} \int_{a_{2k}}^{a_{2k-1}} \sqrt{|P(\tau)|}\, d\tau = (\nu+\tfrac{1}{2})\pi - \frac{i}{2}\sum_{n=1}^{\infty} \lambda_{k,\nu}^{-n} \int_{\gamma_k} [(-1)^{k+1} A(x)]^{-n} p_n(x)\, ds. \tag{4.18}$$

However, we felt that he gave only a sketch of the proof. Furthermore, since then, M. A. Evgrafov and M. V. Fedorjuk [3] explained again exactly the same method, and they gave only the first approximations of $\lambda_{k,\nu}$, i.e.

$$\lambda_{k,\nu}=\frac{(\nu+\frac{1}{2})\pi}{\int_{a_{2k}}^{a_{2k-1}}\sqrt{|P(\tau)|}\,d\tau}+O\left(\frac{1}{\nu}\right). \tag{4.19}$$

In addition, they remarked that we need a more elaborate analysis to obtain higher order approximations. Therefore, we felt that it might be worth-while to present another proof for the results of M. V. Fedorjuk. By using an idea due to N. Fröman [5] we can prove that

$$\int_{\gamma_k}\mathcal{A}(x)^{-2n}p_{2n}(x)\,dx=0 \quad (n=1,2,\dots). \tag{4.20}$$

Hence Theorem 2 gives us the same results as those of M. V. Fedorjuk. We shall give a proof of (4.20) in Section 10 of Chapter III. The analysis of M. V. Fedorjuk is based on the computation of the so-called Stokes multipliers around a transition point after another. Our analysis is based on the computation of the Stokes multipliers around a pair of transition points a_{2k-1} and a_{2k} after another pair.

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II. Proof of Theorem 1

5. Proof of Lemma 2 (Part I)

We shall prove Lemma 2 of Section 2 in Sections 5 and 6. First of all, we shall construct some particular trajectories of the autonomous system (2.13)

$$dx/dt=i\mathcal{A}(\bar{x}),$$

where t is a real independent variable and $^{-}$ denotes the complex conjugate, and prove the following lemma:

LEMMA 6. *For each* $h=1,2,\dots,m$ *the system* (2.13) *has a unique trajectory* $\mathcal{T}_h$

$$x=x_h(t) \tag{5.1}$$

such that $x_h(t)$ *is defined for* $t_h<t<0$ *if* $h=2$ *or* $3 \bmod 4$ *and* $x_h(t)$ *is defined for* $0<t<t_h$ *if* $h=0$ *or* $1 \bmod 4$, *where* $|t_h|=+\infty$ *if* $m=1,2$, *and* $|t_h|<+\infty$ *otherwise, and such that*

(i) $\lim_{t\to 0} x_h(t)=a_h$;

(ii) $\operatorname{Im} x_h(t)>0$ *for* $t_h<t<0$ *cr* $0<t<t_h$*;*

(iii) $\lim_{t\to t_h} |x_h(t)| = +\infty$*;*

(iv) *there is an integer* k *such that*

$$\lim_{t\to t_h} \arg x_h(t) = \frac{(2k-1)\pi}{(m+2)},$$

and that

(a) $1\leqslant k\leqslant (m+3)/2$ *(m = odd),* $1\leqslant k\leqslant 1+m/2$ *(m = even);*

(b) k *is even if* $h=2$ *or* $3 \bmod 4$*;*
k *is odd if* $h=0$ *or* $1 \bmod 4$.

In Lemma 6, the integers k are not exactly specified. If this lemma is proved, we can determine the exact values of the integers k by using the fact that the trajectories $\mathcal{J}_h$ cannot intersect each other. We can prove the following lemma.

LEMMA 7. *The trajectories* $\mathcal{J}_h$ *satisfy the conditions*

$$\lim_{t\to t_h} \arg x_h(t) = h\pi/(m+2) \quad \text{if} \quad h \text{ is odd,} \tag{5.2}$$

and

$$\lim_{t\to t_h} \arg x_h(t) = (h+1)\pi/(m+2) \quad \text{if} \quad h \text{ is even.} \tag{5.3}$$

Now we shall prove Lemma 6. Let

$$\mathcal{D}_{2j-1} = \{x;\, |x-a_{2j-1}|<\delta,\quad |\arg(x-a_{2j-1})|<\pi\},$$
$$\mathcal{D}_{2j} = \{x;\, |x-a_{2j}|<\delta,\quad |\arg(x-a_{2j})-\pi|<\pi\},$$

where δ is a small positive number. Then $\mathcal{D}_h$ are contained in Ω_0 if δ is sufficiently small. Hence the functions

$$F_h(x) = \int_{a_h}^{x} \mathcal{A}(\tau)\, d\tau \qquad (h=1,2,\dots,m) \tag{5.4}$$

are holomorphic in $\mathcal{D}_h$ respectively. Let $x=x(t)$ $(t_0\leqslant t\leqslant t_1)$ be a trajectory of (2.13) which is contained in $\mathcal{D}_h$ for some h. Then

$$dF_h(x(t))/dt = \mathcal{A}(x(t)) \quad dx(t)/dt = i\,|\mathcal{A}(x(t))|^2.$$

Hence $\operatorname{Re} F_h(x(t))$ is constant. On the other hand, if a curve $x=x(\tau)$ $(\tau_0\leqslant\tau\leqslant\tau_1)$ satisfies the conditions:

(i) $x(\tau) \in \mathcal{D}_h$ for some h;

(ii) $dx(\tau)/d\tau \neq 0$ for $\tau_0 \leqslant \tau \leqslant \tau_1$;

(iii) $\operatorname{Re} F_h(x(\tau)) \equiv$ constant for $\tau_0 \leqslant \tau \leqslant \tau_1$;

then this curve is a trajectory of (2.13). In fact

$$\alpha(\tau) = -i\, dF_h(x(\tau))/d\tau$$

is real and different from zero for $\tau_0 \leqslant \tau \leqslant \tau_1$. Let

$$t = \int_{\tau_0}^{\tau} \alpha(s)\, |\mathcal{A}(x(s))|^{-2}\, ds \quad (\tau_0 \leqslant \tau \leqslant \tau_1). \tag{5.5}$$

Then the function $x = x(\tau(t))$ satisfies the differential equation (2.13), where $\tau(t)$ is the inverse function of (5.5).

Now we shall study the function F_h in the neighborhood of a_h. It is easily seen that in the interval $a_{2j-1} < x < a_{2(j-1)}$ we have

$$\mathcal{A}(x) \begin{cases} >0 & \text{if } j \text{ is odd,} \\ <0 & \text{if } j \text{ is even,} \end{cases}$$

where $a_0 = +\infty$. Therefore

$$\mathcal{A}(x) = \begin{cases} (-1)^{j+1}(x-a_{2j-1})^{\frac{1}{2}}\{\phi_{2j-1} + O(x-a_{2j-1})\} & \text{in } \mathcal{D}_{2j-1}, \\ (-1)^{j+1}(x-a_{2j})^{\frac{1}{2}}\{i\phi_{2j} + O(x-a_{2j})\} & \text{in } \mathcal{D}_{2j}, \end{cases} \tag{5.6}$$

where

$$\arg[(x-a_h)^{\frac{1}{2}}] = \tfrac{1}{2}\arg(x-a_h)$$

and

$$\phi_h = \sqrt{\prod_{n \neq h} |a_h - a_n|} > 0.$$

Hence

$$\left.\begin{aligned} F_{2j-1} &= (-1)^{j+1}(x-a_{2j-1})^{\frac{3}{2}}\{\tfrac{2}{3}\phi_{2j-1} + O(x-a_{2j-1})\} \quad \text{in } \mathcal{D}_{2j-1}, \\ F_{2j} &= (-1)^{j+1}(x-a_{2j})^{\frac{3}{2}}\{\tfrac{2}{3}i\phi_{2j} + O(x-a_{2j})\} \quad \text{in } \mathcal{D}_{2j}, \end{aligned}\right\} \tag{5.7}$$

where $\arg[(x-a_h)^{\frac{3}{2}}] = \frac{3}{2}\arg(x-a_h)$. From (5.7) we can conclude that there are two trajectories in $\mathcal{D}_h$ which approach the point a_h. Two such trajectories can be given by

$$\theta = \frac{\pi}{3} + O(r) \quad \text{and} \quad \theta = -\frac{\pi}{3} + O(r) \qquad \text{in } \mathcal{D}_{2j-1}$$

and

$$\theta = \frac{2\pi}{3} + O(r) \quad \text{and} \quad \theta = \frac{4\pi}{3} + O(r) \qquad \text{in } \mathcal{D}_{2j}$$

if we put $x - a_h = re^{i\theta}$. Let us denote by $\mathcal{J}_h$ the one of these two trajectories in $\mathcal{D}_h$

which lies in the upper half-plane in the neighborhood of a_h. If b_h is a point on the trajectory $\mathcal{J}_h$, this trajectory is determined by the solution of (2.13) which satisfies the initial condition $x(0)=b_h$. Let $x=y_h(t)$ be this unique solution. If we put

$$y_h(t)-a_h=r(t)e^{i\theta(t)},$$

then we get
$$\theta(t)=\frac{\pi}{3}+O(r(t)) \quad \text{if } h \text{ is odd} \tag{5.8}$$

and
$$\theta(t)=\frac{2\pi}{3}+O(r(t)) \quad \text{if } h \text{ is even.} \tag{5.9}$$

From (2.13), (5.6), (5.8) and (5.9), we derive

$$\frac{dr(t)}{dt}=\begin{cases}(-1)^{j+1}r(t)^{\frac{1}{2}}\phi_{2j-1}\{1+O(r(t))\} & \text{in } \mathcal{D}_{2j-1},\\ (-1)^{j+1}r(t)^{\frac{1}{2}}\phi_{2j}\{-1+O(r(t))\} & \text{in } \mathcal{D}_{2j}.\end{cases}$$

Hence in $\mathcal{D}_h$ we have

$$\frac{dr}{dt}\begin{cases}>0 & \text{if } h=0 \text{ or } 1 \bmod 4,\\ <0 & \text{if } h=2 \text{ or } 3 \bmod 4.\end{cases}$$

On the other hand, since $dt/dr=O(r^{-\frac{1}{2}})$, the trajectory $x=y_h(t)$ arrives at a_h from b_h within a finite time-interval. Thus we can construct the unique trajectories $\mathcal{J}_h$ which satisfy the conditions (i) and (ii) of Lemma 6 in the neighborhood of the point a_h.

Now we shall complete the proof of Lemma 6 for the case $h=3 \bmod 4$. Other cases can be treated in the same manner. Assume that the trajectory

$$x=x_h(t) \quad t_h<t<0 \tag{5.10}$$

satisfies the conditions (i) and (ii) of Lemma 6. Furthermore, assume that this trajectory cannot be extended without violating the condition (ii). We want to prove that the trajectory (5.10) satisfies the condition (iii) of Lemma 6.

To do this, first of all, assume that there is a decreasing sequence of values $\hat{t}_n$ of t such that $\lim_{n\to\infty}\hat{t}_n=t_h$, and that $\lim_{n\to\infty}x_h(\hat{t}_n)=\xi$ exists and $\operatorname{Im}\xi>0$. If, in this case, $|t_h|<+\infty$, then we have $\lim_{t\to t_h}x_h(t)=\xi$. This contradicts the assumption that trajectory (5.10) cannot be extended without violating the condition (ii) of Lemma 6. Hence we must have $t_h=-\infty$. Now assume that $t_h=-\infty$. Then since $\operatorname{Im} x_h(t)>0$ for $0>t>-\infty$ and $\operatorname{Im} x_h(0)=\operatorname{Im} a_h=0$, this trajectory must approach a limit cycle as t tends to $-\infty$. Otherwise, we must have $\lim_{n\to\infty}x_h(t_n)=\xi=\infty$. This limit cycle must be in the upper half-plane. However, this is impossible, since there is no zero of $\mathcal{A}(\bar{\xi})$ in the upper half-plane. (Poincaré–Bendixson Theorem [1; pp. 389–403].)

Secondly, assume that there is a decreasing sequence of values $\hat{t}_n$ of t such that $\lim_{n\to+\infty} \hat{t}_n = t_h$, and that $\lim_{n\to+\infty} x_h(\hat{t}_n) = \xi$ exists and $\operatorname{Im} \xi = 0$. Then we can prove that

$$\lim_{t\to t_h} x_h(t) = \xi \quad \text{and} \quad |t_h| < +\infty. \tag{5.11}$$

In fact, if $a_{2p+1} < \xi < a_{2p}$, then $\mathcal{A}(\xi)$ is real and different from zero. Hence the assertion (5.11) can be easily verified. In case when $\xi = a_p$, let us consider the function

$$F_p(x) = \int_{a_p}^{x} \mathcal{A}(\tau)\, d\tau$$

in the upper half-plane Ω. The function F_p is holomorphic in Ω and $\operatorname{Re} F_p(x_h(t))$ is constant for $t_h < t < 0$. On the other hand, $F_p(x_h(\hat{t}_n))$ tends to zero as n tends to infinity. Hence

$$\operatorname{Re} F_p(x_h(t)) \equiv 0 \quad \text{for} \quad t_h < t < 0.$$

Therefore, in the neighborhood of a_p, the trajectory (5.10) must coincide with the trajectory $\mathcal{J}_p$. This implies (5.11). We shall now prove that ξ is not in the interval $a_{2p} < x < a_{2p-1}$. To do this, assume that $a_{2p} < \xi < a_{2p-1}$. Then consider the function

$$F_{2p}(x) = \int_{a_{2p}}^{x} \mathcal{A}(\tau)\, d\tau$$

in Ω. The function F_{2p} is holomorphic in Ω and $\operatorname{Re} F_{2p}(x_h(t))$ is constant for $t_h < t < 0$. It is easily seen that $\mathcal{A}(x)$ is purely imaginary for $a_{2p} < x < a_{2p-1}$. Hence $\operatorname{Re} F_{2p}(x_h(\hat{t}_n))$ tends to zero as n tends to infinity, and we have

$$\operatorname{Re} F_{2p}(x_h(t)) \equiv 0 \quad \text{for} \quad t_h < t < 0.$$

However, for sufficiently large n, it is easily verified that

$$\operatorname{Re} F_{2p}(x_h(\hat{t}_n)) \neq 0.$$

This is a contradiction. Thus we have proved (5.11). Moreover, we also have shown that

$$a_{2p+1} \leqslant \xi \leqslant a_{2p} \quad \text{for some } p. \tag{5.12}$$

Now we shall derive a contradiction from the conditions (i), (ii), (5.11) and (5.12). In fact, since (5.10) is a trajectory of (2.13), then $x = \overline{x_h(-t)}$ $(0 < t < t_h)$ is also a trajectory of (2.13). Hence if the conditions (i), (ii), (5.11), and (5.12) are satisfied, we have a closed trajectory of (2.13). By modifying this closed trajectory at a_h and ξ, if necessary, we can construct a simple closed curve in Ω_0 so that $\arg \mathcal{A}(\bar{x})$ increases by

2π if x goes along this closed curve counterclockwise. This means that the index of this closed curve with respect to the vector field $i\mathcal{A}(\bar{x})$ is one. (See, for example, [1; pp. 398-400].) This implies that

$$\int d\log \mathcal{A}(x) = -2\pi i$$

if the integration is taken along this closed curve counterclockwise. However, every residue of $\mathcal{A}'(x)/\mathcal{A}(x)$ is $\frac{1}{2}$. Hence $\int d\log \mathcal{A}(x) > 0$. Thus we arrive at a contradiction. Consequently, we proved that

$$\lim_{t\to t_h} |x_h(t)| = +\infty. \tag{5.13}$$

In order to prove the condition (iv) of Lemma 6, we shall consider the equation (2.13) at $x=\infty$. We know that $\mathcal{A}(x)$ has the form

$$\mathcal{A}(x) = x^{\frac{1}{2}m}\left\{1 + \sum_{h=1}^{\infty} b_h(a)\, x^{-h}\right\}$$

in the neighborhood of $x=\infty$. Let us define $F_\infty(x)$ by

$$F_\infty(x) = \begin{cases} x^{\frac{1}{2}m+1}\left\{\dfrac{2}{m+2} + \displaystyle\sum_{l=1}^{\infty} \dfrac{2}{m+2-2l}\, b_l(a)\, x^{-l}\right\} & \text{if } m \text{ is odd,} \\ x^{\frac{1}{2}m+1}\left\{\dfrac{2}{m+2} + \displaystyle\sum_{2l\neq m+2} \dfrac{2}{m+2-2l}\, b_l(a)\, x^{-l}\right\} + b_{1+m/2}(a)\log x & \text{if } m \text{ is even,} \end{cases}$$

where $\arg F_\infty(x) = 0$ for large positive x. Then the function F_∞ is holomorphic in Ω. Furthermore, $\operatorname{Re} F_\infty(x_h(t))$ is constant for $t_h < t < 0$. In the neighborhood of $x=\infty$, any curve defined by $\operatorname{Re} F_\infty(x) = \text{constant}$ has one of the following forms

$$\theta = (2k-1)\pi/(m+2) + O(r^{-1})$$

if $x = re^{i\theta}$ and $0<\theta<\pi$, where

$$1 \leq k \leq (m+3)/2 \quad (m=\text{odd}), \qquad 1 \leq k \leq 1+m/2 \quad (m=\text{even}).$$

On the other hand, along such a curve, we have

$$\frac{dr}{dt} = r^{\frac{1}{2}m}\left\{\sin\left(\frac{m+2}{2}\theta\right) + O(r^{-1})\right\} = r^{\frac{1}{2}m}\{(-1)^{k+1} + O(r^{-1})\}.$$

Hence

$$\frac{dr}{dt}\begin{cases} >0 & \text{if } k \text{ is odd,} \\ <0 & \text{if } k \text{ is even.} \end{cases}$$

Moreover, $dt/dr = O(r^{-\frac{1}{2}m})$. This completes the proof of Lemma 6 for the case $h = 3 \bmod 4$.

6. Proof of Lemma 2 (Part II)

Now we shall construct the curve $\mathcal{C}_{k,1}(\xi)$ of Lemma 2 for $\xi \in I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$ when k is an even integer. The case when k is odd can be treated in the same manner. Let Ω_k be the simply connected subdomain of the upper half-plane Ω which is bounded by $\mathcal{J}_{2k}$, I_{2k-1}, and $\mathcal{J}_{2k-1}$. On the other hand, for each point x_0 in the interior of Ω, we denote by $x=\omega(t, x_0)$ the trajectory of the system (2.13) such that $x_0=\omega(0, x_0)$.

Let $x_0 \in \Omega_k$, and assume that the trajectory $\omega(t, x_0)$ is defined for $t_0'<t<t_0$, where $-t_0'$ and t_0 may be $+\infty$. Then $\omega(t, x_0)$ has the following properties:

$$\lim_{t\to t_0} \omega(t, x_0)=\infty, \quad \lim_{t\to t_0} \arg \omega(t, x_0)=(2k+1)\pi/(m+2) \tag{6.1}$$

and

$$\lim_{t\to t_0'} \omega(t, x_0)=\infty, \quad \lim_{t\to t_0'} \arg \omega(t, x_0)=(2k-1)\pi/(m+2). \tag{6.2}$$

In fact, the trajectory $\omega(t, x_0)$ should be in the interior of Ω_k for $t_0'<t<t_0$. Then by using the Poincaré–Bendixson Theorem [1; pp. 389–403] and Lemma 7, we can prove (6.1) and (6.2).

Now for $\xi \in I_{2k-1}$ we define $\tilde{\mathcal{C}}_{k,1}(\xi)$ by

$$x=\tilde{z}_k(s;\xi)=\begin{cases} \xi+is & (0\leqslant s\leqslant s_1),\\ \omega(s-s_1, \tilde{z}_k(s_1;\xi)) & (s_1\leqslant s\leqslant s_2),\\ \tilde{z}_k(s_2;\xi)+(s-s_2)\exp\{2k\pi i/(m+2)\} & (s_2\leqslant s<+\infty).\end{cases} \tag{6.3}$$

(See Fig. 1.)

If s_1 and $1/s_2$ are sufficiently small, the function

$$F(t)=\mathrm{Re}\int_0^t \mathcal{A}(\tilde{z}_k(s;\xi))\,\dot{\tilde{z}}_k(s;\xi)\,ds$$

is nondecreasing. In fact,

$$F(t)=\begin{cases} -\mathrm{Im}\,[\mathcal{A}(\xi+it)] & (0\leqslant t\leqslant s_1),\\ 0 & (s_1\leqslant t\leqslant s_2),\\ \mathrm{Re}\,[\mathcal{A}(\tilde{z}_k(t;\xi))\exp\{2k\pi i/(m+2)\}] & (s_2\leqslant t<+\infty).\end{cases}$$

Notice that, if k is even, we have

$$\mathcal{A}(x)=\begin{cases} \sqrt{|P(x)|} & \text{if } x\in I_{2k},\\ -i\sqrt{|P(x)|} & \text{if } x\in I_{2k-1},\\ -\sqrt{|P(x)|} & \text{if } x\in I_{2(k-1)},\end{cases}$$

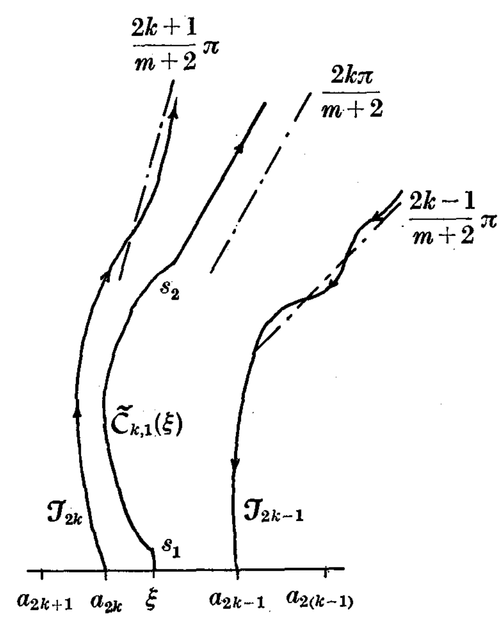

Fig. 1.

where $\sqrt{|P(x)|}>0$. Hence, if s_1 is sufficiently small, we have

$$-\operatorname{Im}[\mathcal{A}(\xi+it)]>\tfrac{1}{2}\sqrt{|P(x)|}>0 \quad \text{for} \quad 0\leqslant t\leqslant s_1.$$

On the other hand, if s_2 is sufficiently large, we have

$$k\pi<\arg[\mathcal{A}(\tilde{z}_k(t;\xi))\exp\{2k\pi i/(m+2)\}]<(k+\tfrac{1}{2})\pi$$

for $s_2\leqslant t<+\infty$. Hence

$$\operatorname{Re}[\mathcal{A}(\tilde{z}_k(t;\xi))\exp\{2k\pi i/(m+2)\}]>0.$$

Thus we get $\dot{F}(t)\geqslant 0$ for all t.

For $\xi\in I_{2k}$, let $\tilde{C}_{k,1}(\xi)$ be defined hy

$$x=\tilde{z}_k(s;\xi)=\begin{cases}\omega(s,\xi) & (0\leqslant s\leqslant s_1),\\ \tilde{z}_k(s_1;\xi)+(s-s_1) & (s_1\leqslant s\leqslant s_2),\\ \omega(s-s_2,\tilde{z}_k(s_2;\xi)) & (s_2\leqslant s\leqslant s_3),\\ \tilde{z}_k(s_3;\xi)+(s-s_3)\overline{\mathcal{A}(\tilde{z}_k(s_3;\xi))} & (s_3\leqslant s\leqslant s_4),\\ \omega(s-s_4,\tilde{z}_k(s_4;\xi)) & (s_4\leqslant s\leqslant s_5),\\ \tilde{z}_k(s_5;\xi)+(s-s_5)\exp\{2k\pi i/(m+2)\} & (s_5\leqslant s<+\infty).\end{cases} \tag{6.4}$$

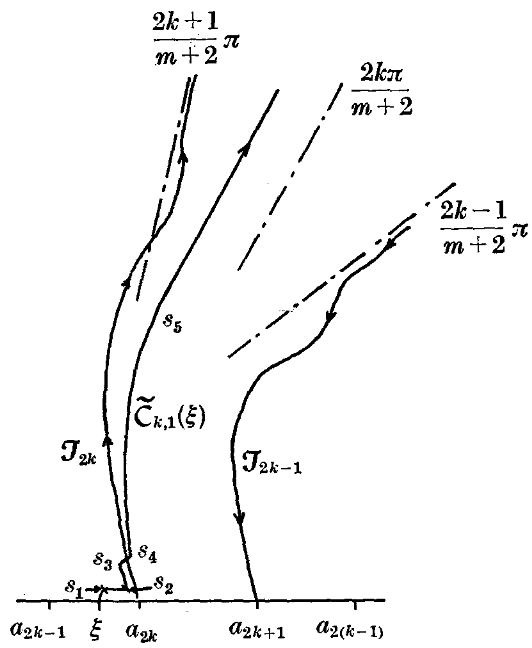

Fig. 2.

(See Fig. 2.) We can determine s_1, s_2, s_3, s_4, and s_5 so that the function

$$\operatorname{Re}\int_0^t \mathcal{A}(\tilde{z}_k(s;\xi))\,\dot{\tilde{z}}(s;\xi)\,ds$$

is nondecreasing.

Finally, for $\xi\in I_{2(k-1)}$, let $\tilde{C}_{k,1}(\xi)$ be defined by

$$x=\tilde{z}_k(s;\xi)=\begin{cases}\omega(-s,\xi) & (0\leqslant s\leqslant s_1),\\ \tilde{z}_k(s_1;\xi)-(s-s_1) & (s_1\leqslant s\leqslant s_2),\\ \omega(s_2-s,\tilde{z}_k(s_2;\xi)) & (s_2\leqslant s\leqslant s_3),\\ \tilde{z}_k(s_3;\xi)+(s-s_3)\,\overline{\mathcal{A}(\tilde{z}_k(s_3;\xi))} & (s_3\leqslant s\leqslant s_4),\\ \omega(s-s_4,\tilde{z}_k(s_4;\xi)) & (s_4\leqslant s\leqslant s_5),\\ \tilde{z}_k(s_5;\xi)+(s-s_5)\exp\{2k\pi i/(m+2)\} & (s_5\leqslant s<+\infty).\end{cases}\tag{6.5}$$

(See Fig. 3.)

Since the mapping $z=\int_\xi^x\mathcal{A}(t)\,dt$ is conformal, we can find $C_{k,1}(\xi)$ and $\theta_0(\xi)$ by modifying $\tilde{C}_{k,1}(\xi)$ in a suitable manner.

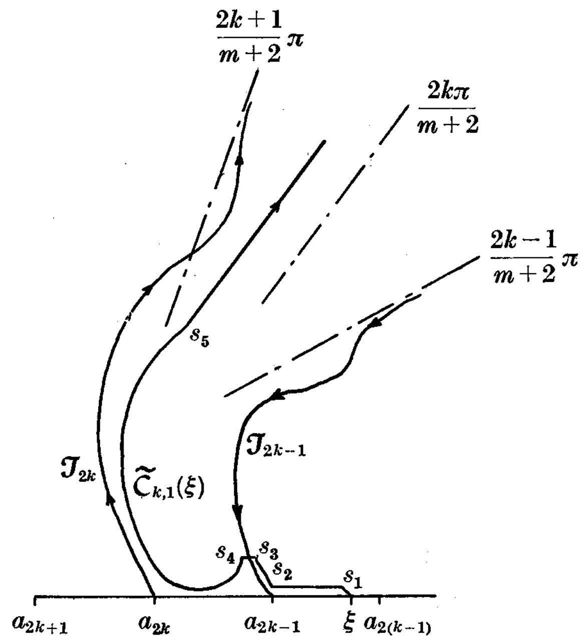

Fig. 3.

7. Proof of Lemma 4

Let
$$\Omega(\delta)=\{x;x\in\Omega,|x-a_h|\geqslant\delta>0,(h=1,2,\ldots,m)\},\tag{7.1}$$
where Ω is the upper half-plane. Then
$$|\mathcal{A}(x)|\geqslant\delta^{\frac{1}{2}m},\quad |\arg\mathcal{A}(x)|\leqslant\gamma\quad\text{for}\quad x\in\Omega(\delta),\tag{7.2}$$
where γ is a positive constant. By using the Borel–Ritt Theorem [11; p. 47] we can construct a function $g(u,x)$ of two complex variables (u,x) so that

(i) g is holomorphic for
$$x\in\Omega(\delta),\quad |u|\geqslant u_0>0,\quad |\arg u|\leqslant\gamma+2\pi;\tag{7.3}$$

(ii) g satisfies the inequalities
$$\left.\begin{aligned}&\left|g(u,x)-\sum_{n=1}^{N}u^{-n}p_n(x)\right|\leqslant E_N(\delta)\,|u|^{-N-1},\\&\left|\partial g(u,x)/\partial u-\sum_{n=1}^{N}(-n)\,u^{-n-1}p_n(x)\right|\leqslant E_N(\delta)\,|u|^{-N-2},\\&\left|\partial g(u,x)/\partial x-\sum_{n=1}^{N}u^{-n}p_n'(x)\right|\leqslant E_N(\delta)\,|u|^{-N-1}\end{aligned}\right\}\tag{7.4}$$
for (7.3) and $N=1,2,\ldots,$ where E_N are positive constants depending of δ.

Put
$$q(x, \lambda) = g\,((-1)^{k+1} \lambda A(x), x). \tag{7.5}$$

Then we can prove the following lemma.

LEMMA 8. *The function $q(x, \lambda)$ is holomorphic in (x, λ) for*

$$x \in \Omega(\delta), \quad |\lambda| \geq u_0 \delta^{-\frac{1}{2}m}, \quad |\arg \lambda| \leq \theta_0(\xi). \tag{7.6}$$

Moreover, q satisfies the inequalities

$$\left.\begin{aligned} &\left| q(x, \lambda) - \sum_{n=1}^{N} [(-1)^{k+1} \lambda A(x)]^{-n} p_n(x) \right| \leq B_N(\delta) \, |\lambda A(x)|^{-N-1} \\ &\left| q'(x, \lambda) - \sum_{n=1}^{N} [(-1)^{k+1} \lambda A(x)]^{-n} (-\tfrac{1}{2} n R(x) p_n(x) + p_n'(x)) \right| \leq B_N(\delta) \, |\lambda A(x)|^{-N-1} \end{aligned}\right\} \tag{7.7}$$

for (7.6) *and $N = 1, 2, \dots,$ where B_N are positive constants depending on δ.*

In fact, if (x, λ) is in the region (7.6), then the inequalities (7.2) imply that

$$|\lambda A(x)| \geq u_0 \quad \text{and} \quad |\arg [(-1)^{k+1} \lambda A(x)]| \leq \gamma + 2\pi.$$

Hence q is holomorphic for (7.6). The first inequality of (7.7) is then derived from the first inequality of (7.4). On the other hand, the formula

$$q'(x, \lambda) = (-1)^{k+1} \lambda A'(x) \, \partial g((-1)^{k+1} \lambda A(x), x)/\partial u + \partial g((-1)^{k+1} \lambda A(x), x)/\partial x$$

and the second and the third inequalities of (7.4) imply the second inequality of (7.7). Here we used the identity

$$A'(x) = \tfrac{1}{2} A(x) \, R(x)$$

and the fact that $|R(x)|$ is bounded in $\Omega(\delta)$.

Let us transform the Riccati equation (2.3) by

$$w = (-1)^{k+1} \lambda A(x) + \tfrac{1}{4} R(x) + q(x, \lambda) + \lambda A(x) \, v. \tag{7.8}$$

Then we have
$$v' = L(x, \lambda) + ((-1)^k \, 2 \lambda A(x) - 2 q(x, \lambda)) v - \lambda A(x) \, v^2, \tag{7.9}$$

where
$$\begin{aligned} L(x, \lambda) = &-(\lambda A(x))^{-1} [q'(x, \lambda) + \tfrac{1}{16} R(x)^2 - \tfrac{1}{4} R'(x) \\ &+ 2((-1)^{k+1} \lambda A(x) + \tfrac{1}{4} R(x)) \, q(x, \lambda) + q(x, \lambda)^2]. \end{aligned} \tag{7.10}$$

By using Lemma 8 and the definitions (2.9) and (2.10) of the functions $p_n(x)$, we can prove that

$$|L(x, \lambda)| \leq H_N(\delta) \, |\lambda A(x)|^{-N} \quad (N = 1, 2, \dots) \tag{7.11}$$

for (7.6), where H_N are positive constants depending on δ.

For each ξ in $I_{2k}\cup I_{2k-1}\cup I_{2(k-1)}$ there exists a positive constant $\delta(\xi)$ such that the curve $C_{k,1}(\xi)$ is contained in $\Omega(\delta(\xi))$. We have

$$z_k(s;\xi)\in\Omega(\delta(\xi)) \quad \text{for} \quad 0\leqslant s<+\infty. \tag{7.12}$$

Hence we have

$$|L(z_k(s;\xi),\lambda)|\leqslant H_N(\delta(\xi))|\lambda A(z_k(s;\xi))|^{-N} \quad (N=1,2,\ldots) \tag{7.13}$$

for $0\leqslant s<+\infty$ and $|\lambda|\geqslant u_0\delta(\xi)^{-\frac{1}{2}m}$, and $|\arg\lambda|\leqslant\theta_0(\xi)$.

Assume that a function

$$v=v(s;\lambda;\xi) \tag{7.14}$$

satisfies the differential equation

$$dv/ds=\dot z_k(s;\xi)\,[L(z,\lambda)+((-1)^k 2\lambda A(z)-2q(z,\lambda))\,v-\lambda A(z)\,v^2], \tag{7.15}$$

where $z=z_k(s;\xi)$, and the inequalities

$$|v(s,\lambda;\xi)|\leqslant M_n(\xi)\,|\lambda A(z_k(s;\xi))|^{-N} \quad (N=1,2,\ldots) \tag{7.16}$$

hold for

$$0\leqslant s<+\infty, \quad |\lambda|\geqslant M_0(\xi), \quad |\arg\lambda|\leqslant\theta_0(\xi), \tag{7.17}$$

where M_N are positive constants depending on ξ. Then

$$w_{k,1}(s,\lambda;\xi)=(-1)^{k+1}\lambda A(z_k(s;\xi))-\tfrac{1}{4}R(z_k(s;\xi))+q(z_k(s;\xi),\lambda)+\lambda A(z_k(s;\xi))\,v(s,\lambda;\xi)$$

satisfies all the conditions of Lemma 4. Hence we shall construct a solution (7.14) of (7.15) which satisfies the conditions (7.16). To do this let us reduce the equation (7.15) to an integral equation

$$v(s,\lambda;\xi)$$
$$=\int_{+\infty}^{s}[L(z,\lambda)-2q(z,\lambda)\,v(\sigma,\lambda;\xi)-\lambda A(z)\,v(\sigma,\lambda;\xi)^2]\exp\left\{(-1)^k 2\lambda\int_0^s A(\mathfrak{z})\,\dot{\mathfrak{z}}\,d\tau\right\}\dot z\,d\sigma, \tag{7.18}$$

where $z=z_k(\sigma;\xi)$, $\mathfrak{z}=z_k(\tau;\xi)$ We shall solve this integral equation using successive approximations.

Let us denote by $F(s,\lambda,v;\xi)$ the right-hand member of (7.18). Then successive approximations are defined by

$$\left.\begin{aligned} &v_0(s,\lambda;\xi)=0 \quad (0\leqslant s<+\infty),\\ &v_n(s,\lambda;\xi)=F(s,\lambda,v_{n-1};\xi) \quad (n\geqslant 1).\end{aligned}\right\} \tag{7.19}$$

In order to prove that the approximations are well defined for (7.17) if M_0 is sufficiently large, we remark that the real part of $(-1)^k\lambda\int_0^t A(z)\dot{z}ds$ is nondecreasing, where $z=z_k(s;\xi)$. Hence

$$\left|\exp\left\{(-1)^k 2\lambda\int_0^s A(z)\,\dot{z}\,d\tau\right\}\right|\leqslant 1 \quad \text{for} \quad s\leqslant\sigma<+\infty. \tag{7.20}$$

On the other hand, we have

$$|L(z,\lambda)|\leqslant H_{N+2}(\delta(\xi))\,|\lambda A(z)|^{-N-2} \tag{7.21}$$

and

$$|q(z,\lambda)|\leqslant|p_1(z)|\,|\lambda A(z)|^{-1}+B_1(\delta(\xi))\,|\lambda A(z)|^{-2} \tag{7.22}$$

for $0\leqslant s<+\infty$ and $|\lambda|\geqslant u_0\,\delta(\xi)^{-\frac{1}{2}m}$, $|\arg\lambda|\leqslant\theta_0(\xi)$. Therefore, if

$$|v(s,\lambda;\xi)|\leqslant K\,|\lambda A(z_k(s;\xi))|^{-N} \tag{7.23}$$

for (7.17) and some positive number K and some positive integer $N\geqslant 4$, we have

$$|F(s,\lambda,v;\xi)|\leqslant\int_s^{+\infty}[(H_{N+2}(\delta)+2\,B_1(\delta)\,K)\,|\lambda A(z)|^{-N-2}$$
$$+2\,K\,|p_1(z)|\,|\lambda A(z)|^{-N-1}+K^2|\lambda A(z)|^{-2N+1}]\,|\dot{z}|\,d\tau, \tag{7.24}$$

where $z=z_k(\tau;\xi)$.

Notice that, for large values of s, we have

$$z_k(s;\xi)=z_0+s\exp\left(2k\pi i/(m+2)\right),$$

where z_0 is some constant. Hence, for large values of s, we get

$$\tfrac{1}{2}s\leqslant|z_k(s;\xi)|\leqslant 2s.$$

On the other hand, if $x\in\Omega$ and $|x|$ is sufficiently large, we have

$$\tfrac{1}{2}|x|^{\frac{1}{2}m}\leqslant|A(x)|\leqslant 2\,|x|^{\frac{1}{2}m}.$$

Moreover, (2.6) and (2.9) imply that

$$p_1(x)=O(x^{-2})$$

as x tends to ∞. Thus, we can conclude that F is well defined for (7.17). Furthermore, if M_0 is sufficiently large, the function F satisfies the inequality

$$|F(s,\lambda,v;\xi)|\leqslant K\,|\lambda A(z_k(s;\xi))|^{-N} \tag{7.25}$$

for (7.17). Now fix N and K, and choose M_0. Then the successive approximations v_n can be well defined for (7.17) and they satisfy the inequalities

$$|v_n(s, \lambda; \xi)| \leqslant K |\lambda A(z_k(s; \xi))|^{-N} \tag{7.26}$$

for (7.17). The uniform convergence of the sequence $\{v_n\}$ can be proved in a similar manner. Thus we get a solution (7.14) of (7.18) which satisfies the condition (7.16) for a particular N. Now it is still necessary to prove that we can choose a constant $M_0(\xi)$ independent of N. Since $A(x) \neq 0$ in $\Omega(\delta)$, this can be accomplished by choosing $M_N(\xi)$ in a suitable manner. This completes the proof of Lemma 4.

8. Proof of Theorem 1

In this section we shall evaluate the quantities $f_{-k}(x, \lambda)$ and $f'_{-k}(x, \lambda)$ for each fixed x in $I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$. The quantities f_k and f'_k can be treated in the same manner.

Assume that ξ is given in $I_{2k} \cup I_{2k-1} \cup I_{2(k-1)}$. For $0 \leqslant s < +\infty$, $|\lambda| \geqslant M_0(\xi)$, $|\arg \lambda| \leqslant \theta_0(\xi)$, let us define a function $G(s, \lambda; \xi)$ by

$$G = [f'_{-k}(z, \lambda) - w_{k,1}(s, \lambda; \xi) f_{-k}(z, \lambda)] \exp \left\{ \int_0^s w_{k,1}(\sigma, \lambda; \xi) \dot{z}_k(\sigma; \xi) \, d\sigma \right\},$$

where $z = z_k(s; \xi)$. Then it is easily verified that $dG/ds = 0$ except at points of discontinuity of $\dot{z}_k(s; \xi)$. Hence G is piecewise constant with respect to s. Since G is continuous in s, this function G must be constant with respect to s. This constant can be computed by letting s tend to $+\infty$. First of all, from Lemma 2 we derive

$$\lim_{s \to +\infty} z_k(s; \xi) = \infty, \qquad \lim_{s \to +\infty} \arg z_k(s; \xi) = 2\pi k/(m+2).$$

Since f_{-k} is subdominant for $\lambda > 0$ in the sector (1.17-(-k))

$$|\arg x - 2\pi k/(m+2)| < \pi/(m+2),$$

then, if $\theta_0(\xi) > 0$ is small and $|\arg \lambda| \leqslant \theta_0(\xi)$, we have

$$\lim_{s \to +\infty} f_{-k}(z_k, \lambda) = 0 \quad \text{and} \quad \lim_{s \to +\infty} f'_{-k}(z_k, \lambda) = 0.$$

Furthermore, if we use the asymptotic representation (1.15-(-k)) of f_{-k} and the inequality (2.18) of Lemma 4, we can prove that

$$\lim_{s \to +\infty} w_{k,1}(s, \lambda; \xi) f_{-k}(z_k, \lambda) = 0.$$

On the other hand, it is easily verified that

$$\lim_{s\to+\infty} \arg [A(z_k)\dot{z}_k(s;\xi)] = k\pi,$$

since $z_k(s;\xi) = z_0 + s \exp(2\pi k i/(m+2))$ for large positive values of s, where z_0 is a certain constant. Thus we get

$$\lim_{s\to+\infty} \exp\left\{\int_0^s w_{k,1}(\sigma,\lambda;\xi)\,\dot{z}_k(\sigma;\xi)\,d\sigma\right\} = 0.$$

Therefore, we have $G(s,\lambda;\xi) = 0$ identically. This in turn implies that

$$f'_{-k}(z,\lambda) - w_{k,1}(s,\lambda;\xi) f_{-k}(z,\lambda) = 0$$

for
$$0 \leqslant s < +\infty, \quad |\lambda| \geqslant M_0(\xi), \quad |\arg\lambda| \leqslant \theta_0(\xi), \tag{8.1}$$

where $z = z_k(s;\xi)$. Consider now the function

$$H(s,\lambda;\xi) = f_{-k}(z_k,\lambda) \exp\left\{-\int_0^s w_{k,1}(\sigma,\lambda;\xi)\,\dot{z}_k(\sigma;\xi)\,d\sigma\right\}.$$

It is easily verified that $dH/ds = 0$ except at points of discontinuity of $\dot{z}_k(s;\xi)$. However H is continuous in s. Thus we conclude that $H(s,\lambda;\xi)$ is independent of s. Let us denote by $C(\lambda;\xi)$ the values of H. Then we get

$$f_{-k}(z_k(s;\xi),\lambda) = C(\lambda;\xi) \exp\left\{\int_0^s w_{k,1}(\sigma,\lambda;\xi)\,\dot{z}_k(\sigma;\xi)\,d\sigma\right\},$$

$$f'_{-k}(z_k(s;\xi),\lambda) = C(\lambda;\xi)\, w_{k,1}(s,\lambda;\xi) \exp\left\{\int_0^s w_{k,1}(\sigma,\lambda;\xi)\,\dot{z}_k\,d\sigma\right\}$$

for (8.1). By putting $s = 0$, we get

$$f_{-k}(\xi,\lambda) = C(\lambda;\xi),$$

$$f'_{-k}(\xi,\lambda) = C(\lambda,\xi)\, w_{k,1}(0,\lambda;\xi)$$

for
$$|\lambda| \geqslant M_0(\xi), \quad |\arg\lambda| \leqslant \theta_0(\xi). \tag{8.2}$$

We can evaluate the quantity $C(\lambda;\xi)$ if we compute $\lim_{s\to+\infty} H(s,\lambda;\xi)$. To do this, first of all, we shall rewrite the asymptotic representation (1.15-(-k)) of the function f_{-k} using the following identities;

$$\varrho\omega^k A_m(\varrho\omega^k x, \varrho\omega^k a) = \begin{cases} (-1)^k \lambda A_m(x,a) & (m = \text{odd}), \\ (-1)^k \lambda A_m(x,a) + \lambda\alpha_k \left\{\dfrac{1}{x+\varrho^{-1}\omega^{-k}} - \dfrac{1}{x+1}\right\} & (m = \text{even}). \end{cases}$$

Let
$$\gamma(\lambda)=\begin{cases}(\varrho\omega^{-k})^{-\frac{1}{4}m} & (m=\text{odd}),\\ (\varrho\omega^{-k})^{-\frac{1}{4}m-\lambda\alpha_k} & (m=\text{even}).\end{cases}$$

Then the function f_{-k} has the asymptotic representation

$$f_{-k}=\gamma(\lambda)\,x^{-\frac{1}{4}m}\,\{1+O(x^{-\frac{1}{2}})\}\,\exp\left\{(-1)^{k+1}\lambda\int_0^x A_m(t,a)\,dt\right\} \tag{8.3}$$

uniformly on each compact set of the λ-plane as x tends to infinity in the sector

$$\left|\arg\varrho+\arg x-\frac{2\pi k}{(m+2)}\right|\leqslant\frac{3\pi}{(m+2)}-\delta_0, \tag{8.4}$$

where the path of integration should be taken in the sector (8.4). In deriving (8.3), we used the identity

$$\left\{\frac{\varrho\omega^{-k}x+1}{x+1}\right\}^{\lambda\alpha_k}=\exp\left[\lambda\alpha_k\int_0^x\left\{\frac{1}{t+\varrho^{-1}\omega^k}-\frac{1}{t+1}\right\}dt\right].$$

Secondly, we remark that

$$\mathcal{B}(\xi+)/\mathcal{B}(z)=\exp\left[-\frac{1}{4}\int_0^s R(z_k(\sigma;\xi))\,\dot{z}_k(\sigma;\xi)\,d\sigma\right],$$

where $z=z_k(s;\xi)$, and that the integral

$$\int_0^{+\infty}Q_{k,1}(s,\lambda;\xi)\,\dot{z}_k(s;\xi)\,ds$$

exists for (8.2).

Finally, we shall write the integral

$$\int_0^s A(z_k(\sigma;\xi))\,\dot{z}_k(\sigma;\xi)\,d\sigma$$

in the form

$$\int_0^s A(z_k(\sigma;\xi))\,\dot{z}_k(\sigma;\xi)\,d\sigma=\int_0^{z_k(s;\xi)}A(t)\,dt-\int_0^{\xi}A(\tau+)\,d\tau,$$

where the first integral is taken along a path in the sector (8.4), and the second integral is taken along the real axis. Notice that

$$\int_0^{\infty}\{A_m(t,a)-A(t)\}\,dt$$

exists if the integral is taken along a path in the sector (8.4). Furthermore, it is easily verified that this integral is equal to

$$\int_0^{+\infty}\{\mathcal{A}_m(\tau,a)-\mathcal{A}(\tau+)\}\,d\tau$$

which is taken along the real axis. This completes the proof of Theorem 1 for f_{-k}.

III. Proof of Theorem 2

9. Wronskians

Hereafter we assume that m is even, and that

$$k=1,2,\dots,\tfrac{1}{2}m. \tag{9.1}$$

Let us fix a point ξ_h in each interval I_h, where $h=1,2,\dots,m-1$.

LEMMA 9. *If we denote by $\Delta_k(\lambda)$ the Wronskian of f_k and f_{-k}, i.e.*

$$\Delta_k(\lambda)=\begin{vmatrix} f_k(x,\lambda) & f_{-k}(x,\lambda)\\ f'_k(x,\lambda) & f'_{-k}(x,\lambda)\end{vmatrix},$$

we find that

$$\Delta_k(\lambda)=2i\varrho^{1-2\lambda\alpha_k}\exp\{(-1)^{k+1}\lambda L_k+O(\lambda^{-1})\} \tag{9.2}$$

as λ tends to ∞ in the sector $|\arg\lambda|\leqslant\theta_0=\min_h\theta_0(\xi_h)$, where

$$L_k=2\sum_{j=1}^{k-1}(-1)^{j+1}\int_{a_{2j+1}}^{a_{2j}}\sqrt{|P(\tau)|}\,d\tau+\int_0^{a_1}[\mathcal{A}(\tau+)+\mathcal{A}(\tau-)]\,dt$$
$$+\int_0^{+\infty}[(\mathcal{A}_m(\tau,a)-\mathcal{A}(\tau+))+(\mathcal{A}_m(\tau,a)-\mathcal{A}(\tau-))]\,d\tau, \tag{9.3}$$

and $\sqrt{|P(\tau)|}\geqslant 0$, and τ is a real variable.

Remark. This lemma verifies that f_k and f_{-k} are linearly independent for large λ in the sector $|\arg\lambda|\leqslant\theta_0$.

In order to prove this lemma, let us put $x=\xi_{2k-1}$ in (4.2). Then we derive

$$\Delta_k(\lambda)=f_k(\xi,\lambda)f_{-k}(\xi,\lambda)[(-1)^{k+1}\lambda\{\mathcal{A}(\xi+)-\mathcal{A}(\xi-)\}+O(\lambda^{-1})] \tag{9.4}$$

from Lemmas 4 and 5 and Theorem 1, where $\xi=\xi_{2k-1}$. Now, using Theorem 1, we get

$$f_k(\xi,\lambda)f_{-k}(\xi,\lambda)$$
$$=C_k(\lambda)C_{-k}(\lambda)\mathcal{B}(\xi-)^{-1}\mathcal{B}(\xi+)^{-1}\exp\left[(-1)^{k+1}\lambda\int_0^{\xi}\{\mathcal{A}(\tau+)+\mathcal{A}(\tau-)\}\,d\tau+O(\lambda^{-1})\right] \tag{9.5}$$

and

$$C_k(\lambda)\,C_{-k}(\lambda)$$
$$=\varrho^{-\frac{1}{2}m-2\lambda\alpha_k}\exp\left[(-1)^{k+1}\lambda\int_0^{+\infty}\{(\mathcal{A}_m(\tau,a)-\mathcal{A}(\tau+))+(\mathcal{A}_m(\tau,a)-\mathcal{A}(\tau-))\}\,d\tau\right]. \quad (9.6)$$

On the other hand, it is easily verified that, for $\xi=\xi_{2k-1}$, we have

$$\left.\begin{aligned}&\mathcal{A}(\xi+)=(-1)^{k+1}i\sqrt{|P(\xi)|},\\&\mathcal{A}(\xi-)=(-1)^{k+1}(-i)\sqrt{|P(\xi)|},\\&\mathcal{B}(\xi-)\,\mathcal{B}(\xi+)=\sqrt{|P(\xi)|}.\end{aligned}\right\} \quad (9.7)$$

Therefore $\qquad \mathcal{B}(\xi-)^{-1}\mathcal{B}(\xi+)^{-1}(\mathcal{A}(\xi+)-\mathcal{A}(\xi-))=(-1)^{k+1}2i.$

Finally, since

$$\mathcal{A}(\tau+)=\mathcal{A}(\tau-)=\mathcal{A}(\tau)=(-1)^j\sqrt{|P(\tau)|}\ \text{ for }\ \tau\in I_{2j}$$

and
$$\mathcal{A}(\tau+)=-\mathcal{A}(\tau-)\quad\text{for}\quad \tau\in I_{2j-1},$$

we find that
$$\int_\xi^{a_1}[\mathcal{A}(\tau+)+\mathcal{A}(\tau-)]\,d\tau=2\sum_{j=1}^{k-1}(-1)^j\int_{a_{2j+1}}^{a_{2j}}\sqrt{|P(\tau)|}\,d\tau.$$

Hence

$$\int_0^\xi[\mathcal{A}(\tau+)+\mathcal{A}(\tau-)]\,d\tau=\int_0^{a_1}[\mathcal{A}(\tau+)+\mathcal{A}(\tau-)]\,d\tau-2\sum_{j=1}^{k-1}(-1)^j\int_{a_{2j+1}}^{a_{2j}}\sqrt{|P(\tau)|}\,d\tau.$$

This completes the proof of Lemma 9.

LEMMA 10. *If we denote by $D_{k,1}(\lambda)$ and $D_{k,2}(\lambda)$ the Wronskians of f_k, f_{k+1} and f_{-k}, f_{-k-1} respectively, i.e.*

$$D_{k,1}(\lambda)=\begin{vmatrix}f_k & f_{k+1}\\ f'_k & f'_{k+1}\end{vmatrix},\qquad D_{k,2}(\lambda)=\begin{vmatrix}f_{-k} & f_{-k-1}\\ f'_{-k} & f'_{-k-1}\end{vmatrix},$$

then we find that

$$D_{k,1}(\lambda)=2\varrho\omega^{k-m/4}\omega^{\lambda\alpha_k},\qquad D_{k,2}(\lambda)=2\varrho\omega^{-k+m/4}\omega^{-\lambda\alpha_k}.$$

We shall compute $D_{k,1}(\lambda)$. The Wronskian $D_{k,2}(\lambda)$ is computed in the same manner. To compute $D_{k,1}$, we use the asymptotic representations (1.15) and the identity

$$\varrho\omega^k\mathcal{A}_m(\varrho\omega^k x,\varrho\omega^k a)=(-1)^k\lambda\mathcal{A}_m(x,a)+\lambda\alpha_k\left\{\frac{1}{x+\varrho^{-1}\omega^{-k}}-\frac{1}{x+1}\right\}.$$

We get

$$D_{k,1}(\lambda)=(\varrho\omega^k)^{-\frac{1}{4}m}(\varrho\omega^{k+1})^{-\frac{1}{4}m}\lambda\begin{vmatrix}1 & 1\\ (-1)^{k+1} & (-1)^{k+2}\end{vmatrix}\lim_{x\to\infty}\left(\frac{\varrho\omega^k x+1}{x+1}\right)^{-\lambda\alpha_k}\left(\frac{\varrho\omega^{k+1}x+1}{x+1}\right)^{\lambda\alpha_k}$$

$$=(-1)^k\,2\,\varrho\omega^{-m(2k+1)/4}\,\omega^{\lambda\alpha_k}=2\,\varrho\omega^{k-m/4}\,\omega^{\lambda\alpha_k},$$

where x tends to infinity along the direction $\arg x=-2\pi k/m+2$. Here we used the identity $\alpha_{k+1}=-\alpha_k$.

LEMMA 11. *Let*

$$\Delta_{k,1}(\lambda)=\begin{vmatrix}f_{-k}(x,\lambda) & f_{k+1}(x,\lambda)\\ f'_{-k}(x,\lambda) & f'_{k+1}(x,\lambda)\end{vmatrix}$$

and

$$\Delta_{k,2}(\lambda)=\begin{vmatrix}f_k(x,\lambda) & f_{-k-1}(x,\lambda)\\ f'_k(x,\lambda) & f'_{-k-1}(x,\lambda)\end{vmatrix}.$$

Then

$$\Delta_{k,1}(\lambda)=J_k(\lambda)\,D_{k,1}(\lambda)+E_{k,1}(\lambda), \tag{9.8}$$

and

$$\Delta_{k,2}(\lambda)=D_{k,2}(\lambda)/J_k(\lambda)+E_{k,2}(\lambda), \tag{9.9}$$

where

$$J_k(\lambda)=(-1)^k(\omega^{2k})^{\frac{1}{4}m+\lambda\alpha_k}\exp\{M_k(\lambda)\}, \tag{9.10}$$

$$M_k(\lambda)=(-1)^{k+1}\,2i\lambda\sum_{j=1}^{k}(-1)^j\int_{a_{2j}}^{a_{2j-1}}\sqrt{|P(\tau)|}\,d\tau$$
$$+\int_0^{+\infty}Q_{k,2}(s,\lambda;\xi_{2k})\,\dot{\zeta}_k(s;\xi_{2k})\,ds-\int_0^{+\infty}Q_{k,1}(s,\lambda;\xi_{2k})\,\dot{z}_k(s;\xi_{2k})\,ds, \tag{9.11}$$

and for $\operatorname{Re}\lambda\geqslant M_0$, $|\operatorname{Im}\lambda|\leqslant r_0$, *we have*

$$|E_{k,n}(\lambda)|\leqslant K_0\left|\varrho\exp\left\{-2\lambda\int_{\xi_{2k}}^{a_{2k}}\sqrt{|P(\tau)|}\,dt\right\}\right|\quad(n=1,2), \tag{9.12}$$

if K_0, M_0 *and* $1/r_0$ *are sufficiently large positive numbers.*

We shall prove this lemma in a number of steps. First of all, we shall prove the following lemma.

LEMMA 11. *We have the identity*

$$f_{-k}(\xi_{2k},\lambda)=J_k(\lambda)\,f_k(\xi_{2k},\lambda). \tag{9.13}$$

In fact, by using Theorem 1, we get

$$\frac{f_{-k}(\xi, \lambda)}{f_k(\xi, \lambda)} = \frac{C_{-k}(\lambda)\, \mathcal{B}(\xi+)^{-1} \exp\left[(-1)^{k+1}\lambda \int_0^{\xi} \mathcal{A}(\tau+)\, d\tau - \int_0^{+\infty} Q_{k,1}(s, \lambda; \xi)\, \dot{z}_k(s; \xi)\, ds\right]}{C_k(\lambda)\, \mathcal{B}(\xi-)^{-1} \exp\left[(-1)^{k+1}\lambda \int_0^{\xi} \mathcal{A}(\tau-)\, d\tau - \int_0^{+\infty} Q_{k,2}(s, \lambda; \xi)\, \dot{\zeta}_k(s; \xi)\, ds\right]}, \tag{9.14}$$

where $\xi = \xi_{2k}$. Furthermore,

$$\int_0^{+\infty} [\mathcal{A}_m(\tau, a) - \mathcal{A}(\tau+)]\, d\tau - \int_0^{+\infty} [\mathcal{A}_m(\tau, a) - \mathcal{A}(\tau-)]\, d\tau = -\int_0^{+\infty} [\mathcal{A}(\tau+) - \mathcal{A}(\tau-)]\, d\tau.$$

On the other hand, it is easily seen that

$$\mathcal{A}(\tau+) = \mathcal{A}(\tau-) = \mathcal{A}(\tau) = (-1)^j \sqrt{|P(\tau)|} \quad \text{for} \quad \tau \in I_{2j}$$

and
$$\mathcal{A}(\tau+) = -\mathcal{A}(\tau-) = (-1)^{j-1} i \sqrt{|P(\tau)|} \quad \text{for} \quad \tau \in I_{2j-1}.$$

Hence
$$\int_{\xi_{2k}}^{+\infty} [\mathcal{A}(\tau+) - \mathcal{A}(\tau-)]\, d\tau = 2i \sum_{j=1}^{k} (-1)^{j-1} \int_{a_{2j}}^{a_{2j-1}} \sqrt{|P(\tau)|}\, d\tau.$$

Notice also that

$$\arg \mathcal{B}(\xi_{2k}+) = 2k(\pi/4) = \tfrac{1}{2} k\pi, \quad \arg \mathcal{B}(\xi_{2k}-) = 2k(-\pi/4) = -\tfrac{1}{2} k\pi.$$

Therefore
$$\mathcal{B}(\xi_{2k}-)/\mathcal{B}(\xi_{2k}+) = (-1)^k.$$

This completes the proof of Lemma 12.

LEMMA 13. *We have the inequality*

$$|Q_{k,1}(0, \lambda; \xi_{2k}) - Q_{k,2}(0, \lambda; \xi_{2k})| \leq K_1 \left|\lambda \exp\left\{-2\lambda \int_{\xi}^{a_{2k}} \sqrt{|P(\tau)|} \cdot d\tau\right\}\right|, \tag{9.15}$$

where K_1 is a suitable positive constant.

In order to prove this lemma, we shall again compute the Wronskian $\Delta_k(\lambda)$ of f_k and f_{-k} by putting $x = \xi_{2k}$ in (4.2). Since $\mathcal{A}(\xi+) = \mathcal{A}(\xi-)$ for $\xi \in I_{2k}$, we have

$$\Delta_k(\lambda) = \{Q_{k,1}(0, \lambda; \xi) - Q_{k,2}(0, \lambda; \xi)\}\, f_k(\xi, \lambda)\, f_{-k}(\xi, \lambda), \tag{9.16}$$

where $\xi = \xi_{2k}$. Hence

$$Q_{k,1}(0, \lambda; \xi_{2k}) - Q_{k,2}(0, \lambda; \xi_{2k}) = \Delta_k(\lambda)/f_k(\xi_{2k}, \lambda)\, f_{-k}(\xi_{2k}, \lambda). \tag{9.17}$$

On the other hand, from the formula (9.4) we derive

$$|Q_{k,1}(0,\lambda;\xi_{2k}) - Q_{k,2}(0,\lambda;\xi_{2k})| \leqslant K\,|\lambda| \left| \frac{f_k(\xi_{2k-1},\lambda)\,f_{-k}(\xi_{2k-1},\lambda)}{f_k(\xi_{2k},\lambda)\,f_{-k}(\xi_{2k},\lambda)} \right|$$

$$= |\lambda| \left| \exp\left[(-1)^{k+1}\lambda \int_{\xi_{2k}}^{\xi_{2k-1}} (A(\tau+) + A(\tau-))\,d\tau + O(1) \right] \right|,$$

where K is a suitable positive constant. Since

$$A(\tau+) = -A(\tau-) \quad \text{for} \quad \tau \in I_{2k-1}$$

and

$$A(\tau+) = A(\tau-) = (-1)^k \sqrt{|P(\tau)|} \quad \text{for} \quad \tau \in I_{2k},$$

we have

$$\int_{\xi_{2k}}^{\xi_{2k-1}} (A(\tau+) + A(\tau-))\,d\tau = \int_{\xi_{2k}}^{a_{2k}} (A(\tau+) + A(\tau-))\,d\tau = (-1)^k 2 \int_{\xi_{2k}}^{a_{2k}} \sqrt{|P(\tau)|}\,d\tau.$$

This completes the proof of Lemma 13.

LEMMA 14. *We find that*

$$f'_{-k}(\xi_{2k},\lambda) = J_k(\lambda)\,f'_k(\xi_{2k},\lambda)\quad\{1 + F_k(\lambda)\}, \tag{9.18}$$

where

$$|F_k(\lambda)| \leqslant K_2 \left| \exp\left\{ -2\lambda \int_{\xi_{2k}}^{a_{2k}} \sqrt{|P(\tau)|}\;d\tau \right\} \right|$$

and K_2 is a suitable positive constant.

In fact, we have

$$f'_{-k}(\xi,\lambda) = [(-1)^{k+1}\lambda A(\xi+) - \tfrac{1}{4}R(\xi) + Q_{k,1}(0,\lambda;\xi)]\,J_k(\lambda)\,f_k(\xi,\lambda)$$

where $\xi = \xi_{2k}$. Here we used Theorem 1 and Lemma 12. Then using $A(\xi_{2k}+) = A(\xi_{2k}-)$ and Lemma 13, we can prove (9.18) without any difficulty.

Now we shall prove (9.8). The formula (9.9) can be proved in the same manner. Letting $x = \xi_{2k}$ in (4.5) and using Lemmas 12 and 14, we get

$$\Delta_{k,1}(\lambda) = J_k(\lambda)\,D_{k,1}(\lambda) - J_k(\lambda)\,f'_k(\xi_{2k},\lambda)\,f_{k+1}(\xi_{2k},\lambda)\,F_k(\lambda).$$

It is easily seen that $J_k(\lambda)$ is bounded with respect to λ for $\operatorname{Re}\lambda \geqslant M_0$, $|\operatorname{Im}\lambda| \leqslant r_0$. On the other hand, Theorem 1 implies that

$$f'_k(\xi,\lambda)\,f_{k+1}(\xi,\lambda) = \{(-1)^{k+1}\lambda A(\xi-) - \tfrac{1}{4}R(\xi) + Q_{k,2}(0,\lambda;\xi)\}\,f_k(\xi,\lambda)\,f_{k+1}(\xi,\lambda)$$

and $$f_k(\xi,\lambda)\, f_{k+1}(\xi,\lambda) = C_k(\lambda)\, C_{k+1}(\lambda)\, \mathcal{B}(\xi-)^{-2} \exp\,[O(\lambda^{-1})] = O(\varrho^{-\frac{1}{2}m}),$$

where $\xi=\xi_{2k}$. This completes the proof of Lemma 11.

10. Proof of Theorem 2

We shall use the notations which were introduced in Section 4. We have already shown that large positive eigenvalues of Problem (P) are zeros of the function (4.16). Using Lemmas 10 and 11, and the formula (4.9), we find that

$$H_1(\lambda)\begin{pmatrix} D_{0,2}(\lambda) \\ -D_{0,1}(\lambda)\end{pmatrix} = J_1^{-1}(D_{0,2}-J_1 D_{0,1})\begin{pmatrix} D_{1,2} \\ -J_1 D_{1,1}\end{pmatrix} + \varrho^2\begin{pmatrix} U_1(\lambda) \\ V_1(\lambda)\end{pmatrix},$$

where $$|U_1(\lambda)|+|V_1(\lambda)| = O\left(\exp\left[-2\lambda\int_{\xi_2}^{a_2} \sqrt{|P(\tau)|}\, d\tau\right]\right)$$

for $\operatorname{Re}\lambda \geqslant M_0$, $|\operatorname{Im}\lambda| \leqslant r_0$. In the same manner we have

$$H_2 H_1\begin{pmatrix} D_{0,2} \\ -D_{0,1}\end{pmatrix} = (J_1 J_2)^{-1}(D_{0,2}-J_1 D_{0,1})(D_{1,2}-J_1 J_2 D_{1,1})\begin{pmatrix} D_{2,2} \\ -J_2 D_{2,1}\end{pmatrix} + \varrho^3\begin{pmatrix} U_2(\lambda) \\ V_2(\lambda)\end{pmatrix},$$

where $$|U_2(\lambda)|+|V_2(\lambda)| = O\left(\sum_{j=1}^{2} \exp\left[-2\lambda\int_{\xi_{2j}}^{a_{2j}} \sqrt{|P(\tau)|}\, d\tau\right]\right)$$

for $\operatorname{Re}\lambda \geqslant M_0$, $|\operatorname{Im}\lambda| \leqslant r_0$. In general, we have

$$H(\lambda)\begin{pmatrix} D_{0,2}(\lambda) \\ -D_{0,1}(\lambda)\end{pmatrix} = \left\{\prod_{h=1}^{\frac{1}{2}m-1} J_h(\lambda)\right\}^{-1} \prod_{k=1}^{\frac{1}{2}m-1}(D_{k-1,2}-J_{k-1}J_k D_{k-1,1})\begin{pmatrix} D_{\frac{1}{2}m-1,2} \\ -J_{\frac{1}{2}m-1} D_{\frac{1}{2}m-1,1}\end{pmatrix}$$
$$+\varrho^{\frac{1}{2}m}\begin{pmatrix} U_{\frac{1}{2}m-1}(\lambda) \\ V_{\frac{1}{2}m-1}(\lambda)\end{pmatrix}, \tag{10.1}$$

where $$|U_{\frac{1}{2}m-1}(\lambda)|+|V_{\frac{1}{2}m-1}(\lambda)| = O\left(\sum_{j=1}^{\frac{1}{2}m-1} \exp\left[-2\lambda\int_{\xi_{2j}}^{a_{2j}} \sqrt{|P(\tau)|}\, d\tau\right]\right) \tag{10.2}$$

for $\operatorname{Re}\lambda \geqslant M_0$, $|\operatorname{Im}\lambda| \leqslant r_0$. Thus we have proved that *almost all positive eigenvalues of Problem (P) must satisfy the equation*

$$(D_{\frac{1}{2}m,1} D_{\frac{1}{2}m-1,2} - J_{\frac{1}{2}m-1} D_{\frac{1}{2}m-1,1} D_{\frac{1}{2}m,2}) \prod_{k=1}^{\frac{1}{2}m-1}(D_{k-1,2}-J_{k-1}J_k D_{k-1,1}) = \lambda U(\lambda), \tag{10.3}$$

where $J_0(\lambda)\equiv 1$ *and*
$$U(\lambda) = O(\exp(-\beta\lambda)) \tag{10.4}$$

as λ tends to ∞ in the strip $\operatorname{Re}\lambda\geqslant 0$, $|\operatorname{Im}\lambda|\leqslant r_0$, *the quantity β being a suitable positive constant.*

Now by using Lemma 10 and the formula (9.10), we shall rewrite the equation (10.3). In fact, we get

$$D_{k,1}(\lambda)=\omega^{2\lambda\alpha_k+2k-m/2}D_{k,2}(\lambda).$$

Hence, if we use the identity $\alpha_{k+1}=-\alpha_k$, we get

$$D_{0,2}(\lambda)-J_1(\lambda)\,D_{0,1}(\lambda)=D_{0,2}(\lambda)\,\{1+\exp\,[M_1(\lambda)]\} \tag{10.5}$$

and

$$D_{k-1,2}(\lambda)-J_{k-1}(\lambda)\,J_k(\lambda)\,D_{k-1,1}(\lambda)$$
$$=D_{k-1,2}(\lambda)\,\{1+\exp\,[M_{k-1}(\lambda)+M_k(\lambda)]\}\quad (k=2,3,\dots,-1+m/2). \tag{10.6}$$

In the same manner, if we use the identities

$$\alpha_{\frac12 m}=-\alpha_{\frac12 m-1},\qquad (\omega^{m+2})^{\lambda\alpha_{\frac12 m}}=\exp\,[2\pi i\lambda\alpha_{\frac12 m}],\qquad (\omega^{m+2})^{\frac12 m}=(-1)^{\frac12 m},$$

we have

$$D_{\frac12 m,1}(\lambda)\,D_{\frac12 m-1,2}(\lambda)-J_{\frac12 m-1}(\lambda)\,D_{\frac12 m-1,2}(\lambda)\,D_{\frac12 m,2}(\lambda)$$
$$=D_{\frac12 m,1}(\lambda)\,D_{\frac12 m-1,2}(\lambda)\,\{1+\exp\,[-2\pi i\lambda\alpha_{\frac12 m}+M_{\frac12 m-1}(\lambda)]\}. \tag{10.7}$$

Thus we have proved that *almost all positive eigenvalues must satisfy the equation*

$$\{1+\exp\,[-2\pi i\lambda\alpha_{\frac12 m}+M_{\frac12 m-1}]\}\,(1+\exp\,[M_1])\prod_{k=2}^{\frac12 m-1}(1+\exp\,[M_{k-1}+M_k])=V(\lambda), \tag{10.8}$$

where

$$V(\lambda)=O(\exp\,(-\beta\lambda)) \tag{10.9}$$

as λ tends to ∞ in the strip $\operatorname{Re}\lambda\geqslant 0$, $|\operatorname{Im}\lambda|\leqslant r_0$.

Finally, using the definitions (9.11) of the quantities $M_k(\lambda)$, we shall derive Theorem 2. It is easily seen that

$$M_1(\lambda)=-2i\lambda\int_{a_2}^{a_1}\sqrt{|P(\tau)|}\,d\tau+\int_0^{+\infty}Q_{1,2}(s,\lambda;\xi_2)\,\dot\zeta_1(s;\xi_2)\,ds-\int_0^{+\infty}Q_{1,1}(s,\lambda;\xi_2)\,\dot z_1(s;\xi_2)\,ds,$$

and

$$M_{k-1}(\lambda)+M_k(\lambda)=-2i\lambda\int_{a_{2k}}^{a_{2k-1}}\sqrt{|P(\tau)|}\,d\tau+\int_0^{+\infty}Q_{k-1,2}(s,\lambda;\xi_{2(k-1)})\,\dot\zeta_{k-1}(s;\xi_{2(k-1)})\,ds$$
$$-\int_0^{+\infty}Q_{k-1,1}(s,\lambda;\xi_{2(k-1)})\,\dot z_{k-1}(s;\xi_{2(k-1)})\,ds+\int_0^{+\infty}Q_{k,2}(s,\lambda;\xi_{2k})\,\dot\zeta_k(s;\xi_{2k})\,ds$$
$$-\int_0^{+\infty}Q_{k,1}(s,\lambda;\xi_{2k})\,\dot z_k(s;\xi_{2k})\,ds\qquad (k=2,3\dots,-1+m/2).$$

On the other hand, notice that

$$2\pi i b_{1+m/2}(a)=\int A(t)\,dt,$$

where the integration should be taken along a large circle in the counterclockwise sense. Hence we have

$$2\pi i b_{1+m/2}(a)=\int_{a_m}^{a_1}[A(\tau-)-A(\tau+)]\,d\tau=2i\sum_{j=1}^{\frac{1}{2}m}(-1)^j\int_{a_{2j}}^{a_{2j-1}}\sqrt{|P(\tau)|}\,d\tau.$$

Therefore

$$-2\pi i\lambda\alpha_{m/2}=2\pi i(-1)^{1+m/2}\lambda b_{1+m/2}(a)=2i(-1)^{1+m/2}\lambda\sum_{j=1}^{\frac{1}{2}m}(-1)^j\int_{a_{2j}}^{a_{2j-1}}\sqrt{|P(\tau)|}\,d\tau.$$

Thus we have

$$\begin{aligned}-2\pi i\lambda\alpha_{m/2}&+M_{-1+m/2}(\lambda)\\&=-2i\lambda\int_{a_m}^{a_{m-1}}\sqrt{|P(\tau)|}\,d\tau+\int_0^{+\infty}Q_{\frac{1}{2}m-1,2}(s,\lambda;\xi_{m-2})\,\dot{\zeta}_{\frac{1}{2}m-1}(s;\xi_{m-2})\,ds\\&\qquad-\int_0^{+\infty}Q_{\frac{1}{2}m-1,1}(s,\lambda;\xi_{m-2})\,\dot{z}_{\frac{1}{2}m-1}(s;\xi_{m-2})\,ds.\end{aligned}$$

For each $k=1,2,\dots,-1+m/2$, let Γ_k be a circle which encircles the points $a_1,a_2,\dots,a_{2k}$, while the points $a_{2k+1},\dots,a_m$ lie outside the curve Γ_k. Then by using Lemmas 4 and 5 and the definitions (3.4) and (3.5) of $Q_{k,1}$ and $Q_{k,2}$, we find that

$$M_1(\lambda)\cong-2i\lambda\int_{a_1}^{a_2}\sqrt{|P(\tau)|}\,d\tau+\sum_{n=1}^{\infty}\lambda^{-n}\int_{\Gamma_1}A(x)^{-n}p_n(x)\,dx,$$

$$\begin{aligned}M_{k-1}(\lambda)+M_k(\lambda)\cong-2i\lambda\int_{a_{2k}}^{a_{2k-1}}\sqrt{|P(\tau)|}\,d\tau+\sum_{n=1}^{\infty}\lambda^{-n}\Big\{\int_{\Gamma_k}[(-1)^{k+1}A(x)]^{-n}p_n(x)\,dx\\+(-1)^n\int_{\Gamma_{k-1}}[(-1)^{k+1}A(x)]^{-n}p_n(x)\,dx\Big\}\quad(k=2,3,\dots,-1+m/2).\end{aligned}$$

and

$$-2\pi i\lambda\alpha_{m/2}+M_{-1+m/2}(\lambda)\cong-2i\lambda\int_{a_m}^{a_{m-1}}\sqrt{|P(\tau)|}\,d\tau+\sum_{n=1}^{\infty}\lambda^{-n}\int_{\Gamma_{\frac{1}{2}m-1}}[(-1)^{m/2}A(x)]^{-n}p_n(x)\,dx$$

as λ tends to ∞, where $\cong$ indicates an asymptotic expansion in powers of λ^{-1} in the usual sense, and every integral should be taken in the counterclockwise sense. Then if we use (4.20), Theorem 2 can easily be proved. In fact,

$$\int_{\Gamma_k} A(x)^{-n} p_n(x)\,dx=0 \quad \text{for even } n,$$

$$\int_{\Gamma}-\int_{\Gamma_{k-1}}=\int_{\gamma_k}, \quad \int_{\Gamma_{\frac{1}{2}m-1}}=-\int_{\gamma_{\frac{1}{2}m}}.$$

Finally we shall prove (4.20). Put

$$r(x,\lambda)=\lambda A(x)+\sum_{n=1}^{\infty}(\lambda A(x))^{-2n+1}p_{2n-1}(x)$$

and

$$s(x,\lambda)=-\tfrac{1}{4}R(x)+\sum_{n=1}^{\infty}(\lambda A(x))^{-2n}p_{2n}(x).$$

Then the formal series $r+s$ is a formal solution of the Riccati equation (2.3). Since r contains only odd powers of λ, we get

$$r'+2rs=0. \tag{10.10}$$

This formal identity was obtained by N. Fröman [5]. It can be written as

$$s=-\tfrac{1}{2}(\log r)'. \tag{10.11}$$

Hence we get the formal identity

$$\int_{\gamma_k} s(x,\lambda)\,dx=0 \pmod{\pi i}, \tag{10.12}$$

which completes the proof of (4.20).

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