AUTOMORPHISMS AND INVARIANT STATES OF OPERATOR ALGEBRAS

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1. Introduction

Let $\mathfrak{A}$ be a von Neumann algebra and G a group of *-automorphisms of $\mathfrak{A}$ with fixed point algebra $\mathcal{B}$ in $\mathfrak{A}$. If $\mathfrak{A}$ is semi-finite and $\mathcal{B}$ contains the center of $\mathfrak{A}$ the normal G-invariant states of $\mathfrak{A}$ were analysed in [3], [12], [13]. In the present paper we shall extend these studies to the general situation, in which the center is not necessarily left fixed by G. The main result, from which the rest follows, states that if $\mathfrak{A}$ is semi-finite and ω a faithful normal G-invariant state of $\mathfrak{A}$, and if G acts ergodicly on the center of $\mathfrak{A}$, then there exists a faithful normal G-invariant semi-finite trace τ of $\mathfrak{A}$ which is unique up to a scalar multiple, and a positive self-adjoint operator $B \in L^1(\mathfrak{A}, \tau)$ affiliated with $\mathcal{B}$ such that $\omega(A) = \tau(BA)$ for all $A \in \mathfrak{A}$. For example, if G is ergodic on $\mathfrak{A}$ then ω is a trace, hence $\mathfrak{A}$ is finite. As an application to C^*-algebras we show that if $\mathcal{A}$ is an asymptotically abelian C^*-algebra (more specifically G-abelian) and ϱ is an extremal G-invariant state, then either the weak closure of its representation, viz $\pi_\varrho(\mathcal{A})''$, is of type III, or the cyclic vector x_ϱ such that $\varrho(A) = (\pi_\varrho(A)x_\varrho, x_\varrho)$, $A \in \mathcal{A}$, is a trace vector for the commutant of $\pi_\varrho(\mathcal{A})$. This has previously been shown for invariant factor states [12].

The basic technical tool used in this paper is the theory of Tomita [15] and Takesaki [14] on the modular automorphisms associated with faithful normal states of von Neumann algebras. It will, however, mainly be applied to semi-finite algebras. We recall from [14] that if $\mathfrak{A}$ is a von Neumann algebra with a separating and cyclic vector x_0 then the *-operation S: $Ax_0 \to A^*x_0$ is a pre-closed conjugate linear operator with polar decomposition $S = J\Delta^{\frac{1}{2}}$, where J is a conjugation of the underlying Hilbert space, and Δ is a positive self-adjoint operator—the modular operator defined by x_0. The modular automorphism σ_t of $\mathfrak{A}$ associated with x_0 (or rather the state ω_{x_0}) is given by $\sigma_t(A) = \Delta^{it}A\Delta^{-it}$. Furthermore, J satisfies the relation $J\mathfrak{A}J = \mathfrak{A}'$. For details and further results from this

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theory we refer the reader to the notes of Takesaki [14]. For other references on von Neumann algebras the reader is referred to the book of Dixmier [1].

In most of the discussion we shall study faithful normal G-invariant states of $\mathfrak{A}$. If a normal G-invariant state ω is not faithful then its support E belongs to $\mathcal{B}$, hence we can restrict attention to the von Neumann algebra $E\mathfrak{A}E$ and the automorphisms $EAE \to Eg(A)E$, $g \in G$, of this von Neumann algebra, and then apply the results for faithful states.

2. Automorphisms of von Neumann algebras

In this section we prove the main results concerning invariant states of von Neumann algebras.

LEMMA 1. *Let $\mathfrak{A}$ be a von Neumann algebra and let G be a group of unitary operators such that $U\mathfrak{A}U^{-1} = \mathfrak{A}$ for $U \in G$. Suppose x_0 is a separating and cyclic vector for $\mathfrak{A}$ such that $Ux_0 = x_0$ for $U \in G$, and let Δ be its modular operator. Suppose $\Delta^{it} = \Gamma(t)\Gamma'(t)$, where $\Gamma(t)$ (resp. $\Gamma'(t)$) is a strongly continuous one-parameter unitary group in $\mathfrak{A}$ (resp. $\mathfrak{A}'$). If $U\Gamma(t)U^{-1} = \Gamma(t)$ and $U\Gamma'(t)U^{-1} = \Gamma'(t)$ for all t and $U \in G$, then $\mathfrak{A}$ has a faithful normal G-invariant semi-finite trace.*

This lemma follows from the proof of [14, Theorem 14.1], because the trace constructed in that proof will clearly be G-invariant.

LEMMA 2. (1) *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Suppose x_0 is a separating and cyclic vector for $\mathfrak{A}$, and let Δ be its modular operator. Suppose U is a unitary operator on $\mathcal{H}$ such that $U\mathfrak{A}U^{-1} = \mathfrak{A}$ and $Ux_0 = x_0$. Then $U\Delta = \Delta U$ and $UJ = JU$.*

Proof. As in the proof of [14, Theorem 12.1] $\mathfrak{A}$ is made into a generalized Hilbert algebra via the representation $A \to x_0(A) = Ax_0$ with multiplication $x_0(A)x_0(B) = x_0(AB)$ and involution $x_0(A)^{\#} = x_0(A^*)$, $A \in \mathfrak{A}$. The unitary operator U defines an isometric *-automorphism of the generalized Hilbert algebra $\mathfrak{A}$ by $Ux_0(A) = x_0(UAU^{-1})$, which extends to an isometry of the domain $\mathcal{D}^{\#}$ of $\Delta^{\frac{1}{2}}$ onto itself, cf. [14, Theorem 7.1]. Now for $A \in \mathfrak{A}$ we have

$$J\Delta^{\frac{1}{2}}x_0(A) = x_0(A)^{\#} = A^*x_0 = U^{-1}(UAU^{-1})^*x_0 = U^{-1}J\Delta^{\frac{1}{2}}UAU^{-1}x_0$$
$$= (U^{-1}JU)(U^{-1}\Delta^{\frac{1}{2}}U)x_0(A).$$

Since the generalized Hilbert algebra $\mathfrak{A}$ is dense in the Hilbert space $\mathcal{D}^{\#}$ [14, Lemma 3.4] we have that $J\Delta^{\frac{1}{2}}x = (U^{-1}JU)(U^{-1}\Delta^{\frac{1}{2}}U)x$ for all $x \in \mathcal{D}^{\#}$. Hence from the uniqueness of polar decomposition we have $J = U^{-1}JU$ and $\Delta^{\frac{1}{2}} = U^{-1}\Delta^{\frac{1}{2}}U$, hence $\Delta = U^{-1}\Delta U$.

(1) A partial result in this direction has been obtained by Winnink [17, Lemma IV. 5].

We next show our main result. In the theorem we assume that the group G of automorphisms of $\mathfrak{A}$ acts ergodicly on the center $\mathcal{C}$ of $\mathfrak{A}$, i.e. $\mathcal{B} \cap \mathcal{C} = \mathbf{C}$, where $\mathcal{B}$ is the fixed points of G in $\mathfrak{A}$. This assumption is made mainly for convenience and is analogous to that of studying factors rather than general von Neumann algebras.

THEOREM 1. *Let $\mathfrak{A}$ be a semi-finite von Neumann algebra and G a group of *-automorphisms of $\mathfrak{A}$ acting ergodicly on the center of $\mathfrak{A}$. Suppose ω is a faithful normal G-invariant state of $\mathfrak{A}$. Then there exists up to a scalar multiple a unique faithful normal G-invariant semi-finite trace τ of $\mathfrak{A}$, and there is a positive self-adjoint operator $B \in L^1(\mathfrak{A}, \tau)$ affiliated with the fixed point algebra $\mathcal{B}$ of G in $\mathfrak{A}$ such that $\omega(A) = \tau(BA)$ for all $A \in \mathfrak{A}$.*

Proof. Uniqueness. Suppose φ is another normal G-invariant semi-finite trace of $\mathfrak{A}$. Then it is an easy consequence of the Radon–Nikodym theorem for normal traces [1, Ch. III, § 4] that its Radon–Nikodym derivative with respect to τ will be affiliated with both $\mathcal{B}$ and the center of $\mathfrak{A}$, so it is a scalar by hypothesis. Thus $\varphi = \mu\tau$, with $\mu \geqslant 0$.

Existence. We first make a digression. Since G is ergodic on the center $\mathcal{C}$ of $\mathfrak{A}$ it follows that $\mathfrak{A}$ is either of type I, II_1, or II_∞. In the type I and II_1 cases it is easy to show the existence of the invariant trace τ, and we may even weaken the assumptions and only assume that ω is a normal G-invariant state of $\mathcal{C}$. (I am indebted to G. Elliott and R. Kadison for valuable comments on these cases.) Indeed, since G is ergodic on $\mathcal{C}$, ω is faithful on $\mathcal{C}$. Suppose first $\mathfrak{A}$ is of type I. Let E be an abelian projection in $\mathfrak{A}$ with central carrier I. Let ψ be a faithful normal center valued trace of $\mathfrak{A}$ such that $\psi(E) = I$ [1, Ch. III, § 4]. If g is a *-automorphism of $\mathfrak{A}$ then $g(E)$ is an abelian projection in $\mathfrak{A}$ with central carrier I, hence $g(E)$ is equivalent to E [1, Ch. III, § 3]. Thus $I = \psi(E) = \psi(g(E)) = g^{-1}(\psi(g(E)))$. Now $g^{-1}\psi g$ is a faithful normal center valued trace on $\mathfrak{A}$ which coincides with ψ on E. Therefore they are equal, hence ψ is G-invariant. Then $\omega \circ \psi$ is a faithful normal G-invariant semi-finite trace of $\mathfrak{A}$. Note that if $\mathfrak{A}$ is finite there exists a unique faithful normal center valued trace ψ of $\mathfrak{A}$ such that $\psi(I) = I$. By uniqueness ψ is G-invariant, and the proof is completed as in the type I case. Thus all that remains is the II_∞ case. Since the type I and II_1 cases come under the argument we shall give, we only assume $\mathfrak{A}$ is semi-finite.

Considering the Gelfand–Naimark–Segal construction for ω we may assume $\omega = \omega_{x_0}$ with x_0 a separating and cyclic unit vector for $\mathfrak{A}$ in the underlying Hilbert space $\mathcal{H}$, and that there is a unitary representation $g \to U_g$ of G on $\mathcal{H}$ such that $U_g x_0 = x_0$ and $U_g A U_g^{-1} = g(A)$ for all $g \in G$, $A \in \mathfrak{A}$.

Let E_0 be the orthogonal projection on the subspace of $\mathcal{H}$ consisting of all vectors $y \in \mathcal{H}$

such that $U_g y = y$ for all $g \in G$. Then $E_0 x_0 = x_0$, so $E_0 \neq 0$. From the ergodic theorem [11, § 144] there exists a net $\{\Sigma_i \lambda_i^\alpha U_{g_i^\alpha}\}_{\alpha \in K}$ in conv $(U_g: g \in G)$ which converges strongly to E_0. By [7, Theorem 2] there exists a unique faithful normal G-invariant projection map Φ of $\mathfrak{A}$ onto $\mathcal{B}$, and by [2, Corollary 1] we have

$$\Phi(A) = \text{strong} \lim_\alpha \sum_i \lambda_i^\alpha U_{g_i^\alpha} A U_{g_i^\alpha}^{-1} \tag{1}$$

for all $A \in \mathfrak{A}$.

Let Tr be a faithful normal semi-finite trace of $\mathfrak{A}$ [1, p. 99], and let H be a positive self-adjoint operator in $L^1(\mathfrak{A}, \mathrm{Tr})$ such that $\omega_{x_0}(A) = \mathrm{Tr}(HA)$ for all $A \in \mathfrak{A}$ [1, p. 107]. Let Δ be the modular operator and J the unitary involution defined by x_0. By Lemma 2 $U_g \Delta^{it} = \Delta^{it} U_g$ and $U_g J = J U_g$ for all $g \in G$. By [14, Corollary 14.1 and end of § 14] $\Delta^{it} = H^{it} J H^{it} J$ so that $H^{it} = J H^{-it} J \Delta^{it}$ (recall that $J \mathfrak{A} J = \mathfrak{A}'$). Thus for $g \in G$ we have

$$U_g H^{it} U_g^{-1} = J U_g H^{-it} U_g^{-1} J \Delta^{it}.$$

Therefore we have from (1) that

$$\Phi(H^{it}) = J \Phi(H^{-it}) J \Delta^{it}.$$

Let $B_t = \Phi(H^{it})$. Then $B_t \in \mathcal{B}$, and furthermore

$$B_t = J B_t^* J J H^{it} J H^{it},$$

so that $B_t H^{-it} = J B_t^* H^{it} J \in \mathfrak{A} \cap \mathfrak{A}' = \mathcal{C}$, where $\mathcal{C}$ is the center of $\mathfrak{A}$. Therefore $B_t = C_t H^{it}$ with $C_t \in \mathcal{C}$.

Let F_t be the range projection of B_t. Then $F_t \in \mathcal{B}$. But F_t is also the range projection of C_t, hence belongs to $\mathcal{C}$, so that $F_t \in \mathcal{B} \cap \mathcal{C}$, which equals the scalar operators by assumption. Thus either $F_t = 0$ or $F_t = I$. Since Φ is strongly continuous on bounded sets and $H^{it} \to I$ strongly as $t \to 0$, $B_t = \Phi(H^{it}) \to I$ strongly as $t \to 0$. Therefore there is a neighborhood $\mathcal{N}$ of 0 in $\mathbf{R}$ such that $F_t = I$ for $t \in \mathcal{N}$. Let $B_t = V_t |B_t|$ and $C_t = U_t |C_t|$ be the polar decompositions of B_t and C_t. Then V_t and U_t are unitary operators in $\mathcal{B}$ and $\mathcal{C}$ respectively for $t \in \mathcal{N}$. Since $B_t = V_t |B_t| = C_t H^{it} = U_t H^{it} |C_t|$ it follows from the uniqueness of polar decomposition of an operator that $V_t = U_t H^{it}$ and $|B_t| = |C_t|$ for all t. Therefore there is a number $\lambda_t \geq 0$ such that $B_t = \lambda_t V_t = \lambda_t U_t H^{it}$, and $\lambda_t > 0$ for $t \in \mathcal{N}$.

The map $t \to V_t$ is strongly continuous for $t \in \mathcal{N}$. Indeed, $t \to B_t$ is strongly continuous, and so is $t \to B_{-t} = B_t^*$. Since $\|B_t\| \leq 1$, $t \to \lambda_t = |B_t| = (B_t^* B_t)^{\frac{1}{2}}$ is strongly continuous [6]. Therefore $t \to V_t = \lambda_t^{-1} B_t$ is strongly continuous for $t \in \mathcal{N}$.

We next want to define V_t for those t for which $B_t = 0$. Let $\lambda \in \mathcal{N}$, $\lambda \neq 0$, and let $N = [-\lambda, \lambda]$. Consider V_t as only defined for $t \in N$. If $s \notin N$ with $s > 0$ let t be the largest number in N such that $s = tn$ with n a positive integer. Let $V_s = (V_t)^n$. If $s < 0$ let $V_s = V_{-s}^*$. We show that $s \to V_s$ is strongly continuous for $s \neq n\lambda$ and continuous from below (resp. above) if $s=n\lambda$, $n>0$ (resp. $n<0$). Indeed, it suffices to show this for $s>0$. Let $s=nt$ with t the largest number in N which divides s in an integer. Since the function $t\to nt$ is open and continuous there exists a neighborhood $\mathcal{U}_s$ of s such that if $s'\in\mathcal{U}_s$ then $s'=nt'$ with t' in a neighborhood of t. Assume first $t\neq\lambda$. Let $s'\in\mathcal{U}_s$, so $s'=nt'$, $t'\in N$. If $s'=(n+k)t_1$ with $t_1\in N$, k a positive integer, then $t_1<t$. If $s'=(n-1)t_2$, $t_2\in N$, then if s' is sufficiently close to s it follows from the above argument that $s=(n-1)t_3$ with $t_3\in N$. But then $t_3>t$ contradicting the maximality of t. Therefore s' is not of the form $(n-1)t_2$ with $t_2\in N$. If $s'=(n-k)t_2$ with $t_2\in N$, $n-k>1$, then also $s'=(n-1)t_1$ with $t_1=(n-k)(n-1)^{-1}t_2<t_2$, so $t_1\in N$, a case which is ruled out. Therefore there is a neighborhood $\mathcal{V}_s$ of s such that if $s'\in\mathcal{V}_s$ then $s'=nt'$ with t' in a neighborhood of t, and t' is the largest number in N which divides s' in an integer. If $s=n\lambda$ then the same holds for $s'\in\mathcal{W}_s=\{s'\in\mathcal{V}_s: s'\leqslant s\}$. Now let $x_1, \dots, x_r$ be r vectors in $\mathcal{H}$ and $\varepsilon>0$. Since $t\to V_t$ is strongly continuous for $t\in N$, so is $t\to V_t^n$. Therefore, if $\mathcal{X}_s$ is a sufficiently small neighborhood of s contained in $\mathcal{V}_s$ (or in $\mathcal{W}_s$ if $s=n\lambda$) then $\|(V_s-V_{s'})x_j\|=\|(V_t^n-V_{t'}^n)x_j\|<\varepsilon$ for $s'\in\mathcal{X}_s$. Thus $s\to V_s$ is strongly continuous for $s\neq n\lambda$ and strongly continuous from below for $s=n\lambda$, as asserted.

Let $s=nt$, $t\in N$. Then $V_t=U_tH^{it}$ with $U_t\in\mathfrak{C}$, and $V_s=V_t^n=U_t^nH^{is}$. Hence if $A\in\mathfrak{A}$ we have $V_sAV_s^{-1}=H^{is}AH^{-is}$. Note that

$$V_sV_{s'}AV_{s'}^{-1}V_s^{-1}=H^{i(s+s')}AH^{-i(s+s')}=V_{s+s'}AV_{s+s'}^{-1}.$$

Now $V_sV_{s'}V_{s+s'}^{-1}=\gamma(s,\ s')I$ with $\gamma(s,\ s')$ in the circle group T_1, because $V_sV_{s'}V_{s+s'}^{-1}\in\mathfrak{B}\cap\mathfrak{C}=\mathfrak{C}$. One can easily show that γ: $\mathbf{R}\times\mathbf{R}\to T_1$ is a Borel map. Furthermore, since $V_t=H^{is}U_t^n$ all the V_s commute with each other. Therefore it is trivial to show that

$$\gamma(s_2, s_3)\gamma(s_1+s_2, s_3)^{-1}\gamma(s_1, s_2+s_3)\gamma(s_1, s_2)^{-1}=1$$

for all $s_1, s_2, s_3\in\mathbf{R}$. Thus γ is a 2-cocycle as a cochain on $\mathbf{R}$ with coefficients in T_1 (with trivial action on T_1) in the usual cohomology theory of groups cf. [10]. Since $H^2(\mathbf{R}, T_1)=0$ [10, Theorem 11.5] γ is a 2-coboundary, so there is a function $\xi(s)$ on $\mathbf{R}$ with values in T_1 such that $\gamma(s,s')=\xi(s)^{-1}\xi(s')^{-1}\xi(s+s')$, and as pointed out by Kadison [5, p. 197] it follows from [9, Théorème 2] that $\xi(s)$ can be chosen as a Borel function. Since $\gamma(s,\ -s)=1$ and we may normalize ξ so that $\xi(0)=1$, we have that $\xi(s)^{-1}=\xi(-s)$.

We next show that $\xi(s)$ is continuous at 0, and for this we modify the proof of [4, Theorem 22.18]. Let W_0 be a symmetric neighborhood of 1 in T_1, and let W be a symmetric neighborhood of 1 in T_1 such that $W^3\subset W_0$. Since T_1 is compact there is a finite subset $y_1, \dots, y_r\in T_1$ such that $T_1=\bigcup_{n=1}^r Wy_n$. Now γ is continuous in a neighborhood of 0 in $\mathbf{R}\times\mathbf{R}$. Let A be an open symmetric neighborhood of 0 in $\mathbf{R}$ such that if $a, b\in A$ then $\gamma(a,\ -b)\in W$. We have that $A=\bigcup_{n=1}^r(\xi^{-1}(Wy_n)\cap A)$. Since $\xi(s)$ is Borel by the preceding paragraph, we have at least one value of n for which $\xi^{-1}(Wy_n)\cap A$ is Borel measurable and has positive Lebesgue measure. By [4, Corollary 20.17] there is a neighborhood V of 0 in $\mathbf{R}$ such that

$$V\subset(\xi^{-1}(Wy_n)\cap A)-(\xi^{-1}(Wy_n)\cap A).$$

Let $s\in V$. Let $a, b\in\xi^{-1}(Wy_n)\cap A$ be such that $s=a-b$. Then $\xi(a)=w_1y_n$, $\xi(b)=w_2y_n$ with $w_1, w_2\in W$. Thus we have

$$\xi(s)=\xi(a-b)=\gamma(a,\ -b)\ \xi(a)\xi(b)^{-1}=\gamma(a,\ -b)w_1w_2^{-1}\in W^3\subset W_0.$$

Thus ξ is continuous at 0 as asserted.

Let $\Gamma(s)=\xi(s)V_s$. Then

$$\begin{aligned}\Gamma(s+s')=\xi(s+s')V_{s+s'}&=\xi(s+s')\gamma(s,s')^{-1}V_sV_{s'}\\&=\xi(s+s')\xi(s)\xi(s')\xi(s+s')^{-1}V_sV_{s'}=\Gamma(s)\Gamma(s'),\end{aligned}$$

so that $s\to\Gamma(s)$ is a one-parameter unitary representation in $\mathfrak{B}$, which is strongly continuous at 0, hence strongly continuous everywhere. Furthermore, if $A\in\mathfrak{A}$ then

$$\Gamma(s)A\Gamma(-s)=V_sAV_s^{-1}=H^{is}AH^{-is}=\Delta^{is}A\Delta^{-is}.$$

Let $\Gamma'(s)=\Gamma(-s)\Delta^{is}$. Then $s\to\Gamma'(s)$ is a strongly continuous one parameter unitary group in $\mathfrak{A}'$, and $\Delta^{is}=\Gamma(s)\Gamma'(s)$ for all $s\in R$. Therefore the assumptions in Lemma 1 are satisfied, so $\mathfrak{A}$ has a faithful normal G-invariant semi-finite trace τ. Let B be the positive self-adjoint operator in $L^1(\mathfrak{A},\tau)$ such that $\omega(A)=\tau(BA)$ for $A\in\mathfrak{A}$. Then if $g\in G$ we have

$$\tau(U_gBU_g^{-1}A)=\tau(BU_g^{-1}AU_g)=\omega(U_g^{-1}AU_g)=\omega(A)=\tau(BA).$$

By the uniqueness of B, $B=U_gBU_g^{-1}$ for all $g\in G$, hence B is affiliated with $\mathfrak{B}$. This completes the proof of the theorem.

We note that the converse of the theorem is a triviality.

COROLLARY 1.(1) *Let assumptions and notation be as in Theorem 1. Then $\mathfrak{B}$ is semi-finite.*

Proof. By Theorem 1 $\omega(A)=\tau(BA)$ for $A\in\mathfrak{A}$, with B affiliated with $\mathfrak{B}$. Thus the modular automorphism σ_t of ω is $\sigma_t(A)=B^{it}AB^{-it}$. Since B is affiliated with $\mathfrak{B}$, σ_t is also the modular automorphism of ω restricted to $\mathfrak{B}$. Since $\sigma_t|\mathfrak{B}$ is inner, $\mathfrak{B}$ is semi-finite by [14, Theorem 14.1].

The next two corollaries are direct generalizations of theorems of Hugenholtz [3] and the author [12], see also [1, p. 101, Théorème 7].

COROLLARY 2. *Let $\mathfrak{A}$ be a semi-finite von Neumann algebra and G an ergodic group of *-automorphisms of $\mathfrak{A}$. Suppose ω is a faithful normal G-invariant state of $\mathfrak{A}$. Then $\mathfrak{A}$ is finite and ω is a trace.*

(1) This corollary also follows from [16].

Proof. Let τ and B be as in Theorem 1. Since $\mathcal{B}=\mathbf{C}I$, B is a scalar λI, $\lambda>0$. Thus $\omega(A)=\lambda\tau(A)$ is a finite trace of $\mathfrak{A}$. In particular $\mathfrak{A}$ is finite.

A more direct proof of this corollary can be obtained if we notice that if $B_t=C_tH^{it}$ as in the proof of Theorem 1, then B_t is a scalar, hence $H^{it}AH^{-it}=A$ for t in a neighborhood of 0 for all $A\in\mathfrak{A}$. Thus H is affiliated with the center of $\mathfrak{A}$, so $\omega(A)=\operatorname{Tr}(HA)$ is a trace on $\mathfrak{A}$.

COROLLARY 3. *Let $\mathfrak{A}$ be a von Neumann algebra acting on a Hilbert space $\mathcal{H}$. Let G be a group of unitary operators on $\mathcal{H}$ such that $U\mathfrak{A}U^{-1}=\mathfrak{A}$ for $U\in G$. Suppose there exists a unit vector $x_0\in\mathcal{H}$ such that*

(i) *x_0 is cyclic for $\mathfrak{A}$,*

(ii) *$\mathbf{C}x_0$ is the set of vectors in $\mathcal{H}$ invariant under G.*

Then $\mathfrak{A}$ is of type III if and only if x_0 is not a trace vector for $\mathfrak{A}'$.

Proof. Let $F=[\mathfrak{A}'x_0]$. Then F is the support of the G-invariant state ω_{x_0}, so $F\in\mathcal{B}$—the fixed point algebra of G in $\mathfrak{A}$, Since x_0 is cyclic for $\mathfrak{A}$, it is separating for $\mathfrak{A}'$, hence $\mathfrak{A}'\cong\mathfrak{A}'F$. Thus $\omega_{x_0}|\mathfrak{A}'$ is a trace if and only if $\omega_{x_0}|\mathfrak{A}'F$ is a trace. If $\mathfrak{A}$ is of type III then so is $\mathfrak{A}'$, hence $\omega_{x_0}|\mathfrak{A}'$ is not a trace. Conversely, assume $\omega_{x_0}|\mathfrak{A}'$ is not a trace, hence $\omega_{x_0}|\mathfrak{A}'F$ is not a trace. We show that under this assumption $F\mathfrak{A}F$ is of type III, hence $\mathfrak{A}'F$ is of type III, so that $\mathfrak{A}'$ is of type III, and therefore $\mathfrak{A}$ is of type III. We may therefore assume $F=I$, i.e. we assume x_0 is separating and cyclic for $\mathfrak{A}$. Let E_0 be the one dimensional projection on $\mathbf{C}x_0$. By (ii) and the ergodic theorem [11, § 144] $E_0\in\operatorname{conv}(U\colon U\in G)^-$, so $E_0\in\mathcal{B}'$. Thus ω_{x_0} is a faithful homomorphism of $\mathcal{B}$ onto $\mathbf{C}$, so $\mathcal{B}=\mathbf{C}I$. Since the central projections in $\mathfrak{A}$ on the different type portions of $\mathfrak{A}$ are invariant under the automorphisms, they are in $\mathcal{B}=\mathbf{C}I$. Therefore $\mathfrak{A}$ is either semi-finite or of type III. If $\mathfrak{A}$ is semi-finite then by Corollary 2 $\mathfrak{A}$ is finite and ω_{x_0} is a trace. Since x_0 is separating and cyclic for $\mathfrak{A}$, $\omega_{x_0}|\mathfrak{A}'$ is also a trace, contradicting our hypothesis. Therefore $\mathfrak{A}$ is of type III.

Remark. If the von Neumann algebra $\mathfrak{A}$ is not semi-finite we can obtain an analogue of Theorem 1 as follows. Suppose ω and ϱ are normal G-invariant states of $\mathfrak{A}$ with ω faithful. Then there exists a positive self-adjoint operator H affiliated with $\mathcal{B}$ such that $\varrho(A)=\omega(HAH)$ for all $A\in\mathfrak{A}$. Indeed by [7], see also [2], there exists a unique faithful normal G-invariant projection Φ of $\mathfrak{A}$ onto $\mathcal{B}$ such that $\varrho=(\varrho|\mathcal{B})\circ\Phi$. By the Radon–Nikodym Theorem for von Neumann algebras [14, Theorem 15.1] there exists a positive self-adjoint operator H affiliated with $\mathcal{B}$ such that $\varrho(B)=\omega(HBH)$ for $B\in\mathcal{B}$, hence $\varrho(A)=\varrho(\Phi(A))=\omega(H\Phi(A)H)$ for $A\in\mathfrak{A}$. But the state $A\to\omega(HAH)$ is normal and G-invariant. Hence $\varrho(A)=\omega(H\Phi(A)H)=\omega(HAH)$, $A\in\mathfrak{A}$, as asserted.

3. Asymptotically abelian C^*-algebras

It was shown in [12] that the specialization of Corollary 3 to factors was applicable to describe the types of invariant factor states of asymptotically abelian C^*-algebras. We can now give a criterion valid for all extremal invariant states, and this can be done for the most general of the different notions of asymptotic abelianness, namely that of G-abelian introduced by Lanford and Ruelle [8]; see [2] for the other notions.

Let $\mathcal{A}$ be a C^*-algebra and G a group of $*$-automorphisms of $\mathcal{A}$. We say $\mathcal{A}$ is *G-abelian* if for each G-invariant state ϱ of $\mathcal{A}$ and all self-adjoint operators A, $B \in \mathfrak{A}$ we have

$$0 = \inf \{ |\varrho([A', B])| : A' \in \operatorname{conv}(g(A) : g \in G)\}.$$

Let $\varrho(A) = (\pi_\varrho(A) x_\varrho, x_\varrho)$ be its Gelfand–Naimark–Segal decomposition, and $g \to U_g$ a unitary representation of G on the Hilbert space $\mathcal{H}_\varrho$ such that $U_g x_\varrho = x_\varrho$, and $\pi_\varrho(g(A)) = U_g \pi_\varrho(A) U_g^{-1}$, $A \in \mathcal{A}$. Then ϱ is extremal invariant if and only if x_ϱ is up to a scalar multiple the unique vector $y \in \mathcal{H}_\varrho$ such that $U_g y = y$ for all $g \in G$. We thus have the following immediate consequence of Corollary 3.

COROLLARY 4. *Let $\mathcal{A}$ be a C^*-algebra and G a group of $*$-automorphisms of $\mathcal{A}$. Suppose $\mathcal{A}$ is G-abelian and that ϱ is an extremal G-invariant state of $\mathcal{A}$. Then $\pi_\varrho(\mathcal{A})''$ is a von Neumann algebra of type III if and only if ω_{x_ϱ} is not a trace when restricted to $\pi_\varrho(\mathcal{A})'$.*

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