LOCALIZATION OF SHEAVES AND COUSIN COMPLEXES

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Introduction

One of the main difficulties in the theory of duality for coherent sheaves on schemes, or on analytic spaces, is the problem of joining locally defined objects of the derived category of the category of sheaves to a global object. Grothendieck presented a solution in the algebraic case (Hartshorne [4]) by showing that there is a category of complexes of sheaves, the injective Cousin complexes, which is equivalent to a subcategory of the derived category. It is then possible to join together locally defined objects of this subcategory.

The Cousin complexes are characterized in (Hartshorne [4]) by means of local cohomology. However, the procedure is not subject to immediate generalization, since it depends strongly on the special topological properties of the underlying space of a locally noetherian scheme. The purpose of this paper is to investigate the problem without restrictive hypotheses concerning the underlying space.

In section 1 we study localization in a category, in the sense of Gabriel [1], and its relation to local cohomology. For convenience we consider only categories of sheaves and localizing subcategories defined by subsets, though categories with injective envelopes may be treated in the same manner. In section 2 the results are extended to the category of complexes of sheaves. Also, Cousin complexes with respect to a filtration of the space are defined and some of their general properties are studied.

In section 3 we introduce a class of filtrations of the space, the admissible filtrations. The main result is Theorem 3.9, which shows that a subcategory of the category of Cousin complexes with respect to an admissible filtration is equivalent to a subcategory of the derived category.

In particular, when applied to locally noetherian schemes and filtrations defined by a codimension function (Hartshorne [4], V § 7), Theorem 3.9 implies that the category of all Cousin complexes is equivalent to a subcategory of the derived category.

For information concerning the derived category and the derived functors we refer to (Hartshorne [4]).

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§ 1. Z-pure and Z-closed Modules

Let $(X, \mathcal{O}_X)$ be a ringed space, Z a subset of X (not necessarily closed or locally closed). The category of $\mathcal{O}_X$-Modules is denoted by $C(X)$; those $\mathcal{O}_X$-Modules whose support is contained in Z form a full subcategory $C_Z(X)$ of $C(X)$. We recall that the support of a sheaf of abelian groups $\mathcal{F}$ on X, written Supp $(\mathcal{F})$, is the set of points $x \in X$ such that $\mathcal{F}_x \neq 0$; it is not necessarily closed in X.

PROPOSITION 1.1. *If $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ is an exact sequence of $\mathcal{O}_X$-Modules, then $\mathcal{F}$ is in $C_Z(X)$ if and only if $\mathcal{F}'$, $\mathcal{F}''$ are in $C_Z(X)$.*

In fact, Supp $(\mathcal{F})$ is the union of Supp $(\mathcal{F}')$ and Supp $(\mathcal{F}'')$.

It follows from Proposition 1.1 that Ker (u), Im (u), and Coker (u) are in $C_Z(X)$ whenever u is a morphism of $C_Z(X)$. Hence $C_Z(X)$ is an abelian category, and the natural embedding functor $C_Z(X) \to C(X)$ is exact.

PROPOSITION 1.2. *If $\mathcal{F}$ is an $\mathcal{O}_X$-Module, there is a largest member $\Gamma_Z(\mathcal{F})$ in the family of sub-Modules $\mathcal{E}$ of $\mathcal{F}$ such that* Supp $(\mathcal{E}) \subset Z$.

Proof. Let $\mathcal{E}$ be a sub-Module of $\mathcal{F}$ such that Supp $(\mathcal{E}) \subset Z$. For each open subset U of X, $\Gamma(U, \mathcal{E})$ is contained in the family $\Gamma_{Z \cap U}(U, \mathcal{F}|U)$ of sections $f \in \Gamma(U, \mathcal{F})$ having the property Supp $(\mathcal{O}_U \cdot f) \subset Z \cap U$. It is easy to see, however, that the mapping $U \to \Gamma_{Z \cap U}(U, \mathcal{F}|U)$ is a sub-Module of $\mathcal{F}$ with support in Z.

If $\mathcal{F}$ is an object of $C_Z(X)$, each morphism $\mathcal{F} \to \mathcal{G}$ factors through $\Gamma_Z(\mathcal{G})$. Hence $\mathcal{F} \mapsto \Gamma_X(\mathcal{F})$ is a functor from $C(X)$ to $C_Z(X)$; in fact, it is the right adjoint of the embedding functor $C_Z(X) \to C(X)$. Clearly, $\Gamma_Z(\mathcal{F}) = \mathcal{F}$ if and only if $\mathcal{F}$ is an object of $C_Z(X)$.

The functor Γ_Z is left exact; its right derived functors are denoted by $\mathcal{H}^i_Z$ for $i > 0$, $\mathcal{H}^0_Z = \Gamma_Z$. More generally, for each subset Z' of Z, we write $\Gamma_{Z/Z'}(\mathcal{F}) = \Gamma_Z(\mathcal{F})/\Gamma_{Z'}(\mathcal{F})$ and denote by $\mathcal{H}^i_{Z/Z'}(i \geqslant 0)$ the right derived functors of $\Gamma_{Z/Z'}$. In particular, we have the functor $\Gamma_{X/Z}(\mathcal{F}) = \mathcal{F}/\Gamma_Z(\mathcal{F})$ and its right derived functors $\mathcal{H}^i_{X/Z}$ $(i \geqslant 0)$; also, $\mathcal{H}^i_{Z/\phi} = \mathcal{H}^i_Z$ for each $i \geqslant 0$.

PROPOSITION 1.3. *Let Z, Z', Z'' be subsets of X such that $Z \supset Z' \supset Z''$. For each $\mathcal{O}_X$-Module $\mathcal{F}$ there is a long exact sequence*

$$0 \to \mathcal{H}^0_{Z'/Z''}(\mathcal{F}) \to \mathcal{H}^0_{Z/Z''}(\mathcal{F}) \to \mathcal{H}^0_{Z/Z'}(\mathcal{F})$$
$$\to \mathcal{H}^1_{Z'/Z''}(\mathcal{F}) \to \mathcal{H}^1_{Z/Z''}(\mathcal{F}) \to \mathcal{H}^1_{Z/Z'} \to \dots$$

functorial in $\mathcal{F}$. *In particular, there is a functorial exact sequence*

$$0 \to \Gamma_Z(\mathcal{F}) \to \mathcal{F} \xrightarrow{\varrho_{X/Z}(\mathcal{F})} \mathcal{H}^0_{Z/Z}(\mathcal{F}) \to \mathcal{H}^1_Z(\mathcal{F}) \to 0$$

and a family of isomorphisms of functors

$$\mathcal{H}^i_{Z/Z}(\mathcal{F}) \to \mathcal{H}^{i+1}_Z(\mathcal{F}), \quad i \geqslant 1.$$

Indeed, the long exact sequence is defined by the exact sequence of functors

$$0 \to \Gamma_{Z'/Z''}(\mathcal{F}) \to \Gamma_{Z/Z''}(\mathcal{F}) \to \Gamma_{Z/Z'}(\mathcal{F}) \to 0.$$

A homomorphism $f\colon \mathcal{M} \to \mathcal{N}$ of $\mathcal{O}_X$-Modules is called a *Z-isomorphism* if Ker (f) and Coker (f) are in $C_Z(X)$; this means that $f_x\colon \mathcal{M}_x \to \mathcal{N}_x$ is bijective for each $x \in X - Z$. By Proposition 1.3 the canonical homomorphism $\varrho_{X/Z}(\mathcal{F})\colon \mathcal{F} \to \mathcal{H}^0_{X/Z}(\mathcal{F})$ is a Z-isomorphism for each $\mathcal{O}_X$-Module $\mathcal{F}$.

LEMMA 1.4. *For each* $\mathcal{O}_X$*-Module* $\mathcal{F}$ *we have* $\Gamma_Z(\mathcal{H}^0_{X/Z}(\mathcal{F})) = 0$. *If* $\mathcal{F}$ *is in* $C_Z(X)$, *then* $\mathcal{H}^0_{X/Z}(\mathcal{F}) = \mathcal{H}^1_Z(\mathcal{F}) = 0$.

Proof. Since any $\mathcal{O}_X$-Module $\mathcal{F}$ may be embedded into an injective $\mathcal{O}_X$-Module $\mathcal{J}$, in which case $\mathcal{H}^0_{X/Z}(\mathcal{F})$ is a sub-Module of $\mathcal{H}^0_{X/Z}(\mathcal{J})$, it is sufficient to prove the first assertion for an injective Module. If $\mathcal{F}$ is injective, $\varrho_{X/Z}\colon \mathcal{F} \to \mathcal{H}^0_{X/Z}(\mathcal{F})$ is an epimorphism, so that $\mathcal{G} = (\varrho_{X/Z})^{-1}(\Gamma_Z(\mathcal{H}^0_{X/Z}(\mathcal{F})))$ is an extension of $\Gamma_Z(\mathcal{H}^0_{X/Z}(\mathcal{F}))$ by $\Gamma_Z(\mathcal{F})$. Then $\mathcal{G}$ is in $C_Z(X)$ by Proposition 1.1, so $\mathcal{G} = \Gamma_Z(\mathcal{F})$ and therefore $\Gamma_Z(\mathcal{H}^0_{X/Z}(\mathcal{F})) = 0$.

If Supp $(\mathcal{F}) \subset Z$, then $\Gamma_Z(\mathcal{F}) \to \mathcal{F}$ is an isomorphism, so that $\mathcal{H}^0_{X/Z}(\mathcal{F}) \to \mathcal{H}^1_Z(\mathcal{F})$ is bijective by Proposition 1.3. As $\mathcal{H}^1_Z(\mathcal{F})$ is in $C_Z(X)$, we must have $\mathcal{H}^0_{X/Z}(\mathcal{F}) = \Gamma_Z(\mathcal{H}^0_{X/Z}(\mathcal{F})) = 0$.

THEOREM 1.5. *For each* $\mathcal{O}_X$*-Module* $\mathcal{F}$ *the following conditions are equivalent:*

(a) $\varrho_{X/Z}(\mathcal{F})\colon \mathcal{F} \to \mathcal{H}^0_{X/Z}(\mathcal{F})$ *is a monomorphism (an isomorphism).*

(a′) $\Gamma_Z(\mathcal{F}) = 0$ *(and* $\mathcal{H}^1_Z(\mathcal{F}) = 0$*).*

(b) *For each object* $\mathcal{E}$ *of* $C_Z(X)$, $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = 0$ *(and* $\mathrm{Ext}^1_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F}) = 0$*).*

(c) *For each Z-isomorphism* $f\colon \mathcal{M} \to \mathcal{N}$ *of* $\mathcal{O}_X$*-Modules the map*

$$\mathrm{Hom}_{\mathcal{O}_X}(f, \mathcal{F})\colon \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{N}, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{M}, \mathcal{F})$$

is injective (bijective).

Proof. The equivalence of (a) and (a') is an immediate consequence of 1.3. To prove that (a) and (a') imply (b), let $\mathcal{E}$ be an object of $C_Z(X)$. Each homomorphism $\mathcal{E}\to\mathcal{J}$ factors through $\Gamma_Z(\mathcal{J})$, so $\mathrm{Hom}_{O_X}(\mathcal{E}, \mathcal{J})=0$ whenever $\Gamma_Z(\mathcal{J})=0$. For the second part, let $\mathcal{G}$ be an extension of $\mathcal{E}$ by $\mathcal{J}$. Then $\mathcal{H}^0_{X/Z}(\mathcal{J})\to\mathcal{H}^0_{X/Z}(\mathcal{G})$ is an isomorphism, since $\mathcal{H}^0_{X/Z}(\mathcal{E})=0$ by Lemma 1.4. If $\mathcal{J}\to\mathcal{H}^0_{X/Z}(\mathcal{J})$ is bijective, $\mathcal{J}\to\mathcal{G}\to\mathcal{H}^0_{X/Z}(\mathcal{G})$ is an isomorphism, so the extension splits.

Next we prove that (c) follows from (b). Let f: $\mathcal{M}\to\mathcal{N}$ be a Z-isomorphism of O_X-Modules. If $\mathrm{Hom}_{O_X}(\mathcal{E}, \mathcal{J})=0$ for each object $\mathcal{E}$ of $C_Z(X)$, the kernel of the map $\mathrm{Hom}_{O_X}(f, \mathcal{J})$ is $\mathrm{Hom}_{O_X}(\mathrm{Coker}\,(f), \mathcal{J})=0$. If, in addition, f is an epimorphism, the cokernel of $\mathrm{Hom}_{O_X}(f, \mathcal{J})$ is a sub-Module of $\mathrm{Hom}_{O_X}(\mathrm{Ker}\,(f), \mathcal{J})=0$, so that $\mathrm{Hom}_{O_X}(f, \mathcal{J})$ is bijective. Hence, it is enough to show that $\mathrm{Hom}_{O_X}(f, \mathcal{J})$ is bijective if f is a monomorphism and $\mathrm{Hom}_{O_X}(\mathcal{E}, \mathcal{J})=\mathrm{Ext}^1_{O_X}(\mathcal{E}, \mathcal{J})=0$ for each object $\mathcal{E}$ of $C_Z(X)$. But this follows from the long exact sequence of the functor Ext.

It remains to show that (c) implies (a'). Applying (c) to the trivial Z-isomorphism $0\to\Gamma_Z(\mathcal{J})$ we find that $\mathrm{Hom}_{O_X}(\Gamma_Z(\mathcal{J}), \mathcal{J})=0$, whence $\Gamma_Z(\mathcal{J})=0$. Furthermore, if the map $\mathrm{Hom}_{O_X}(\mathcal{H}^0_{X/Z}(\mathcal{J}), \mathcal{J})\to\mathrm{Hom}_{O_X}(\mathcal{J}, \mathcal{J})$ is surjective, the extension

$$0\to\mathcal{J}\to\mathcal{H}^0_{X/Z}(\mathcal{J})\to\mathcal{H}^1_Z(\mathcal{J})\to 0$$

splits. But, by Lemma 1.4, $\mathcal{H}^0_{X/Z}(\mathcal{J})$ has no nontrivial direct factor with support in Z, so that $\mathcal{H}^1_Z(\mathcal{J})=0$.

Definition 1.6. An O_X-Module $\mathcal{J}$ is *Z-pure* (*Z-closed*) if the equivalent conditions (a), (a'), (b), (c) of Theorem 1.5 are satisfied.

PROPOSITION 1.7. *For each O_X-Module $\mathcal{J}$, $\mathcal{H}^0_{X/Z}(\mathcal{J})$ is Z-pure; if $\mathcal{J}$ is Z-pure, $\mathcal{H}^0_{X/Z}(\mathcal{J})$ is Z-closed.*

Proof. The first assertion has already been proved (Lemma 1.4). If $\mathcal{J}$ is Z-pure, the monomorphism $\varrho_{X/Z}(\mathcal{J})$: $\mathcal{J}\to\mathcal{H}^0_{X/Z}(\mathcal{J})=\mathcal{G}$ defines an isomorphism

$$\mathcal{H}^0_{X/Z}(\varrho_{X/Z}(\mathcal{J}))\colon \mathcal{H}^0_{X/Z}(\mathcal{J})\to\mathcal{H}^0_{X/Z}(\mathcal{G}),$$

since $\mathcal{H}^0_{X/Z}(\mathcal{H}^1_Z(\mathcal{J}))=0$. Thus it is enough to show that $\varrho_{X/Z}(\mathcal{G})=\mathcal{H}^0_{X/Z}(\varrho_{X/Z}(\mathcal{J}))$. As $\mathcal{H}^0_{X/Z}(\mathcal{G})$ is Z-pure, it suffices to note, by 1.5 (c), that they induce the same homomorphism from $\mathcal{J}$ to $\mathcal{H}^0_{X/Z}\mathcal{G})$.

In virtue of Proposition 1.7 we can associate with each O_X-Module $\mathcal{J}$ functorially a Z-closed Module $\mathcal{H}^0_{X/Z}(\mathcal{H}^0_{X/Z}(\mathcal{J}))$. It is called the *$Z$-closure* of $\mathcal{J}$ and denoted by $Cl_{X/Z}(\mathcal{J})$. Composing the homomorphisms $\varrho_{X/Z}(\mathcal{J})$ and $\varrho_{X/Z}(\mathcal{H}^0_{X/Z}(\mathcal{J}))$ we obtain a natural transfor-

mation $\mu_{X/Z}$ from the identity functor to the functor $Cl_{X/Z}$. The homomorphism $\mu_{X/Z}(\mathcal{F})$: $\mathcal{F}\to Cl_{X/Z}(\mathcal{F})$ is a Z-isomorphism for each $\mathcal{O}_X$-Module $\mathcal{F}$; if $\mathcal{F}$ is Z-closed, $\mu_{X/Z}(\mathcal{F})$ is an isomorphism.

The following proposition gives a direct characterization of the functor $Cl_{X/Z}$.

PROPOSITION 1.8. *Each homomorphism from an $\mathcal{O}_X$-Module $\mathcal{F}$ to a Z-closed $\mathcal{O}_X$-Module $\mathcal{G}$ has a unique factorization*

$$\mathcal{F}\xrightarrow{\mu_{X/Z}(\mathcal{F})} Cl_{X/Z}(\mathcal{F})\to\mathcal{G}.$$

This is an immediate consequence of 1.5 (c), since $\mu_{X/Z}(\mathcal{F})$ is a Z-isomorphism.

If the full subcategory of $C(X)$ formed by Z-closed $\mathcal{O}_X$-Modules is denoted by $C_{X/Z}(X)$, Proposition 1.8 means that $Cl_{X/Z}$ is the left adjoint of the natural embedding functor $C_{X/Z}(X)\to C(X)$.

COROLLARY 1.9. *If f: $\mathcal{F}\to\mathcal{F}'$ is a Z-isomorphism of $\mathcal{O}_X$-Modules,*

$$Cl_{X/Z}(f)\colon Cl_{X/Z}(\mathcal{F})\to Cl_{X/Z}(\mathcal{F}')$$

is an isomorphism.

Indeed, f induces an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}',\mathcal{G})\to\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})$$

of functors of Z-closed Modules $\mathcal{G}$.

It follows from 1.9 that a Z-isomorphism of Z-closed Modules is an isomorphism.

Remark. The functor $Cl_{X/Z}$ is the localization functor of Gabriel [1]. Similar results are obtained if $C_Z(X)$ is replaced by any localizing subcategory of $C(X)$. More generally, the construction of the localization functor applies in any category with injective envelopes.

The connection of localization with local cohomology has also been studied in [2] (IV, 5.9) in the case of a locally noetherian scheme. Additional information concerning local cohomology is given in [3], [4], [5].

§ 2. Cousin complexes

Let $(X, \mathcal{O}_X)$ be a ringed space, $Z^\cdot=(Z^p)_{p\in Z}$ a family of subsets of X such that $Z^p\supset Z^{p+1}$ for each $p\in Z$, and $Z^p=X$ for some $p\in Z$. In other words, $Z^\cdot$ is a strictly exhaustive decreasing filtration of X.

By a complex $\mathcal{F}^\cdot$ we shall always mean a complex of $\mathcal{O}_X$-Modules. The category whose objects are complexes, and whose morphisms are homotopy equivalence classes of morphisms of complexes, is denoted by $K(X)$.

Definition 2.1. A complex $\mathcal{F}^\cdot$ *has supports in* $Z^\cdot$, or is a *complex with supports in* $Z^\cdot$, if $\mathrm{Supp}\,(\mathcal{F}^p) \subset Z^p$ for each $p \in Z$.

The complexes with supports in $Z^\cdot$ form a full subcategory $K_{Z^\cdot}(X)$ of $K(X)$.

PROPOSITION 2.2. *Each complex* $\mathcal{L}^\cdot$ *has a subcomplex* $\mathcal{F}^\cdot$ *with supports in* $Z^\cdot$ *such that each morphism from a complex with supports in* $Z^\cdot$ *to* $\mathcal{L}^\cdot$ *factors through* $\mathcal{F}^\cdot$.

A complex $\mathcal{F}^\cdot$ having the desired property is given by $\mathcal{F}^p = \mathrm{Ker}\,(\Gamma_{Z^p}(\mathcal{L}^p) \to \Gamma_{Z/X^{p-1}}(\mathcal{L}^{p+1}))$ for $p \in Z$. It is necessarily unique, and functorial in $\mathcal{L}^\cdot$; we denote it by $\Gamma_{Z^\cdot}(\mathcal{L}^\cdot)$. It is easy to see that the functor $\Gamma_{Z^\cdot}$ preserves homotopy classes of morphisms of complexes; hence it defines a functor from $K(X)$ to $K_{Z^\cdot}(X)$, which is also denoted by $\Gamma_{Z^\cdot}$. This is the right adjoint of the embedding functor $K_{Z^\cdot}(X) \to K(X)$.

PROPOSITION 2.3. *There is an isomorphism*

$$\mathcal{H}^p(\Gamma_{Z^\cdot}(\mathcal{L}^\cdot)) \tilde{\to} \mathrm{Im}\,(\mathcal{H}^p(\Gamma_{Z^p}(\mathcal{L}^\cdot)) \to \mathcal{H}^p(\Gamma_{Z^{p-1}}(\mathcal{L}^\cdot)))$$

of functors of $\mathcal{L}^\cdot$ *for each* $p \in Z$.

Proof. It follows immediately from the definition of $\Gamma_{Z^\cdot}(\mathcal{L}^\cdot) = \mathcal{F}^\cdot$ that

$$\mathcal{Z}^p(\mathcal{F}^\cdot) = \mathcal{Z}^p(\Gamma_{Z^p}(\mathcal{L}^\cdot))$$

and

$$\mathcal{B}^p(\mathcal{F}^\cdot) = \mathcal{B}^p(\Gamma_{Z^{p-1}}(\mathcal{L}^\cdot)) \cap \mathcal{Z}^p(\Gamma_{Z^p}(\mathcal{L}^\cdot)).$$

Hence the kernel of

$$\eta : \mathcal{Z}^p(\mathcal{F}^\cdot) \to \mathcal{H}^p(\Gamma_{Z^{p-1}}(\mathcal{L}^\cdot))$$

is $\mathcal{B}^p(\mathcal{F}^\cdot)$, so $\mathcal{H}^p(\mathcal{F}^\cdot)$ is isomorphic to

$$\mathrm{Im}\,\eta = \mathrm{Im}\,(\mathcal{H}^p(\Gamma_{Z^p}(\mathcal{L}^\cdot)) \to \mathcal{H}^p(\Gamma_{Z^{p-1}}(\mathcal{L}^\cdot))).$$

Definition 2.4. A complex $\mathcal{F}^\cdot$ is $Z^\cdot$*-closed*, if the Module $\mathcal{F}^p$ is Z^{p+1}-closed for each $p \in Z$.

The full subcategory of $K(X)$ formed by $Z^\cdot$-closed complexes is denoted by $K_{X/Z^\cdot}(X)$.

THEOREM 2.5. *For each complex* $\mathcal{L}^\cdot$ *there is a* $Z^\cdot$*-closed complex* $\mathcal{F}^\cdot$ *and a morphism* $f : \mathcal{L}^\cdot \to \mathcal{F}^\cdot$ *such that each morphism* g *from* $\mathcal{L}^\cdot$ *to a* $Z^\cdot$*-closed complex* $\mathcal{G}^\cdot$ *has a unique factorization*

$$\mathcal{L}^\cdot \xrightarrow{f} \mathcal{F}^\cdot \xrightarrow{u} \mathcal{G}^\cdot.$$

Moreover, the homotopy class of u *depends only on the homotopy class of* g.

Proof. We define the complex $\mathcal{F}^\cdot$ and the morphism f by induction. As the filtration $Z^\cdot$ is strictly exhaustive, we must set $\mathcal{F}^p = 0$ for p small enough. After defining $\mathcal{F}^{p-1}$, $f^{p-1} : \mathcal{L}^{p-1} \to \mathcal{F}^{p-1}$, and $d^{p-2} : \mathcal{F}^{p-2} \to \mathcal{F}^{p-1}$, we write

$$\mathcal{E}^p = \text{Coker } (d^{p-2}) \oplus_{\mathcal{L}^{p-1}} \mathcal{L}^p,$$

$\mathcal{F}^p = Cl_{X/Z^{p+1}}(\mathcal{E}^p)$, and define the homomorphisms f^p, d^{p-1} by composing $\mathcal{L}^p \to \mathcal{E}^p$, and $\mathcal{F}^{p-1} \to \mathcal{E}^p$, with $\mathcal{E}^p \to \mathcal{F}^p$.

If u is a morphism of $\mathcal{F}^\cdot$ into a $Z^\cdot$-closed complex $\mathcal{G}^\cdot$, and $g = u \circ f$, then u^{p-1} induces homomorphism Coker $(d^{p-2}) \to \mathcal{G}^p$ such that the diagram

$$\begin{array}{ccc} \mathcal{L}^{p-1} & \longrightarrow & \mathcal{L}^p \\ \downarrow & & \downarrow g^p \\ \text{Coker } (d^{p-2}) & \longrightarrow & \mathcal{G}^p \end{array}$$

commutes. Hence there is a unique homomorphism v^p: $\mathcal{E}^p \to \mathcal{G}^p$ such that g^p has the factorization

$$\mathcal{L}^p \longrightarrow \mathcal{E}^p \xrightarrow{v^p} \mathcal{G}^p$$

and

$$\begin{array}{ccc} \mathcal{F}^{p-1} & \longrightarrow & \mathcal{E}^p \\ \downarrow & & \downarrow v^p \\ \mathcal{G}^{p-1} & \longrightarrow & \mathcal{G}^p \end{array}$$

commutes. As $\mathcal{G}^p$ is Z^{p+1}-closed, v^p has a unique factorization

$$\mathcal{E}^p \longrightarrow \mathcal{F}^p \xrightarrow{u^p} \mathcal{G}^p.$$

This proves that the morphism u is uniquely defined by $g = u \circ f$. The same procedure also gives an inductive construction of u for any morphism g: $\mathcal{L}^\cdot \to \mathcal{G}^\cdot$.

For the last assertion it is sufficient to show that u is null homotopic whenever $u \circ f$ is null homotopic.

Let $k = (k^p)$, k^p: $\mathcal{L}^p \to \mathcal{G}^{p-1}$, be a family of homomorphisms such that

$$u^p \circ f^p = d^{p-1} \circ k^p + k^{p+1} \circ d^p$$

for each $p \in Z$. We define inductively homomorphisms h^p: $\mathcal{F}^p \to \mathcal{G}^{p-1}$ satisfying the conditions $k^p = h^p \circ f^p$, and

$$u^p = d^{p-1} \circ h^p + h^{p+1} \circ d^p$$

for each $p \in Z$. We recall that $\mathcal{F}^p = 0$ for p small enough.

After defining h^p we have

$$u^p \circ d^{p-1} = d^{p-1} \circ u^{p-1} = d^{p-1} \circ h^p \circ d^{p-1}$$

so that $u^p - d^{p-1} \circ h^p$ has a factorization

$$\mathcal{F}^p \longrightarrow \operatorname{Coker}(d^{p-1}) \xrightarrow{l^p} \mathcal{G}^p.$$

Then the diagram

$$\begin{array}{ccc} \mathcal{L}^p & \longrightarrow & \mathcal{L}^{p+1} \\ \downarrow & & \downarrow k^{p+1} \\ \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot) & \xrightarrow{l^p} & \mathcal{G}^p \end{array}$$

commutes, showing the existence of a homomorphism $m^{p+1}: \mathcal{E}^{p+1} \to \mathcal{G}^p$ such that $u^p - d^{p-1} \circ h^p$ and k^{p+1} have the factorizations

$$\mathcal{F}^p \longrightarrow \mathcal{E}^{p+1} \xrightarrow{m^{p+1}} \mathcal{G}^p, \ \mathcal{L}^{p+1} \longrightarrow \mathcal{E}^{p+1} \xrightarrow{m^{p+1}} \mathcal{G}^p.$$

As $\mathcal{G}^p$ is Z^{p+1}-closed, and so Z^{p+2}-closed, m^{p+1} factors through $\mathcal{F}^{p+1}$ defining a homomorphism $h^{p+1}: \mathcal{F}^{p+1} \to \mathcal{G}^p$ with the desired properties. This completes the proof.

It follows from Theorem 2.5 that the embedding functor of the category of $Z^\cdot$-closed complexes and morphisms of complexes into the category of complexes has a left adjoint $\mathcal{L}^\cdot \mapsto \mathcal{F}^\cdot$. The complex $\mathcal{F}^\cdot$ is called the $Z^\cdot$-closure of $\mathcal{L}^\cdot$ and denoted by $Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$. The last assertion of 2.5 shows, in addition, that $Cl_{X/Z^\cdot}$ defines a functor from $K(X)$ to $K_{X/Z^\cdot}(X)$.

PROPOSITION 2.6. *Let $\mathcal{L}^\cdot$ be a complex, $\mathcal{F}^\cdot = Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$, and $f: \mathcal{L}^\cdot \to \mathcal{F}^\cdot$ the canonical morphism. Then*

$$\operatorname{Supp}(\operatorname{Ker}(\mathcal{H}^p(f))) \subset Z^{p+1}, \ \operatorname{Supp}(\operatorname{Coker}(\mathcal{H}^p(f))) \subset Z^{p+2}$$

for each $p \in Z$.

Proof. We use the notation of the proof of Theorem 2.5. For each $p \in Z$, $\mathcal{H}^p(f)$ is a restriction of the homomorphism

$$\mathcal{L}^p/\mathcal{B}^p(\mathcal{L}^\cdot) \to \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot)$$

induced by f^p. This has the factorization

$$\mathcal{L}^p/\mathcal{B}^p(\mathcal{L}^\cdot) \xrightarrow{u} \operatorname{Coker}(\mathcal{F}^{p-1} \longrightarrow \mathcal{E}^p) \xrightarrow{v} \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot)$$

where u is bijective.

As $\operatorname{Im}(\mathcal{F}^{p-1} \to \mathcal{E}^p)$ is mapped onto $\mathcal{B}^p(\mathcal{F}^\cdot)$ by the homomorphism $\mathcal{E}^p \to \mathcal{F}^p$, $\operatorname{Ker}(v)$ is a quotient of $\operatorname{Ker}(\mathcal{E}^p \to \mathcal{F}^p)$ and therefore $\operatorname{Supp}(\operatorname{Ker}(\mathcal{H}^p(f))) = \operatorname{Supp}(\operatorname{Ker}(v)) \subset Z^{p+1}$.

On the other hand, $\operatorname{Coker}(\mathcal{H}^p(f))$ is

$$\operatorname{Ker}(\mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot) \to \mathcal{F}^{p+1})/\operatorname{Im}(\mathcal{Z}^p(\mathcal{L}^\cdot) \to \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot)).$$

It follows from the definition of $\mathcal{E}^{p+1}$ that the sequence

$$\mathcal{Z}^p(\mathcal{L}^\cdot) \to \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot) \to \mathcal{E}^{p+1}$$

is exact. Hence Coker $(\mathcal{H}^p(f))$ is isomorphic to a sub-Module of Ker $(\mathcal{E}^{p+1} \to \mathcal{F}^{p+1})$, whose support is in Z^{p+2}.

Definition 2.7. A complex of O_X-Modules is a *Cousin complex* with respect to the filtration $Z^\cdot$ if it is $Z^\cdot$-closed and has supports in $Z^\cdot$ (cf. Hartshorne [4], p. 241).

The category of Cousin complexes and morphisms of complexes is denoted by Coz $(Z^\cdot; X)$. It is a full subcategory of $K(X)$, since any two homotopic morphisms form a complex with supports in $Z^\cdot$ to a $Z^\cdot$-closed complex are equal.

PROPOSITION 2.8. *Let $\mathcal{F}^\cdot$, $\mathcal{G}^\cdot$ be Cousin complexes. If $f: \mathcal{F}^\cdot \to \mathcal{G}^\cdot$ is a morphism of complexes satisfying the conditions*

$$\text{Supp (Ker } (\mathcal{H}^p(f))) \subset Z^{p+1}, \text{ Supp (Coker } (\mathcal{H}^p(f))) \subset Z^{p+2}$$

for each $p \in Z$, then f is an isomorphism.

Proof. We prove by induction that $f^p: \mathcal{F}^p \to \mathcal{G}^p$ is bijective, and f^{p+1} induces an isomorphism of $\mathcal{B}^{p+1}(\mathcal{F}^\cdot)$ onto $\mathcal{B}^{p+1}(\mathcal{G}^\cdot)$, for each $p \in Z$. This is trivially true if $Z^p = X$.

Let us consider the diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{B}^p(\mathcal{F}^\cdot) & \to & \mathcal{Z}^p(\mathcal{F}^\cdot) & \to & \mathcal{H}^p(\mathcal{F}^\cdot) & \to 0 \\ & \downarrow f'' & & \downarrow f' & & \downarrow \mathcal{H}^p(f) & \\ 0 \to & \mathcal{B}^p(\mathcal{G}^\cdot) & \to & \mathcal{Z}^p(\mathcal{G}^\cdot) & \to & \mathcal{H}^p(\mathcal{G}^\cdot) & \to 0. \end{array}$$

By the induction assumption f'' is bijective. Hence Ker (f') is isomorphic to Ker $(\mathcal{H}^p(f))$, so Ker $(f') = 0$, as $\mathcal{Z}^p(\mathcal{F}^\cdot)$ is Z^{p+1}-pure. Furthermore, the support of Coker (f') is in Z^{p+2}.

On the other hand, the diagram

$$\begin{array}{ccccccc} 0 \to & \mathcal{Z}^p(\mathcal{F}^\cdot) & \to & \mathcal{F}^p & \to & \mathcal{B}^{p+1}(\mathcal{F}^\cdot) & \to 0 \\ & \downarrow f' & & \downarrow f^p & & \downarrow f'' & \\ 0 \to & \mathcal{Z}^p(\mathcal{G}^\cdot) & \to & \mathcal{G}^p & \to & \mathcal{B}^{p+1}(\mathcal{G}^\cdot) & \to 0 \end{array}$$

defines an exact sequence

$$0 \to \text{Ker } (f^p) \to \text{Ker } (f'') \to \text{Coker } (f') \to \text{Coker } (f^p) \to \text{Coker } (f'') \to 0.$$

As $\mathcal{F}^p$ is Z^{p+1}-pure, and Supp (Ker $(f'')) \subset Z^{p+1}$, we have Ker $(f^p)=0$. Then Ker (f'') is isomorphic to a sub-Module of Coker (f'), whose support is in Z^{p+2}; so Ker $(f'')=0$, as $\mathcal{B}^{p+1}(\mathcal{F}^\cdot)$ is Z^{p+2}-pure. Moreover, the support of Coker (f''), and therefore the support of Coker (f^p), is in Z^{p+1}. As $\mathcal{F}^p$ is Z^{p+1}-closed, Coker (f^p) is a direct factor of the Z^{p+1}-pure Module $\mathcal{G}^p$. Hence Coker $(f^p)=0$, and so Coker $(f'')=0$.

Remark. $\mathcal{H}^p(f)=0$ for each $p \in Z$ does not imply $f=0$ even if $f: \mathcal{F}^\cdot \to \mathcal{G}^\cdot$ is a morphism of Cousin complexes.

PROPOSITION 2.9. (i) *If $\mathcal{L}^\cdot$ is a $Z^\cdot$-closed Ccomplex, $\Gamma_{Z^\cdot}(\mathcal{L}^\cdot)$ is a Cousin complex.*

(ii) *If $\mathcal{L}^\cdot$ is a complex with supports in $Z^\cdot$, $Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$ is a Cousin complex.*

Proof. Let $\mathcal{L}^\cdot$ be a $Z^\cdot$-closed complex. Then $\mathcal{F}^p = \mathrm{Ker}\,(\Gamma_{Z^p}(\mathcal{L}^p) \to \Gamma_{X/Z^{p+1}}(\mathcal{L}^{p+1}))$ is Z^{p+1}-pure, and its support is in Z^p for each $p \in Z$. To prove that $\mathcal{F}^p$ is Z^{p+1}-closed, let us consider a short exact sequence of $\mathcal{O}_X$-Modules

$$0 \to \mathcal{F}^p \to \mathcal{G} \to \mathcal{E} \to 0,$$

where Supp $(\mathcal{E}) \subset Z^{p+1}$. As $\mathcal{L}^p$ is Z^{p+1}-closed, the inclusion $\mathcal{F}^p \to \mathcal{L}^p$ may be extended to a homomorphism g from $\mathcal{G}$ to $\mathcal{L}^p$. As Supp $(\mathcal{G}) \subset Z^p$, g factors through $\Gamma_{Z^p}(\mathcal{L}^p)$. Then Im $(g) \subset \mathcal{F}^p$, since $\Gamma_{Z^p}(\mathcal{L}^p)/\mathcal{F}^p$ is Z^{p+1}-pure, so $\mathcal{E}$ is a direct factor of $\mathcal{G}$. This proves (i).

For the proof of (ii) let us consider a complex $\mathcal{L}^\cdot$ with supports in $Z^\cdot$. The natural morphism from $\mathcal{L}^\cdot$ to $Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$ factors through $\mathcal{F}^\cdot = \Gamma_{Z^\cdot}(Cl_{X/Z^\cdot}(\mathcal{L}^\cdot))$, which is a Cousin complex by (i). But then it is seen by the universal property of $Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$ that the inclusion $\mathcal{F}^\cdot \to Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$ is bijective.

PROPOSITION 2.10. *Let $\mathcal{L}^\cdot$ be a complex with supports in $Z^\cdot$, $\mathcal{G}^\cdot$ a Cousin complex, and $g: \mathcal{L}^\cdot \to \mathcal{G}^\cdot$ a morphism of complexes. If*

$$\text{Supp (Ker } (\mathcal{H}^p(g))) \subset Z^{p+1}, \quad \text{Supp (Coker } (\mathcal{H}^p(g))) \subset Z^{p+2}$$

for each $p \in Z$, then $\mathcal{G}^\cdot$ is isomorphic to $Cl_{X/Z^\cdot}(\mathcal{L}^\cdot)$.

In fact, g has a factorization

$$\mathcal{L}^\cdot \xrightarrow{f} Cl_{X/Z^\cdot}(\mathcal{L}) \xrightarrow{u} \mathcal{G}^\cdot,$$

where f satisfies the same conditions as g by Proposition 2.6. It follows that Proposition 2.8 may be applied to u.

§ 3. Admissible filtrations

Let $(X, \mathcal{O}_X)$ be a ringed space, $Z^\cdot = (Z^p)_{p\in Z}$ a strictly exhaustive decreasing filtration of X.

Definition 3.1. Let $\mathcal{F}$ bea n $\mathcal{O}_X$-Module, Z a subset of X. Then $\mathcal{F}$ is called *Z-acyclic* if $\mathcal{H}^i_Z(\mathcal{F})=0$ for each integer $i>0$. If $\mathcal{F}$ is Z^p-acyclic for each $p\in Z$, $\mathcal{F}$ is called *$Z^\cdot$-acyclic.*

Let Z, Z' be subsets of X such that $Z\supset Z'$. If $\mathcal{F}$ is a Z-acyclic and Z'-acyclic Module, then $\mathcal{H}^0_{Z/Z'}(\mathcal{F})=\Gamma_{Z/Z'}(\mathcal{F})$ and $\mathcal{H}^i_{Z/Z'}(\mathcal{F})=0$ for $i>0$ by Proposition 1.3. In particular, if $\mathcal{F}$ is Z-acyclic, $\mathcal{H}^0_{X/Z}(\mathcal{F})=\Gamma_{X/Z}(\mathcal{F})$ and $\mathcal{H}^i_{X/Z}(\mathcal{F})=0$ for $i>0$.

Example. A flabby $\mathcal{O}_X$-Module is Z-acyclic for each subset Z of X.

To prove this, it suffices to show that if

$$0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$$

is an exact sequence of $\mathcal{O}_X$-Modules where $\mathcal{F}'$ is flabby, then $\Gamma_Z(\mathcal{F})\to\Gamma_Z(\mathcal{F}'')$ is surjective. In fact, let s be a section of $\Gamma_Z(\mathcal{F}'')$ over an open subset U of X, so there is an open subset V of U such that $U-V\subset Z$ and the restriction $s\,|\,V=0$. We may assume that s comes from a section t of $\mathcal{F}$ over U. Then $t\,|\,V$ is a section of $\mathcal{F}'$, and if $t'\in\Gamma(U, \mathcal{F}')$ is an extension of $t\,|\,V$, then $t-t'$ is a section of $\Gamma_X(\mathcal{F})$, which is mapped onto s.

PROPOSITION 3.2. *Let $0\to\mathcal{F}'\to\mathcal{F}\to\mathcal{F}''\to 0$ be an exact sequence of $\mathcal{O}_X$-Modules. If $\mathcal{F}'$ is $Z^\cdot$-acyclic, then $\mathcal{F}$ is $Z^\cdot$-acyclic if and only if $\mathcal{F}''$ is.*

This is recorded only for reference.

Definition 3.3. A filtration $Z^\cdot=(Z^p)_{p\in Z}$ of X is *admissible*, if $\Gamma_{Z^p}(\mathcal{J})$ is $Z^\cdot$-acyclic for each $p\in Z$ and for each injective $\mathcal{O}_X$-Module $\mathcal{J}$.

Examples. If Z is a closed subset of X, $\Gamma_Z(\mathcal{F})$ is flabby for each flabby $\mathcal{O}_X$-Module $\mathcal{F}$. Hence any filtration of X by closed subsets is admissible.

Let us assume, for the moment, that X is a locally noetherian space and that each closed irreducible subset of X has a generic point. If Z is a subset of X stable under specialization, then $\Gamma_Z(\mathcal{F})$ is flabby for each flabby $\mathcal{O}_X$-Module $\mathcal{F}$. To prove this, we may assume X noetherian. If f is a section of $\Gamma_Z(\mathcal{F})$ over an open subset U of X, $Y=\mathrm{Supp}\,(\mathcal{O}_U\cdot f)$ has a finite number of irreducible components $Y_i (1\leqslant i\leqslant n)$ closed in U. Since a generic point of $\overline{Y}_i$ is contained in $U\cap\overline{Y}_i=Y_i\subset Z$, it follows by assumption that $\overline{Y}_i\subset Z$ for $1\leqslant i\leqslant n$. But then $Z'=\overline{Y}_1\cup\ldots\cup\overline{Y}_n$ is a closed subset of Z, and f is a section of the flabby sub-Module $\Gamma_{Z'}(\mathcal{F})$ of $\Gamma_Z(\mathcal{F})$.

It follows, in particular, that if X is a locally noetherian scheme, then any filtration of X by subsets stable under specialization is admissible.

PROPOSITION 3.4. *If $Z^{\cdot}$ is an admissible filtration of X and if $\mathcal{F}$ is a $Z^{\cdot}$-acyclic O_X-Module, then $\Gamma_{Z^p}(\mathcal{F})$ and $\Gamma_{X/Z^p}(\mathcal{F})$ are $Z^{\cdot}$-acyclic for each $p \in Z$. Moreover, $\Gamma_{Z^p/Z^q}(\mathcal{F})$ is $Z^{\cdot}$-acyclic for $p \leqslant q$.*

Proof. By proposition 3.2 it is enough to show that $\Gamma_{Z^p}(\mathcal{F})$ is $Z^{\cdot}$-acyclic for each $p \in Z$. If $\mathcal{J}^{\cdot}$ is an injective resolution of $\mathcal{F}$, then $\mathcal{L}^{\cdot} = \Gamma_{Z^p}(\mathcal{J}^{\cdot})$ is a *resolution* of $\Gamma_{Z^p}(\mathcal{F})$ by $Z^{\cdot}$-acyclic Modules, so that $\mathcal{H}^i_{Z^q}(\Gamma_{Z^p}(\mathcal{F}))$ is the ith cohomology Module of $\Gamma_{Z^q}(\mathcal{L}^{\cdot})$ for each $q \in Z$ and for each integer $i \geqslant 0$. But $\Gamma_{Z^q}(\mathcal{L}^{\cdot}) = \Gamma_{Z^r}(\mathcal{J}^{\cdot})$, where $r = \sup(p, q)$, so it is cohomologically trivial.

COROLLARY 3.5. *With the hypotheses of Proposition* 3.4, *$Cl_{X/Z^p}(\mathcal{F}) = \Gamma_{X/Z^p}(\mathcal{F})$ for each $p \in Z$.*

In fact, $\mathcal{H}^0_{X/Z^p}(\mathcal{F}) = \Gamma_{X/Z^p}(\mathcal{F})$ is Z^p-pure and Z^p-acyclic, hence Z^p-closed.

We denote by $D(X)$ the derived category of $C(X)$. Its objects are complexes of O_X-Modules. Those complexes which are bounded below form a full subcategory $D^+(X)$ of $D(X)$. We recall that each additive functor $F\colon C(X) \to C(X)$ has a right derived functor $\mathbf{R}^+F\colon D^+(X) \to D^+(X)$, as $C(X)$ has enough injective objects (Hartshorne [4], I, 5.3).

If Z, Z' are subsets of X such that $Z \supset Z'$, we denote by $\mathcal{H}^i_{Z/Z'}(\mathcal{L}^{\cdot})$ the ith cohomology Module of $\mathbf{R}^+\Gamma_{Z/Z'}(\mathcal{L}^{\cdot})$ for each object $\mathcal{L}^{\cdot}$ of $D^+(X)$ and for each $i \in Z$. We note that, for example, $\mathcal{H}^i_{Z^p/Z^q}(\mathcal{L}^{\cdot}) = \mathcal{H}^i(\Gamma_{Z^p/Z^q}(\mathcal{L}^{\cdot}))$ for integers i, p, q such that $p \leqslant q$, if $\mathcal{L}^{\cdot}$ is a complex of $Z^{\cdot}$-acyclic Modules.

In the rest of this section we shall assume that $Z^{\cdot}$ is an admissible strictly exhaustive filtration of X. All complexes are understood to be bounded below.

PROPOSITION 3.6. *Let $\mathcal{L}^{\cdot}$ be a complex of $Z^{\cdot}$-acyclic Modules. The following conditions are equivalent:*

(a) *$\Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot})$ is a complex of $Z^{\cdot}$-acyclic Modules and the canonical morphism $\Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot}) \to \mathcal{L}^{\cdot}$ is a quasi-isomorphism.*

(b) $\mathcal{H}^i_{X/Z^p}(\mathcal{L}^{\cdot}) = 0$ *for* $i, p \in Z, i \geqslant p$.

(b′) $\mathcal{H}^i_{Z^p/Z^{p+1}}(\mathcal{L}^{\cdot}) = 0$ *for* $i, p \in Z, i > p$.

Proof. If $\mathcal{F}^{\cdot} = \Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot})$ is a complex of $Z^{\cdot}$-acyclic Modules and if $\mathcal{F}^{\cdot} \to \mathcal{L}^{\cdot}$ is a quasi-isomorphism, then $\mathcal{H}^i_{X/Z^p}(\mathcal{L}^{\cdot}) = \mathcal{H}^i(\Gamma_{X/Z^p}(\mathcal{F}^{\cdot})) = 0$ for $i \geqslant p$, as $\Gamma_{X/Z^p}(\mathcal{F}^i) = 0$. Hence (a) implies (b).

By Proposition 1.3 (b′) is a consequence of (b). Conversely, it follows from (b′) by

induction that $\mathcal{H}^i_{Z^{p-r}/Z^p}(\mathcal{L}^\cdot)=0$ for $i\geqslant p$, $r>0$. But $Z^{p-r}=X$ for r large enough, so (b′) implies (b).

Finally, let us assume (b) and (b′). Then $\mathcal{H}^i_{Z^p}(\mathcal{L}^\cdot)\to\mathcal{H}^i(\mathcal{L}^\cdot)$ is surjective for $i=p$ and bijective for $i>p$; so $\mathcal{J}^\cdot\to\mathcal{L}^\cdot$ is a quasi-isomorphism by Proposition 2.3. Further, we note that, by (b′),

$$\Gamma_{Z^p/Z^{p+1}}(\mathcal{L}^p)\to\Gamma_{Z^p/Z^{p+1}}(\mathcal{L}^{p+1})\to\dots$$

is a resolution of $\mathcal{J}^p/\Gamma_{Z^{p+1}}(\mathcal{L}^p)$ by $Z^\cdot$-acyclic Modules, and it follows that $\mathcal{J}^p/\Gamma_{Z^{p+1}}(\mathcal{L}^p)$ is $Z^\cdot$-acyclic. Then $\mathcal{J}^p$ is $Z^\cdot$-acyclic for each $p\in Z$ by Proposition 3.2, so (a) holds.

PROPOSITION 3.7. *Let $\mathcal{L}^\cdot$ be a complex of $Z^\cdot$-acyclic Modules. The following conditions are equivalent:*

(a) *$Cl_{X/Z}.(\mathcal{L}^\cdot)$ is a complex of $Z^\cdot$-acyclic Modules and the canonical morphism $\mathcal{L}^\cdot\to Cl_{X/Z}.(\mathcal{L}^\cdot)$ is a quasi-isomorphism.*

(b) $\mathcal{H}^i_{Z^p}(\mathcal{L}^\cdot)=0$ *for* $i,p\in Z, i<p$.

(b′) $\mathcal{H}^{p-1}_{Z^p}(\mathcal{L}^\cdot)=0$ *for* $p\in Z$.

Moreover, if these conditions are satisfied, $\mathcal{L}^\cdot\to Cl_{X/Z}.(\mathcal{L}^\cdot)$ is an epimorphism of complexes.

Proof. If $\mathcal{J}^\cdot=Cl_{X/Z}.(\mathcal{L}^\cdot)$ is a complex of $Z^\cdot$-acyclic Modules and if $\mathcal{L}^\cdot\to\mathcal{J}^\cdot$ is a quasi-isomorphism, then $\mathcal{H}^i_{Z^p}(\mathcal{L}^\cdot)=\mathcal{H}^i(\Gamma_{Z^p}(\mathcal{J}^\cdot))=0$ for $i<p$, as $\Gamma_{Z^p}(\mathcal{J}^i)=0$. Hence (a) implies (b), and (b′) is a trivial consequence of (b).

It remains to show that (b′) implies (a). The notations being as in the proof of Theorem 2.5, let $b^{p-1}\colon\mathcal{J}^{p-1}\to\mathcal{E}^p$ be the canonical homomorphism, and let $c^p\colon\mathcal{E}^p\to\mathcal{L}^{p+1}$ denote the unique homomorphism such that $c^p\circ b^{p-1}=0$ and $d^p\colon\mathcal{L}^p\to\mathcal{L}^{p+1}$ has the factorization

$$\mathcal{L}^p\longrightarrow\mathcal{E}^p\xrightarrow{c^p}\mathcal{L}^{p+1},$$

for each $p\in Z$. We prove by induction that $\mathcal{J}^{p-1}$, $\mathcal{E}^p$ are $Z^\cdot$-acyclic, and that the natural morphism from $\mathcal{L}^\cdot$ into the complex $\mathcal{J}^\cdot_{(p)}$:

$$\dots\longrightarrow\mathcal{J}^{p-2}\longrightarrow\mathcal{J}^{p-1}\xrightarrow{b^{p-1}}\mathcal{E}^p\ {}^{c^p}\ \mathcal{L}^{p+1}\longrightarrow\mathcal{L}^{p+2}\longrightarrow\dots$$

is a surjective quasi-isomorphism for each $p\in Z$. This is true for p small enough, as $\mathcal{L}^\cdot$ is bounded below.

If $\mathcal{E}^p$ is $Z^\cdot$-acyclic, then $\mathcal{J}^p=Cl_{X/Z^{p+1}}(\mathcal{E}^p)=\Gamma_{X/Z^{p+1}}(\mathcal{E}^p)$ is $Z^\cdot$-acyclic, and $\mathcal{E}^p\to\mathcal{J}^p$ is surjective. If $\mathcal{L}^p\to\mathcal{E}^p$ is also surjective, then

$$\mathcal{E}^{p+1} = (\mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot)) \oplus_{\mathcal{E}^p} \mathcal{L}^{p+1},$$

so $\mathcal{E}^{p+1}$ is the quotient of $\mathcal{L}^{p+1}$ by the image of

$$\operatorname{Ker}(\mathcal{E}^p \to \mathcal{F}^p/\mathcal{B}^p(\mathcal{F}^\cdot)) = \operatorname{Im}(b^{p-1}) + \Gamma_{Z^{p+1}}(\mathcal{E}^p),$$

that is, $\mathcal{E}^{p+1} = \mathcal{L}^{p+1}/c^p(\Gamma_{Z^{p+1}}(\mathcal{E}^p))$.

By the induction assumption, $\mathcal{F}^\cdot_{(p)}$ is a complex of $Z^\cdot$-acyclic Modules, and $\mathcal{L}^\cdot \to \mathcal{F}^\cdot_{(p)}$ is a quasi-isomorphism, so $\mathcal{H}^p_{Z^{p+1}}(\mathcal{L}^\cdot) = 0$ is the pth cohomology Module of the complex $\Gamma_{Z^{p+1}}(\mathcal{F}^\cdot_{(p)})$:

$$\ldots \to 0 \to \Gamma_{Z^{p+1}}(\mathcal{E}^p) \to \Gamma_{Z^{p+1}}(\mathcal{L}^{p+1}) \to \ldots.$$

Hence $\operatorname{Ker}(c^p) \cap \Gamma_{Z^{p+1}}(\mathcal{E}^p) = 0$, and therefore $\mathcal{F}^\cdot_{(p+1)}$ is the quotient of $\mathcal{F}^\cdot_{(p)}$ by the trivial complex of $Z^\cdot$-acyclic Modules

$$\ldots \to 0 \to \Gamma_{Z^{p+1}}(\mathcal{E}^p) \to c^p(\Gamma_{Z^{p+1}}(\mathcal{E}^p)) \to 0 \to \cdots.$$

This completes the proof of the induction step.

We note that the final assertion of the proposition is evident from the proof.

Definition 3.8. A complex $\mathcal{F}^\cdot$ of $\mathcal{O}_X$-Modules is called *Cohen–Macaulay* with respect to the filtration $Z^\cdot$, if it is bounded below and if

$$\mathcal{H}^i_{X/Z^p}(\mathcal{F}^\cdot) = 0 \quad \textit{for} \quad i \geqslant p,$$

$$\mathcal{H}^i_{Z^p}(\mathcal{F}^\cdot) = 0 \quad \textit{for} \quad i < p.$$

The full subcategory of $D^+(X)$ formed by Cohen–Macaulay complexes is denoted by $D^+(X)_{CM(Z^\cdot)}$.

We also denote by $\operatorname{Acz}(Z^\cdot; X)$ the full subcategory of $\operatorname{Coz}(Z^\cdot; X)$ which consists of Cousin complexes of $Z^\cdot$-acyclic Modules. By Propositions 3.6 and 3.7 objects of $\operatorname{Acz}(Z^\cdot; X)$ are Cohen–Macaulay complexes.

THEOREM 3.9. *Let $(X, \mathcal{O}_X)$ be a ringed space, $Z^\cdot$ an admissible strictly exhaustive filtration of X. Then the natural functor*

$$Q\colon \operatorname{Acz}(Z^\cdot; X) \to D^+(X)_{CM(Z^\cdot)}$$

defines an equivalence of categories.

Proof. We have to show that Q is fully faithful and that each Cohen–Macaulay complex is isomorphic in $D^+(X)$ to an object of $\operatorname{Acz}(Z^\cdot; X)$.

Let $\mathcal{F}^\cdot$, $\mathcal{G}^\cdot$ be Cousin complexes of $Z^\cdot$-acyclic Modules. If g is a quasi-isomorphism of $\mathcal{G}^\cdot$ into an injective complex $\mathcal{J}^\cdot$, then each morphism u from $\mathcal{F}^\cdot$ to $\mathcal{G}^\cdot$ *in the derived*

category is represented by a morphism of complexes $f: \mathcal{F}^{\cdot} \to \mathcal{J}^{\cdot}$. Since $\mathcal{F}^{\cdot}$ has supports in $Z^{\cdot}$, f factors through $\mathcal{L}^{\cdot} = \Gamma_{Z^{\cdot}}(\mathcal{J}^{\cdot})$, and so it defines a morphism f' from $\mathcal{F}^{\cdot}$ to $Cl_{X/Z^{\cdot}}(\mathcal{L}^{\cdot})$. Likewise, g induces a morphism of complexes $\mathcal{G}^{\cdot} \to Cl_{X/Z^{\cdot}}(\mathcal{L}^{\cdot})$, which is a quasi-isomorphism by Propositions 3.6 and 3.7, hence an isomorphism by Proposition 2.8. Composing its inverse isomorphism with f' we obtain a morphism of complexes $u': \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$ representing u. In fact, the difference $f - g \circ u'$ factors through the kernel $\mathcal{E}^{\cdot}$ of the canonical morphism $\mathcal{L}^{\cdot} \to Cl_{X/Z^{\cdot}}(\mathcal{L}^{\cdot})$. But $\mathcal{E}^{\cdot}$ is acyclic, hence the inclusion of $\mathcal{E}^{\cdot}$ into the injective complex $\mathcal{J}^{\cdot}$ is homotopic to zero, and so f is homotopic to $g \circ u'$.

On the other hand, let us assume that $u, u': \mathcal{F}^{\cdot} \to \mathcal{G}^{\cdot}$ are morphisms of complexes, which define the same morphism in $D(X)$. Then the composite morphisms $g \circ u$, $g \circ u'$ from $\mathcal{F}^{\cdot}$ to $\mathcal{J}^{\cdot}$ are homotopic, and so are the morphisms $f, f': \mathcal{F}^{\cdot} \to Cl_{X/Z^{\cdot}}(\mathcal{L}^{\cdot})$ induced by them. But then $f = f'$, so $u = u'$. Hence we have shown that the functor Q is fully faithful in the category $\mathrm{Acz}\,(Z^{\cdot}; X)$.

Finally, to prove that each Cohen–Macaulay complex $\mathcal{L}^{\cdot}$ is isomorphic to an object of $\mathrm{Acz}\,(Z^{\cdot}; X)$, we may assume $\mathcal{L}^{\cdot}$ injective. But then $\Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot}) \to \mathcal{L}^{\cdot}$ and $\Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot}) \to \mathcal{F}^{\cdot} = Cl_{X/Z^{\cdot}}(\Gamma_{Z^{\cdot}}(\mathcal{L}^{\cdot}))$ are isomorphisms in the derived category, by Propositions 3.6 and 3.7, and $\mathcal{F}^{\cdot}$ is a Cousin complex of $Z^{\cdot}$-acyclic Modules.

Remark. The inverse equivalence of categories

$$E: D^{+}(X)_{CM(Z^{\cdot})} \to \mathrm{Acz}\,(Z^{\cdot}; X)$$

is given by $E(\mathcal{F}^{\cdot})^{p} = \mathcal{H}^{p}_{Z^{p}/Z^{p+1}}(\mathcal{F}^{\cdot})$ (cf. Hartshorne [4], p. 241).

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