STABILITY OF UNFOLDINGS IN SPACE AND TIME

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Introduction

In René Thom's catastrophe theory, gradient models for natural pheonomena are given locally by stable unfoldings, whose unfolding space (the space parametrized by the unfolding parameters) corresponds to the control space of the gradient models. Thom's celebrated list of the seven elementary catastrophes is in fact a classification of stable unfoldings of low unfolding dimension.

However, the equivalence relation on unfoldings used in Thom's classification and in defining the stability notion used there is fairly coarse; the diffeomorphisms used in defining this equivalence notion can operate on the unfolding space via an arbitrary local diffeomorphism. This means that in the mathematical description of a gradient model, all of the control parameters are treated as being interchangeable. In particular, when the control space is space-time, no distinction is made between the spatial coordinates and the time coordinate. Hence Thom's list can give the same mathematical description to physical events which an observer would see as being quite different.

The purpose of this paper is to develop mathematically a stability theory for unfoldings based on a finer equivalence notion ((r, s)-equivalence) than the ordinary one, in which some of the unfolding parameters are treated as being "more important" than the others. Such a theory can be applied in catastrophe theory to give an adequate mathematical description of spatio-temporal events in nature. The theory developed here generalizes the ordinary theory of stable unfoldings (for which see [11]).

The paper is organized as follows: § 1 contains preliminaries, lemmas which will be applied throughout the paper. In particular some useful corollaries of the Malgrange pre-

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(1) The title of this paper before publication was "(r, s)-Stability of Unfoldings".

paration theorem are proved. § 2 is a quick review of the ordinary theory of stable unfoldings, for reference and for comparison with the results on (r, s)-stability. In § 3 several (r, s)-stability notions are defined and the theory of (r, s)-stability is developed in analogy to the results of § 2 for ordinary stability; in particular the equivalence of the different definitions of (r, s)-stability is proved. § 4 treats the problem of classifying (r, s)-stable unfoldings; in particular, an algorithm is developed for finding all (r, s)-stable unfoldings of a given germ. In § 5 the classification is carried out for (3, 1)- and (1, 3)-stability and analoga to Thom's list are computed for these two cases. Finally, § 6 contains pictures of the (3, 1)-stable unfoldings.

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§ 1. Preliminaries

In this section we define notation and collect some basic results for future reference.

Definition 1.1. We denote by $\mathcal{E}(n, p)$ the set of germs at $0 \in \mathbf{R}^n$ of smooth mappings from $\mathbf{R}^n$ to $\mathbf{R}^p$. If $p=1$ we write simply $\mathcal{E}(n)$ for $\mathcal{E}(n, 1)$. The $\mathbf{R}$-algebra structure of $\mathbf{R}$ induces a natural $\mathbf{R}$-algebra structure on $\mathcal{E}(n)$. The ring $\mathcal{E}(n)$ has a unique maximal ideal $\mathfrak{m}(n)$, consisting of those germs $f \in \mathcal{E}(n)$ such that $f(0)=0$.

If $f \in \mathcal{E}(n, p)$ and $1 \leqslant i \leqslant p$, then we shall often for convenience write f_i to denote the germ in $\mathcal{E}(n)$ of the composition $y_i \circ f$, where y_i: $\mathbf{R}^p \to \mathbf{R}$ is the i-th coordinate function of $\mathbf{R}^p$.

If g is an element of $\mathcal{E}(n, p)$ and if $g(0)=0$, then for any r the germ g induces a canonical $\mathbf{R}$-linear map g^*: $\mathcal{E}(p, r) \to \mathcal{E}(n, r)$ defined by setting $g^*(f) = f \circ g$ for $f \in \mathcal{E}(p, r)$. If $r=1$ then g^* is a homomorphism of $\mathbf{R}$-algebras.

Definition 1.2. Let k be a non-negative integer. We denote by $J^k(n, p)$ the set of k-jets at 0 of germs in $\mathcal{E}(n, p)$. The jet space $J^k(n, p)$ is a finite-dimensional $\mathbf{R}$-vector space of dimension $p \binom{n+k}{k}$.

For each k there is a canonical $\mathbf{R}$-linear projection π_k: $\mathcal{E}(n, p) \to J^k(n, p)$ which assigns to each germ in $\mathcal{E}(n, p)$ its k-jet at 0. Similarly, for each k and q with $q \geqslant k$ there is a linear projection $\pi_{q,k}$: $J^q(n, p) \to J^k(n, p)$ defined by forgetting the higher-order terms.

We define $J_0^k(n, p) := \{z \in J^k(n, p) \mid \pi_{k,0}(z) = 0\}$. This is a subspace of $J^k(n, p)$ of codimension p. For each k there is a canonical projection ϱ_k: $J^k(n, p) \to J_0^k(n, p)$ defined by "forgetting" the zero-order terms. More specifically, if $f \in \mathcal{E}(n, p)$ and if $z = \pi_k(f)$ we have $\varrho_k(z) = \pi_k(f - f(0))$. For any k we define a projection $\pi_{0,k}: \mathcal{E}(n, p) \to J_0^k(n, p)$ by setting $\pi_{0,k} = \varrho_k \pi_k$.

Since $J^k(n, p)$ and $J_0^k(n, p)$ are finite-dimensional real vector spaces, they have a natural C^∞ differentiable structure.

Suppose $g \in \mathcal{E}(n, p)$ and $g(0) = 0$. Then for any $f \in \mathcal{E}(p, r)$, the k-jet at 0 of the composition $f \circ g$ depends only on the k-jets of f and of g. Hence g induces for each k a linear map ${}_k g^*: J^k(p, r) \to J^k(n, r)$ defined by setting ${}_k g^*(\pi_k(f)) = \pi_k(g^*(f))$ for $f \in \mathcal{E}(p, r)$. Similarly, if $z \in J_0^k(n, p)$ then z induces a linear map $z^*: J^k(p, r) \to J^k(n, r)$ defined by setting $z^* = {}_k g^*$, where $g \in \mathcal{E}(n, p)$ is any germ such that $\pi_k(g) = z$.

Let U be an open subset of $\mathbf{R}^n$ and let $f: U \to \mathbf{R}^p$ be a smooth mapping. We define for each k a smooth mapping $J^k f: U \to J^k(n, p)$, called the k-jet section of f, as follows: For each $x \in U$ define a germ $f_{(x)} \in \mathcal{E}(n, p)$ by setting $f_{(x)}(z) = f(x + z)$ for z near 0 in $\mathbf{R}^n$. Now define $J^k f$ by setting $J^k f(x) = \pi_k(f_{(x)})$.

Since clearly the *germ* of $J^k f$ at any point $x \in U$ depends only on the *germ* of f at x, we may also in the same way associate to every germ $g \in \mathcal{E}(n, p)$ a germ $J^k g$ which is the germ at 0 of a smooth mapping from $\mathbf{R}^n$ to $J^k(n, p)$.

Remark: Where no confusion can result, we shall often for convenience use the same symbols to denote functions, their germs at 0, and their k-jets at 0. Similarly we shall often use the same symbol to denote a point $c \in \mathbf{R}^p$ and to denote a constant mapping whose value everywhere is c (or to denote the germ or the k-jet of such a mapping).

Definition 1.3: We define $L(n) = \{\varphi \in \mathcal{E}(n, n) \mid \varphi(0) = 0$ and φ is non-singular at $0\}$. We can make $L(n)$ into a group by taking as the group operation the composition of germs in $\mathcal{E}(n, n)$. The group $L(n)$ is the group of germs of local diffeomorphisms of $\mathbf{R}^n$ at 0. Observe that whether or not a germ $\varphi \in \mathcal{E}(n, n)$ belongs to $L(n)$ depends only on the 1-jet of φ.

If k is a non-negative integer, we set $L^k(n) := \pi_k(L(n)) \subseteq J_0^k(n, n)$. If φ and ψ are elements of $L(n)$, then the k-jet of $\varphi \circ \psi$ depends only on the k-jets of φ and of ψ. Hence the group operation of $L(n)$ induces in a natural way a group operation on $L^k(n)$.

$L^k(n)$ is an open subset of $J_0^k(n, p)$, and hence has a natural C^∞ differentiable structure. One easily sees that with respect to this differentiable structure $L^k(n)$ is a Lie group.

$L(n)$ acts on $\mathfrak{m}(n)$ on the right, and $L(1)$ acts on $\mathfrak{m}(n)$ on the left, the group action in both cases being given by composition of germs. We may combine these two actions to obtain an action of $L(1) \times L(n)$ on $\mathfrak{m}(n)$ "on both sides"; formally we can write this action as an action from the right if we define $f \cdot (\psi, \varphi) := \psi^{-1} \circ f \circ \varphi$ for $f \in \mathfrak{m}(n)$, $\varphi \in L(n)$, $\psi \in L(1)$.

The group actions defined above induce smooth actions of the groups $L^k(n)$, $L^k(1)$ and $L^k(1) \times L^k(n)$ on $J_0^k(n, 1)$.

The following theorems and lemmas will find frequent application in this paper. These are all well-known results, except for the corollaries to the Malgrange Preparation Theorem (Corollaries 1.7 and 1.8), which are new.

LEMMA 1.4. (Nakayama's Lemma). *Let R be a commutative ring with identity and let I be an ideal in R such that $1+z$ is invertible for all $z \in I$. Let A and B be submodules of some R-module M and suppose A is finitely generated over R.*

If

(a) $$B + I \cdot A \supseteq A,$$

then

(b) $$B \supseteq A,$$

and if equality holds in (a), then equality holds in (b).

For a proof, see e.g. [2, p. 281] or [11, Lemma 1.13].

COROLLARY 1.5. *Let A be a finitely generated $\mathcal{E}(n)$ module and let B be a submodule of A such that for some k*

$$\dim_{\mathbf{R}} A/(\mathfrak{m}(n)^{k+1}A + B) \leqslant k.$$

Then $\mathfrak{m}(n)^k A \subseteq B$.

For a proof, see [3, Corollary 1.6] or [11, Corollary 1.14].

THEOREM 1.6 (Malgrange Preparation Theorem). *Let $f \in \mathcal{E}(n, p)$ and suppose $f(0)=0$. Let A be a finitely generated $\mathcal{E}(n)$-module and suppose $\dim_{\mathbf{R}} A/f^*(\mathfrak{m}(p))A$ is finite. Then A is finitely generated as an $\mathcal{E}(p)$ module via f^*.*

This is Mather's version of the theorem ([3, p. 132]). For a proof, see e.g. [3, pp. 131–134], or see [1, Ch. V], or see the articles of Wall, Nirenberg, Łojiasiewicz, Mather and Glaeser in [10, pp. 90–132].

COROLLARY 1.7. *Suppose we are given (for $i=1, 2, ..., k$) germs $f_i \in \mathcal{E}(n, p_i)$ with $f_i(0)=0$, such that for each i, $1 \leqslant i \leqslant k-1$, there is a germ $g_i \in \mathcal{E}(p_{i+1}, p_i)$, with $g_i(0)=0$, such that $f_i = g_i f_{i+1}$.*

Let C be a finitely-generated $\mathcal{E}(n)$ module. Then for each i we may also consider C as an $\mathcal{E}(p_i)$ module via f_i^*.

Let B be an $\mathcal{E}(n)$-submodule of C and for each i, $1 \leqslant i \leqslant k$, let A_i be a finitely generated $\mathcal{E}(p_i)$ submodule of C.

If

(a) $$A_1 + A_2 + \ldots + A_k + B + \mathfrak{m}(p_1)C = C,$$

then

(b) $$A_1 + A_2 + \ldots + A_k + B = C$$

(*Note:* When $k=0$, equation (a) reduces to: $B + \mathfrak{m}(n)C = C$.)

Proof. The proof is by induction on k. If $k=0$, then (b) follows by Nakayama's Lemma.

Suppose now $k>0$ and suppose the statement has been proved for all smaller values of k. Let $\bar{C}=C/B$; then $\bar{C}$ is a finitely generated $\mathcal{E}(n)$-module but we may also consider $\bar{C}$ as an $\mathcal{E}(p_k)$ module via f_k^*. Let $\pi: C\to\bar{C}$ be the projection, and for each i, $1\leqslant i\leqslant k$, let $A_i'=\pi(A_i)$ and let A_i'' be the $\mathcal{E}(p_k)$ submodule of $\bar{C}$ generated by A_i'. (Note that A_i' is an $\mathcal{E}(p_i)$ submodule of $\bar{C}$).

Since $\mathfrak{m}(p_1)C\subseteq\mathfrak{m}(p_k)C$, it follows from (a) that $A_1''+\ldots+A_k''+\mathfrak{m}(p_k)\bar{C}=\bar{C}$, and since each A_i'' is a finitely generated $\mathcal{E}(p_k)$ module, this equation implies that $\dim_{\mathbf{R}}\bar{C}/\mathfrak{m}(p_k)\bar{C}$ is finite. Hence by Theorem 1.6, $\bar{C}$ is finitely generated as an $\mathcal{E}(p_k)$ module.

Now equation (a) implies that $A_1'+A_2'+\ldots+A_k'+\mathfrak{m}(p_1)\bar{C}=\bar{C}$, and since $\bar{C}$ is finitely generated over $\mathcal{E}(p_k)$ we may apply the induction assumption for the case $k-1$ to conclude that $A_1'+\ldots+A_k'=\bar{C}$. But this clearly implies (b). Q.E.D.

Corollary 1.7 is a generalisation of [3, Lemma, p. 134].

COROLLARY 1.8. *Let k be an integer, $k\geqslant 1$, and suppose we are given, for $1\leqslant i\leqslant k$, germs $f_i\in\mathcal{E}(n,p_i)$, with $f_i(0)=0$, such that for each i, $1\leqslant i\leqslant k-1$, there is a germ $g_i\in\mathcal{E}(p_{i+1},p_i)$, with $g_i(0)=0$, such that $f_i=g_if_{i+1}$.*

Let C be a finitely generated $\mathcal{E}(n)$ module. Let B be an $\mathcal{E}(n)$ submodule of C, and for each i, $1\leqslant i\leqslant k$, let A_i be a finitely generated $\mathcal{E}(p_i)$ submodule of C generated by d_i elements over $\mathcal{E}(p_i)$.

Suppose

(a) $$A_1+A_2+\ldots+A_k+B+\mathfrak{m}(p_1)C+\mathfrak{m}(p_2)^{d_1+1}C=C.$$

Then

(b) $$A_1+\ldots+A_k+B=C$$

and

(c) $$\mathfrak{m}(p_2)^{d_1}C\subseteq A_2+\ldots+A_k+B+\mathfrak{m}(p_1)C.$$

(*Note:* When $k=1$, then (a) reduces to $A_1+B+\mathfrak{m}(p_1)C+\mathfrak{m}(n)^{d_1+1}C=C$ and (c) reduces to: $\mathfrak{m}(n)^{d_1}C\subseteq B+\mathfrak{m}(p_1)C$).

Proof. Let A_1' be the $\mathcal{E}(p_2)$ submodule of C generated by A_1 and let $B'=B+\mathfrak{m}(p_1)C$ considered as an $\mathcal{E}(n)$ submodule of C. Clearly (a) implies $A_1'+A_2+\ldots+A_k+B'+\mathfrak{m}(p_2)C=C$ and by Corollary 1.7 we get $A_1'+A_2+\ldots+A_k+B'=C$. Hence since A_1' is finitely generated over $\mathcal{E}(p_2)$, it follows that if we set $\bar{C}=C/(A_2+\ldots+A_k+B')$, then $\bar{C}$ is a finitely generated $\mathcal{E}(p_2)$ module.

Let $\pi: C\to\bar{C}$ be the projection and let $\bar{A}_1=\pi(A_1)$. From (a) it follows that $\bar{A}_1+\mathfrak{m}(p_2)^{d_1+1}\bar{C}=\bar{C}$. Since $\bar{A}_1$ is generated by d_1 elements over $\mathcal{E}(p_1)$ and since $\mathfrak{m}(p_1)\bar{C}=0$, it follows that $\dim_{\mathbf{R}}\bar{C}/\mathfrak{m}(p_2)^{d_1+1}\bar{C}\leqslant d_1$. Hence by Corollary 1.5 $\mathfrak{m}(p_2)^{d_1}\bar{C}=0$; this implies (c). And (c) and (a) together imply $A_1+A_2+\ldots+A_k+B+\mathfrak{m}(p_1)C=C$; by Corollary 1.7, (b) then follows. Q.E.D.

Corollary 1.8 is a generalisation of [3, Theorem 1.13].

THEOREM 1.9 (Thom's Transversality Lemma). *Let U be an open subset of $\mathbf{R}^n$. Let k be a non-negative integer and let N be a smoothly immersed submanifold of $J^k(n, p)$. Let B be the set of all smooth mappings $f: U \to \mathbf{R}^p$ such that the mapping $J^k f$ is transversal to N everywhere on U. Then B is a countable intersection of open dense subsets of $C^\infty(U, \mathbf{R}^p)$, the space of all smooth mappings from U into $\mathbf{R}^p$.*

In particular, since $C^\infty(U, \mathbf{R}^p)$ is a Baire space, B is dense.

(*Note:* in this paper we take the weak C^∞-topology on $C^\infty(U, \mathbf{R}^p)$. A basis for this topology consists of all sets of the form $\{h \in C^\infty(U, \mathbf{R}^p) \mid J^r(g-h)(L) \subseteq W\}$, where L is any compact subset of U, r is any non-negative integer, W is any open neighbourhood of 0 in $J^r(n, p)$ and g is any element of $C^\infty(U, \mathbf{R}^p)$. The weak C^∞-topology is *not* the same as the Whitney topology, which is often used by other authors).

For the proof, see e.g. [6] and [11, Theorem 1.22 and Corollary 1.23].

We conclude this section with a very useful lemma of Mather's.

LEMMA 1.10. *Let $F \in \mathcal{E}(n+1)$ and let $F(0)=0$. Suppose there are germs $\xi \in \mathcal{E}(n+1, n)$ and $\eta \in \mathcal{E}(n+2)$ such that for x near 0 in $\mathbf{R}^n$ and for t near 0 in $\mathbf{R}$ the following equation holds:*

$$\text{(a)} \qquad \frac{\partial F(x,t)}{\partial t} = \sum_{j=1}^{n} \frac{\partial F(x,t)}{\partial x_j} \xi_j(x,t) + \eta(F(x,t), x, t).$$

(*Remark: Here $\xi_j = y_j \circ \xi$, where y_j is the j-th coordinate function on $\mathbf{R}^n$. See Def. 1.1.*)

Then there exist germs $\varphi \in \mathcal{E}(n+1, n)$ and $\lambda \in \mathcal{E}(n+2)$ such that for x near 0 in $\mathbf{R}^n$ and s near 0 in $\mathbf{R}$

$$\text{(b)} \qquad \varphi(x, 0) = x \text{ and } \lambda(s, x, 0) = s,$$

and such that for x near 0 in $\mathbf{R}^n$ and t near 0 in $\mathbf{R}$ we have

$$\text{(c)} \qquad F(\varphi(x, t), t) = \lambda(F(x, 0), x, t).$$

Moreover we may choose φ and λ (in fact uniquely) such that

$$\text{(d)} \qquad \frac{\partial \varphi_j(x,t)}{\partial t} = -\xi_j(\varphi(x,t), t) \quad (j = 1, \ldots, n)$$

and

$$\frac{\partial \lambda}{\partial t}(s, x, t) = \eta(\lambda(s, x, t), \varphi(x, t), t),$$

for $t \in \mathbf{R}$, $x \in \mathbf{R}^n$, $s \in \mathbf{R}$.

For the proof, see [11, Lemma 1.29]. This lemma is an easy corollary of a lemma of Mather's ([3, p. 144] (or see [11, Lemma 1.27])).

§ 2. Stability of unfoldings

In this section we collect, for future reference, some of the basic results in the theory of unfoldings. A more detailed discussion of the subject, and the proofs of the theorems listed here, can be found in [11].

To keep the notation manageable, it will be convenient to agree on some notational conventions. We shall be considering germs in $\mathcal{E}(n+r)$ for some given n and r. We shall denote the standard coordinates on $\mathbf{R}^n$ by $x_1, \ldots, x_n$ and the standard coordinates on $\mathbf{R}^r$ by $u_1, \ldots, u_r$, and we shall denote elements of $\mathbf{R}^{n+r}$ by pairs (x, u) where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^r$. Occasionally we shall also be considering germs in $\mathcal{E}(n+r+s)$ for some given n, r and s; in that case we shall take coordinates $v_1, \ldots, v_s$ on $\mathbf{R}^s$ and we shall denote elements of $\mathbf{R}^{n+r+s}$ by triples (x, u, v) where $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$.

We shall apply similar notational conventions to mappings. For example, if $\Phi \in \mathcal{E}(p, n+r)$ and we write $\Phi = (\varphi, \psi)$, this will mean $\varphi \in \mathcal{E}(p, n)$, $\psi \in \mathcal{E}(p, r)$ and for y near 0 in $\mathbf{R}^p$ we have $\Phi(y) = (\varphi(y), \psi(y)) \in \mathbf{R}^{n+r}$.

We shall identify $\mathbf{R}^n$ with the subspace $\mathbf{R}^n \times \{0\}$ of $\mathbf{R}^n \times \mathbf{R}^r = \mathbf{R}^{n+r}$, and similarly we identify $\mathbf{R}^{n+r}$ with the subspace $\mathbf{R}^{n+r} \times \{0\}$ of $\mathbf{R}^{n+r+s}$. Also, we shall consider $\mathcal{E}(r)$ to be embedded as a subring in $\mathcal{E}(n+r)$, via the injective ring homomorphism π^*, where π: $\mathbf{R}^n \times \mathbf{R}^r \to \mathbf{R}^r$ is the projection onto the second factor. An element φ of $\mathcal{E}(n+r)$ is in $\mathcal{E}(r)$ if and only if φ does not depend on $x_1, \ldots, x_n$; in this case we shall generally write "$\varphi(u)$" to abbreviate "$\varphi(x, u)$", where $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^r$.

Similarly, for any p we may consider $\mathcal{E}(r, p)$ to be embedded in $\mathcal{E}(n+r, p)$.

In a similar way, we may identify $\mathcal{E}(n)$ with a subring of $\mathcal{E}(n+r)$, and we may identify $\mathcal{E}(n)$, $\mathcal{E}(n+r)$, $\mathcal{E}(r+s)$, and $\mathcal{E}(s)$ with subrings of $\mathcal{E}(n+r+s)$.

Finally, some algebraic notation: Let R be a ring, M an R-module, and let S be a subring of R (so that M is also an S-module). If $a_1, \ldots, a_k \in M$, we shall denote by $\langle a_1, \ldots, a_k \rangle_S$ the S-submodule of M generated by $a_1, \ldots, a_k$.

Frequently we shall use an abbreviated version of this notation: Suppose $f \in \mathcal{E}(n+r)$, and suppose S is a subring of $\mathcal{E}(n+r)$. Then we shall write $\langle \partial f/\partial x \rangle_S$ as an abbreviation for the S-submodule $\langle \partial f/\partial x_1, \ldots, \partial f/\partial x_n \rangle_S$ of $\mathcal{E}(n+r)$ and we shall write $\langle \partial f/\partial u \rangle_S$ as an abbreviation for $\langle \partial f/\partial u_1, \ldots, \partial f/\partial u_r \rangle_S$. We shall use a similar abbreviated notation to denote modules generated by derivatives of germs in $\mathcal{E}(n)$ or $\mathcal{E}(n+r+s)$; the meaning of the notation will always be clear.

Definition 2.1. Let $\eta \in \mathfrak{m}(n)$. An *r-dimensional unfolding* of η is a germ $f \in \mathcal{E}(n+r)$ such that $f \mid \mathbf{R}^n = \eta$.

One may think of an r-dimensional unfolding of η as being an r-parameter family of germs in $\mathcal{E}(n)$ which contains the given germ $\eta \in \mathfrak{m}(n)$ at $0 \in \mathbf{R}^r$. This implicitly understood structure of an unfolding is reflected in the following definition of equivalence of unfoldings.

Definition 2.2. Let f and g be germs in $\mathfrak{m}(n+r)$. To f we associate a germ $F \in \mathcal{E}(n+r, 1+r)$, defined by $F(x, u) = (f(x, u), u) \in \mathbf{R} \times \mathbf{R}^r$, for $x \in \mathbf{R}^n$ and $u \in \mathbf{R}^r$. Similarly, to g we associate a germ $G \in \mathcal{E}(n+r, 1+r)$ defined by $G(x, u) = (g(x, u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$.

We say that f and g are *equivalent as r-dimensional unfoldings* (or *r-equivalent*) if there are germs $\Phi \in L(n+r)$, $\psi \in L(r)$ and $\Lambda \in L(1+r)$ such that the following diagram commutes:

$$\begin{array}{ccccc} \mathbf{R}^{n+r} & \xrightarrow{F} & \mathbf{R}^{1+r} & \xrightarrow{p_2} & \mathbf{R}^r \\ \Phi\downarrow & & \Lambda\downarrow & & \psi\downarrow \\ \mathbf{R}^{n+r} & \xrightarrow{G} & \mathbf{R}^{1+r} & \xrightarrow{p_2} & \mathbf{R}^r \end{array}$$

where p_2: $\mathbf{R} \times \mathbf{R}^r \to \mathbf{R}^r$ is the projection onto the second factor. Such a triple (Φ, ψ, Λ) is called an *r-equivalence* from f to g.

Note that if (Φ, ψ, Λ) is an r-equivalence, then $\Phi = (\varphi, \psi)$ for some germ $\varphi \in \mathcal{E}(n+r, n)$.

Definition 2.3. Let U be an open subset of $\mathbf{R}^p$. Let f: $U \to \mathbf{R}$ be a smooth function and let $z \in U$. We define a *germ* $f_z \in \mathfrak{m}(p)$ by setting $f_z(y) = f(z+y) - f(z)$ for all y near 0 in $\mathbf{R}^p$.

Definition 2.4. Let U and V be open subsets of $\mathbf{R}^{n+r}$, and let f: $U \to \mathbf{R}$ and g: $V \to \mathbf{R}$ be smooth functions. Let $(x, u) \in U$ and let $(y, w) \in V$. We say f at (x, u) is *r-equivalent* to g at (y, w) if the germs $f_{(x,u)}$ and $g_{(y,w)}$ in $\mathfrak{m}(n+r)$ are equivalent as r-dimensional unfoldings.

We can now define stability for unfoldings. There are several ways of doing this, but the different stability notions we define below will all turn out to be equivalent to each other.

Definition 2.5. Let $f \in \mathfrak{m}(n+r)$. We say f is *weakly stable* as an r-dimensional unfolding if for every open neighbourhood U of 0 in $\mathbf{R}^{n+r}$ and for every representative function f': $U \to \mathbf{R}$ of the germ f, the following holds:

For any smooth function h: $U \to \mathbf{R}$, there is a real number $\varepsilon > 0$ such that if t is any real number with $|t| < \varepsilon$, then there is a point $(x, u) \in U$ such that $f' + th$ at (x, u) is r-equivalent to f' at 0.

Definition 2.6. Let $f \in \mathfrak{m}(n+r)$. We say f is *strongly stable* as an r-dimensional unfolding if for any open neighbourhood U of 0 in $\mathbf{R}^{n+r}$ and any representative function f' of f

defined on U, there is a neighbourhood V of f' in $C^\infty(U, \mathbf{R})$ (with the weak C^∞ topology) such that for any function $g' \in V$ there is a point (x, u) in U such that g' at (x, u) is r-equivalent to f' at 0.

Definition 2.7. Let $f \in \mathfrak{m}(n+r)$. Define $F \in \mathcal{E}(n+r, 1+r)$ by setting $F(x,u) = (f(x,u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$. We say f is *infinitesimally stable* as an r-dimensional unfolding (or *r-infinitesimally stable*) if

(a) $$\mathcal{E}(n+r) = \langle \partial f/\partial x \rangle_{\mathcal{E}(n+r)} + \langle \partial f/\partial u \rangle_{\mathcal{E}(r)} + F^*\mathcal{E}(1+r).$$

(Geometrically this condition means roughly that the "tangent space" at f to the r-equivalence class of f is maximal, i.e. is equal to the "tangent space" to $\mathfrak{m}(n+r)$).

Condition 2.7 (a) can be reformulated in a slightly simpler form (Theorem 2.9):

Definition 2.8. Let $f \in \mathfrak{m}(n+r)$ be an r-dimensional unfolding of $\eta \in \mathfrak{m}(n)$. If $1 \leqslant i \leqslant r$, we set $\alpha_i(f) = \partial f/\partial u_i \,|\, \mathbf{R}^n \in \mathcal{E}(n)$.

We define

$$W_f := \langle \alpha_1(f), \ldots, \alpha_r(f) \rangle_{\mathbf{R}} \subseteq \mathcal{E}(n).$$

THEOREM 2.9. *Let $f \in \mathfrak{m}(n+r)$ unfold $\eta \in \mathfrak{m}(n)$. Then f is r-infinitesimally stable if and only if*

(a) $$\mathcal{E}(n) = \langle \partial \eta/\partial x \rangle_{\mathcal{E}(n)} + W_f + \eta^*\mathcal{E}(1).$$

This theorem follows from the Malgrange Preparation Theorem. For the proof, see [11, Lemma 4.9].

THEOREM 2.10. *Let f and g be elements of $\mathfrak{m}(n+r)$. If f and g are r-equivalent, and if f is r-infinitesimally stable, then g is r-infinitesimally stable.*

Proof. See [11, Corollary 4.10, Lemma 4.3 and Chapter 3]. (This theorem can also be proved directly from the definition of infinitesimal stability.)

THEOREM 2.11. *Let $f \in \mathfrak{m}(n+r)$. The following statements are equivalent:*

(a) *f is weakly stable as an r-dimensional unfolding.*
(b) *f is strongly stable as an r-dimensional unfolding.*
(c) *f is infinitesimally stable as an r-dimensional unfolding.*

Proof. See [11, Theorem 4.11]. The most important consequence of this result is that it equates the *geometrically* defined stability notions (Definitions 2.5 and 2.6) with easily verifiable algebraic conditions. (2.7 (a) or 2.9 (a)).

Definition 2.12. A germ $f\in\mathfrak{m}(n+r)$ is said to be *stable as an r-dimensional unfolding* (or *r-stable*) if f fulfills one (and hence all) of the equivalent conditions 2.11 (a), 2.11 (b) or 2.11 (c).

Note: In future, in using the terms "stable", "equivalent", etc., we shall often omit reference to r when no confusion can result. For example, if f is an r-dimensional unfolding of η, we shall write "f is a stable unfolding of η" to mean f is r-stable.

A natural question to ask now is the following: Given a germ $\eta\in\mathfrak{m}(n)$, when does η have stable unfoldings and what are they? The following results give a complete answer to this question. The proofs can all be found in [11]; the method of proof was in most cases suggested by the work of Mather, who proved analogous results for a slightly simpler case in [4].

Definition 2.13. Let $\eta\in\mathfrak{m}(n)$. Let k be a non-negative integer. We say η is *right k-determined* if for every germ $\mu\in\mathfrak{m}(n)$ with $\pi_k(\mu)=\pi_k(\eta)$ there is a germ $\varphi\in L(n)$ such that $\eta=\mu\varphi$.

We say η is *right-left k-determined* if for every germ $\mu\in\mathfrak{m}(n)$ with $\pi_k(\mu)=\pi_k(\eta)$ there is a germ $\varphi\in L(n)$ and a germ $\lambda\in L(1)$ such that $\eta=\lambda\mu\varphi$.

Clearly, if η is right k-determined then η is right-left k-determined.

LEMMA 2.14. *Let $\eta\in\mathfrak{m}(n)$. If η is right-left k-determined, then η is right $(k+2)$-determined.*

Proof. [11, Corollary 2.12].

Definition 2.15. A germ $\eta\in\mathfrak{m}(n)$ is said to be *finitely determined* if η is right k-determined for some non-negative integer k (or equivalently, by 2.14., if η is right-left k-determined for some k).

Definition 2.16. Let $\eta\in\mathfrak{m}(n)$. We define

$$\tau(\eta)=\dim_{\mathbf{R}}\mathcal{E}(n)/\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)},$$
$$\sigma(\eta)=\dim_{\mathbf{R}}\mathcal{E}(n)/(\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}+\eta^*\mathcal{E}(1)).$$

If k is a non-negative integer, we set

$$\tau_k(\eta)=\dim_{\mathbf{R}}\mathcal{E}(n)/(\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}+\mathfrak{m}(n)^k),$$
$$\sigma_k(\eta)=\dim_{\mathbf{R}}\mathcal{E}(n)/(\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}+\eta^*\mathcal{E}(1)+\mathfrak{m}(n)^k).$$

Clearly $\tau_k(\eta)\leqslant\tau(\eta)$ and $\sigma_k(\eta)\leqslant\sigma(\eta)$ for all k.

Note that $\tau_k(\eta)$ and $\sigma_k(\eta)$ obviously depend only on the k-jet of η. We call $\sigma(\eta)$ the codimension of η (or more specifically, the right-left codimension of η).

THEOREM 2.17. (Tougeron). *Let $\eta \in \mathfrak{m}(n)$. The following conditions are equivalent:*

(a) *η is finitely determined,*

(b) $\tau(\eta) < \infty$,

(c) $\sigma(\eta) < \infty$,

(d) *the numbers $\tau_k(\eta)$, for k a non-negative integer, are bounded,*

(e) *the numbers $\sigma_k(\eta)$, for k a non-negative integer, are bounded,*

(f) *for some non-negative integer k*

$$\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)} + \eta^*\mathcal{E}(1),$$

(g) *for some non-negative integer k*

$$\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)}.$$

Proof. See [11, Corollary 2.17.] (conditions (d) and (e) above are not given in Corollary 2.17 of [11], but (d) obviously follows from (b) above and it implies 2.17 (d) of [11] and (e) follows from (c) above and implies 2.17 (e) of [11]). The proof uses results of Mather ([4, Prop. 1] (or see [11, Theorem 2.6]) and [3, § 7] (or see [11, Lemma 2.8])). This result is due in part to Tougeron [8].

Definition 2.18. If $z \in J_0^k(n, 1)$, we set $\tau(z) := \tau_k(\eta)$, where η is any germ in $\mathfrak{m}(n)$ such that $\pi_k(\eta) = z$. Clearly $\tau(z)$ is well-defined independently of the choice of η.

We set $Z_k := \{z \in J_0^k(n, 1) \mid \tau(z) \geqslant k\}$.

LEMMA 2.19. (a). *If $\eta \in \mathfrak{m}(n)$ is finitely determined, then for k sufficiently large, $\pi_k(\eta) \notin Z_k$.*

(b) *If $\eta \in \mathfrak{m}(n)$ and if, for some positive k, $\pi_k(\eta) \notin Z_k$, then $\mathfrak{m}(n)^{k-1} \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)}$ (and so η is finitely determined).*

(c) *Z_k is an algebraic subset of $J_0^k(n, 1)$.*

Proof. (a) follows easily from Theorem 2.17 (d); (b) follows from Corollary 1.5; for (c) see [11, Prop. 2.22].

THEOREM 2.20. *Let $\eta \in \mathfrak{m}(n)$. Then η has stable unfoldings if and only if η is finitely determined. The minimal unfolding dimension of a stable unfolding of η is $\sigma(\eta)$. In fact if $\mu_1, ..., \mu_r \in \mathcal{E}(n)$ are a basis of the* **R**-*vector space $\mathcal{E}(n)/(\langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)} + \eta^*\mathcal{E}(1))$, and if $f \in \mathfrak{m}(n+r)$ is defined by $f(x, u) = \eta(x) + u_1\mu_1(x) + ... + u_r\mu_r(x)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, then f is a stable unfolding of η of minimal unfolding dimension.*

Proof. This follows easily from 2.9 and 2.17.

THEOREM 2.21. *Let $\eta \in \mathfrak{m}(n)$. If $f \in \mathcal{E}(n+r)$ and $g \in \mathcal{E}(n+r)$ are r-stable unfoldings of η, then f and g are r-equivalent.*

Proof. See [11, Theorem 3.20 (see also Theorem 3.22 (b) and Def. 3.6)].

Definition 2.22. Let $g \in \mathfrak{m}(p)$. The q-dimensional *constant unfolding* of g is the germ $f \in \mathcal{E}(p+q)$ defined by $f(x, u) := g(x)$ for $x \in \mathbf{R}^p$, $u \in \mathbf{R}^q$.

COROLLARY 2.23. *Let $\eta \in \mathfrak{m}(n)$, and let $f \in \mathcal{E}(n+r)$ and $g \in \mathcal{E}(n+s)$ be unfoldings of η. If f is r-stable and g is s-stable, and if $s \leqslant r$, then f is r-equivalent to the $r-s$ dimensional constant unfolding of g.*

Proof. From Theorem 2.9 it follows trivially that the $r-s$ dimensional constant unfolding of g is an r-stable unfolding of η. The corollary then follows immediately from Theorem 2.21.

Theorem 2.20 and Corollary 2.23 together completely describe all stable unfoldings of a *given* germ η.

A related but somewhat more general question is to ask for a classification of all stable unfoldings (without specifying η). René Thom's celebrated list of the seven elementary catastrophes gives a partial answer to this question; Thom's list classifies the r-stable unfoldings for $r \leqslant 4$. We shall state this theorem below:

Definition 2.24. Let $\mu \in \mathfrak{m}(n)$ and let $g \in \mathcal{E}(n+r)$ unfold μ. Let $\eta \in \mathfrak{m}(n+q)$ and let $f \in \mathcal{E}(n+q+r+s)$ unfold η. We say f *reduces to* g if f is $r+s$-equivalent to an unfolding $g' \in \mathcal{E}(n+q+r+s)$ of the form

(a) $$g'(x, y, u, v) = g(x, u) + Q(y) \qquad (x \in \mathbf{R}^n,\ y \in \mathbf{R}^q,\ u \in \mathbf{R}^r,\ v \in \mathbf{R}^s),$$

where Q is a non-degenerate quadratic form on $\mathbf{R}^q$.

If $q+s$ is positive (i.e. non zero), we say f reduces *properly* to g. If f has no proper reductions, we say f is an *irreducible* unfolding of η.

Definition 2.25. Let $f \in \mathcal{E}(n+r)$ unfold $\eta \in \mathfrak{m}(n)$. We say f has a *simple singularity* at 0 if f reduces to the trivial unfolding $0 \in \mathfrak{m}(0)$.

THEOREM 2.26. (Thom's list of the seven elementary catastrophes). *Let $f \in \mathcal{E}(n+r)$ be a stable unfolding of a germ $\eta \in \mathfrak{m}(n)^2$, and suppose $r \leqslant 4$. Then either f has a simple singularity at 0, or f reduces to a unique one of the following seven stable and irreducible unfoldings g_i of germs μ_i:*

Name	μ_i	g_i	Unfolding dimension
Fold	$\mu_1(x)=x^3$	$g_1(x, u)=x^3+ux$	1
Cusp	$\mu_2(x)=x^4$	$g_2(x, u, v)=x^4+ux^2+vx$	2
Swallowtail	$\mu_3(x)=x^5$	$g_3(x, u, v, w)=x^5+ux^3+vx^2+wx$	3
Butterfly	$\mu_4(x)=x^6$	$g_4(x, u, v, w, t)=x^6+ux^4+vx^3+wx^2+tx$	4
Hyperbolic umbilic (wave crest)	$\mu_5(x, y)=x^3+y^3$	$g_5(x, y, u, v, w)=x^3+y^3+uxy+vx+wy$	3
Elliptic umbilic (hair)	$\mu_6(x, y)=x^3-xy^2$	$g_6(x, y, u, v, w)=x^3-xy^2+u(x^2+y^2)+vx+wy$	3
Parabolic umbilic (mushroom)	$\mu_7(x, y)=x^2y+y^4$	$g_7(x, y, u, v, w, t)=x^2y+y^4+ux^2+vy^2+wx+ty$	4

Moreover, if f reduces to one of the g_i's, then f does not have a simple singularity at 0.

Proof. See [11, Chapter 5]. (This theorem is Theorem 5.6 of [11].)

Thom's list is of course well-known particularly because of its relevance to Thom's catastrophe theory. For a discussion of the relationship of this formulation of Thom's list to the theory of catastrophes, see the appendix to [11].

§ 3. (r, s)-stability of unfoldings

In this section we shall investigate a generalisation of the stability notions defined in § 2, and shall prove for this generalised stability notion analogues to some of the theorems quoted in § 2. The analogue to Theorem 2.26 (Thom's list) will be proved in § 5.

The generalisation we shall consider here may appear rather artificial and uninteresting mathematically; however it was motivated by important considerations in the theory of catastrophes. For a discussion of these motivations, see § 5.

We shall retain in this section the notational conventions introduced at the beginning of § 2.

Definition 3.1. Let f and g be germs in $\mathfrak{m}(n+r+s)$. To f we associate a germ $F\in\mathcal{E}(n+r+s, 1+r+s)$, defined by $F(x, u, v)=(f(x, u, v), u, v)\in\mathbf{R}\times\mathbf{R}^r\times\mathbf{R}^s$, for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$,

$v \in \mathbf{R}^s$. Similarly, to g we associate a germ $G \in \mathcal{E}(n+r+s, 1+r+s)$, defined by $G(x, u, v) = (g(x, u, v), u, v)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$.

We say f and g are *(r, s)-equivalent* if there are germs $\Phi \in L(n+r+s)$, $\Lambda \in L(1+r+s)$, $\psi \in L(r+s)$ and $\varrho \in L(s)$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \mathbf{R}^{n+r+s} & \xrightarrow{F} & \mathbf{R}^{1+r+s} & \xrightarrow{p} & \mathbf{R}^{r+s} & \xrightarrow{q} & \mathbf{R}^s \\ \Phi \downarrow & & \Lambda \downarrow & & \psi \downarrow & & \varrho \downarrow \\ \mathbf{R}^{n+r+s} & \xrightarrow{G} & \mathbf{R}^{1+r+s} & \xrightarrow{p} & \mathbf{R}^{r+s} & \xrightarrow{q} & \mathbf{R}^s \end{array}$$

where $p\colon \mathbf{R}^{1+r+s} \to \mathbf{R}^{r+s}$ is the projection onto the second factor and $q\colon \mathbf{R}^{r+s} \to \mathbf{R}^s$ is the projection onto the second factor.

Such a quadruple $(\Phi, \psi, \varrho, \Lambda)$ is called an (r, s)-equivalence from f to g.

Note that if $(\Phi, \psi, \varrho, \Lambda)$ is an (r, s)-equivalence, then $\psi = (\gamma, \varrho)$ for some germ $\psi \in \mathcal{E}(r+s, r)$ and $\Phi = (\varphi, \gamma, \varrho)$ for some germ $\varphi \in \mathcal{E}(n+r+s, n)$. Moreover $\Lambda = (\lambda, \gamma, \varrho)$ for some germ $\lambda \in \mathcal{E}(1+r+s)$.

Definition 3.2. Let U and V be open subsets of $\mathbf{R}^{n+r+s}$ and let $f\colon U \to \mathbf{R}$ and $g\colon V \to \mathbf{R}$ be smooth functions. Let $(x, u, v) \in U$ and let $(y, w, t) \in V$. We say f at (x, u, v) is *(r, s)-equivalent* to g at (y, w, t) if the germs $f_{(x,u,v)}$ and $g_{(y,w,t)}$ in $\mathfrak{m}(n+r+s)$ are (r, s)-equivalent. (See Def. 2.3 for the definition of $f_{(x,u,v)}$ and $g_{(y,w,t)}$).

We now define stability notions as before:

Definition 3.3. Let $f \in \mathfrak{m}(n+r+s)$. We say f is *weakly (r, s)-stable* if for every open neighbourhood U of 0 in $\mathbf{R}^{n+r+s}$ and for every representative function $f'\colon U \to \mathbf{R}$ of the germ f, the following holds: For any smooth function $h\colon U \to \mathbf{R}$ there is a real number $\varepsilon > 0$ such that if t is any real number with $|t| < \varepsilon$, then there is a point $(x, u, v) \in U$ such that $f' + th$ at (x, u, v) is (r, s)-equivalent to f' at 0.

Definition 3.4. Let $f \in \mathfrak{m}(n+r+s)$. We say f is *strongly (r, s)-stable* if for any open neighbourhood U of 0 in R^{n+r+s} and any representative function f' of f defined on U, there is a neighbourhood V of f' in $C^\infty(U, \mathbf{R})$ (with the weak C^∞-topology) such that for any function $g' \in V$ there is a point (x, u, v) in U such that g' at (x, u, v) is (r, s)-equivalent to f' at 0.

Definition 3.5. Let $f \in \mathfrak{m}(n+r+s)$. Define $F \in \mathcal{E}(n+r+s, 1+r+s)$ by setting $F(x, u, v) = (f(x, u, v), u, v)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. We say f is *infinitesimally (r, s)-stable* if

(a) $\mathcal{E}(n+r+s) = \langle \partial f/\partial x \rangle_{\mathcal{E}(n+r+s)} + \langle \partial f/\partial u \rangle_{\mathcal{E}(r+s)} + \langle \partial f/\partial v \rangle_{\mathcal{E}(s)} + F^*\mathcal{E}(1+r+s).$

(Again, as in the case of ordinary infinitesimal stability, one may interpret this condition geometrically as saying roughly that the "tangent space" at f to the (r, s)-equivalence class of f is maximal, i.e. is equal to the "tangent space" to $\mathfrak{m}(n+r+s)$).

Remark. Obviously if $r=0$ or if $s=0$, then (r, s)-equivalence and the (r, s)-stability notions are the same as ordinary $r+s$-equivalence and the ordinary $r+s$-stability notions which we defined in § 2, so that (r, s)-stability is in fact a generalisation of ordinary stability of unfoldings.

Moreover, it is clear that if two germs are (r, s)-equivalent, then they are certainly $r+s$-equivalent, and if a germ satisfies any of the (r, s)-stability conditions, then it is $r+s$-stable. We shall make frequent use of this fact in what follows. For example if $f \in \mathcal{E}(n+r+s)$ unfolds $\eta \in \mathfrak{m}(n)$ and if f is (r, s)-stable in any of the senses defined above, then η must be finitely determined.

The following theorem, which is a slightly strengthened analogue to Theorem 2.9, provides us with additional criteria for infinitesimal (r, s)-stability:

THEOREM 3.6. *Let $f \in \mathcal{E}(n+r+s)$ unfold $\eta \in \mathfrak{m}(n)$. Suppose η is finitely determined and choose an integer k such that $\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)}$. Let $q=k(s+1)$. Let $f_0 = f\,|\,\mathbf{R}^{n+r}$ and define $F_0 \in \mathcal{E}(n+r, 1+r)$ by setting $F_0(x, u) = (f(x, u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$.*

Then the following statements are equivalent:

(a) *f is infinitesimally (r, s)-stable,*

(b) $\mathcal{E}(n+r) = \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial f_0/\partial u\rangle_{\mathcal{E}(r)} + \langle \partial f/\partial v\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}} + F_0^*\mathcal{E}(1+r)$,

(c) $$\mathcal{E}(n+r) = \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial f_0/\partial u\rangle_{\mathcal{E}(r)} + \langle \partial f/\partial v\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}} + F_0^*\mathcal{E}(1+r) + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r) + \mathfrak{m}(n+r)^q.$$

(*Note:* If η is not finitely determined then neither (a) nor (b) can occur (because by restricting to $\mathbf{R}^n$ it follows from (a) or (b) that the codimension of η is finite) and (c) is meaningless because q is not defined.)

Proof of Theorem 3.6. (a) ⇒ (b): Let $\alpha\colon \mathcal{E}(n+r+s) \to \mathcal{E}(n+r)$ be defined by $\alpha(g) := g\,|\,\mathbf{R}^{n+r}$ for $g \in \mathcal{E}(n+r+s)$. Applying the homomorphism α to both sides of equation 3.5. (a) yields (b) immediately.

(b) ⇒ (c) is trivial.

(c) ⇒ (a): Since $\eta \in \mathfrak{m}(n)$, clearly $f_0 \in \mathfrak{m}(n+r)$ and hence $f_0^q \in \mathfrak{m}(n+r)^q$. So $F_0^*\mathcal{E}(1+r) \subseteq \langle 1, f_0, \ldots, f_0^{q-1}\rangle_{\mathcal{E}(r)} + \mathfrak{m}(n+r)^q$. Hence on the right hand side of (c) we may replace the summand $F_0^*\mathcal{E}(1+r)$ by $\langle 1, f_0, \ldots, f_0^{q-1}\rangle_{\mathcal{E}(r)}$, which is finitely generated over $\mathcal{E}(r)$. (This step will allow us to apply Corollary 1.8 later on in the proof).

Since $f_0|\mathbf{R}^n=\eta$ it follows that $\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}\subseteq\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\mathfrak{m}(r)\,\mathcal{E}(n+r)$. Therefore $\mathfrak{m}(n)^k\subseteq\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\mathfrak{m}(r)\,\mathcal{E}(n+r)$ and since $\mathfrak{m}(n+r)^k\subseteq\mathfrak{m}(n)^k+\mathfrak{m}(r)\,\mathcal{E}(n+r)$ it follows that $\mathfrak{m}(n+r)^k\subseteq\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\mathfrak{m}(r)\,\mathcal{E}(n+r)$ and hence $\mathfrak{m}(n+r)^q=\mathfrak{m}(n+r)^{k(s+1)}\subseteq\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)$. We may now therefore drop the term $\mathfrak{m}(n+r)^q$ from the right-hand side of (c).

To summarize: We have shown that (c) implies

(d)
$$\mathcal{E}(n+r)=\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\langle\partial f_0/\partial u\rangle_{\mathcal{E}(r)}+\langle 1,f_0,\dots,f_0^{\,q-1}\rangle_{\mathcal{E}(r)}+\langle\partial f/\partial v\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}}+\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r).$$

We claim this implies

(e)
$$\mathcal{E}(n+r+s)=\langle\partial f/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial f/\partial u\rangle_{\mathcal{E}(r+s)}+\langle 1,f,\dots,f^{q-1}\rangle_{\mathcal{E}(r+s)}+\langle\partial f/\partial v\rangle_{\mathcal{E}(s)}+\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s)+\mathfrak{m}(s)\,\mathcal{E}(n+r+s).$$

This is so because if we let $\alpha\colon\mathcal{E}(n+r+s)\to\mathcal{E}(n+r)$ be the restriction homomorphism then α applied to equation (e) yields equation (d), so (e) holds modulo the kernel of α. But the kernel of α is $\mathfrak{m}(s)\,\mathcal{E}(n+r+s)$, which is contained in both sides of (e). Hence (e) is valid.

Since $\langle\partial f/\partial v\rangle_{\mathcal{E}(s)}$ is generated by s elements over $\mathcal{E}(s)$, we may by Corollary 1.8 drop the terms $\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s)+\mathfrak{m}(s)\,\mathcal{E}(n+r+s)$ from the right-hand expression in (e), and in the resulting equation we may on the right replace $\langle 1,f,\dots,f^{q-1}\rangle_{\mathcal{E}(r+s)}$ by $F^*\mathcal{E}(1+r+s)$, which is bigger. But this yields equation 3.5 (a), so f is infinitesimally (r,s)-stable. Q.E.D.

COROLLARY 3.7. *Let n, r, s and t be non-negative integers. Let $f\in\mathfrak{m}(n+r+s)$ and let $g\in\mathfrak{m}(n+r+s+t)$, and suppose $g|\mathbf{R}^{n+r+s}=f$.*

If f is infinitesimally (r,s)-stable, then g is infinitesimally $(r,s+t)$-stable.

Proof. We take coordinates $w_1,\dots,w_t$ on $\mathbf{R}^t$,and the usual coordinates on $\mathbf{R}^{n+r+s}$. Let $h=g|\mathbf{R}^{n+r}$ and define $H\in\mathcal{E}(n+r,1+r)$ by $H(x,u)=(h(x,u),u)$ for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$.

Let
$$A=\langle\partial h/\partial x\rangle_{\mathcal{E}(n+r)}+\langle\partial h/\partial u\rangle_{\mathcal{E}(r)}+\langle\partial g/\partial v\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}}+H^*\mathcal{E}(1+r)\subseteq\mathcal{E}(n+r).$$

By Theorem 3.6 (b), g is infinitesimally $(r,s+t)$-stable if and only if (*) $\mathcal{E}(n+r)=A+\langle\partial g/\partial w\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}}$. But since f is infinitesimally (r,s)-stable, and since $f|\mathbf{R}^{n+r}=g|\mathbf{R}^{n+r}=h$ and $(\partial f/\partial v_i)|\mathbf{R}^{n+r}=(\partial g/\partial v_i)|\mathbf{R}^{n+r}$ for $1\leqslant i\leqslant s$, it follows from Theorem 3.6 (b) that $\mathcal{E}(n+r)=A$, so clearly (*) holds, and we are done.

THEOREM 3.8. *Let f and g be elements of $\mathfrak{m}(n+r+s)$. If f is infinitesimally (r, s)-stable and if f is (r, s)-equivalent to g, then g is infinitesimally (r, s)-stable.*

Proof. Define F and G in $\mathcal{E}(n+r+s, 1+r+s)$ by $F(x, u, v)=(f(x, u, v), u, v)$ and $G(x, u, v)=(g(x, u, v), u, v)$ for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$, $v\in\mathbf{R}^s$. Let $(\Phi, \psi, \varrho, \Lambda)$ be an (r, s)-equivalence from f to g. Then $F=\Lambda^{-1}G\Phi$.

We let t be the coordinate of $\mathbf{R}$, so we have coordinates $t, u_1, ..., u_r, v_1, ..., v_s$ on $\mathbf{R}^{1+r+s}$ Recall there exist germs $\varphi\in\mathcal{E}(n+r+s, n)$ and $\gamma\in\mathcal{E}(r+s, r)$ such that $\Phi=(\varphi, \gamma, \varrho)$ and $\psi=(\gamma, \varrho)$. Moreover clearly there is a germ $\lambda'\in\mathcal{E}(1+r+s)$ such that $\Lambda^{-1}=(\lambda', \psi^{-1})$. Since $F=\Lambda^{-1}G\Phi$ it follows that $f=\lambda' G\Phi$ or, more explicitly:

(a) $\quad f(x, u, v)=\lambda'(g(\varphi(x, u, v), \gamma(u, v), \varrho(v)), \gamma(u, v), \varrho(v))$ for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$, $v\in\mathbf{R}^s$.

Let μ: $\mathcal{E}(n+r+s)\to\mathcal{E}(n+r+s)$ be the map given by multiplication with the germ $(\partial\lambda'/\partial t)\circ G\circ\Phi\in\mathcal{E}(n+r+s)$.

From (a) one easily calculates that

(b) $\quad \dfrac{\partial f}{\partial x_i}\in\mu\left(\left\langle\dfrac{\partial g}{\partial x}\circ\Phi\right\rangle_{\mathcal{E}(n+r+s)}\right)$ for $1\leqslant i\leqslant n$,

(c) $\quad \dfrac{\partial f}{\partial u_j}\in\mu\left(\left\langle\dfrac{\partial g}{\partial x}\circ\Phi\right\rangle_{\mathcal{E}(n+r+s)}+\left\langle\dfrac{\partial g}{\partial u}\circ\Phi\right\rangle_{\mathcal{E}(r+s)}\right)+\Phi^*G^*\mathcal{E}(1+r+s)$ for $1\leqslant j\leqslant r$,

(d) $\quad \dfrac{\partial f}{\partial v_k}\in\mu\left(\left\langle\dfrac{\partial g}{\partial x}\circ\Phi\right\rangle_{\mathcal{E}(n+r+s)}+\left\langle\dfrac{\partial g}{\partial u}\circ\Phi\right\rangle_{\mathcal{E}(r+s)}+\left\langle\dfrac{\partial g}{\partial v}\circ\Phi\right\rangle_{\mathcal{E}(s)}\right)+\Phi^*G^*\mathcal{E}(1+r+s)$ for $1\leqslant k\leqslant s$.

Moreover since $F=\Lambda^{-1}G\Phi$ we have $F^*\mathcal{E}(1+r+s)=\Phi^*G^*(\Lambda^{-1})^*\mathcal{E}(1+r+s)$, but $\Lambda^{-1}\in L(1+r+s)$, so $(\Lambda^{-1})^*\mathcal{E}(1+r+s)=\mathcal{E}(1+r+s)$ and hence

(e) $$F^*\mathcal{E}(1+r+s)=\Phi^*G^*\mathcal{E}(1+r+s).$$

If f is infinitesimally (r, s)-stable then

$$\mathcal{E}(n+r+s)=\langle\partial f/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial f/\partial u\rangle_{\mathcal{E}(r+s)}+\langle\partial f/\partial v\rangle_{\mathcal{E}(s)}+F^*\mathcal{E}(1+r+s).$$

From this it follows, using (b), (c), (d), and (e), that

(f) $\quad \mathcal{E}(n+r+s)\subseteq\mu\left(\left\langle\dfrac{\partial g}{\partial x}\circ\Phi\right\rangle_{\mathcal{E}(n+r+s)}+\left\langle\dfrac{\partial g}{\partial u}\circ\Phi\right\rangle_{\mathcal{E}(r+s)}+\left\langle\dfrac{\partial g}{\partial v}\circ\Phi\right\rangle_{\mathcal{E}(s)}\right)+\Phi^*G^*\mathcal{E}(1+r+s).$

Clearly $\Phi^*\mathcal{E}(r+s)=\mathcal{E}(r+s)$ and $\Phi^*\mathcal{E}(s)=\mathcal{E}(s)$; therefore

$$\left\langle\frac{\partial g}{\partial v}\circ\Phi\right\rangle_{\mathcal{E}(s)}=\Phi^*(\langle\partial g/\partial v\rangle_{\mathcal{E}(s)})\quad\text{and}\quad\left\langle\frac{\partial g}{\partial u}\circ\Phi\right\rangle_{\mathcal{E}(r+s)}=\Phi^*(\langle\partial g/\partial u\rangle_{\mathcal{E}(r+s)}).$$

Moreover $\langle(\partial g/\partial x)\circ\Phi\rangle_{\mathcal{E}(n+r+s)}=\Phi^*(\langle\partial g/\partial x\rangle_{\mathcal{E}(n+r+s)})$. Finally, since μ is multiplication by $(\partial\lambda'/\partial t)\circ G\circ\Phi$, which is an element of $\Phi^*G^*\mathcal{E}(1+r+s)$, we have $\Phi^*G^*\mathcal{E}(1+r+s)=\mu\Phi^*G^*\mathcal{E}(1+r+s)$. Using these facts, it follows from (f) that

$$\text{(g)}\qquad \mathcal{E}(n+r+s)\subseteq\mu\Phi^*(\langle\partial g/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial g/\partial u\rangle_{\mathcal{E}(r+s)}+\langle\partial g/\partial v\rangle_{\mathcal{E}(s)}+G^*\mathcal{E}(1+r+s)).$$

But $\mu\Phi^*$ is a bijection of $\mathcal{E}(n+r+s)$ onto itself, so applying $(\mu\Phi^*)^{-1}$ to equation (g) we get:

$$\mathcal{E}(n+r+s)=(\mu\Phi^*)^{-1}\mathcal{E}(n+r+s)\subseteq\langle\partial g/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial g/\partial u\rangle_{\mathcal{E}(r+s)}+\langle\partial g/\partial v\rangle_{\mathcal{E}(s)}+G^*\mathcal{E}(1+r+s),$$

and hence, by definition 3.5, g is infinitesimally (r, s)-stable. Q.E.D.

The next few definitions and theorems will be devoted to showing that infinitesimal (r, s)-stability can be expressed as a certain transversality condition.

Definition 3.9. Let n and r be non-negative integers. We define $L(n, r)$ to be the set of triples $(\Phi, \psi, \Lambda)\in L(n+r)\times L(r)\times L(1+r)$, such that the following diagram commutes:

$$\begin{array}{ccccc}
\mathbf{R}^{n+r} & \xrightarrow{\ p\ } & \mathbf{R}^{r} & \xleftarrow{\ p'\ } & \mathbf{R}^{1+r}\\
\Phi\downarrow & & \psi\downarrow & & \Lambda\downarrow\\
\mathbf{R}^{n+r} & \xrightarrow{\ p\ } & \mathbf{R}^{r} & \xleftarrow{\ p'\ } & \mathbf{R}^{1+r}
\end{array}$$

where p and p' are projections onto the second factor.

Clearly $L(n, r)$ is a subgroup of $L(n+r)\times L(r)\times L(1+r)$. Moreover $L(n, r)$ acts on $\mathfrak{m}(n+r)$ from the right if we define $f\cdot(\Phi, \psi, \Lambda)$, for $f\in\mathfrak{m}(n+r)$ and $(\Phi, \psi, \Lambda)\in L(n, r)$, as follows: Define $F\in\mathcal{E}(n+r, 1+r)$ by setting $F(x, u)=(f(x, u), u)$ for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$. Let p_1: $\mathbf{R}^{1+r}\to\mathbf{R}$ be the projection onto the first factor.

Now set

$$f\cdot(\Phi, \psi, \Lambda):=p_1\circ(\Lambda^{-1}F\Phi).$$

Remark. It is easy to see that $L(n, r)$ is in fact just the group of r-equivalences of r-dimensional unfoldings, and $f\cdot(\Phi, \psi, \Lambda)=g$ if and only if (Φ, ψ, Λ) is an r-equivalence from g to f.

Definition 3.10. Let n, r and q be non-negative integers. We set $L^q(n, r)=\{(\Phi', \psi', \Lambda')\in L^q(n+r)\times L^q(r)\times L^q(1+r)\mid$ there exists $(\Phi, \psi, \Lambda)\in L(n, r)$ such that $\Phi'=\pi_q(\Phi)$, $\psi'=\pi_q(\psi)$ and $\Lambda'=\pi_q(\Lambda)\}$. Clearly $L^q(n, r)$ is a closed Lie subgroup of $L^q(n+r)\times L^q(r)\times L^q(1+r)$.

If $f\in\mathfrak{m}(n+r)$ and $(\Phi, \psi, \Lambda)\in L(n, r)$ then one easily convinces oneself that the q-jet of $f\cdot(\Phi, \psi, \Lambda)$ depends only on the q-jets of f, Φ, ψ, and Λ. Hence the group action of

$L(n, r)$ on $\mathfrak{m}(n+r)$ induces a well-defined right group action of $L^q(n, r)$ on $J_0^q(n+r, 1)$, such that for $f \in \mathfrak{m}(n+r)$ and $(\Phi, \psi, \Lambda) \in L(n, r)$ we have $\pi_q(f) \cdot (\pi_q(\Phi), \pi_q(\psi), \pi_q(\Lambda)) = \pi_q(f \cdot (\Phi, \psi, \Lambda))$. Moreover one readily checks that the action of $L^q(n, r)$ on $J_0^q(n+r, 1)$ is a Lie-group action, i.e. smooth.

Remark. If $z \in J_0^q(n+r)$ we shall denote the $L^q(n, r)$ orbit of z by $zL^q(n, r)$ and we shall denote the tangent space at z to $zL^q(n, r)$ by $T_z zL^q(n, r)$. Since $J_0^q(n+r, 1)$ is a finite dimensional real vector space we may identify its tangent space at any point with $J_0^q(n+r, 1)$ itself; in particular, we may identify $T_z zL^q(n, r)$ with a linear subspace of $J_0^q(n+r, 1)$. With this identification in mind, we have the following lemma:

LEMMA 3.11. *Let $f \in \mathfrak{m}(n+r)$ and let $z = \pi_q(f)$. Define $F \in \mathcal{E}(n+r, 1+r)$ by setting $F(x, u) = (f(x, u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$. Then*

$$T_z zL^q(n, r) = \pi_q(\langle \partial f/\partial x \rangle_{\mathfrak{m}(n+r)} + \langle \partial f/\partial u \rangle_{\mathfrak{m}(r)} + F^* \mathfrak{m}(1+r))$$

Proof. Let $w: \mathbf{R} \to L^q(n, r)$ be an arbitrary smooth map such that $w(0)$ is the identity. Then it is easy to see that one can find smooth maps $\gamma: \mathbf{R}^{n+r} \times \mathbf{R} \to \mathbf{R}^n$, $\delta: \mathbf{R}^r \times \mathbf{R} \to \mathbf{R}^r$ and $\lambda: \mathbf{R}^{1+r} \times \mathbf{R} \to \mathbf{R}$ with the following properties:

Firstly, if we define, for each $t \in \mathbf{R}$, germs $\Phi_t \in \mathcal{E}(n+r, n+r)$, $\psi_t \in \mathcal{E}(r, r)$ and $\Lambda_t \in \mathcal{E}(1+r, 1+r)$ by the equations $\Phi_t(x, u) = (x + \gamma(x, u, t), u + \delta(u, t))$; $\psi_t(u) = u + \delta(u, t)$; and $\Lambda_t(s, u) = (s + \lambda(s, u, t), u + \delta(u, t))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$ and $s \in \mathbf{R}$, then for each $t \in \mathbf{R}$ the triple $(\Phi_t, \psi_t, \Lambda_t)$ is in $L(n, r)$ and (*) $w(t) = (\pi_q(\Phi_t), \pi_q(\psi_t), \pi_q(\Lambda_t))$. And secondly, $\gamma(x, u, 0) = \delta(u, 0) = \lambda(s, u, 0) = 0$ for all $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, and $s \in \mathbf{R}$.

Conversely, if we are given arbitrary maps γ, δ, and λ satisfying all properties above except (*), then equation (*) can be used to define a smooth map $w: \mathbf{R} \to L^q(n, r)$ such that $w(0)$ is the identity.

Now $T_z zL^q(n, r)$ consists of all tangent vectors of the form $\partial(z \cdot w(t))/\partial t|_{t=0}$, where w is an arbitrary map as above. But if we have maps γ, δ, and λ as above, and define germs Φ_t, ψ_t, and Λ_t as above, and if we let $p_1: \mathbf{R}^{1+r} \to \mathbf{R}$ be the projection onto the first factor, then by straight computation (and using the fact that Φ_0, ψ_0, and Λ_0 are germs of identity mappings) one sees that

$$\frac{\partial}{\partial t}(z \cdot w(t))|_{t=0} = \pi_q\left(\frac{\partial}{\partial t}(f \cdot (\Phi_t, \psi_t, \Lambda_t))|_{t=0}\right) = \pi_q\left(\frac{\partial}{\partial t}(p_1 \Lambda_t^{-1} F \Phi_t)|_{t=0}\right)$$

$$= \pi_q\left(\left(\left(\frac{\partial}{\partial t} p_1 \Lambda_t^{-1}\right)\Big|_{t=0} \circ F\right) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\left(\frac{\partial \gamma_i}{\partial t}\Big|_{t=0}\right) + \sum_{j=1}^{r} \frac{\partial f}{\partial u_j}\left(\frac{\partial \delta_j}{\partial t}\Big|_{t=0}\right)\right).$$

Since $p_1 \Lambda_t^{-1}\Lambda_t = p_1$ we have

$$0 = \frac{\partial}{\partial t} p_1\bigg|_{t=0} = \left(\frac{\partial}{\partial t} p_1 \Lambda_t^{-1}\right)\bigg|_{t=0} + p_1\left(\frac{\partial}{\partial t}\Lambda_t\right)\bigg|_{t=0},$$

and hence

$$\left(\frac{\partial}{\partial t} p_1 \Lambda_t^{-1}\right)\bigg|_{t=0} = -\frac{\partial \lambda}{\partial t}\bigg|_{t=0}.$$

But now the lemma follows immediately because on the one hand the condition $(\Phi_t, \psi_t, \Lambda_t) \in L(n, r)$ implies that for all t we have $\gamma(0, 0, t) = \delta(0, t) = \lambda(0, 0, t) = 0$ and hence $(\partial \gamma_i/\partial t)|_{t=0} \in \mathfrak{m}(n+r)$, $(\partial \delta_j/\partial t)|_{t=0} \in \mathfrak{m}(r)$ and $(\partial \lambda/\partial t)|_{t=0} \in \mathfrak{m}(1+r)$; but on the other hand this is clearly the only restriction on the derivatives with respect to t at $t=0$, so one can choose γ, δ and λ such that the $(\partial \gamma_i/\partial t)|_{t=0}$ are arbitrary elements of $\mathfrak{m}(n+r)$, the $(\partial \delta_j/\partial t)|_{t=0}$ are arbitrary elements of $\mathfrak{m}(r)$, and $(\partial \lambda/\partial t)|_{t=0}$ is an arbitrary element of $\mathfrak{m}(1+r)$. Q.E.D.

Definition 3.12. Let $f \in \mathfrak{m}(n+r+s)$ and let q be a non-negative integer. Let $i: \mathbf{R}^{n+r} \to \mathbf{R}^{n+r+s}$ be the canonical inclusion given by $i(x, u) = (x, u, 0)$ for $(x, u) \in \mathbf{R}^{n+r}$. Let $f_0 = f|\mathbf{R}^{n+r} = i^*f$. We define

$$M^q_{r,s}(f) := (\varrho_q \circ {}_q i^*)^{-1}(\pi_q(f_0) L^q(n, r)) \subseteq J^q(n+r+s, 1),$$

and we shall say f is (r, s) *q-transversal* if the map-germ $J^q f$ is transversal at 0 to $M^q_{r,s}(f)$. (See definition 1.2 for the definitions of ϱ_q, ${}_q i^*$, and $J^q f$).

Remark. Since $\varrho_q \circ {}_q i^*: J^q(n+r+s, 1) \to J^q_0(n+r, 1)$ is a projection of real vector spaces, $M^q_{r,s}(f)$ is obviously an immersed submanifold of $J^q(n+r+s, 1)$, so it makes sense to speak of transversality).

LEMMA 3.13. *Let $f \in \mathfrak{m}(n+r+s)$. Let $f_0 = f|\mathbf{R}^{n+r}$ and define $F_0 \in \mathcal{E}(n+r, 1+r)$ by $F_0(x, u) = (f_0(x, u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$. Let q be a non-negative integer. Then f is (r, s) q-transversal if and only if*

(a) $\mathcal{E}(n+r) = \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial f_0/\partial u\rangle_{\mathcal{E}(r)} + \langle \partial f/\partial v|\mathbf{R}^{n+r}\rangle_{\mathbf{R}} + F_0^*\mathcal{E}(1+r) + \mathfrak{m}(n+r)^{q+1}.$

Proof. Let $A \subseteq J^q(n+r+s, 1)$ be the image under the differential of $J^q f$ of the tangent space at 0 to $\mathbf{R}^{n+r+s}$. (Here again we identify A, which is in fact a subspace of the *tangent space* to $J^q(n+r+s, 1)$ at $\pi_q(f)$, with a subspace of $J^q(n+r+s, 1)$ itself, in the obvious way; we may do this because $J^q(n+r+s, 1)$ is a finite-dimensional real vector space).

A is generated by the partial derivatives of $J^q f$ with respect to the coordinate axes of $\mathbf{R}^{n+r+s}$, all evaluated at 0.

Recall the definition of $J^q f$: choose a representative f' of f, defined on a neighbourhood U of 0 in $\mathbf{R}^{n+r+s}$. For any $y \in U$ define $f_{(y)} \in \mathcal{E}(n+r+s)$ by $f_{(y)}(z)=f'(y+z)$ for $z \in \mathbf{R}^{n+r+s}$, and set $J^q f(y)=\pi_q(f_{(y)})$. From this definition it is clear that if t denotes any one of the coordinates of $\mathbf{R}^{n+r+s}$, then

$$\left.\frac{\partial J^q f}{\partial t}\right|_{y=0} = \pi_q\left(\frac{\partial f}{\partial t}\right) \in J^q(n+r+s, 1).$$

Hence $A = \pi_q(\langle \partial f/\partial x\rangle_{\mathbf{R}} + \langle \partial f/\partial u\rangle_{\mathbf{R}} + \langle \partial f/\partial v\rangle_{\mathbf{R}})$.

Now let B be the tangent space to $M^q_{r,s}(f)$ at $\pi_q(f)=J^q f(0)$ (as usual we shall identify B with a subspace of $J^q(n+r+s, 1)$). Then from the definition of $M^q_{r,s}(f)$ and from Lemma 3.11 it is clear that $B=\pi_q(\langle \partial f_0/\partial x\rangle_{\mathfrak{m}(n+r)} + \langle \partial f_0/\partial u\rangle_{\mathfrak{m}(r)} + F_0^*\mathfrak{m}(1+r) + \langle 1\rangle_{\mathbf{R}} + \mathfrak{m}(s)\,\mathcal{E}(n+r+s))$ (where the expression in parentheses is of course to be considered as a subset of $\mathcal{E}(n+r+s)$).

The germ f is by definition (r, s) q-transversal if and only if $J^q(n+r+s, 1) = \pi_q\mathcal{E}(n+r+s) = A+B$, and this is clearly equivalent (using the fact that $F_0^*\mathfrak{m}(1+r) + \langle 1\rangle_{\mathbf{R}} = F_0^*\,\mathcal{E}(1+r)$ and the fact that if t is any coordinate of $\mathbf{R}^{n+r}$, then $\partial f/\partial t - \partial f_0/\partial t \in \mathfrak{m}(s)\,\mathcal{E}(n+r+s)$) to

(b) $$\mathcal{E}(n+r+s) = \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial f_0/\partial u\rangle_{\mathcal{E}(r)} + \langle \partial f/\partial v\rangle_{\mathbf{R}} + F_0^*\,\mathcal{E}(1+r) + \mathfrak{m}(s)\,\mathcal{E}(n+r+s) + \mathfrak{m}(n+r+s)^{q+1}.$$

Let α: $\mathcal{E}(n+r+s) \to \mathcal{E}(n+r)$ be defined by setting $\alpha(g)=g\,|\,\mathbf{R}^{n+r}$ for $g \in \mathcal{E}(n+r+s)$. Then the kernel of α is $\mathfrak{m}(s)\,\mathcal{E}(n+r+s)$ and since $\mathfrak{m}(s)\,\mathcal{E}(n+r+s)$ is contained in both sides of (b), equation (b) holds if and only if the equation which results when we apply α to both sides of (b) is valid. But doing this clearly yields equation (a), so we are done. Q.E.D.

As a corollary, we have

THEOREM 3.14. *Let $f \in \mathfrak{m}(n+r+s)$. Then f is infinitesimally (r, s)-stable if and only if f is (r, s) q-transversal for all non-negative integers q.*

Proof. "if": Let β: $\mathcal{E}(n+r) \to \mathcal{E}(n)$ be given by restriction, i.e. let $\beta(g)=g\,|\,\mathbf{R}^n$ for $g \in \mathcal{E}(n+r)$. Let $\eta = f\,|\,\mathbf{R}^n$.

If f is (r, s) q-transversal for all q, then 3.13 (a) holds for all q, and applying β we find that

$$\mathcal{E}(n) = \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)} + \langle \partial f/\partial u\,|\,\mathbf{R}^n,\ \partial f/\partial v\,|\,\mathbf{R}^n\rangle_{\mathbf{R}} + \eta^*\mathcal{E}(1) + \mathfrak{m}(n)^{q+1}$$

for arbitrary q. This implies $\sigma_q(\eta) \leqslant r+s$ for all q (see definition 2.16) and hence by Theorem 2.17 (c) η is finitely determined and for some integer k, $\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)}$. Equation 3.13 (a) holds in particular for $q=k(s+1)-1$ so by Theorem 3.6, f is infinitesimally (r, s)-stable.

"Only if": If f is infinitesimally (r, s)-stable then equation 3.6 (b) holds, but this clearly implies that equation 3.13 (a) holds for any q, so f is (r, s) q-transversal for all q.

We are now in position to prove the analogue for (r, s)-stability of Theorem 2.11.

THEOREM 3.15. *Let $f \in \mathfrak{m}(n+r+s)$ unfold $\eta \in \mathfrak{m}(n)$. The following statements are equivalent:*

(a) *f is infinitesimally (r, s)-stable.*
(b) *f is strongly (r, s)-stable.*
(c) *f is weakly (r, s)-stable.*

Proof. The method of proof is essentially the same as for Theorem 2.11 (see [11, Th. 4.11]), although a small amount of extra work is needed here.

Proof that (a) $\Rightarrow$ (b). Suppose f is infinitesimally (r, s)-stable. Let a neighbourhood U of $0 \in \mathbf{R}^{n+r+s}$ and a representative $f': U \to \mathbf{R}$ of f be given. We must find a neighbourhood of f' in $C^\infty(U, \mathbf{R})$ such that any mapping in this neighbourhood is, at some point of U, (r, s)-equivalent to f' at 0.

We begin by defining some notation. We define a function $\Gamma: U \times C^\infty(U, \mathbf{R}) \to \mathfrak{m}(n+r+s)$ as follows: If $(y, w, z) \in U$ and $h \in C^\infty(U, \mathbf{R})$, define $\Gamma(y, w, z, h)$ by setting $\Gamma(y, w, z, h)(x, u, v) = h(x+y, u+w, v+z) - h(y, w, z)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. For $(y, w, z) \in U$ and $h \in C^\infty(U, \mathbf{R})$, define $\gamma(y, w, z, h) = \Gamma(y, w, z, h) \,|\, \mathbf{R}^{n+r}$ and define $\zeta(y, w, z, h) = \Gamma(y, w, z, h) \,|\, \mathbf{R}^n$. Note that $\Gamma(0, 0, 0, f') = f$ and $\zeta(0, 0, 0, f') = \eta$.

Since η is finitely determined there is a positive integer k such that $\pi_k(\eta) \notin Z_k$ (see Definition 2.18). Since Z_k is an algebraic and hence closed subset of $J_0^k(n, 1)$ (by Lemma 2.19 (c)) and since $\pi_k \circ \zeta: U \times C^\infty(U, \mathbf{R}) \to J_0^k(n, 1)$ is continuous, it follows that there is a neighbourhood $U_1 \subseteq U$ of 0 in $\mathbf{R}^{n+r+s}$ and a neighbourhood V_1 of f' in $C^\infty(U, \mathbf{R})$ such that if $(y, w, z) \in U_1$ and $h \in V_1$ then $\pi_k \zeta(y, w, z, h) \notin Z_k$ and hence (applying Lemma 2.19 (b)) $\mathfrak{m}(n)^k \subseteq \langle \partial \zeta(y, w, z, h)/\partial x \rangle_{\mathfrak{m}(n)}$.

Let $q = k(s+1)$. If g is any germ in $\mathfrak{m}(n+r+s)$, we set $g_0 = g \,|\, \mathbf{R}^{n+r}$ and we let $A(g)$ be the finite subset of $J^{q-1}(n+r, 1)$ consisting of the $q-1$-jets of the following elements of $\mathcal{E}(n+r)$: all elements of the form (monomial in x_i and u_j of degree $<q$ times some $\partial g_0/\partial x_i$); all elements of the form (monomial in u_j of degree $<q$ times some $\partial g_0/\partial u_j$); the germs $\partial g/\partial v_l \,|\, \mathbf{R}^{n+r}$ for $1 \leqslant l \leqslant s$; and all germs of the form (monomial in u_j of degree $<q$ times g_0^i for some i, $0 \leqslant i < q$). The elements of $A(g)$ depend linearly on $\pi_q(g)$.

Since f is infinitesimally (r, s)-stable, it clearly follows from Theorem 3.6 that $A(f)$ generates $J^{q-1}(n+r, 1)$ over $\mathbf{R}$. Since $\pi_q \circ \Gamma: U \times C^\infty(U, \mathbf{R}) \to J^q(n+r+s, 1)$ is continuous, this implies that there is a neighbourhood $U_2 \subseteq U$ of 0 in $\mathbf{R}^{n+r+s}$ and a neighbourhood V_2

of f' in $C^\infty(U, \mathbf{R})$ such that if $(y, w, z) \in U_2$ and $h \in V_2$, then $A(\Gamma(y, w, z, h))$ generates $J^{q-1}(n+r, 1)$ over $\mathbf{R}$.

Choose a compact neighbourhood K of 0 in $\mathbf{R}^{n+r+s}$ such that $K \subseteq U_1 \cap U_2$, and choose a real number $c > 0$ small enough, so that the closed $n+r+s$-cube $[-c, c]^{n+r+s}$ is contained in K.

It is not very difficult to see, finally, that there is a neighbourhood V_3 of f' in $C^\infty(U, \mathbf{R})$ and a neighbourhood V_4 of 0 in $C^\infty(U, \mathbf{R})$ such that if $g \in V_4$ and if $h \in V_3$ and $(y, w, z) \in K$, then $\pi_{q-1}(\gamma(y, w, z, g))$ can be written as a real linear combination of elements of $A(\Gamma(y, w, z, h))$ in such a way, that the coefficients have absolute value less than c.

Now choose a neighbourhood V of 0 in $C^\infty(U, \mathbf{R})$ such that $V \subseteq V_4$ and such that if $h \in V$, then for any real number $t \in [0, 1]$, the mapping $f' + th$ is in $V_1 \cap V_2 \cap V_3$.

We shall show that if $h \in V$, then $f' + h$ is at some point of K (r, s)-equivalent to f' at 0. This will prove (a) $\Rightarrow$ (b). So suppose $h \in V$. Let a point $(y, w, z) \in K$ and a real number $a \in [0, 1]$ be given. Define a germ $H \in \mathfrak{m}(n+r+s+1)$ by setting

$$H(x, u, v, t) = (f' + (a+t)h)(y+x, w+u, z+v) - (f' + (a+t)h)(y, w, z)$$
$$\text{for } x \in \mathbf{R}^n,\ u \in \mathbf{R}^r,\ v \in \mathbf{R}^s \text{ and } t \in \mathbf{R}$$

and define a germ $\bar{H} \in \mathcal{E}(n+r+s+1, 1+r+s+1)$ by setting

$$\bar{H}(x, u, v, t) = (H(x, u, v, t), u, v, t) \text{ for } x \in \mathbf{R}^n,\ u \in \mathbf{R}^r,\ v \in \mathbf{R}^s,\ t \in \mathbf{R}.$$

Let $\mu = H \,|\, \mathbf{R}^n$, and let $H_0 = H \,|\, \mathbf{R}^{n+r}$ and $H_1 = H \,|\, \mathbf{R}^{n+r+s}$. Observe that $\mu = \zeta(y, w, z, f' + ah)$ and $H_1 = \Gamma(y, w, z, f' + ah)$. By the choice of V and K, and since $(y, w, z) \in K$ and $h \in V$ and $0 \leq a \leq 1$, it is clear that $\mathfrak{m}(n)^k \subseteq \langle \partial\mu/\partial x \rangle_{\mathfrak{m}(n)}$ and that $A(H_1)$ generates $J^{q-1}(n+r, 1)$ over $\mathbf{R}$.

Hence

$$\text{(d)}\quad \mathcal{E}(n+r) = \langle \partial H_0/\partial x \rangle_{\mathcal{E}(n+r)} + \langle \partial H_0/\partial u \rangle_{\mathcal{E}(r)} + \langle \partial H_1/\partial v \,|\, \mathbf{R}^{n+r} \rangle_{\mathbf{R}} + \langle 1, H_0, \ldots, H_0^{q-1} \rangle_{\mathcal{E}(r)} + \mathfrak{m}(n+r)^q$$

Since $\mathfrak{m}(n)^k \subseteq \langle \partial\mu/\partial x \rangle_{\mathfrak{m}(n)}$ it follows as in the proof of Theorem 3.6 ((c) $\Rightarrow$ (a)) that

$$\text{(e)}\qquad \mathfrak{m}(n+r)^q \subseteq \langle \partial H_0/\partial x \rangle_{\mathfrak{m}(n+r)} + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r).$$

In particular, in (d) we may replace the term $\mathfrak{m}(n+r)^q$ by $\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)$. From the resulting equation it follows, by Corollary 1.8 (c), that

$$\mathfrak{m}(r)^s\mathcal{E}(n+r) \subseteq \langle \partial H_0/\partial x \rangle_{\mathcal{E}(n+r)} + \langle \partial H_0/\partial u \rangle_{\mathcal{E}(r)} + \langle 1, H_0, \ldots, H_0^{q-1} \rangle_{\mathcal{E}(r)},$$

and from this and (e) we get:

$$\text{(f)}\qquad \mathfrak{m}(n+r)^q \subseteq \langle \partial H_0/\partial x \rangle_{\mathfrak{m}(n+r)} + \langle \partial H_0/\partial u \rangle_{\mathfrak{m}(r)} + \langle 1, H_0, \ldots, H_0^{q-1} \rangle_{\mathfrak{m}(r)}.$$

Now clearly $(\partial H/\partial t)|\mathbf{R}^{n+r}=\gamma(y, w, z, h)$, so, since $(y, w, z)\in K$ and $h\in V$ (and hence $h\in V_4$ and $f'+ah\in V_3$), it follows that $\pi_{q-1}((\partial H/\partial t)|\mathbf{R}^{n+r})$ can be written as a linear combination of elements of $A(H_1)$ with coefficients of absolute value $<c$.

This, together with (f), clearly implies that we can find germs $\xi_1, ..., \xi_n\in\mathcal{E}(n+r)$, germs $\chi_1, ..., \chi_r\in\mathcal{E}(r)$, real numbers $\beta_1, ..., \beta_s$, and a germ $\omega\in\mathcal{E}(1+r)$ such that (if we define $\bar{H}_0\in\mathcal{E}(n+r, 1+r)$ by $\bar{H}_0(x, u)=(H_0(x, u), u)$)

(g)
$$\frac{\partial H}{\partial t}\Big|\mathbf{R}^{n+r}=\sum_{i=1}^{n}\xi_i\frac{\partial H_0}{\partial x_i}+\sum_{j=1}^{r}\chi_j\frac{\partial H_0}{\partial u_j}+\sum_{l=1}^{s}\beta_l\frac{\partial H}{\partial v_l}\Big|\mathbf{R}^{n+r}+\bar{H}_0^*(\omega),$$

and such that $|\xi_i(0, 0)|<c$, (for $1\leqslant i\leqslant n$); $|\chi_j(0)|<c$ (for $1\leqslant j\leqslant r$); $|\beta_l|<c$ (for $1\leqslant l\leqslant s$); and $|\omega(0, 0)|<c$.

Define $\alpha\in\mathcal{E}(n+r+s+1)$ by setting

$$\alpha=\frac{\partial H}{\partial t}-\sum_{i=1}^{n}\xi_i\frac{\partial H}{\partial x_i}-\sum_{j=1}^{r}\chi_j\frac{\partial H}{\partial u_j}-\sum_{l=1}^{s}\beta_l\frac{\partial H}{\partial v_l}-\bar{H}^*(\omega).$$

By (g) we have $\alpha|\mathbf{R}^{n+r}=0$ so $\alpha\in\mathfrak{m}(s+1)\mathcal{E}(n+r+s+1)$.

By (d), H_1 is infinitesimally (r, s)-stable, so by Corollary 3.7 H is infinitesimally $(r, s+1)$-stable. Hence equation 3.5 (a) holds (with appropriate substitutions) and multiplying by $\mathfrak{m}(s+1)\mathcal{E}(n+r+s+1)$ we find

$$\begin{aligned}\mathfrak{m}(s+1)\mathcal{E}(n+r+s+1)=\langle\partial H/\partial x\rangle_{\mathfrak{m}(s+1)\mathcal{E}(n+r+s+1)}+\langle\partial H/\partial u\rangle_{\mathfrak{m}(s+1)\mathcal{E}(r+s+1)}\\+\langle\partial H/\partial v\rangle_{\mathfrak{m}(s+1)}+\langle\partial H/\partial t\rangle_{\mathfrak{m}(s+1)}+\bar{H}^*(\mathfrak{m}(s+1)\mathcal{E}(1+r+s+1)).\end{aligned}$$

So there are germs $\xi_1', ..., \xi_n'\in\mathfrak{m}(s+1)\mathcal{E}(n+r+s+1)$, germs $\chi_1', ..., \chi_r'\in\mathfrak{m}(s+1)\mathcal{E}(r+s+1)$, germs $\beta_1', ..., \beta_s'\in\mathfrak{m}(s+1)$, a germ $\delta\in\mathfrak{m}(s+1)$ and a germ $\omega'\in\mathfrak{m}(s+1)\mathcal{E}(1+r+s+1)$ such that

$$\alpha=\sum_{i=1}^{n}\xi_i'\frac{\partial H}{\partial x_i}+\sum_{j=1}^{r}\chi_j'\frac{\partial H}{\partial x_j}+\sum_{l=1}^{s}\beta_l'\frac{\partial H}{\partial v_l}+\bar{H}^*(\omega')+\delta\frac{\partial H}{\partial t}.$$

Since $\delta\in\mathfrak{m}(s+1)$ it follows that $1-\delta$ is a unit of $\mathcal{E}(s+1)$. Define germs $\xi_i''\in\mathcal{E}(n+r+s+1)$ (for $1\leqslant i\leqslant n$); germs $\chi_j''\in\mathcal{E}(r+s+1)$ (for $1\leqslant j\leqslant r$); germs $\beta_l''\in\mathcal{E}(s+1)$ (for $1\leqslant l\leqslant s$) and a germ $\omega''\in\mathcal{E}(1+r+s+1)$ by setting $\xi_i''=(\xi_i+\xi_i')/(1-\delta)$; $\chi_j''=(\chi_j+\chi_j')/(1-\delta)$; $\beta_l''=(\beta_l+\beta_l')/(1-\delta)$; and $\omega''=(\omega+\omega')/(1-\delta)$. From the two different expressions we have for α, it easily follows that

(h)
$$\frac{\partial H}{\partial t}=\sum_{i=1}^{n}\xi_i''\frac{\partial H}{\partial x_i}+\sum_{j=1}^{r}\chi_j''\frac{\partial H}{\partial u_j}+\sum_{l=1}^{s}\beta_l''\frac{\partial H}{\partial v_l}+\bar{H}^*(\omega'').$$

(Note that $\bar{H}^*$ is a homomorphism of $\mathcal{E}(s+1)$-modules!)

Moreover $|\xi_i''(0, 0, 0, 0)| < c$; $|\chi_j''(0, 0, 0)| < c$; $|\beta_l''(0, 0)| < c$; and $|\omega''(0, 0, 0, 0)| < c$. By Lemma 1.10 there is a germ $\Phi \in \mathcal{E}(n+r+s+1, n+r+s)$ and a germ $\lambda \in \mathcal{E}(1+n+r+s+1)$ such that

(i) $\Phi(x, u, v, 0) = (x, u, v)$ and $\lambda(\tau, x, u, v, 0) = \tau$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $\tau \in \mathbf{R}$,

(j) $H(\Phi(x, u, v, t), t) = \lambda(H(x, u, v, 0), x, u, v, t)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$,

and finally, if we write $\Phi = (\varphi, \sigma, \varrho)$ for some $\varphi \in \mathcal{E}(n+r+s+1, n)$, $\sigma \in \mathcal{E}(n+r+s+1, r)$ and $\varrho \in \mathcal{E}(n+r+s+1, s)$, then

(k)
$$\frac{\partial \varphi_i}{\partial t}(x, u, v, t) = -\xi_i''(\varphi(x, u, v, t), \sigma(x, u, v, t), \varrho(x, u, v, t), t) \quad (i = 1, \ldots, n),$$

$$\frac{\partial \sigma_j}{\partial t}(x, u, v, t) = -\chi_j''(\sigma(x, u, v, t), \varrho(x, u, v, t), t) \quad (j = 1, \ldots, r),$$

$$\frac{\partial \varrho_l}{\partial t}(x, u, v, t) = -\beta_l''(\varrho(x, u, v, t), t) \quad (l = 1, \ldots, s),$$

and

$$\frac{\partial \lambda}{\partial t}(\tau, x, u, v, t) = \omega''(\lambda(\tau, x, u, v, t), \sigma(x, u, v, t), \varrho(x, u, v, t), t),$$

for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $\tau \in \mathbf{R}$ and $t \in \mathbf{R}$.

By (i) $\varrho(x, u, v, 0) = v$ and hence does not depend on x or u. Moreover by (k) $\partial\varrho/\partial t$ depends only on the value of ϱ and not directly on x or u, so by the uniqueness of solutions of ordinary differential equations with given initial conditions, ϱ does not depend on x or u, i.e. $\varrho \in \mathcal{E}(s+1, s)$. Using this fact it then follows by the same argument that $\sigma \in \mathcal{E}(r+s+1, r)$. And, then, by the same argument again, it follows from the last equation (k) that $\lambda \in \mathcal{E}(1+r+s+1)$.

Now choose representative functions for the germs φ, σ, ϱ, and λ; for convenience we use the same names for the representatives as for the germs.

Then we can choose suitable neighbourhoods W_1 of 0 in $\mathbf{R}^{n+r+s}$, W_2 of 0 in $\mathbf{R}^{r+s}$, W_3 of 0 in $\mathbf{R}^s$, W_4 of 0 in $\mathbf{R}^{1+r+s}$ and a neighbourhood T of 0 in $\mathbf{R}$ such that for $t \in T$ we may define functions $\Phi_t: W_1 \to \mathbf{R}^{n+r+s}$; $\psi_t: W_2 \to \mathbf{R}^{r+s}$; $\varrho_t: W_3 \to \mathbf{R}^s$, and $\lambda_t: W_4 \to \mathbf{R}$ by the equations $\Phi_t(x, u, v) = (\varphi(x, u, v, t), \sigma(u, v, t), \varrho(v, t))$; $\psi_t(u, v) = (\sigma(u, v, t), \varrho(v, t))$; $\varrho_t(v) = \varrho(v, t)$; and $\lambda_t(\tau, u, v) = \lambda(\tau, u, v, t)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $\tau \in \mathbf{R}$.

From (i) it follows that if the W_i and T are chosen small enough then Φ_t, ψ_t and ϱ_t

will be diffeomorphisms into for every $t\in T$, and for every $(\tau, u, v)\in W_4$ and every $t\in T$, $(\partial\lambda_t/\partial\tau)(\tau, u, v)\neq 0$.

Now if one chooses a suitably small neighbourhood $U_{(y,w,z)}$ of $(y, w, z)\in\mathbf{R}^{n+r+s}$ then one can for each $(y', w', z')\in U_{(y,w,z)}$ and for each $t\in T$ define germs $\Phi'_{t,y',w',z'}\in L(n+r+s)$; $\psi'_{t,y',w',z'}\in L(r+s)$; $\varrho'_{t,y',w',z'}\in L(s)$ and $\Lambda'_{t,y',w',z'}\in L(1+r+s)$ by the equations: $\Phi'_{t,y',w',z'}(x, u, v)=\Phi_t(x+y'-y,\ u+w'-w,\ v+z'-z)-\Phi_t(y'-y,\ w'-w,\ z'-z)$; $\psi'_{t,y',w',z'}(u, v)=\psi_t(u+w'-w,\ v+z'-z)-\psi_t(w'-w, z'-z)$; $\varrho'_{t,y',w',z'}(v)=\varrho_t(v+z'-z)-\varrho_t(z'-z)$; and $\Lambda'_{t,y',w',z'}(\tau, u, v)=(\lambda_t(\tau+(f'+ah)(y', w', z')-(f'+ah)(y, w, z),\ w'-w+u,\ z'-z+v)+(f'+(a+t)h)(y, w, z)-(f'+(a+t)h)((y, w, z)+\Phi_t(y'-y, w'-w, z'-z)),\ \psi'_{t,y',w',z'}(u, v))$ for $x\in\mathbf{R}^n, u\in\mathbf{R}^r$, $v\in\mathbf{R}^s$, $\tau\in\mathbf{R}$. (That these germs are in fact in $L(n+r+s)$ etc. is immediately clear from the defining equations and the properties of Φ_t, ψ_t, ϱ_t and Λ_t; to show that $\Lambda'_{t,y',w',z'}(0)=0$ one must also use (j) and the definition of H).

Moreover, from (j) and the definition of H one can easily check by direct computation that if $U_{(y,w,z)}$ and T were chosen small enough, then for each $(y',w',z')\in U_{(y,w,z)}$ and each $t\in T$, the quadruple $(\Phi'_{t,y',w',z'}, \psi'_{t,y',w',z'}, \varrho'_{t,y',w',z'}, \Lambda'_{t,y',w',z'})$ is an (r, s)-equivalence from $f'+ah$ at (y', w', z') to $f'+(a+t)h$ at $(y, w, z)+\Phi_t(y'-y, w'-w, z'-z)$.

To simplify the notation, we shall write $d(y', w', z', t)$ for $(y, w, z)+\Phi_t(y'-y, w'-w, z'-z)$.

Because for t small enough Φ_t: $W_1\to\mathbf{R}^{n+r+s}$ is a diffeomorphism into, it is easily seen that for any t near enough to 0 and for (y'', w'', z'') near enough to (y, w, z), there is a (y', w', z') in $U_{(y,w,z)}$ such that $(y'', w'', z'')=d(y', w', z', t)$.

From this, and by composing equivalences it follows that given any t_1 and t_2 sufficiently near 0 and any (y'', w'', z'') sufficiently near (y, w, z), there is a point $(\bar{y}, \bar{w}, \bar{z})$ such that $f'+(a+t_1)h$ at (y'', w'', z'') is (r, s)-equivalent to $f'+(a+t_2)h$ at $(\bar{y}, \bar{w}, \bar{z})$. In fact we need only choose (y', w', z') such that $(y'', w'', z'')=d(y', w', z', t_1)$ and then set $(\bar{y}, \bar{w}, \bar{z})=d(y', w', z', t_2)$.

Moreover, from this, and the definition of d, and because of the equations (k) and the fact that the absolute values of the ξ''_i, χ''_j, β''_l and ω'' at 0 are less than c, it follows that if t_1 and t_2 are small enough, and if (y'', w'', z'') is close enough to (y, w, z) then for t between t_1 and t_2 each coordinate of $\partial d(y', w', z', t)/\partial t$ will be smaller than c in absolute value and hence corresponding coordinates of (y'', w'', z'') and $(\bar{y}, \bar{w}, \bar{z})$ will differ by at most $c|t_1-t_2|$.

Now since (y, w, z) and a were arbitrary it follows by the compactness of K and of $[0, 1]$ that there is a real number $\varepsilon>0$ such that if $|t|<\varepsilon$, then for any $(x, u, v)\in K$ and any $b\in[0, 1]$ there is an $(x', u', v')\in\mathbf{R}^{n+r+s}$ such that $f'+bh$ at (x, u, v) is (r, s)-equivalent to $f'+(b+t)h$ at (x', u', v') and such that corresponding coordinates of (x, u, v) and (x', u', v') differ by at most $c|t|$.

Now it is easy to define by induction a sequence of real numbers $0=t_0<t_1<\dots<t_m=1$ and a corresponding sequence of points $p_0=0, p_1, p_2, \dots, p_m \in \mathbf{R}^{n+r+s}$ such that $f'+t_i h$ at p_i is (r, s)-equivalent to $f'+t_{i+1}h$ at p_{i+1} (for $0 \leqslant i < m$) and such that corresponding coordinates of p_i and p_{i+1} differ by at most $c(t_{i+1}-t_i)$. Since $[-c, c]^{n+r+s} \subseteq K$ this implies each p_i will be in K so that the induction can always be carried out as long as $t_i \leqslant 1$.

And then it follows that $f'=f'+t_0 h$ at $0=p_0$ is (r, s)-equivalent to $f'+h$ at $p_m \in K \subseteq U$, so we are done proving (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (c) is trivial.

Proof that (c) $\Rightarrow$ (a). Suppose f is weakly (r, s)-stable. We shall use Thom's transversality lemma to find a germ which is (r, s)-equivalent to f and which is infinitesimally (r, s)-stable; this will imply f is infinitesimally (r, s)-stable.

Choose a neighbourhood U of 0 in $\mathbf{R}^{n+r+s}$ and a representative function $f'\colon U \to \mathbf{R}$ of f. If q is a non-negative integer and if $t \in \mathbf{R}$ and $t \neq 0$, we define $V_t^q = \{h \in C^\infty(U, \mathbf{R}) \mid J^q(f'+th)$ is transversal to $M_{r,s}^q(f')$ everywhere on $U\}$. By Thom's transversality lemma (Theorem 1.9), for each q and t the set V_t^q is a countable intersection of open dense subsets of $C^\infty(U, \mathbf{R})$.

Let $V = \bigcap_{q=0}^{\infty} \bigcap_{k=1}^{\infty} V_{1/k}^q$. Then V is also a countable intersection of open dense subsets of $C^\infty(U, \mathbf{R})$ and in particular, since $C^\infty(U, \mathbf{R})$ is a Baire space, V is dense and hence non-empty. Choose an $h \in V$.

Since f is weakly (r, s)-stable it follows that if we choose an integer k sufficiently large, then $f'+k^{-1}h$ at some point (y, w, z) of U will be (r, s)-equivalent to f' at 0. Or in other words, if we define $g \in \mathfrak{m}(n+r+s)$ by $g(x, u, v) = (f'+k^{-1}h)(y+x, w+v, z+u) - (f'+k^{-1}h)(y, w, z)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, then g is (r, s)-equivalent to f. It easily follows that $f \mid \mathbf{R}^{n+r}$ and $g \mid \mathbf{R}^{n+r}$ are r-equivalent and hence for any q we have $M_{r,s}^q(f) = M_{r,s}^q(g)$.

Moreover since $h \in V$ it follows that for any q, $J^q(f'+k^{-1}h)$ is transversal to $M_{r,s}^q(f)$ at (y, w, z); by the definition of $M_{r,s}^q(f)$ it is clear that then $J^q g$ is transversal to $M_{r,s}^q(f) = M_{r,s}^q(g)$ at 0. Hence g is (r, s) q-transversal for all q, so by Theorem 3.14 g is infinitesimally (r, s)-stable. But g is (r, s)-equivalent to f so by Theorem 3.8 f is infinitesimally (r, s)-stable. Q.E.D.

As a consequence of this theorem we may make the following definition:

Definition 3.16. A germ $f \in \mathfrak{m}(n+r+s)$ will be said to be (r, s)-*stable* if any of the equivalent conditions 3.15 (a), (b), or (c) holds.

Here again it is now natural to pose the question of determining the (r, s)-stable unfoldings of a given germ η, and of classifying (r, s)-stable unfoldings in general. Clearly, since any (r, s)-stable unfolding is also $(r+s)$-stable, η can have (r, s)-stable unfoldings only if it is finitely determined, and if f is an (r, s)-stable unfolding of η, then $r+s$ must be $\geqslant \operatorname{codim}(\eta)$.

Conversely, if η is finitely determined, then clearly η has (r, s)-stable unfoldings for sufficiently large r and s. In fact, if s is given, then there is an r_0 such that η has (r, s)-stable unfoldings for all $r \geqslant r_0$.

For one can obtain an (r, s)-stable unfolding of η by first finding an r-stable unfolding f of η; the s-dimensional constant unfolding of f will then be (r, s)-stable. From this it is clear that the minimal value for r_0 above is at most $\sigma(\eta)$ (and, of course, at least $\sigma(\eta)-s$).

However, the problem of determining in general whether for given r *and* s a given germ η has (r, s)-stable unfoldings is somewhat more difficult than in the case of ordinary stability. So is the problem of classifying the (r, s)-stable unfoldings, for, as we shall see, an (r, s)-stable unfolding of a germ η, for given r and s, need not be uniquely determined up to (r, s)-equivalence.

These are the questions which will concern us in the following sections.

§ 4. Classifying (r, s)-stable unfoldings

In this section we shall develop an algorithm for finding (up to (r, s)-equivalence) all (r, s)-stable unfoldings of a given germ η, for given r and s. In particular this algorithm will also enable us to tell, given r and s, whether or not η has any (r, s)-stable unfoldings.

In constructing this algorithm, we shall make use of what we know about ordinary stability (in particular the fact that we know all *r-stable* unfoldings of a given germ η for given r), and we shall make use of the fact that we can at least recognize (r, s)-stable unfoldings when we see them. Given r and s, we begin by taking a standard $(r+s)$-stable unfolding of η. This unfolding will of course be $(r+s)$-equivalent to any (r, s)-stable unfoldings that η may have, but this is not enough because (r, s)-equivalence is finer than $(r+s)$-equivalence. The idea is to alter the standard unfolding in a canonical way so as to generate a *set* of unfoldings of η such that any (r, s)-stable unfolding of η will be (r, s)-equivalent to some unfolding in this set; unfortunately, not every unfolding in the set will in fact be (r, s)-stable, but using Theorem 3.6 we shall be able to tell which ones are. Of course we shall devote a fair amount of effort to ensuring that the set of unfoldings which our algorithm produces will be reasonably small, so that computations using this algorithm will be possible in practice and not just in theory.

In this section we shall be working, unless otherwise stated, with a fixed germ $\eta \in \mathfrak{m}(n)$, and we shall assume fixed non-negative integers r and s have been given (so they need not be specifically mentioned in the notation which we shall introduce).

Definition 4.1. For the purposes of *this definition*, we shall in departure from our usual convention denote the standard coordinates of $\mathbf{R}^{r+s}$ by $w_1, \ldots, w_{r+s}$.

We denote by S_{r+s} the symmetric group on $r+s$ letters. If $\sigma \in S_{r+s}$, we denote by ω_σ the germ in $L(r+s)$ given by $\omega_\sigma(w_1, \ldots, w_{r+s}) = (w_{\sigma(1)}, \ldots, w_{\sigma(r+s)})$, and if $f \in \mathfrak{m}(n+r+s)$ we denote by f_σ the germ in $\mathfrak{m}(n+r+s)$ defined by

$$f_\sigma(x, w) = f(x, \omega_\sigma(w)) \text{ for } x \in \mathbf{R}^n,\ w \in \mathbf{R}^{r+s}.$$

Let $k \leqslant \min(r, s)$ and suppose we are given integers $1 \leqslant i_1 < i_2 < \ldots < i_k \leqslant r$ and integers $1 \leqslant j_1 < j_2 < \ldots < j_k \leqslant s$. We denote by $\sigma(i_1, \ldots, i_k; j_1, \ldots, j_k) \in S_{r+s}$ the product of transpositions $(i_1, r+j_1)(i_2, r+j_2) \ldots (i_k, r+j_k)$. We let $T \subseteq S_{r+s}$ be the set of all such $\sigma(i_1, \ldots, i_k; j_1, \ldots, j_k)$. Note that every element of T is of order 2.

Note that (with our usual coordinates $u_1, \ldots, u_r$ on $\mathbf{R}^r$ and $v_1, \ldots, v_s$ on $\mathbf{R}^s$) if $f \in \mathfrak{m}(n+r+s)$ and if $\sigma \in T$, then f_σ is just f preceded by an element of $L(r+s)$ which simply interchanges some of the u_i's with an equal number of the v_j's.

LEMMA 4.2. *Let f and $g \in \mathfrak{m}(n+r+s)$ be $(r+s)$-stable unfoldings of η. Then there exists a $\sigma \in T$ and a germ $\beta \in \mathcal{E}(r+s, s)$ such that $\beta(0) = 0$, $\beta | 0 \times \mathbf{R}^s$ is non-singular, and if we define $h \in \mathfrak{m}(n+r+s)$ by $h(x, u, v) = f_\sigma(x, u, \beta(u, v))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, then h is (r, s)-equivalent to g.*

Proof. Since g and f are both $(r+s)$-stable, g is $(r+s)$-equivalent to f. Let (Φ, ψ, Λ) be an $(r+s)$-equivalence from g to f. Now since $\psi \in L(r+s)$ it is clear that for some suitably chosen $\sigma \in T$, if we set $\psi_\sigma = \omega_\sigma \circ \psi$ and we write $\psi_\sigma = (\gamma_\sigma, \delta_\sigma)$, where $\gamma_\sigma \in \mathcal{E}(r+s, r)$ and $\delta_\sigma \in \mathcal{E}(r+s, s)$, then $\gamma_\sigma | \mathbf{R}^r \times 0$ and $\delta_\sigma | 0 \times \mathbf{R}^s$ will be non-singular.

Moreover if we write $\Phi = (\varphi, \psi)$ for $\varphi \in \mathcal{E}(n+r+s, n)$ and if we write $\Lambda = (\lambda, \psi)$ for $\lambda \in \mathcal{E}(1+r+s)$, and if we set $\Phi_\sigma = (\varphi, \psi_\sigma)$ and $\Lambda_\sigma = (\lambda, \psi_\sigma)$, then $(\Phi_\sigma, \psi_\sigma, \Lambda_\sigma)$ is an $(r+s)$-equivalence from g to f_σ (since σ is of order 2).

Define $\varrho \in L(s)$ by setting $\varrho(v) = \delta_\sigma(0, v)$ for $v \in \mathbf{R}^s$. Set $\Phi' = (\varphi, \gamma_\sigma, \varrho)$; $\psi' = (\gamma_\sigma, \varrho)$ and $\Lambda' = (\lambda, \gamma_\sigma, \varrho)$; by the choice of σ the germs Φ', ψ' and Λ' are non-singular. Set $\Phi'' = \Phi_\sigma(\Phi')^{-1}$; $\psi'' = \psi_\sigma(\psi')^{-1}$ and $\Lambda'' = \Lambda_\sigma(\Lambda')^{-1}$. Clearly $(\Phi'', \psi'', \Lambda'') \in L(n, r+s)$; let $h = f_\sigma \cdot (\Phi'', \psi'', \Lambda'')$. Suppose $(u, v) \in \mathbf{R}^{r+s}$. Let $(u', v') = (\psi')^{-1}(u, v)$ and let $(u'', v'') = \psi''(u, v)$. Then $(u, v) = \psi'(u', v')$ so $u = \gamma_\sigma(u', v')$; but $(u'', v'') = \psi_\sigma(u', v')$ so also $u'' = \gamma_\sigma(u', v') = u$. Hence ψ'' has the form $\psi''(u, v) = (u, \beta(u, v))$ where β is some germ in $\mathcal{E}(r+s, s)$. A similar argument shows that $\Phi''(x, u, v) = (x, u, \beta(u, v))$ and $\Lambda''(t, u, v) = (t, u, \beta(u, v))$ for $x \in \mathbf{R}^n$, $t \in \mathbf{R}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$ (note also that the germ β is in all three cases the same because $(\Phi'', \psi'', \Lambda'')$ is an $(r+s)$-equivalence). So clearly h has the form $h(x, u, v) = f_\sigma(x, u, \beta(u, v))$. Moreover since

$\psi'' \in L(r+s)$ it follows that $\beta(0)=0$ and $\beta|0\times \mathbf{R}^s$ is non-singular. And finally it is clear that $(\Phi', \psi', \varrho, \Lambda')$ is an (r, s)-equivalence from g to h. This completes the proof.

Our goal now is to strengthen Lemma 4.2 by showing that the germ β above can be chosen to be of a fairly simple form.

Definition 4.3. Let f and $g \in \mathfrak{m}(n+r+s)$. A *smooth homotopy from f to g* is a map from $[0, 1] \subseteq \mathbf{R}$ to $\mathfrak{m}(n+r+s)$, mapping $t \in [0, 1]$ to $H_t \in \mathfrak{m}(n+r+s)$, such that $H_0=f$, $H_1=g$ and for every $t_0 \in [0, 1]$ there is a germ $K_{t_0} \in \mathfrak{m}(n+r+s+1)$ such that for $x\in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v\in\mathbf{R}^s$ and t near 0 in $\mathbf{R}$ we have (*) $K_{t_0}(x, u, v, t)=H_{t_0+t}(x, u, v)$ (whenever $t_0+t\in[0, 1]$). (What (*) means precisely is that if K'_{t_0} is any representative of K_{t_0}, then for t near 0 in $\mathbf{R}$ the germ at 0 of $K'_{t_0}|\mathbf{R}^{n+r+s}$ is H_{t_0+t}).

Definition 4.4. Let f and $g\in\mathfrak{m}(n+r+s)$ be (r, s)-stable germs. A *stable homotopy* from f to g is a smooth homotopy $t \to H_t$ from f to g such that for every $t\in[0, 1]$ the germ H_t is (r, s)-stable. If a stable homotopy from f to g exists then we say f and g are *stably homotopic*.

LEMMA 4.5. *Let f and $g\in\mathfrak{m}(n+r+s)$. Suppose there is a smooth homotopy $t\to H_t$ from f to g such that for every $t_0\in[0, 1]$ we can choose a germ $K_{t_0}\in\mathfrak{m}(n+r+s+1)$, with $K_{t_0}(x, u, v, t)= H_{t_0+t}(x, u, v)$ for $x\in\mathbf{R}^n$, $u\in\mathbf{R}^r$, $v\in\mathbf{R}^s$, $t\in\mathbf{R}$ near 0, such that*

$$\text{(a)}\qquad \frac{\partial K_{t_0}}{\partial t} \in \langle \partial K_{t_0}/\partial x\rangle_{\mathfrak{m}(n+r+s)\,\mathcal{E}(n+r+s+1)} + \langle \partial K_{t_0}/\partial u\rangle_{\mathfrak{m}(r+s)\,\mathcal{E}(r+s+1)} + \langle \partial K_{t_0}/\partial v\rangle_{\mathfrak{m}(s)\,\mathcal{E}(s+1)} + \bar{K}^*_{t_0}\mathcal{E}(1+r+s+1),$$

where $\bar{K}_{t_0}\in\mathcal{E}(n+r+s+1,\ 1+r+s+1)$ is given by $\bar{K}_{t_0}(x, u, v, t) = (K_{t_0}(x, u, v, t), u, v, t)$ for $(x, u, v, t)\in\mathbf{R}^{n+r+s+1}$.

Then f is (r, s)-equivalent to g.

Proof. Let $t_0\in[0, 1]$ be given and choose K_{t_0} as above satisfying equation (a). Then there are germs $\xi_1, ..., \xi_n\in\mathfrak{m}(n+r+s)\,\mathcal{E}(n+r+s+1)$, germs $\chi_1, ..., \chi_r\in\mathfrak{m}(r+s)\,\mathcal{E}(r+s+1)$, germs $\delta_1, ..., \delta_s\in\mathfrak{m}(s)\,\mathcal{E}(s+1)$ and a germ $\mu\in\mathcal{E}(1+r+s+1)$ such that

$$\frac{\partial K_{t_0}}{\partial t} = \sum_{i=1}^{n}\frac{\partial K_{t_0}}{\partial x_i}\xi_i + \sum_{j=1}^{r}\frac{\partial K_{t_0}}{\partial u_j}\chi_j + \sum_{l=1}^{s}\frac{\partial K_{t_0}}{\partial v_l}\delta_l + \bar{K}^*_{t_0}(\mu).$$

By Lemma 1.10 there is a germ $\Phi\in\mathcal{E}(n+r+s+1, n+r+s)$ and a germ $\lambda\in\mathcal{E}(1+n+r+s+1)$ such that

$$\text{(b)}\qquad \begin{aligned} &\Phi(x, u, v, 0) = (x, u, v)\\ &\lambda(\tau, x, u, v, 0) = \tau \end{aligned} \qquad \text{for } x\in\mathbf{R}^n,\ u\in\mathbf{R}^r,\ v\in\mathbf{R}^s,\ \tau\in\mathbf{R}.$$

(c) $K_{t_0}(\Phi(x, u, v, t), t) = \lambda(K_{t_0}(x, u, v, 0), x, u, v, t)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$.

(d) If we write $\Phi = (\varphi, \gamma, \varrho)$ where $\varphi \in \mathcal{E}(n+r+s, n)$, $\gamma \in \mathcal{E}(n+r+s, r)$ and $\varrho \in \mathcal{E}(n+r+s, s)$ then

$$\frac{\partial \varphi_i}{\partial t}(x, u, v, t) = -\xi_i(\varphi(x, u, v, t), \gamma(x, u, v, t), \varrho(x, u, v, t), t) \quad (i = 1, ..., n),$$

$$\frac{\partial \gamma_j}{\partial t}(x, u, v, t) = -\chi_j(\gamma(x, u, v, t), \varrho(x, u, v, t), t) \quad (j = 1, ..., r),$$

$$\frac{\partial \varrho_l}{\partial t}(x, u, v, t) = -\delta_l(\varrho(x, u, v, t), t) \quad (l = 1, ..., s),$$

and

$$\frac{\partial \lambda}{\partial t}(\tau, x, u, v, t) = \mu(\lambda(\tau, x, u, v, t), \gamma(x, u, v, t), \varrho(x, u, v, t), t)$$

for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $\tau \in \mathbf{R}$, $t \in \mathbf{R}$.

By (d) $\partial \varrho / \partial t$ depends only on t and the value of ϱ, but not directly on x and u; moreover by (b) $\varrho(x, u, v, 0) = v$ and so does not depend on x and u. The uniqueness of solutions of ordinary differential equations with given initial conditions implies that ϱ does not depend on x and u, i.e. $\varrho \in \mathcal{E}(s+1, s)$. Hence $\partial \gamma / \partial t$ depends only on v and t and the value of γ; and $\gamma(x, u, v, 0) = u$ and so does not depend on x. By the same argument as before, $\gamma \in \mathcal{E}(r+s+1, r)$. And now the same argument again shows $\lambda \in \mathcal{E}(1+r+s+1)$.

Now we claim that for all t near 0 we have $\Phi(0, 0, 0, t) = 0$. By (b) this is true when $t = 0$. Moreover by (d), since the ξ_i, χ_j and δ_l are in $\mathfrak{m}(n+r+s)\,\mathcal{E}(n+r+s+1)$ it follows that whenever $\Phi(x, u, v, t) = 0$ then $(\partial \Phi / \partial t)(x, u, v, t) = 0$. Hence clearly $\Phi(0, 0, 0, t) = 0$ for all t. And from this and the fact that $K_{t_0}(0, 0, 0, t) = 0$ for all t it follows, by (c), that also $\lambda(0, 0, 0, t) = 0$ for all t.

Choose representatives for φ, γ, ϱ and λ defined near 0 (we shall retain the same names for the representatives as for the germs). For t near 0 in $\mathbf{R}$ define germs $\Phi_t \in \mathcal{E}(n+r+s, n+r+s)$, $\psi_t \in \mathcal{E}(r+s, r+s)$, $\varrho_t \in \mathcal{E}(s, s)$ and $\Lambda_t \in \mathcal{E}(1+r+s, 1+r+s)$ by setting $\Phi_t(x, u, v) = (\varphi(x, u, v, t),\ \gamma(u, v, t),\ \varrho(v, t))$; $\psi_t(u, v) = (\gamma(u, v, t), \varrho(v, t))$; $\varrho_t(v) = \varrho(v, t)$; and $\Lambda_t(\tau, u, v) = (\lambda(\tau, u, v, t), \psi_t(u, v))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Clearly (b) implies that for t near 0, the germs Φ_t, ψ_t, ϱ_t and Λ_t are non-singular; moreover we have seen above that their value at 0 is 0, for t near 0. Hence they are in $L(n+r+s)$, $L(r+s)$, $L(s)$ and $L(1+r+s)$ respectively, and from (c) it is clear that for t near 0, $(\Phi_t, \psi_t, \varrho_t, \Lambda_t)$ is an (r, s)-equivalence from H_{t_0} to H_{t_0+t}.

Hence for any $t_0 \in [0, 1]$, H_{t_0} is (r, s)-equivalent to H_t for all t sufficiently near t_0. By the compactness of $[0, 1]$ it follows that $f = H_0$ is (r, s)-equivalent to $g = H_1$. Q.E.D.

COROLLARY 4.6. *Let f and $g \in \mathfrak{m}(n+r+s)$ be (r, s)-stable germs. Suppose $f|\mathbf{R}^{n+r} = g|\mathbf{R}^{n+r}$ and suppose the map $t \to (1-t)f+tg$ is a stable homotopy from f to g. Then f is (r, s)-equivalent to g.*

Proof. Let $t_0 \in [0, 1]$. Define $K_{t_0} \in \mathfrak{m}(n+r+s+1)$ by setting $K_{t_0}(x, u, v, t) = (1-t-t_0)f(x, u, v) + (t+t_0)g(x, u, v)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$.

By hypothesis, $K_{t_0}|\mathbf{R}^{n+r+s} = (1-t_0)f + t_0 g$ is (r, s)-stable. Hence K_{t_0} is $(r, s+1)$-stable, so

(a) $$\mathcal{E}(n+r+s+1) = \langle \partial K_{t_0}/\partial x\rangle_{\mathcal{E}(n+r+s+1)} + \langle \partial K_{t_0}/\partial u\rangle_{\mathcal{E}(r+s+1)} + \langle \partial K_{t_0}/\partial v\rangle_{\mathcal{E}(s+1)} + \langle \partial K_{t_0}/\partial t\rangle_{\mathcal{E}(s+1)} + \bar{K}^*_{t_0}(\mathcal{E}(1+r+s+1)),$$

where $\bar{K}_{t_0} \in \mathcal{E}(n+r+s+1, 1+r+s+1)$ is defined by $\bar{K}_{t_0}(x, u, v, t) = (K_{t_0}(x, u, v, t), u, v, t)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$.

Since $g|\mathbf{R}^{n+r} = f|\mathbf{R}^{n+r}$, it follows that $\partial K_{t_0}/\partial t = g - f \in \mathfrak{m}(s)\,\mathcal{E}(n+r+s+1)$, and from the equation which results when both sides of equation (a) are multiplied by $\mathfrak{m}(s)$, it is clear that there is a germ $\alpha \in \mathfrak{m}(s)\,\mathcal{E}(s+1)$ such that $\partial K_{t_0}/\partial t - \alpha(\partial K_{t_0}/\partial t)$ is contained in the right-hand side of 4.5 (a). But $1-\alpha$ is a unit of $\mathcal{E}(s+1)$, so $\partial K_{t_0}/\partial t$ itself is contained in the right-hand side of 4.5 (a). Since this is true for any $t_0 \in [0, 1]$, it follows by Lemma 4.5 that f is (r, s)-equivalent to g. Q.E.D.

We can strengthen this result:

COROLLARY 4.7. *Let f and $g \in \mathfrak{m}(n+r+s)$ be (r, s)-stable germs such that $f|\mathbf{R}^{n+r} = g|\mathbf{R}^{n+r}$. Then f is (r, s)-equivalent to g.*

Proof. Let $h = f|\mathbf{R}^{n+r} = g|\mathbf{R}^{n+r}$. Define $H \in \mathcal{E}(n+r, 1+r)$ by $H(x, u) = (h(x, u), u)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$. Let $C = \mathcal{E}(n+r)/(\langle \partial h/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial h/\partial u\rangle_{\mathcal{E}(r)} + H^*\mathcal{E}(1+r))$, and let $p: \mathcal{E}(n+r) \to C$ be the projection.

Since f is (r, s)-stable, it follows by Theorem 3.6 (b) that C is generated over $\mathbf{R}$ by $p(\partial f/\partial v_1|\mathbf{R}^{n+r}), \ldots, p(\partial f/\partial v_s|\mathbf{R}^{n+r})$, and hence C is a finite dimensional vector space over $\mathbf{R}$ of some dimension $d \leq s$. Choose a basis $\alpha_1, \ldots, \alpha_d$ of C. Since the $p(\partial f/\partial v_i|\mathbf{R}^{n+r})$ $(i = 1, \ldots, s)$ generate C, it is clear from linear algebra that there is a matrix $A = (a_{ij}) \in GL(s)$ such that $\sum_{j=1}^{s} a_{ij} p(\partial f/\partial v_j|\mathbf{R}^{n+r}) = \alpha_i$ for $i = 1, \ldots, d$. Define $f' \in \mathfrak{m}(n+r+s)$ by $f'(x, u, v) = f(x, u, vA)$; clearly f' is (r, s)-equivalent to f and $p(\partial f'/\partial v_i|\mathbf{R}^{n+r}) = \alpha_i$ for $i = 1, \ldots, d$. Moreover $f'|\mathbf{R}^{n+r} = f|\mathbf{R}^{n+r} = h$. Similarly there is a germ $g' \in \mathfrak{m}(n+r+s)$ such that $g'|\mathbf{R}^{n+r} = h$ and g' is (r, s)-equivalent to g and $p(\partial g'/\partial v_i|\mathbf{R}^{n+r}) = \alpha_i$ for $i = 1, \ldots, d$.

For $t \in [0, 1]$, set $H_t = (1-t)f' + tg'$. Clearly, for any $t \in [0, 1]$ we have $H_t|\mathbf{R}^{n+r} = h$ and

$p(\partial H_t/\partial v_i|\mathbf{R}^{n+r})=\alpha_i$, for $i=1, \ldots, d$. Hence by Theorem 3.6 (b), H_t is (r, s)-stable for every $t\in[0, 1]$, so by Corollary 4.6 f' is (r, s)-equivalent to g', and this implies f is (r, s)-equivalent to g. Q.E.D.

COROLLARY 4.8. *Let f and $g\in\mathfrak{m}(n+r+s)$ and suppose f is (r, s)-stable. Suppose $g-f\in\mathfrak{m}(s)\,\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s)+\mathfrak{m}(s)^2\mathcal{E}(n+r+s)$. Then g is (r, s)-stable and is (r, s)-equivalent to f.*

Proof. Let $\eta=f|\mathbf{R}^n=g|\mathbf{R}^n$. Since f is (r, s)-stable, η is finitely determined, so for some integer k we have $\mathfrak{m}(n)^k\subseteq\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}$. Let $q=k(s+1)$.

Let $h=f|\mathbf{R}^{n+r}$. Since f is (r, s)-stable it follows from Theorem 3.6 (c) that

$$\text{(a)}\qquad \mathcal{E}(n+r)=\langle\partial h/\partial x\rangle_{\mathcal{E}(n+r)}+\langle\partial h/\partial u\rangle_{\mathcal{E}(r)}+\langle 1, h, \ldots, h^{q-1}\rangle_{\mathcal{E}(r)}+\langle\partial f/\partial v|\mathbf{R}^{n+r}\rangle_{\mathbf{R}}+\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)+\mathfrak{m}(n+r)^q.$$

Since $g-f\in\mathfrak{m}(s)\,\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s)+\mathfrak{m}(s)^2\mathcal{E}(n+r+s)$ it follows that for each i, $1\leqslant i\leqslant s$, we have $\partial g/\partial v_i|\mathbf{R}^{n+r}-\partial f/\partial v_i|\mathbf{R}^{n+r}\in\mathfrak{m}(r)^s\mathcal{E}(n+r)$; moreover from (a) it follows by Corollary 1.8 (c) that

$$\mathfrak{m}(r)^s\mathcal{E}(n+r)\subseteq\langle\partial h/\partial x\rangle_{\mathcal{E}(n+r)}+\langle\partial h/\partial u\rangle_{\mathcal{E}(r)}+\langle 1, h, \ldots, h^{q-1}\rangle_{\mathcal{E}(r)}+\mathfrak{m}(n+r)^q.$$

Hence clearly (a) still holds if we replace $\langle\partial f/\partial v|\mathbf{R}^{n+r}\rangle_{\mathbf{R}}$ by $\langle\partial g/\partial v|\mathbf{R}^{n+r}\rangle_{\mathbf{R}}$; since $g|\mathbf{R}^{n+r}=h$ this implies, by Theorem 3.6 (c), that g is (r, s)-stable. And by Corollary 4.7 it then follows that f is (r, s)-equivalent to g.

LEMMA 4.9. *Let f and $g\in\mathfrak{m}(n+r+s)$ be (r, s)-stable germs. Suppose $f-g\in\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s)$ and suppose the map $t\to(1-t)f+tg$ is a stable homotopy from f to g. Then f is (r, s)-equivalent to g.*

Proof. Suppose $t_0\in[0, 1]$. Let $H=(1-t_0)f+t_0g$. Define $K_{t_0}\in\mathfrak{m}(n+r+s+1)$ by $K_{t_0}(x, u, v, t)=(1-t_0-t)f(x, u, v)+(t_0+t)g(x, u, v)$ for $(x, u, v, t)\in\mathbf{R}^{n+r+s+1}$.

Let $\mu=H|\mathbf{R}^n$; since H is by hypothesis stable there is a k such that $\mathfrak{m}(n)^k\subseteq\langle\partial\mu/\partial x\rangle_{\mathcal{E}(n)}$, and if we set $q=k(s+1)$ we have

$$\text{(a)}\qquad \mathcal{E}(n+r+s)=\langle\partial H/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial H/\partial u\rangle_{\mathcal{E}(r+s)}+\langle\partial H/\partial v\rangle_{\mathcal{E}(s)}+\langle 1, H, \ldots, H^{q-1}\rangle_{\mathcal{E}(r+s)}+\mathfrak{m}(n+r+s)^q$$

(here we have used the fact that $H^q\in\mathfrak{m}(n+r+s)^q$).

As in the proof of Theorem 3.6 ((c)$\Rightarrow$(a)), $\mathfrak{m}(n)^k\subseteq\langle\partial\mu/\partial x\rangle_{\mathcal{E}(n)}$ implies $\mathfrak{m}(n+r+s)^q\subseteq\langle\partial H/\partial x\rangle_{\mathcal{E}(n+r+s)}+\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s)$, so on the right of (a) we may replace the term $\mathfrak{m}(n+r+s)^q$ by $\mathfrak{m}(r+s)^{s+1}$.

By Corollary 1.8 the resulting equation implies

(b) $$\mathcal{E}(n+r+s)=\langle\partial H/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial H/\partial u\rangle_{\mathcal{E}(r+s)}+\langle\partial H/\partial v\rangle_{\mathcal{E}(s)}+\langle 1, H, \ldots, H^{q-1}\rangle_{\mathcal{E}(r+s)},$$

and

(c) $$\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s)\subseteq\langle\partial H/\partial x\rangle_{\mathcal{E}(n+r+s)}+\langle\partial H/\partial u\rangle_{\mathcal{E}(r+s)} +\langle 1, H, \ldots, H^{q-1}\rangle_{\mathcal{E}(r+s)}+\mathfrak{m}(s)\,\mathcal{E}(n+r+s).$$

Let $A=\langle\partial K_{t_0}/\partial x\rangle_{\mathcal{E}(n+r+s+1)}+\langle\partial K_{t_0}/\partial u\rangle_{\mathcal{E}(r+s+1)}+\langle 1, K_{t_0}, \ldots, K_{t_0}^{q-1}\rangle_{\mathcal{E}(r+s+1)}$.

We have $K_{t_0}-H=t(g-f)\in\mathfrak{m}(1)\,\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s+1)$. Hence $\partial K_{t_0}/\partial x_i-\partial H/\partial x_i\in\mathfrak{m}(1)\,\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s+1)$ for $i=1, \ldots, n$; $\partial K_{t_0}/\partial u_j-\partial H/\partial u_j\in\mathfrak{m}(1)\,\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s+1)$ for $j=1, \ldots, r$; and $K_{t_0}^l-H^l\in\mathfrak{m}(1)\,\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s+1)$ for $l=1, \ldots, q-1$. Moreover $\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s+1)=\mathfrak{m}(r+s)^s(\mathcal{E}(n+r+s)+\mathfrak{m}(1)\,\mathcal{E}(n+r+s+1))$.

Hence (c) clearly implies

(d) $$\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s+1)\subseteq A+\mathfrak{m}(s)\,\mathcal{E}(n+r+s+1)+\mathfrak{m}(1)\,\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s+1).$$

From (b) it follows that $\mathcal{E}(n+r+s+1)=A+\langle\partial K_{t_0}/\partial v\rangle_{\mathcal{E}(s+1)}+\mathfrak{m}(1)\,\mathcal{E}(n+r+s+1)$ and hence, by Corollary 1.7, we have (e): $\mathcal{E}(n+r+s+1)=A+\langle\partial K_{t_0}/\partial v\rangle_{\mathcal{E}(s+1)}$. Let $C=\mathcal{E}(n+r+s+1)/A$, considered as an $\mathcal{E}(r+s+1)$ module. By (e) it is clear that C is finitely generated over $\mathcal{E}(r+s+1)$, and so $\mathfrak{m}(r+s)^sC$ is also finitely generated over $\mathcal{E}(r+s+1)$. Moreover (d) implies $\mathfrak{m}(r+s)^sC\subseteq\mathfrak{m}(s)\,C+\mathfrak{m}(r+s+1)\,\mathfrak{m}(r+s)^sC$, so by Nakayama's lemma (Lemma 1.4) $\mathfrak{m}(r+s)^sC\subseteq\mathfrak{m}(s)\,C$ and hence $\mathfrak{m}(r+s)^s\mathcal{E}(n+r+s+1)\subseteq A+\mathfrak{m}(s)\,\mathcal{E}(n+r+s+1)$. From this and (e) we find $\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s+1)\subseteq\mathfrak{m}(r+s)A+\mathfrak{m}(s)\,\mathcal{E}(n+r+s+1)=\mathfrak{m}(r+s)A+\langle\partial K_{t_0}/\partial v\rangle_{\mathfrak{m}(s)\mathcal{E}(s+1)}$; hence $\mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s+1)$ is contained in the right-hand side of Equation 4.5 (a). Since this holds for any $t_0\in[0, 1]$, and since for any $t_0\in[0, 1]$ we have $\partial K_{t_0}/\partial t=g-f\in\mathfrak{m}(r+s)^{s+1}$, it follows from Lemma 4.5 that f is (r, s)-equivalent to g.

COROLLARY 4.10. *Let f and $g\in\mathfrak{m}(n+r+s)$ and suppose f is (r, s)-stable. Suppose $g-f\in\mathfrak{m}(r+s)^{s+2}\mathcal{E}(n+r+s)$. Then f is (r, s)-equivalent to g (and g is also (r, s)-stable).*

Proof. If $t\in[0, 1]$, set $H_t=(1-t)f+tg$, and set $J_t=H_t|\mathbf{R}^{n+r}$. Let $f_0=f|\mathbf{R}^{n+r}$. Let $\eta=f|\mathbf{R}^n$. Then η is finitely determined, so for some integer k we have $\mathfrak{m}(n)^k\subseteq\langle\partial\eta/\partial x\rangle_{\mathcal{E}(n)}$. Set $q=k(s+1)$.

Since f is (r, s)-stable we have, by Theorem 3.6 (c):

(a) $$\mathcal{E}(n+r)=\langle\partial f_0/\partial x\rangle_{\mathcal{E}(n+r)}+\langle\partial f_0/\partial u\rangle_{\mathcal{E}(r)}+\langle\partial f/\partial v\,|\,\mathbf{R}^{n+r}\rangle_{\mathbf{R}} +\langle 1, f_0, \ldots, f_0^{q-1}\rangle_{\mathcal{E}(r)}+\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)+\mathfrak{m}(n+r)^q.$$

Let $t \in [0, 1]$. Then $H_t = f + t(g-f)$ and hence $H_t - f \in \mathfrak{m}(r+s)^{s+2}\mathcal{E}(n+r+s)$. This implies that all the first-order partial derivatives of H_t, restricted to $\mathbf{R}^{n+r}$, differ from the corresponding derivatives of f, restricted to $\mathbf{R}^{n+r}$, by an element of $\mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)$, and the powers of J_t differ from corresponding powers of f_0 by an element of $\mathfrak{m}(r)^{s+2}\mathcal{E}(n+r)$. So it clearly follows from (a) that

$$\mathcal{E}(n+r) = \langle \partial J_t/\partial x\rangle_{\mathcal{E}(n+r)} + \langle \partial J_t/\partial u\rangle_{\mathcal{E}(r)} + \langle \partial H_t/\partial v \,|\, \mathbf{R}^{n+r}\rangle_{\mathbf{R}} + \langle 1, J_t, \ldots, J_t^{q-1}\rangle_{\mathcal{E}(r)} + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r) + \mathfrak{m}(n+r)^q,$$

and hence H_t is (r, s)-stable, by Theorem 3.6. This holds for any $t \in [0, 1]$ (so in particular $g = H_1$ is (r, s)-stable), and certainly $g - f \in \mathfrak{m}(r+s)^{s+1}\mathcal{E}(n+r+s)$, so by Lemma 4.9 f is (r, s)-equivalent to g.

With the aid of the preceding lemmas and corollaries, we can now prove a strengthened form of Lemma 4.2:

THEOREM 4.11. *Let $g \in \mathfrak{m}(n+r+s)$ be an (r, s)-stable unfolding of $\eta \in \mathfrak{m}(n)$, and suppose $f \in \mathfrak{m}(n+r+s)$ is an $(r+s)$-stable unfolding of η. Then there exists a permutation $\sigma \in T$ and polynomial map-germs $p, \xi_1, \ldots, \xi_s \in \mathcal{E}(r, s)$, with $p(0) = 0$ and $\xi_i(0) = 0$ for $i = 1, \ldots, s$, such that p has degree at most $s+1$, and $\xi_1, \ldots, \xi_s$ have degree at most $s-1$, and such that if we define $h \in \mathfrak{m}(n+r+s)$ by*

(a) $$h(x, u, v) = f_\sigma(x, u, v + p(u) + \textstyle\sum_{i=1}^{s} v_i \xi_i(u)) \text{ for } x \in \mathbf{R}^n,\ u \in \mathbf{R}^r,\ v \in \mathbf{R}^s,$$

then g is (r, s)-equivalent to h.

(*Remark*. Observe that when $s = 1$, the ξ_i are all 0, so h is of the form $h(x, u, v) = f_\sigma(x, u, v + p(u))$).

Proof. By Lemma 4.2 there is a $\sigma \in T$ and a germ $\beta \in \mathcal{E}(r+s, s)$ such that $\beta(0) = 0$, $\beta \,|\, 0 \times \mathbf{R}^s$ is nonsingular and g is (r, s)-equivalent to the germ $h_1 \in \mathfrak{m}(n+r+s)$ given by $h_1(x, u, v) = f_\sigma(x, u, \beta(u, v))$.

Define $\varrho \in L(s)$ by $\varrho(v) = \beta(0, v)$ and define $\beta' \in \mathcal{E}(r+s, s)$ by $\beta'(u, v) = \beta(u, \varrho^{-1}(v))$. Define $h_2 \in \mathfrak{m}(n+r+s)$ by $h_2(x, u, v) = f_\sigma(x, u, \beta'(u, v)) = h_1(x, u, \varrho^{-1}(v))$; clearly h_2 is (r, s)-equivalent to h_1 and hence to g. Clearly $\beta' \,|\, 0 \times \mathbf{R}^s = id_{\mathbf{R}^s}$, so one can find a germ $\gamma \in \mathcal{E}(r+s, s)$ such that $\beta'(u, v) = v + \gamma(u, v)$ and such that $\gamma(0, v) = 0$ for all $v \in \mathbf{R}^s$. We can find a germ $\chi \in \mathfrak{m}(r)\mathcal{E}(r, s)$, polynomial map germs $\xi_1, \ldots, \xi_s \in \mathfrak{m}(r)\mathcal{E}(r, s)$ of degree $\leqslant s-1$, germs $\mu_1, \ldots, \mu_s \in \mathfrak{m}(r)^s\mathcal{E}(r, s)$, and a germ $\delta \in \mathfrak{m}(s)^2\mathcal{E}(r+s, s)$ such that

$$\gamma(u, v) = \chi(u) + \textstyle\sum_{i=1}^{s} v_i(\xi_i(u) + \mu_i(u)) + \delta(u, v) \text{ for } u \in \mathbf{R}^r, v \in \mathbf{R}^s.$$

Define $\gamma' \in \mathcal{E}(r+s, s)$ by setting $\gamma'(u, v) = \chi(u) + \sum_{i=1}^{s} v_i \xi_i(u)$ for $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, i.e. $\gamma' = \gamma - \delta - \sum_{i=1}^{s} v_i \mu_i$. Define $h_3 \in \mathfrak{m}(n+r+s)$ by setting $h_3(x, u, v) = f_\sigma(x, u, v + \gamma'(u, v))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in R^s$.

Clearly $h_3 - h_2 \in \mathfrak{m}(s)\,\mathfrak{m}(r+s)^s \mathcal{E}(n+r+s) + \mathfrak{m}(s)^2 \mathcal{E}(n+r+s)$. Moreover, h_2 is (r, s)-stable, since h_2 is (r, s)-equivalent to g. By Corollary 4.8 it follows that h_3 is (r, s)-stable, and is (r, s)-equivalent to h_2 and hence to g.

Finally, we can find a polynomial mapping $p \in \mathfrak{m}(r)\,\mathcal{E}(r, s)$ of degree $\leqslant s+1$ and a germ $\nu \in \mathfrak{m}(r)^{s+2} \mathcal{E}(r, s)$ such that $\chi = p + \nu$. Define $h \in \mathfrak{m}(n+r+s)$ by setting $h(x, u, v) = f_\sigma(x, u, v + p(u) + \sum_{i=1}^{s} v_i \xi_i(u))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Obviously $h - h_3 \in \mathfrak{m}(r)^{s+2} \mathcal{E}(n+r+s)$. Since h_3 is (r, s)-stable it follows from Corollary 4.10 that h is (r, s)-equivalent to h_3 and hence to g. This completes the proof.

With the aid of Theorem 4.11 we can now state the algorithm we have sought for determining (up to (r, s)-equivalence) all (r, s)-stable unfoldings of a given germ η.

For any non-negative integer d we may identify the set of polynomial mappings in $\mathfrak{m}(r)\,\mathcal{E}(r, s)$ of degree at most d with the finite-dimensional real vector space $J_0^d(r, s)$. Now suppose $\eta \in \mathfrak{m}(n)$, r and s are given. Choose some $(r+s)$-stable unfolding f of η (assuming one exists); for example, we may take for f a constant unfolding of the minimal stable unfolding of η given by Theorem 2.20. If $\sigma \in T$, and if $p \in J_0^{s+1}(r, s)$ and $\xi_1, \ldots, \xi_s \in J_0^{s-1}(r, s)$, we define an unfolding $H(\sigma, p, \xi_1, \ldots, \xi_s) \in \mathfrak{m}(n+r+s)$ by equation 4.11 (a), i.e. we set $H(\sigma, p, \xi_1, \ldots, \xi_s)(x, u, v) = f_\sigma(x, u, v + p(u) + \sum_{i=1}^{s} v_i \xi_i(u))$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. For each of the finitely many $\sigma \in T$ we can compute, using Theorem 3.6 (c), for which $p \in J_0^{s+1}(r, s)$ and $\xi_1, \ldots, \xi_s \in J_0^{s-1}(r, s)$ the germ $H(\sigma, p, \xi_1, \ldots, \xi_s)$ will be (r, s)-stable. In fact, condition 3.6 (c) holds if and only if $J^{q-1}(n+r, 1)$ is generated over $\mathbf{R}$ by certain finitely many elements which depend algebraically on p and the ξ_i, so this computation is an exercise in linear algebra, and it yields, for each $\sigma \in T$, an algebraic subset A_σ of $J_0^{s+1}(r, s) \times (J_0^{s-1}(r, s))^s$ such that $H(\sigma, p, \xi_1, \ldots, \xi_s)$ is (r, s)-stable if and only if $(p, \xi_1, \ldots, \xi_s) \notin A_\sigma$. And by Theorem 4.11 the set of all (r, s)-stable $H(\sigma, p, \xi_1, \ldots, \xi_s)$ for $\sigma \in T$, $p \in J_0^{s+1}(r, s)$ and $\xi_i \in J_0^{s-1}(r, s)$ contains representatives of every (r, s)-equivalence class of (r, s)-stable unfoldings of η, so by this means we may determine them all.

Of course, this algorithm is somewhat unsatisfactory in that it does not yield a unique representative for each (r, s)-equivalence class, and the set of "standard" (r, s)-stable unfoldings of η which it produces is much larger than one would like (after all, the condition on the p and ξ_i for $H(\sigma, p, \xi_1, \ldots, \xi_s)$ to be (r, s)-stable is an open condition!).

However, in practice one can reduce the size of this set considerably, for one can either make further applications of Lemmas and Corollaries 4.5–4.10 in special cases, or one can often by inspection write down (r, s)-equivalences between some of the "standard" unfoldings given by the algorithm. For example, in the next section we shall compute analogues to Thom's list for the cases of (1, 3) and (3, 1)-stability, and the lists we shall compute will contain only one representative of each equivalence class.

Here we can make one general remark, which somewhat reduces the size of the set of "standard" unfoldings one must consider:

Remark 4.12. If, for some $\sigma \in T$ and $p \in J_0^{s+1}(r, s)$ there are germs $\xi_1, \ldots, \xi_s$ and $\xi'_1, \ldots, \xi'_s \in J_0^{s-1}(r, s)$ such that both $H(\sigma, p, \xi_1, \ldots, \xi_s)$ and $H(\sigma, p, \xi'_1, \ldots, \xi'_s)$ are (r, s)-stable, then $H(\sigma, p, \xi_1, \ldots, \xi_s)$ and $H(\sigma, p, \xi'_1, \ldots, \xi'_s)$ are (r, s)-equivalent. (This is an immediate consequence of Corollary 4.7). Hence for each $\sigma \in T$ there is in fact an algebraic subset $B_\sigma \subseteq J_0^{s+1}(r, s)$ such that we need include only *one* unfolding of the form $H(\sigma, p, \xi_1, \ldots, \xi_s)$ in our "standard" list, for each p in the complement of B_σ.

The algorithm described above involves a computation using Theorem 3.6 (c). This computation can sometimes be simplified slightly if one uses instead the following corollary of Theorem 3.6 (which is really of interest only for this purpose).

COROLLARY 4.13. *Let $f \in \mathfrak{m}(n+r+s)$ unfold $\eta \in \mathfrak{m}(n)$. Suppose η is finitely determined and choose a number k such that $\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x \rangle_{\mathcal{E}(n)}$. Suppose that in fact $\eta \in \mathfrak{m}(n)^d$, for some integer d such that $1 \leqslant d \leqslant k$, and let p be the largest integer such that $p < k(s+1)/(d+1)$. Set $q = k(s+1)$. Let $f_0 = f \mid \mathbf{R}^{n+r}$. Then f is (r, s)-stable if and only if*

(a)
$$\mathcal{E}(n+r) = \langle \partial f_0/\partial x \rangle_{\mathcal{E}(n+r)} + \langle \partial f_0/\partial u \rangle_{\mathcal{E}(r)} + \langle \partial f/\partial v \mid \mathbf{R}^{n+r} \rangle_{\mathbf{R}} + \langle 1, f_0, \ldots, f_0^p \rangle_{\mathcal{E}(r)} + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r) + \mathfrak{m}(n+r)^q.$$

(*Remark*. This corollary is of interest only if one can choose $d \geqslant 2$, for if $\eta \notin \mathfrak{m}(n)^2$, then $\mathcal{E}(n+r) = \langle \partial f_0/\partial x \rangle_{\mathcal{E}(n+r)}$ (since some $\partial f_0/\partial x_i$ is a unit) and hence f is automatically (r, s)-stable by Theorem 3.6 (b)).

Proof. "If" is clear, for obviously (a) implies equation 3.6 (c). To prove "only if" we shall show equation 3.6 (c) implies (a).

First we show that $\eta \in \mathfrak{m}(n)^d$ implies $\eta - d^{-1}\sum_{i=1}^n x_i \partial\eta/\partial x_i \in \mathfrak{m}(n)^{d+1}$. We proceed by induction on d. Surely the claim is true if $d = 1$, as one easily verifies. Suppose $d > 1$ and the claim is true for $d-1$. Let $\alpha = \eta - d^{-1}\sum_{i=1}^n x_i \partial\eta/\partial x_i$. To show $\alpha \in \mathfrak{m}(n)^{d+1}$ it is enough to show that for any j, $1 \leqslant j \leqslant n$, we have $\partial\alpha/\partial x_j \in \mathfrak{m}(n)^d$. But

$$\frac{\partial\alpha}{\partial x_j} = \frac{\partial\eta}{\partial x_j} - \frac{1}{d}\frac{\partial\eta}{\partial x_j} - \frac{1}{d}\sum_{i=1}^n x_i \frac{\partial^2\eta}{\partial x_i \partial x_j} = \frac{d-1}{d}\left(\frac{\partial\eta}{\partial x_j} - \frac{1}{d-1}\sum_{i=1}^n x_i \frac{\partial^2\eta}{\partial x_j \partial x_i}\right),$$

and by the induction hypothesis this is in $\mathfrak{m}(n)^d$, since $\partial\eta/\partial x_j \in \mathfrak{m}(n)^{d-1}$. This proves the claim above.

It follows that $f_0 \in \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(n)^{d+1} + \mathfrak{m}(r)\,\mathcal{E}(n+r)$. Moreover $\mathfrak{m}(r)\mathcal{E}(n+r) = \mathfrak{m}(r) + \mathfrak{m}(r)\,\mathfrak{m}(n)\,\mathcal{E}(n+r)$. Hence we can find a germ $\beta \in \mathfrak{m}(r)$ and a germ $\gamma \in \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(n)^{d+1} + \mathfrak{m}(n)\,\mathfrak{m}(r)\,\mathcal{E}(n+r)$ such that $f_0 = \beta + \gamma$.

Now if $a > p$ then there are polynomials $Q(w, z)$ and $R(w, z)$ in two variables w and z, such that Q has degree at most p in z, and such that $(w+z)^a = Q(w, z) + z^{p+1}R(w, z)$. Hence $f_0^a = (\beta+\gamma)^a = Q(\beta,\gamma) + \gamma^{p+1}R(\beta,\gamma)$. Now $Q(\beta,\gamma) = Q(\beta, f_0 - \beta)$ and hence can be written as a polynomial expression in β and f_0 of degree at most p in f_0; this implies $Q(\beta,\gamma) \in \langle 1, f_0, \ldots, f_0^p\rangle_{\mathcal{E}(r)}$. And clearly

$$\text{(b)}\qquad \begin{aligned} \gamma^{p+1}R(\beta,\gamma) &\in \gamma^{p+1}\mathcal{E}(n+r) \\ &\subseteq \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \textstyle\sum_{i=0}^{p+1} (\mathfrak{m}(n)^{d+1})^i (\mathfrak{m}(n)\,\mathfrak{m}(r))^{p+1-i}\mathcal{E}(n+r) \\ &= \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \textstyle\sum_{i=0}^{p+1} \mathfrak{m}(n)^{p+1+id}\mathfrak{m}(r)^{p+1-i}\mathcal{E}(n+r). \end{aligned}$$

Since $\mathfrak{m}(n)^k \subseteq \langle \partial\eta/\partial x\rangle_{\mathcal{E}(n)}$ we have $\mathfrak{m}(n)^k \subseteq \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(r)\,\mathcal{E}(n+r)$, and hence for any non-negative integer c we have

$$\text{(c)}\qquad \mathfrak{m}(n)^{kc} \subseteq \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(r)^c\mathcal{E}(n+r).$$

By the definition of p we have $p+1 \geqslant k(s+1)/(d+1)$, and since $d \leqslant k$, this is readily seen to imply that for any i, $0 \leqslant i \leqslant p+1$, we have $k(s+1-(p+1-i)) \leqslant p+1+id$. Hence by (c) it follows that $\mathfrak{m}(n)^{p+1+id} \subseteq \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(r)^{s+1-(p+1-i)}\mathcal{E}(n+r)$, for any i, $0 \leqslant i \leqslant p+1$. Therefore it is clear from (b) that $\gamma^{p+1}R(\beta,\gamma) \in \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r)$. So we have shown that for any integer $a > p$, we have

$$\text{(d)}\qquad f_0^a \in \langle 1, f_0, \ldots, f_0^p\rangle_{\mathcal{E}(r)} + \langle \partial f_0/\partial x\rangle_{\mathcal{E}(n+r)} + \mathfrak{m}(r)^{s+1}\mathcal{E}(n+r).$$

Now suppose f is (r, s)-stable. Then Equation 3.6 (c) holds; moreover since $f_0^q \in \mathfrak{m}(n+r)^q$ we may clearly replace the term $F_0^*\mathcal{E}(1+r)$ on the right of 3.6 (c) by $\langle 1, f_0, \ldots, f_0^{q-1}\rangle_{\mathcal{E}(r)}$; but by virtue of (d) we may then replace this term by $\langle 1, f_0, \ldots, f_0^p\rangle_{\mathcal{E}(r)}$; this yields equation (a), so we are done.

Thus far in this section we have restricted our attention to the problem of classifying, for a given r and s, the (r, s)-stable unfoldings of a *given* germ η. We should now like to consider the more general problem of determining all (r, s)-stable unfoldings f, for fixed r and s but without having specified the germ η which f unfolds.

One way to attack this problem is first to try to show that a general (r, s)-stable unfolding f is (r, s)-equivalent to an unfolding f' such that the germ $\eta = f'\,|\,\mathbf{R}^n$ is of some standard form; one can then, knowing η, use the algorithm described after Theorem 4.11 to determine what f' can look like. In other words, one tries first to classify the germs η which have (r, s)-stable unfoldings, and hopes that one can apply the results of this classification to the problem of classifying the unfoldings.

This approach works quite well, at least when $(r+s)$ is sufficiently small; in fact, a similar approach can be used in the case of ordinary stability to prove the validity of Thom's list of the seven elementary catastrophes, Theorem 2.26 (see [11, Chapter 5]). If η has (r, s)-stable unfoldings, then the codimension of η must be at most $r+s$. Mather [4, Chapter II] has classified the germs of codimension $\leqslant 5$ (actually Mather classifies the germs whose *right*-codimension (which is $\tau(\eta)-1$) is $\leqslant 5$, but these are the same germs for which $\sigma(\eta) =$ right-left codim $(\eta) \leqslant 5$); Siersma [5] (see also Siersma, Classification and Deformation of Singularities, thesis, Amsterdam, 1974) extends Mather's classification to germs of right-codimension $\leqslant 8$ and the same methods should work for germs of right-left codimension $\leqslant 8$. For a published version of Mather's classification up to right-left codimension 4, see [11, Th. 5.15]. (Note: Siersma's thesis extends the classification up to right-codimension 9).

To conclude this section, we shall prove some lemmas which will enable us to apply the results of a classification of germs to the problem of classifying (r, s)-stable unfoldings. In the next chapter we shall then use these lemmas to prove, for (r, s)-stability, analoga to Thom's theorem (Theorem 2.26).

Definition 4.14. Let η and $\mu \in \mathfrak{m}(n)$. We say η and μ are *equivalent* if there is a germ $\varphi \in L(n)$ and a germ $\lambda \in L(1)$ such that $\eta = \lambda\mu\varphi$.

Notation: In the remainder of this section, we shall often be considering germs in $\mathfrak{m}(n+d)$ for some n and d, and unfoldings of such germs. We shall take coordinates $x_1, \dots, x_n$, $y_1, \dots, y_d$ on $\mathbf{R}^{n+d}$ and denote elements of $\mathbf{R}^{n+d}$ by pairs (x, y), where $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$.

Definition 4.15. Let $\eta \in \mathfrak{m}(n+d)$ and let $\mu \in \mathfrak{m}(n)$. We shall say η *reduces to* μ if there is a non-degenerate quadratic form Q on $\mathbf{R}^d$ such that η is equivalent to the germ $\mu' \in \mathfrak{m}(n+d)$ given by $\mu'(x, y) = \mu(x) + Q(y)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$. If $d > 0$, we say μ is a *proper* reduction of η. If η has no proper reduction, we say η is *irreducible.*

Definition 4.16. Let $f \in \mathfrak{m}(n+d+r+s)$ and let $g \in \mathfrak{m}(n+r+s)$. We say f *(r, s)-reduces to* g if there is a non-degenerate quadratic form Q on $\mathbf{R}^d$ such that f is (r, s)-equivalent to the germ $g' \in \mathfrak{m}(n+d+r+s)$ given by $g'(x, y, u, v) = g(x, u, v) + Q(y)$, for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$.

If $d > 0$, we say g is a *proper* (r, s)-reduction of f. If f has no proper (r, s)-reduction, we say f is (r, s)-irreducible.

Remark. If $f \in \mathfrak{m}(n+d+r+s)$ unfolds $\eta \in \mathfrak{m}(n+d)$, and if $g \in \mathfrak{m}(n+r+s)$ unfolds $\mu \in \mathfrak{m}(n)$, and if f (r, s)-reduces to g, then one easily sees that η reduces to μ. Hence if η is irreducible, then f is (r, s)-irreducible.

LEMMA 4.17. *Suppose $f \in \mathfrak{m}(n+d+r+s)$ (r, s)-reduces to $g \in \mathfrak{m}(n+r+s)$. Then f is (r, s)-stable if and only if g is (r, s)-stable.*

Proof. Clearly it suffices to give the proof in the case when f is of the form $f(x, y, u, v) = g(x, u, v) + Q(y)$ $(x \in \mathbf{R}^n,\ y \in \mathbf{R}^d,\ u \in \mathbf{R}^r, v \in \mathbf{R}^s)$, where Q is a non-degenerate quadratic form on $\mathbf{R}^d$. Let $F \in \mathcal{E}(n+d+r+s, 1+r+s)$ and $G \in \mathcal{E}(n+r+s, 1+r+s)$ be defined by $F(x, y, u, v) = (f(x, y, u, v), u, v)$ and $G(x, u, v) = (g(x, u, v), u, v)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$, $u \in \mathbf{R}^r$, $v \in R^s$.

By definition 3.5, f is (r, s)-stable if and only if

$$\text{(a)} \qquad \mathcal{E}(n+d+r+s) = \langle \partial f/\partial x\rangle_{\mathcal{E}(n+d+r+s)} + \langle \partial f/\partial y\rangle_{\mathcal{E}(n+d+r+s)} + \langle \partial f/\partial u\rangle_{\mathcal{E}(r+s)} + \langle \partial f/\partial v\rangle_{\mathcal{E}(s)} + F^*\mathcal{E}(1+r+s)$$

and g is (r, s)-stable if and only if

$$\text{(b)} \qquad \mathcal{E}(n+r+s) = \langle \partial g/\partial x\rangle_{\mathcal{E}(n+r+s)} + \langle \partial g/\partial u\rangle_{\mathcal{E}(r+s)} + \langle \partial g/\partial v\rangle_{\mathcal{E}(s)} + G^*\mathcal{E}(1+r+s).$$

Let $\alpha: \mathcal{E}(n+d+r+s) \to \mathcal{E}(n+r+s)$ be the restriction homomorphism given by $\alpha(h) = h \mid \mathbf{R}^n \times 0 \times \mathbf{R}^r \times \mathbf{R}^s$ for $h \in \mathcal{E}(n+d+r+s)$. Since Q is non-degenerate, we clearly have

$$\langle \partial f/\partial y\rangle_{\mathcal{E}(n+d+r+s)} = \langle \partial Q/\partial y\rangle_{\mathcal{E}(n+d+r+s)} = \mathfrak{m}(d)\,\mathcal{E}(n+d+r+s).$$

Applying α to both sides of equation (a) yields equation (b) (hence (a) implies (b)); but since the kernel of α, which is $\mathfrak{m}(d)\,\mathcal{E}(n+d+r+s)$, is contained in both sides of equation (a), it also follows that (b) implies (a). This completes the proof.

The following lemma is a converse to the remark following Definition 4.16.

LEMMA 4.18. *Let $f \in \mathfrak{m}(n+d+r+s)$ be an (r, s)-stable unfolding of $\eta \in \mathfrak{m}(n+d)$, and suppose η reduces to a germ $\mu \in \mathfrak{m}(n)$. Then μ has an $(r+s)$-dimensional unfolding $g \in \mathfrak{m}(n+r+s)$ such that f (r, s)-reduces to g.*

Proof. Since η reduces to μ, there is a non-degenerate quadratic form Q on $\mathbf{R}^d$ and there are germs $\varphi \in L(n+d)$ and $\lambda \in L(1)$ such that $\lambda\eta\varphi = \mu'$, where $\mu' \in \mathfrak{m}(n+d)$ is given by $\mu'(x, y) = \mu(x) + Q(y)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$. Define $\Phi \in L(n+d+r+s)$ by setting $\Phi(x, y, u, v) = (\varphi^{-1}(x, y), u, v)$ for $(x, y) \in \mathbf{R}^{n+d}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, and define $\Lambda \in L(1+r+s)$ by setting

$\Lambda(t, u, v) = (\lambda(t), u, v)$ for $t \in \mathbf{R}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Clearly $(\Phi, id_{\mathbf{R}^{r+s}}, id_{\mathbf{R}^s}, \Lambda)$ is an (r, s)-equivalence from f to some unfolding f' of μ'.

Let $A = \mathcal{E}(n+d)/(\langle \partial\mu'/\partial x\rangle_{\mathcal{E}(n+d)} + \langle \partial\mu'/\partial y\rangle_{\mathcal{E}(n+d)} + \mu'^*\mathcal{E}(1))$. Observe that $\langle \partial\mu'/\partial y\rangle_{\mathcal{E}(n+d)} = \langle \partial Q/\partial y\rangle_{\mathcal{E}(n+d)} = \mathfrak{m}(d)\,\mathcal{E}(n+d)$, since Q is non-degenerate.

Since f' is an (r, s)-stable unfolding of μ' we have codim $(\mu') = \dim_{\mathbf{R}} A \leqslant r+s$. Hence we can find germs $b_1, ..., b_{r+s} \in \mathcal{E}(n+d)$ whose classes in A generate A over $\mathbf{R}$; moreover since $\mathfrak{m}(d)\,\mathcal{E}(n+d)$ is contained in the denominator of A we may in fact choose $b_1, ..., b_{r+s}$ to be in $\mathcal{E}(n)$. Define $h \in \mathfrak{m}(n+r+s)$ by $h(x, u, v) = \mu(x) + u_1 b_1(x) + ... + u_r b_r(x) + v_1 b_{r+1}(x) + ... + v_s b_{r+s}(x)$ for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Define $h' \in \mathfrak{m}(n+d+r+s)$ by $h'(x, y, u, v) = h(x, u, v) + Q(y) = \mu'(x, y) + \sum_{i=1}^r u_i b_i(x) + \sum_{j=1}^s v_j b_{r+j}(x)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. By the choice of the b_i it is clear from Theorem 2.9 that h' is an $(r+s)$-stable unfolding of μ'. Hence by Lemma 4.2 there is a permutation $\sigma \in T$ and a germ $\beta \in \mathcal{E}(r+s, s)$, with $\beta(0) = 0$, such that f' is (r, s)-equivalent to the germ $g' \in \mathfrak{m}(n+d+r+s)$ given by $g'(x, y, u, v) = h'_\sigma(x, y, u, \beta(u, v))$ for $(x, y, u, v) \in \mathbf{R}^{n+d+r+s}$. If we define $g \in \mathfrak{m}(n+r+s)$ by $g(x, u, v) = h_\sigma(x, u, \beta(u, v))$ for $(x, u, v) \in \mathbf{R}^{n+r+s}$, then g unfolds μ and clearly $g'(x, y, u, v) = g(x, u, v) + Q(y)$, so g' (r, s)-reduces to g and hence f (r, s)-reduces to g. This completes the proof.

LEMMA 4.19. *Let $f \in \mathfrak{m}(n+d+r+s)$. Suppose f (r, s)-reduces to $g \in \mathfrak{m}(n+r+s)$, and suppose g is (r, s)-equivalent to $h \in \mathfrak{m}(n+r+s)$. Then f also (r, s)-reduces to h.*

Let $(\Phi, \psi, \varrho, \Lambda)$ be an (r, s)-equivalence from g to h. Then $\Lambda = (\lambda, \psi)$ for some $\lambda \in \mathcal{E}(1+r+s)$. Letting τ denote the first coordinate of $\mathbf{R}^{1+r+s}$, we have $(\partial\lambda/\partial\tau)(0) \neq 0$; without loss of generality we may assume $(\partial\lambda/\partial\tau)(0) > 0$, for if not we may achieve this by replacing h by $-h$ and λ by $-\lambda$, and clearly if f reduces to $-h$ then f also reduces to h.

Since f (r, s)-reduces to g, there is a non-degenerate quadratic form Q on $\mathbf{R}^d$ such that f is (r, s)-equivalent to the germ $g' \in \mathfrak{m}(n+d+r+s)$ given by $g'(x, y, u, v) = g(x, u, v) + Q(y)$. Define $h' \in \mathfrak{m}(n+d+r+s)$ by $h'(x, y, u, v) = h(x, u, v) + Q(y)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Clearly it suffices to show g' is (r, s)-equivalent to h'.

Define $k \in \mathfrak{m}(n+d+r+s)$ by setting

$$k(x, y, u, v) = \lambda(g\Phi^{-1}(x, u, v) + Q(y), \psi^{-1}(u, v)) \text{ for } x \in \mathbf{R}^n,\ y \in \mathbf{R}^d,\ u \in \mathbf{R}^r,\ v \in \mathbf{R}^s.$$

One sees immediately from this definition that k and g' are (r, s)-equivalent; to complete the proof we shall show k is (r, s)-equivalent to h'.

For $t \in [0, 1]$, set $H_t = tk + (1-t)h'$. If $t_0 \in [0, 1]$, define $K_{t_0} \in \mathfrak{m}(n+d+r+s+1)$ by setting $K_{t_0}(x, y, u, v, t) = H_{t_0+t}(x, y, u, v)$ for $x \in \mathbf{R}^n$, $y \in \mathbf{R}^d$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$.

We have $\partial K_{t_0}/\partial t = k - h'$. From the definition of k, and since $h'(x, y, u, v) = \lambda(g\Phi^{-1}(x, u, v), \psi^{-1}(u, v)) + Q(y)$, one easily computes that for $x \in \mathbf{R}^n$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$ and $t \in \mathbf{R}$ we have $(\partial K_{t_0}/\partial t)(x, 0, u, v, t) = 0$ and since $Q \in \mathfrak{m}(d)^2$ we have for $i = 1, ..., d$ that $(\partial^2 K_{t_0}/\partial t \partial y_i)(x, 0, u, v, t) = 0$. Hence $\partial K_{t_0}/\partial t \in \mathfrak{m}(d)^2 \mathcal{E}(n+d+r+s+1)$.

For $i = 1, ..., d$ we have

$$\frac{\partial K_{t_0}}{\partial y_i}(x, y, u, v, t) = \frac{\partial Q}{\partial y_i}(y)\left[(t_0 + t)\frac{\partial \lambda}{\partial \tau}(g\Phi^{-1}(x, u, v) + Q(y), \psi^{-1}(u, v)) + 1 - t - t_0\right].$$

When x, y, u, v and t are 0, the function in square brackets evaluates to $(\partial\lambda/\partial\tau)(0) + 1 - t_0$, which is certainly not 0 since $(\partial\lambda/\partial\tau)(0) > 0$ and $|t_0| \leq 1$. Hence the function in square brackets above is a unit of $\mathcal{E}(n+d+r+s+1)$. It follows that

$$\langle \partial K_{t_0}/\partial y \rangle_{\mathcal{E}(n+d+r+s+1)} = \langle \partial Q/\partial y \rangle_{\mathcal{E}(n+d+r+s+1)} = \mathfrak{m}(d)\, \mathcal{E}(n+d+r+s+1).$$

Therefore $$\partial K_{t_0}/\partial t \in \mathfrak{m}(d)^2\, \mathcal{E}(n+d+r+s+1) = \langle \partial K_{t_0}/\partial y \rangle_{\mathfrak{m}(d)\, \mathcal{E}(n+d+r+s+1)},$$

so the homotopy $t \to H_t$ clearly fulfills condition 4.5 (a), and hence by Lemma 4.5 k is (r, s)-equivalent to h'. This completes the proof.

The following lemma is a converse to Lemma 4.19.

LEMMA 4.20. *Let $f \in \mathfrak{m}(n+r+s)$. Suppose f (r, s)-reduces to $g \in \mathfrak{m}(n_1+r+s)$ and to $h \in \mathfrak{m}(n_2+r+s)$ and suppose $g|\mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^3$* and $h|\mathbf{R}^{n_2} \in \mathfrak{m}(n_2)^3$. *Then $n_1 = n_2$ and g is (r, s)-equivalent to h.*

Proof. The idea of the proof is taken from Thom [7, § 5.2 D, p. 76] and Tougeron [9, Ch. VIII, Th. 3.6 (1), pp. 166–7].

Without loss of generality we may assume $n_1 \geq n_2$; otherwise interchange g and h.

The hypotheses imply $n_1 \leq n$ and $n_2 \leq n$. Let $d_1 = n - n_1$ and let $d_2 = n - n_2$. Then there is a non-degenerate quadratic form Q on $\mathbf{R}^{d_1}$ and a non-degenerate quadratic form R on $\mathbf{R}^{d_2}$ such that f is (r, s)-equivalent to the germs g' and $h' \in \mathfrak{m}(n+r+s)$ given by $g'(x, y, u, v) = g(x, u, v) + Q(y)$ for $x \in \mathbf{R}^{n_1}$, $y \in \mathbf{R}^{d_1}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$ and $h'(w, z, u, v) = h(w, u, v) + R(z)$ for $w \in \mathbf{R}^{n_2}$, $z \in \mathbf{R}^{d_2}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$.

By the Morse lemma, there is a germ $\varphi \in L(d_1)$ such that $Q\varphi(y_1, ..., y_{d_1}) = \sum_{i=1}^{d_1} \pm y_i^2$. Define $g'' \in \mathfrak{m}(n+r+s)$ by setting $g''(x, y, u, v) = g'(x, \varphi y, u, v) = g(x, u, v) + \sum_{i=1}^{d_1} \pm y_i^2$ for $x \in \mathbf{R}^{n_1}$, $y \in \mathbf{R}^{d_1}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Clearly g'' is (r, s)-equivalent to g'. Similarly h' is (r, s)-equivalent to a germ $h'' \in \mathfrak{m}(n+r+s)$ of the form $h''(w, z, u, v) = h(w, u, v) + \sum_{i=1}^{d_2} \pm z_i^2$ for $w \in \mathbf{R}^{n_2}$, $z \in \mathbf{R}^{d_2}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$.

Since g'' and h'' are (r, s)-equivalent, there is a coordinate change on $\mathbf{R}^{n+r+s}$ of the form $(x, y, u, v) \to (x'(x, y, u, v), y'(x, y, u, v), u'(u, v), v'(u, v))$ (where $x \in \mathbf{R}^{n_1}$, $x' \in \mathbf{R}^{n_2}$, $y \in \mathbf{R}^{d_1}$, $y' \in \mathbf{R}^{d_2}$, u, $u' \in \mathbf{R}^r$, v, $v' \in \mathbf{R}^s$), and there is a germ $\lambda \in \mathcal{E}(1+r+s)$ satisfying $(\partial\lambda/\partial\tau)(0) \neq 0$ (where τ denotes the first coordinate of $\mathbf{R}^{1+r+s}$), such that everywhere near 0 in $\mathbf{R}^{n+r+s}$ we have

$$\text{(a)} \qquad h(x', u' v') + \sum_{i=1}^{d_2} \pm y_i'^2 = \lambda(g(x, u, v) + \sum_{i=1}^{d_1} \pm y_i^2, u, v)$$

Define $\bar{g} \in \mathfrak{m}(n_1+r+s)$ by $\bar{g}(x, u, v) = \lambda(g(x, u, v), u, v)$ for $x \in \mathbf{R}^{n_1}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$. Clearly $\bar{g}$ is (r, s)-equivalent to g.

If we differentiate (a) with respect to y_i' and then set $y=0$, we find, for each i, $1 \leq i \leq d_2$, that when $y=0$ we have

$$\text{(b)} \qquad \pm 2y_i' = \frac{\partial\lambda}{\partial\tau}(g(x,u,v),u,v)\left(\sum_{j=1}^{n_1} \frac{\partial g}{\partial x_j}(x,u,v)\frac{\partial x_j}{\partial y_i'}\right) = \sum_{j=1}^{n_1} \frac{\partial \bar{g}}{\partial x_j}(x,u,v)\frac{\partial x_j}{\partial y_i'}$$

If, for some k, $1 \leq k \leq n_1$, we differentiate equation (b) with respect to x_k (we may do this because (b) holds for all x near 0 when $y=0$), and then evaluate at $0 \in \mathbf{R}^{n+r+s}$, then we find, for any i, $1 \leq i \leq d_2$, and for any k, $1 \leq k \leq n_1$, that (c) $(\partial y_i'/\partial x_k)(0) = 0$ (the right-hand side evaluates to 0 because $g \,|\, \mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^3$, which clearly implies also $\bar{g} \,|\, \mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^3$).

But the map of $\mathbf{R}^n$ to itself given by $(x, y) \mapsto (x'(x, y, 0, 0), y'(x, y, 0, 0))$ is non-singular at 0; because of (c) this can only be the case if the matrix $((\partial x_j'/\partial x_k)(0))_{1 \leq j \leq n_2; 1 \leq k \leq n_1}$ has rank n_1. Since by assumption $n_1 \geq n_2$, this implies $n_1 = n_2$.

Moreover it follows that the germ $\Phi \in \mathcal{E}(n_1+r+s, n_1+r+s)$ given by $\Phi(x, u, v) = (x'(x, 0, u, v), u'(u, v), v'(u, v))$ is non-singular at 0, so if we define $\bar{h} \in \mathfrak{m}(n_1+r+s)$ by $\bar{h} = h\Phi$, then $\bar{h}$ is (r, s)-equivalent to h. We shall show $\bar{h}$ is (r, s)-equivalent to $\bar{g}$.

Let $\mu = \bar{h} - \bar{g}$. If $t \in [0, 1]$, define $H_t \in \mathfrak{m}(n_1+r+s)$ by setting $H_t = \bar{g} + t\mu$. If $t_0 \in [0, 1]$, define $K_{t_0} \in \mathfrak{m}(n_1+r+s+1)$ by setting $K_{t_0}(x, u, v, t) = H_{t_0+t}(x, u, v)$ for $x \in \mathbf{R}^{n_1}$, $u \in \mathbf{R}^r$, $v \in \mathbf{R}^s$, $t \in \mathbf{R}$.

From (a) and (b) it follows that $\mu \in (\langle \partial\bar{g}/\partial x \rangle_{\mathcal{E}(n_1+r+s)})^2$. For any i, $1 \leq i \leq n_1$, we have

$$\left(\frac{\partial K_{t_0}}{\partial x_i} - \frac{\partial \bar{g}}{\partial x_i}\right)(x,u,v,t) = (t_0+t)\frac{\partial\mu}{\partial x_i}(x,u,v) \quad \text{for} \quad x \in \mathbf{R}^{n_1}, u \in \mathbf{R}^r, v \in \mathbf{R}^s, t \in \mathbf{R}.$$

If J is the ideal of $\mathcal{E}(n_1+r+s)$ generated by the germs $\partial^2\bar{g}/\partial x_j\partial x_k$, $1 \leq j$, $k \leq n_1$, then clearly $\partial\mu/\partial x_i \in J \cdot \langle \partial\bar{g}/\partial x \rangle_{\mathcal{E}(n_1+r+s)}$ (for $1 \leq i \leq n_1$). Moreover, since $\bar{g} \,|\, \mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^3$ we have $(\partial^2\bar{g}/\partial x_j\partial x_k)(0) = 0$ (for $1 \leq j$, $k \leq n_1$) and hence $J \subseteq \mathfrak{m}(n_1+r+s)$ (note that by a similar argument we also have $\langle \partial\bar{g}/\partial x \rangle_{\mathcal{E}(n_1+r+s)} \subseteq \mathfrak{m}(n_1+r+s)$).

So we have

$$\langle\partial\bar{g}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)} \subseteq \langle\partial K_{t_0}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)} + \langle\partial\mu/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)}$$
$$\subseteq \langle\partial K_{t_0}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)} + \langle\partial\bar{g}/\partial x\rangle_{\mathfrak{m}(n_1+r+s+1)}.$$

By Nakayama's lemma (Lemma 1.4) this implies

$$\langle\partial\bar{g}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)} \subseteq \langle\partial K_{t_0}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)}.$$

Hence we find that

$$\partial K_{t_0}/\partial t = \mu \in (\langle\partial\bar{g}/\partial x\rangle_{\mathcal{E}(n_1+r+s+1)})^2 \subseteq \langle\partial\bar{g}/\partial x\rangle_{\mathfrak{m}(n_1+r+s)\,\mathcal{E}(n_1+r+s+1)} \subseteq \langle\partial K_{t_0}/\partial x\rangle_{\mathfrak{m}(n_1+r+s)\,\mathcal{E}(n_1+r+s+1)}$$

Since this holds for any $t_0 \in [0, 1]$, it follows by Lemma 4.5 that $\bar{g} = H_0$ is (r, s)-equivalent to $\bar{h} = H_1$. Q.E.D.

Remark. It is a well known fact that for a germ $\eta \in \mathfrak{m}(n)^2$ the condition $\eta \in \mathfrak{m}(n)^3$ is equivalent to the condition "η is irreducible" (see e.g. [11, Corollary 5.13 (a)]). Moreover if f is an (r, s)-stable unfolding of η, then by virtue of Lemma 4.18 f is (r, s)-irreducible if and only if η is irreducible. Hence in Lemma 4.20 we may replace the condition $g|\mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^3$ by the condition $g|\mathbf{R}^{n_1} \in \mathfrak{m}(n_1)^2$ and g is (r, s)-irreducible; and similarly for h.

Note also that if $\eta \in \mathfrak{m}(n)$ is non-singular at 0, i.e. $\eta \notin \mathfrak{m}(n)^2$, then any $(r+s)$-dimensional unfolding of η is (r, s)-stable and any two $(r+s)$-dimensional unfoldings of η are (r, s)-equivalent.

So by virtue of Lemmas 4.17–4.20 we have reduced the problem of classifying all (r, s)-stable unfoldings to that of classifying the (r, s)-irreducible (r, s)-stable unfoldings. This simplification will prove useful in the next section.

§ 5. Time-stable and space-stable unfoldings: the "Thom lists"

Thom's celebrated list of the seven elementary catastrophes (Theorem 2.26) is a classification theorem for r-stable unfoldings when $r \leqslant 4$. In this section we shall compute analogous lists for (r, s)-stable unfoldings in two important special cases: the cases of $(3, 1)$-stability and $(1, 3)$-stability.

Why are these cases of particular interest? In fact, why is (r, s)-stability of interest at all? The answer lies of course in the applications to catastrophe theory.

Recall that in Thom's catastrophe theory models for natural processes are obtained in the following way: Two manifolds B and M are given. The manifold B, the "control space", is the space in which the process is observed or in which it takes place; in the applications, either B represents physical space-time or B may be a space of control parameters which govern the event to be described (for example, B may be parametrized by certain physical variables whose effect on the outcome of an experiment is to be described). Usually B is of quite low dimension, generally $\leqslant 4$. The manifold M, the "state space", is parametrized by all the physical variables which are relevant to the process under study and which play a rôle in describing the physical "state" (in a general sense) which reigns at various points of B; the state space can be of very high dimension.

We consider $B \times M$ to be fibred over B via the projection $\pi\colon B \times M \to B$. One may define a physical *process* s to be a subset of $B \times M$; if $b \in B$, then the set $s_b = s \cap (\{b\} \times M)$ (considered as a subset of M) can be interpreted as the set of possible physical states which can reign at the point $b \in B$. A point $b \in B$ is said to be *regular* for the process if the set of possible states "looks the same" everywhere near b, i.e. if there is a neighbourhood U of $b \in B$ and a homeomorphism $h\colon U \times M \to U \times M$ such that $\pi h = \pi$ on $U \times M$ and such that $h(s \cap (U \times M)) = U \times s_b$. The non-regular points of B are called *catastrophe* points. In observing a process occurring in nature, one does not notice continuous changes of state; one only sees something happening if the state changes abruptly. So what one observes in nature is the set of catastrophe points of a process.

In the simplest case (but one which is adequate to explain a large variety of phenomena) one supposes the set s is obtained as follows: One takes a smooth function V: $B \times M \to \mathbf{R}$, and one supposes $s \subseteq \{(b, x) \in B \times M\colon V \mid (\{b\} \times M) \text{ has a local minimum at } (b, x)\}$ (*which* subset of this set one chooses s to be is governed by various conventions which we shall not discuss here). In other words, V is considered as a family of potential functions on M, parametrized by B, and for $b \in B$, the set s_b of possible states at b consists of states at which the potential function above b has a local minimum. Models of this sort are called gradient models.

Naturally it is of great interest to classify such functions V, at least locally; the classification should respect the character of V as a *family* of potential functions on M. That is, we wish to classify the functions V locally up to the action on the left of families of diffeomorphisms of $\mathbf{R}$, parametrized by B, and up to the action on the right of diffeomorphisms Φ of $B \times M$ which have the property that there is a local diffeomorphism ψ of B such that $\pi\Phi = \psi\pi$.

If B has dimension r and M has dimension n, then locally such a function V is just an r-dimensional unfolding of a germ $\eta \in \mathfrak{m}(n)$ and two such functions are locally equivalent in the manner described above exactly when the corresponding unfoldings are r-equivalent.

Before performing the classification we may reasonably make two additional restric-

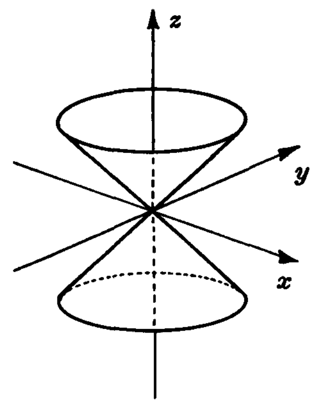


Fig. 1.

tions. Firstly, since B is usually 4-dimensional space-time and in any case is usually of very low dimension, we may assume $r \leqslant 4$. Secondly, to say that a process is "observable in nature" usually means that the process occurs repeatedly or can be evoked again and again in repeated experiments; however, the initial conditions for such a process can never be reproduced exactly. Hence it is reasonable to assume that the function V does not change its appearance under slight perturbations; this corresponds to assuming that the associated unfolding is r-stable.

If we adopt these two additional restrictions, then Thom's list of the seven elementary catastrophes (Theorem 2.26) is precisely a classification theorem of the sort we require.

Unfortunately, however, this classification is inadequate for many applications; it is too coarse. The reason is that in many cases the control space B is not physically isotropic, in the sense that different control parameters need not have the same physical importance. This is the case, for example, when B is space-time; the time-coordinate plays a special rôle; time is not just another spatial dimension. However the equivalences up to which unfoldings are classified in Thom's list can operate on B via an *arbitrary* local diffeomorphism. They take no note of the possible anisotropy of B and they can therefore identify

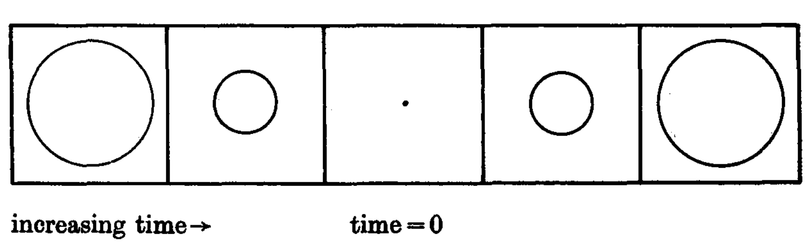


Fig. 2.

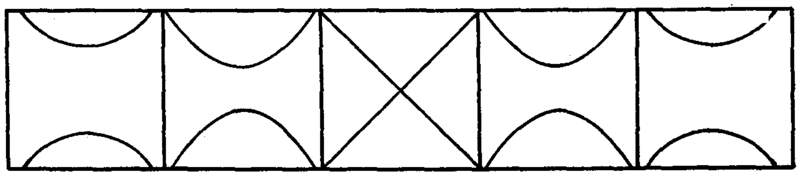


Fig. 3.

two processes which would look entirely different to an observer studying them in nature. An example will make this clear.

For simplicity, we shall take B to be a three-dimensional space-time, with one temporal and two spatial coordinates; locally, B is $\mathbf{R}^3$, with coordinates x, y and z. Suppose we are observing a process whose catastrophe set is the cone $z^2 = x^2 + y^2$ (see figure 1). What we will actually see happening in time is not uniquely determined by this descrip- tion of the catastrophe set; it depends on which direction we choose to be the time direction, or more precisely on how B is foliated into "spatial" planes of constant time. If we choose z to be the time coordinate (so that planes of constant time are those which are parallel to the xy-plane), and if we make a film of what we then observe, then successive frames of the film would look as in figure 2. We should see a bubble collapsing to a point and then expanding again. If instead we choose x as the time coordinate and planes parallel to the yz-plane as planes of constant time, then a film of the event would look as in figure 3, and we should see two hyperbolas approaching each other, merging to form a cross, and then separating again.

Clearly we should say these two events are different, but the unfoldings which generated them would certainly be 3-equivalent, since we have merely interchanged two coordinates of $B = \mathbf{R}^3$.

This example demonstrates that if we wish to obtain, by means of catastrophe theory, an adequate description of events which are seen as developing in time, if we are to be able to describe the spatial configuration at fixed moments of time, then we must classify unfoldings up to equivalences which preserve simultaneity, that is, whose action on B respects the foliation of B into planes of constant time. If we consider the action on B as a change of local coordinates, then the new time coordinate may depend only on the old time coordinate, not on the spatial coordinates. (Note that the new time coordinate need not be the same as the old time coordinate; we may change the time scale). And of course stability must also be defined using this more restricted sort of equivalence of unfoldings.

The equivalence notion which we need for this purpose (when B is 4-dimensional space-time) is obviously (3, 1)-equivalence.

There are other applications of catastrophe theory in which the converse problem arises; that is, one does not need to be able to describe global spatial configurations at individual moments in time, but it is important to be able to say what happens at *fixed points of space* as time progresses. For example, Zeeman has a description of gastrulation in which points of space represent individual cells of the embryo. Each cell undergoes the same temporal development; the change in shape of the embryo as a whole is accounted for by the fact that the temporal development of the cells occurs on a different time scale for different cells. For his model, Zeeman needs to be able to follow the development of individual cells through time; he must be able to identify a given cell at different moments of time. Simultaneity of stages in the development of different cells is not important to the description. For applications of this sort, one needs a classification of unfoldings via equivalences which preserve identity of location, that is, which respect the foliation of space-time into lines of constant position. If we consider the action on space-time as a coordinate change, then the new spatial coordinates may depend only on the old spatial coordinates, but not on the old time coordinate. Again stability must also be defined using this sort of equivalence. The type of equivalence needed here is just (1, 3)-equivalence (when B is 4-dimensional space-time).

One can conceive of other applications of catastrophe theory, in which B need not be space-time but perhaps instead a space of control parameters for a process or a series of experiments, where some of the control parameters might be physically more important than the others or play a different rôle from that of the others; for such applications a classification of (r, s)-stable unfoldings for other values of r and s would be of interest. But clearly the cases of (3, 1)-stability and (1, 3)-stability are the most important, so we make the following definition.

Definition 5.1. A four-dimensional unfolding is said to be *time-stable* if it is (3, 1)-stable; it is said to be *space-stable* if it is (1, 3)-stable.

Similarly we shall say "time-equivalent" and "time-reduces" for "(3, 1)-equivalent" and "(3, 1)-reduces" respectively; "space-equivalent" and "space-reduces" for "(1, 3)-equivalent" and "(1, 3)-reduces" respectively.

Theorems 5.2 and 5.3 below classify the time-stable and the space-stable unfoldings respectively. In stating these theorems we adopt the following notational convention: The unfoldings listed in Theorems 5.2 and 5.3 are germs in $\mathfrak{m}(n+4)$ for some n. We shall use letters x, y etc. to denote the coordinates of $\mathbf{R}^n$. We shall denote the unfolding parameters (i.e. the coordinates of $\mathbf{R}^4$) by u, v, w and t, whereby in both theorems t is to be interpreted as the time-coordinate and u, v, and w as spatial coordinates (so in Theorem 5.2 t corresponds to v_1 in the notation used in previous sections and u, v, w correspond to u_1, u_2, u_3; in Theorem 5.3 t corresponds to u_1 and u, v, w to v_1, v_2, v_3). The proof of both theorems is given after Theorem 5.3.

THEOREM 5.2. *Let $f \in \mathfrak{m}(n+4)$ be a time-stable unfolding of $\eta \in \mathfrak{m}(n)^2$. Then either f has a simple singularity at* 0, *or f time-reduces to a unique one of the following* 12 *unfoldings h_i of germs ν_i:*

Name	ν_i	h_i
Folds: the fold	$\nu_1(x) = x^3$	$h_1(x, u, v, w, t) = x^3 + ux$
bubble collapse	$\nu_2(x) = x^3$	$h_2(x, u, v, w, t) = x^3 + tx + u^2x + v^2x + w^2x$
fission	$\nu_3(x) = x^3$	$h_3(x, u, v, w, t) = x^3 + tx + u^2x + v^2x - w^2x$
fusion	$\nu_4(x) = x^3$	$h_4(x, u, v, w, t) = x^3 + tx + u^2x - v^2x - w^2x$
bubble formation	$\nu_5(x) = x^3$	$h_5(x, u, v, w, t) = x^3 + tx - u^2x - v^2x - w^2x$
Cusps: the cusp	$\nu_6(x) = x^4$	$h_6(x, u, v, w, t) = x^4 + ux^2 + vx$
bec-à-bec	$\nu_7(x) = x^4$	$h_7(x, u, v, w, t) = x^4 + ux^2 + tx + ux + v^2x + w^2x$
bec-à-bec to lip	$\nu_8(x) = x^4$	$h_8(x, u, v, w, t) = x^4 + ux^2 + tx + ux + v^2x - w^2x$
the lip	$\nu_9(x) = x^4$	$h_9(x, u, v, w, t) = x^4 + ux^2 + tx + ux - v^2x - w^2x$
The swallowtail	$\nu_{10}(x) = x^5$	$h_{10}(x, u, v, w, t) = x^5 + ux^3 + vx^2 + wx$
The hyperbolic umbilic	$\nu_{11}(x, y) = x^3 + y^3$	$h_{11}(x, y, u, v, w, t) = x^3 + y^3 + uxy + vx + wy$
The elliptic umbilic	$\nu_{12}(x, y) = x^3 - xy^2$	$h_{12}(x, y, u, v, w, t) = x^3 - xy^2 + u(x^2 + y^2) + vx + wy$

All of the h_i are time-stable and clearly time-irreducible.

THEOREM 5.3. *Let $f \in \mathfrak{m}(n+4)$ be a space-stable unfolding of a germ $\eta \in \mathfrak{m}(n)^2$. Then either f has a simple singularity at* 0, *or f space-reduces to a unique one of the following unfoldings h'_j of germs ν'_j:*

Name	ν'_j	h'_j
Folds:	$\nu'_1(x) = x^3$	$h'_1(x, t, u, v, w) = x^3 + tx$
	$\nu'_2(x) = x^3$	$h'_2(x, t, u, v, w) = x^3 + ux + t^2x$
	$\nu'_3(x) = x^3$	$h'_3(x, t, u, v, w) = x^3 + ux - t^2x$

Name	ν'_j	h'_j
	$\nu'_4(x)=x^3$	$h'_4(x, t, u, v, w)=x^3+ux+vtx+t^3x$
	$\nu'_5(x)=x^3$	$h'_5(x, t, u, v, w)=x^3+ux+vtx+wt^2x+t^4x$
	$\nu'_6(x)=x^3$	$h'_6(x, t, u, v, w)=x^3+ux+vtx+wt^2x-t^4x$
Cusps:	$\nu'_7(x)=x^4$	$h'_7(x, t, u, v, w)=x^4+ux^2+tx^2+tx$
	$\nu'_8(x)=x^4$	$h'_8(x, t, u, v, w)=x^4+ux^2+vtx^2+tx$
	$\nu'_9(x)=x^4$	$h'_9(x, t, u, v, w)=x^4+ux^2+tx^2+vx+t^2x$
	$\nu'_{10}(x)=x^4$	$h'_{10}(x, t, u, v, w)=x^4+ux^2+tx^2+vx+t^3x+wtx$
	$\nu'_{11}(x)=x^4$	$h'_{11}(x, t, u, v, w)=x^4+ux^2+t^2x^2+vx+t^2x+wtx$
	$\nu'_{12}(x)=x^4$	$h'_{12}(x, t, u, v, w)=x^4+ux^2-t^2x^2+vx+t^2x+wtx$
Swallowtails:	$\nu'_{13_c}(x)=x^5$	$h'_{13_c}(x, t, u, v, w)=x^5+ux^3+ctx^3+vtx^3+tx^2+wx+tx$ $(c\in\mathbf{R},\ c\neq\frac{9}{4})$
	$\nu'_{14}(x)=x^5$	$h'_{14}(x, t, u, v, w)=x^5+ux^3+tx^3+vx^2+twx^2+tx$
	$\nu'_{15}(x)=x^5$	$h'_{15}(x, t, u, v, w)=x^5+ux^3-tx^3+vx^2+twx^2+tx$

All of the h'_j are space-stable and clearly space-irreducible.

Remark. Let r and s be arbitrary and suppose $h\in\mathfrak{m}(n+r+s)$ unfolds $\nu\in\mathfrak{m}(n)$. By the remark at the end of § 4, if $\nu\notin\mathfrak{m}(n)^2$ then h is automatically (r, s)-stable and is (r, s)-equivalent to any other $(r+s)$-dimensional unfolding of ν.

Suppose $\nu\in\mathfrak{m}(n)^2$ but h has a simple singularity at 0. Then again h is automatically (r, s)-stable, for ν reduces to the trivial germ $0\in\mathfrak{m}(0)$, so ν is equivalent to a non-degenerate quadratic form on $\mathbf{R}^n$ and it is then easily seen (using Theorem 2.9) that any r-dimensional unfolding of ν is r-stable and hence, by Corollary 3.7, any $(r+s)$-dimensional unfolding of ν is (r, s)-stable. Furthermore, by Lemma 4.18 it follows that h (r, s)-reduces to an $(r+s)$-dimensional unfolding of $0\in\mathfrak{m}(0)$, i.e., to a germ in $\mathfrak{m}(r+s)$. Again any germ in $\mathfrak{m}(r+s)$ is (r, s)-stable and they are all (r, s)-equivalent, for if g' and g'' are in $\mathfrak{m}(r+s)$, and if we define $\Lambda\in L(1+r+s)$ by $\Lambda(\tau, u, v)=(\tau+g''(u, v)-g'(u, v), u, v)$ for $\tau\in\mathbf{R}$, $u\in\mathbf{R}^r$, $v\in\mathbf{R}^s$, then $(id_{\mathbf{R}^{r+s}}, id_{\mathbf{R}^{r+s}}, id_{\mathbf{R}^s}, \Lambda)$ is an (r, s)-equivalence from g' to g''. Hence the classification up to (r, s)-equivalence of unfoldings with a simple singularity is also completely trivial.

Obviously, in Theorems 5.2 and 5.3, if f time-reduces to one of the unfoldings h_i or space-reduces to one of the unfoldings h'_j, then f does not have a simple singularity.

Proof of Theorems 5.2 *and* 5.3. The method of proof is the same for both theorems. If f is time-stable or space-stable, then f is 4-stable and hence f has a simple singularity or f reduces (in the sense of Definition 2.24) to a unique one of the unfoldings g_i in the list of Theorem 2.26. If the latter is the case, then η reduces to a unique one of the germs μ_i in the list of Theorem 2.26 (η cannot reduce to more than one of the μ_i, for then, by Lemmas

4.18 and 4.17, f (4, 0)-reduces to stable unfoldings of more than one of the μ_i and by Corollary 2.23 and Lemma 4.19 it follows that f reduces (in the sense of Definition 2.24) to more than one of the g_i).

By Lemma 4.18, if η reduces to μ_i then f time-reduces (in Theorem 5.2) or space-reduces (in Theorem 5.3) to a four-dimensional unfolding h of μ_i which by Lemma 4.17 must be time-stable (resp. space-stable). Moreover by Lemma 4.19 f also time-(space-) reduces to any other unfolding of μ_i which is time-(space-)equivalent to h, but on the other hand, by Lemma 4.20 the time-(space-)equivalence class of h is uniquely determined by f. Hence to complete the proof we need only show that for each germ μ_i in the list of Theorem 2.26, the lists of Theorems 5.2 and 5.3 contain exactly one representative of each time-(resp. space-)equivalence class of time-(resp. space-)stable unfoldings of μ_i.

To show this, we first apply the algorithm given after Theorem 4.11 to obtain lists containing at least one representative of each of these equivalence classes; special arguments will then be used to reduce these lists to those given by Theorems 5.2 and 5.3; finally we shall show that the size of the lists cannot be reduced further, i.e. that we have a unique representative of each class. The remainder of the proofs of Theorems 5.2 and 5.3 will be conducted separately.

Proof of Theorem 5.2 *continued.* By applying the algorithm given after Theorem 4.11, one finds that up to time-equivalence an arbitrary time-stable unfolding of $\mu_1(x)=x^3$ is either x^3+ux (which is h_1) or is of the form

$$x^3+tx+(Au+Bv+Cw+Du^2+Euv+Fuw+Gv^2+Hvw+Iw^2)x,$$

where $A, B, C, D, E, F, G, H, I \in \mathbf{R}$ and either A, B or C is non-zero or

$$\det\begin{vmatrix} 2D & E & F \\ E & 2G & H \\ F & H & 2I \end{vmatrix} \neq 0.$$

If A, B or C is non-zero then by Lemma 4.9 the unfoldings we get for different values of D, E, F, G, H and I are all time-equivalent, so we may assume $D=E=F=G=H=I=0$; by a linear change of coordinates in uvw-space we may then arrange that $A=1$, $B=C=0$; this gives $x^3+tx+ux$ and changing coordinates by setting $u'=u+t$ (the other coordinates unchanged) gives $x^3+u'x$, which is h_1 again. If $A=B=C=0$, then we have

$$\det\begin{vmatrix} 2D & E & F \\ E & 2G & H \\ F & H & 2I \end{vmatrix} \neq 0;$$

this is the determinant of the Hessian at 0 of the form $Du^2+Euv+Fuw+Gv^2+Hvw+Iw^2$ on $\mathbf{R}^3$, so this quadratic form is non-degenerate and by a linear change of coordinates on uvw-space we may assume $E=F=H=0$, and either $D=G=I=1$; or $D=G=1$, $I=-1$; or $D=1$, $G=I=-1$; or $D=G=I=-1$. These four possibilities give h_2, h_3, h_4, and h_5.

We must show that no two of these unfoldings h_1-h_5 are time-equivalent. To simplify the notation, we denote u, v, and w by u_1, u_2, u_3 respectively. If h_i is time-equivalent to h_j then there is a germ $\lambda\in\mathcal{E}(5)$ and a coordinate change $(x, u_1, u_2, u_3, t)\to(x', u_1', u_2', u_3', t')$ on $\mathbf{R}^5$ such that

$$h_i(x', u_1', u_2', u_3', t')=\lambda(h_j(x, u_1, u_2, u_3, t), u_1, u_2, u_3, t) \tag{a}$$

and such that the u_k' do not depend on x, t' depends only on t, and $(\partial\lambda/\partial\tau)(0)\neq 0$ (where τ is the first coordinate).

Suppose first $j=1$, $i\neq 1$. If we apply the operator $\partial^2/\partial x\partial t$ to (a) and evaluate at $0\in\mathbf{R}^5$, we find $(\partial x'/\partial x)(\partial t'/\partial t)(0)=0$, which is impossible, so h_1 is not time-equivalent to any of the others. So we may suppose $i, j\neq 0$. Then $h_i=x^3+tx+Q_i(u_1, u_2, u_3)x$ and $h_j=x^3+tx+Q_j(u_1, u_2, u_3)x$ where Q_i, Q_j are non-degenerate quadratic forms on $\mathbf{R}^3$. If, for $k=1, 2, 3$, we apply $\partial^3/\partial x^2\partial u_k$ to (a) and evaluate at 0, we find $6(\partial x'/\partial x)^2(\partial x'/\partial u_k)(0)=0$ which implies $(\partial x'/\partial u_k)(0)=0$. From this it follows that if we apply $\partial^3/\partial x\partial u_k\partial u_l$ to (a), for $1\leqslant k, l\leqslant 3$, and evaluate at 0, then we find

$$\frac{\partial x'}{\partial x}(0)\sum_{1\leqslant p,\, q\leqslant 3}\frac{\partial^2 Q_i}{\partial u_p\partial u_q}(0)\frac{\partial u_p'}{\partial u_k}(0)\frac{\partial u_q'}{\partial u_l}(0)=\frac{\partial\lambda}{\partial\tau}(0)\frac{\partial^2 Q_j}{\partial u_k\partial u_l}(0)\quad(\text{for } k, l=1,2,3). \tag{b}$$

Moreover, if we apply $\partial^3/\partial x^3$ to (a) and evaluate at 0, we find $6(\partial x'/\partial x)^3(0)=6(\partial\lambda/\partial\tau)(0)$, so $(\partial x'/\partial x)(0)$ and $(\partial\lambda/\partial\tau)(0)$ have the same sign; hence it follows from (b) that Q_i and Q_j have the same index, so $i=j$. So no two of $h_1, \ldots, h_5$ are time-equivalent.

By applying the algorithm given after Theorem 4.11 and by subsequently reducing the size of the list so obtained via arguments similar to those we used for the unfoldings of μ_1, it is easily verified that up to time-equivalence an arbitrary time-stable unfolding of $\mu_2(x)=x^4$ is either x^4+ux^2+vx (which is h_6) or is of the form $x^4+ux^2+tx+Aux\pm v^2x\pm w^2x$ (where $A\neq 0$) or of the form $x^4+tx^2+Bvx^2\pm u^2x^2\pm w^2x^2+vx$ (for some $B\neq 0$). Unfoldings of the last type are time-equivalent to unfoldings of the second type; for the unfolding $x^4+tx^2+Bvx^2+\varepsilon_u u^2x^2+\varepsilon_w w^2x^2+vx$ (where $\varepsilon_u=\pm 1$, $\varepsilon_w=\pm 1$) becomes

$$x^4+u'x^2+t'x+\frac{1}{B}u'x-\varepsilon_u\frac{|B|}{B}v'^2x-\varepsilon_w\frac{|B|}{B}w'^2x$$

(which is of the second type above) if we change coordinates by setting

$$u=\sqrt{|B|}v';\ v=\frac{1}{B}(u'-\varepsilon_u|B|v'^2-\varepsilon_w|B|w'^2)+t';\ w=\sqrt{|B|}w';\ \text{and } t=-Bt'.$$

Finally, we claim any unfolding of the form $x^4+ux^2+tx+Aux\pm v^2x\pm w^2x$ ($A\neq 0$) is time-equivalent to an unfolding of the form $x^4+ux^2+tx+ux\pm v^2x\pm w^2x$ (and hence clearly to h_7, h_8 or h_9). For suppose we change coordinates by setting $x=\alpha x'$, $u=\beta u'$, $v=\gamma v'$, $w=\delta w'$ and $t=\varepsilon t'$, where $\alpha, \beta, \gamma, \delta, \varepsilon$ are non-zero real numbers. Then $x^4+ux^2+tx+Aux\pm v^2x\pm w^2x$ becomes (*) $\alpha^4x'^4+\alpha^2\beta u'x'^2+\alpha\varepsilon t'x'+A\alpha\beta u'x'\pm\alpha\gamma^2v'^2x'\pm\alpha\delta^2w'^2x'$. If we can choose $\alpha, \beta, \gamma, \delta$, and ε such that (**) $\alpha^4=\alpha^2\beta=\alpha\varepsilon=A\alpha\beta=\pm\alpha\gamma^2=\pm\alpha\delta^2$, then the unfolding (*) will be a real multiple of (and hence time-equivalent to) an unfolding of the form $x'^4+u'x'^2+t'x'+u'x'\pm v'^2x'\pm w'^2x'$ (for some choice of the $\pm$ signs). But equation (**) can be solved by setting $\alpha=A$, $\beta=A^2$, $\gamma=\sqrt{|A^3|}$, $\delta=\sqrt{|A^3|}$, $\varepsilon=A^3$; this proves the claim.

We must show that no two of h_6-h_9 are time-equivalent. If h_6 were time-equivalent to h_i for $i=7$, 8, or 9, then clearly $h_6|_{t=0}$ would be 3-equivalent to $h_i|_{t=0}$. But this is impossible, for one easily checks (using Theorem 2.9) that $h_6|_{t=0}$ is 3-stable and $h_i|_{t=0}$ ($i=7$, 8, or 9) are not.

For $i=7$, 8, 9, the unfolding h_i has the form $x^4+ux^2+tx+ux+Q_i(v,w)x$, where Q_i is a non-degenerate quadratic form in v and w. If $7\leqslant i, j\leqslant 9$, and if h_i is time-equivalent to h_j, then there is a coordinate change $(x,u,v,w,t)\to(x',u',v',w',t')$ on $\mathbf{R}^5$, and there is a germ $\lambda\in\mathcal{E}(5)$, such that

(c) $$h_i(x',u',v',w',t')=\lambda(h_j(x,u,v,w,t),u,v,w,t)$$

and such that u', v', w' do not depend on x and t' depends only on t, and such that $(\partial\lambda/\partial\tau)(0)\neq 0$ (where τ is the first coordinate). If we apply $\partial^5/\partial x^5$ to (c) and evaluate at 0 we find

(d) $$240\left(\frac{\partial x'}{\partial x}(0)\right)^3\frac{\partial^2x'}{\partial x^2}(0)=0 \quad\text{so}\quad \frac{\partial^2x'}{\partial x^2}(0)=0.$$

If we apply $\partial^2/\partial x\partial v$ to (c) and evaluate at 0, we find

(e) $$\frac{\partial x'}{\partial x}(0)\frac{\partial u'}{\partial v}(0)=0 \quad\text{so}\quad \frac{\partial u'}{\partial v}(0)=0;\quad\text{similarly}\quad \frac{\partial u'}{\partial w}(0)=0.$$

If we apply $\partial^4/\partial x^3\partial v$ to (c) and evaluate at 0, then because of (e) we find

(f) $$24\left(\frac{\partial x'}{\partial x}(0)\right)^3\left(\frac{\partial x'}{\partial v}(0)\right)=0,\quad\text{so}\quad \frac{\partial x'}{\partial v}(0)=0;\quad\text{similarly}\quad \frac{\partial x'}{\partial w}(0)=0.$$

If we apply $\partial^4/\partial x^2\partial v^2$ to (c) and evaluate at 0, then because of (d), (e), and (f) we find

$$\text{(g)} \qquad 2\left(\frac{\partial x'}{\partial x}(0)\right)^2 \frac{\partial^2 u'}{\partial v^2}(0)=0; \quad \text{so} \quad \frac{\partial^2 u'}{\partial v^2}(0)=0; \quad \text{similarly} \quad \frac{\partial^2 u'}{\partial v \partial w}(0)=\frac{\partial^2 u'}{\partial w^2}(0)=0.$$

If we apply $\partial^4/\partial x^4$ to (c) and evaluate at 0, we find

$$24\left(\frac{\partial x'}{\partial x}(0)\right)^4 = 24\frac{\partial \lambda}{\partial \tau}(0) \quad \text{so} \quad \frac{\partial \lambda}{\partial \tau}(0)>0.$$

If we apply $\partial^3/\partial x^2\partial u$ to (c) and evaluate at 0, then because of (d) we find

$$2\left(\frac{\partial x'}{\partial x}(0)\right)^2 \frac{\partial u'}{\partial u}(0)=2\frac{\partial \lambda}{\partial \tau}(0) \quad \text{so} \quad \frac{\partial u'}{\partial u}(0)>0.$$

If we apply $\partial^2/\partial x\partial u$ to (c) and evaluate at 0 we find

$$\frac{\partial x'}{\partial x}(0)\frac{\partial u'}{\partial u}(0)=\frac{\partial \lambda}{\partial \tau}(0), \quad \text{so} \quad \frac{\partial x'}{\partial x}(0)>0.$$

Finally if we apply $\partial^3/\partial x\partial v^2$ to (c) and evaluate at 0, then because of (e) and (g) we find

$$\left[\frac{\partial^2 Q_i}{\partial v^2}(0)\left(\frac{\partial v'}{\partial v}(0)\right)^2 + 2\frac{\partial^2 Q_i}{\partial v \partial w}(0)\frac{\partial v'}{\partial v}(0)\frac{\partial w'}{\partial v}(0)+\frac{\partial^2 Q_i}{\partial w^2}(0)\left(\frac{\partial w'}{\partial v}(0)\right)^2\right]\frac{\partial x'}{\partial x}(0)=\frac{\partial \lambda}{\partial \tau}(0)\frac{\partial^2 Q_j}{\partial v^2}(0);$$

by applying $\partial^3/\partial x\partial v\partial w$ and $\partial^3/\partial x\partial w^2$ to (c) and evaluating at 0 we obtain corresponding equations involving $(\partial^2 Q_j/\partial v\partial w)(0)$ and $(\partial^2 Q_j/\partial w^2)(0)$ respectively on the right. Since $(\partial x'/\partial x)(0)$ and $(\partial \lambda/\partial \tau)(0)$ are both positive, and since (because of (e)) the matrix

$$\begin{pmatrix} \frac{\partial v'}{\partial v}(0) & \frac{\partial v'}{\partial w}(0) \\ \frac{\partial w'}{\partial v}(0) & \frac{\partial w'}{\partial w}(0) \end{pmatrix}$$

is non-singular, it follows that Q_i and Q_j have the same index, so $i=j$. Hence no two of the unfoldings h_6-h_9 are time-equivalent.

By applying the algorithm given after Theorem 4.11 and by using arguments similar to those used in classifying the unfoldings of μ_1, one easily verifies that up to time-equivalence h_{10}, h_{11}, and h_{12} are the only time-stable unfoldings of $\mu_3(x)=x^5$, $\mu_5(x,y)=x^3+y^3$ and $\mu_6(x,y)=x^3-xy^2$ respectively; the algorithm also shows that $\mu_4(x)=x^6$ and $\mu_7(x,y)=x^2y+y^4$ have no time-stable unfoldings. This completes the proof of Theorem 5.2.

Proof of Theorem 5.3 *continued*. We first make an observation which will enable us to shorten the computations considerably. To simplify the notation, we shall for the purposes of this remark denote the spatial coordinates u, v and w by v_1, v_2 and v_3 respectively. Let $\mu \in \mathfrak{m}(n)$ for some n and suppose μ has a 4-stable unfolding g of the form $g(x, t, v_1, v_2, v_3) = \mu(x) + \sum_{i=1}^{3} v_i b_i(x)$, where the b_i are in $\mathcal{E}(n)$. Note that g is linear in the v_i and does not depend on t. (By virtue of Theorem 2.20 and the other results of § 2, μ has a 4-stable unfolding of this form exactly when $\sigma(\mu) \leqslant 3$.)

We claim that in this case any space-stable unfolding of μ is space-equivalent to a space-stable unfolding h of the form

$$(*) \qquad h(x, t, v_1, v_2, v_3) = g(x, t, w_1(t, v_1, v_2, v_3), w_2(t, v_1, v_2, v_3), w_3(t, v_1, v_2, v_3)),$$

where each w_i is of the form

$$(**) \qquad w_i(t, v_1, v_2, v_3) = v_i + p_i(t) + \sum_{j=1}^{3} v_j \xi_{ij}(t),$$

the p_i and ξ_{ij} being polynomials in t without constant term, the p_i being of degree at most 4 and the ξ_{ij} of degree at most 2. By Theorem 4.11 we know that any space-stable unfolding f of μ is space-equivalent to a space-stable unfolding h' of the form $h'(x, t, v_1, v_2, v_3) = g_\sigma(x, t, w_1, w_2, w_3)$, where σ is some permutation in T (see Def. 4.1) and the w_i are of the form (**). What we must prove is that we may without loss of generality take σ to be the identity. If σ is not the identity then for some i_0, $1 \leqslant i_0 \leqslant 3$, we have $h'(x, t, v_1, v_2, v_3) = \mu(x) + \sum_{1 \leqslant i \leqslant 3,\, i \neq i_0} w_i(t, v_1, v_2, v_3) b_i(x) + t b_{i_0}(x)$. If in this we replace t by $t + v_{i_0}$ we obtain a space-equivalent unfolding h'' of the form $h''(x, t, v_1, v_2, v_3) = \mu(x) + \sum_{1 \leqslant i \leqslant 3,\, i \neq i_0} w_i(t + v_{i_0}, v_1, v_2, v_3) b_i(x) + v_{i_0} b_{i_0}(x) + t b_{i_0}(x)$. Clearly for suitably chosen polynomials $\xi'_{ij}(t)$ of degree at most 2 and without constant term, and for suitably chosen germs $\gamma_i(t, v_1, v_2, v_3)$ in $\langle v_1 t^3, v_2 t^3, v_3 t^3 \rangle_{\mathcal{E}(4)} + (\langle v_1, v_2, v_3 \rangle_{\mathcal{E}(4)})^2$, and for suitably chosen real numbers c_i, we may write (for $i \neq i_0$)

$$\begin{aligned} w_i(t + v_{i_0}, v_1, v_2, v_3) &= v_i + p_i(t + v_{i_0}) + \sum_{j=1}^{3} v_j \xi_{ij}(t + v_{i_0}) \\ &= v_i + c_i v_{i_0} + p_i(t) + \sum_{j=1}^{3} v_j \xi'_{ij}(t) + \gamma_i(t, v_1, v_2, v_3). \end{aligned}$$

For $i \neq i_0$ set $w'_i(t, v_1, v_2, v_3) = v_i + c_i v_{i_0} + p_i(t) + \sum_{j=1}^{3} v_j \xi'_{ij}(t)$, and set $w'_{i_0}(t, v_1, v_2, v_3) = v_{i_0} + t$. By virtue of Corollary 4.8 h'' is space-equivalent to the unfolding $\bar{h}$ given by $\bar{h}(x, t, v_1, v_2, v_3) = \mu(x) + \sum_{1 \leqslant i \leqslant 3} w'_i(t, v_1, v_2, v_3) b_i(x)$. Now for $i \neq i_0$ set $w''_i(t, v_1, v_2, v_3) = v_i + p_i(t) + \sum_{1 \leqslant j \leqslant 3,\, j \neq i_0} v_j \xi'_{ij}(t) + v_{i_0}(\xi'_{ii_0}(t) - \sum_{1 \leqslant j \leqslant 3,\, j \neq i_0} c_j \xi'_{ij}(t))$ and set $w''_{i_0} = w'_{i_0}$. (Note that each w''_i is of the form (**), and that w''_i is obtained from w'_i by replacing each v_j, for $j \neq i_0$, by $v_j - c_j v_{i_0}$.) If in $\bar{h}$ we replace each $v_j (j \neq i_0)$ by $v_j - c_j v_{i_0}$, then we obtain an unfolding h which is space-equivalent to $\bar{h}$ (and hence to f), and h is of the form (*), for we have $h(x, t, v_1, v_2, v_3) = \mu(x) + \sum_{i=1}^{3} w''_i(t, v_1, v_2, v_3) b_i(x) = g(x, t, w''_1, w''_2, w''_3)$. This completes the proof of the claim above.

We now proceed to find all space-stable unfoldings of the germs μ_i in the list of Theorem 2.26. We shall revert to denoting the spatial coordinates of $\mathbf{R}^4$ by u, v and w (rather than v_1, v_2, v_3). We also agree on the following notational convention: polynomials in t will be denoted by capital letters (P, Q, R, S, A, B, C, D, etc.), and if (for example) P is such a polynomial, then we shall write P_i to denote the coefficient of the term t^i in $P(t)$.

By applying our algorithm (and making use of our observation above) we easily find that any space-stable unfolding of $\mu_1(x)=x^3$ is space-equivalent to an unfolding of the form

$$x^3+ux+P(t)x+Q(t)ux+R(t)vx+S(t)wx, \tag{a}$$

where P, Q, R, and S are polynomials in t without a constant term, P is at most quartic, Q, R, and S are at most quadratic, and where one of the following conditions is satisfied: either (i) $P_1 \neq 0$ or (ii) $P_1=0$, $P_2 \neq 0$ or (iii) $P_1=P_2=0$, $P_3 \neq 0$, and R_1 or S_1 is non-zero, or (iv) $P_1=P_2=P_3=0$, $P_4 \neq 0$, and $R_1S_2-S_1R_2 \neq 0$. In case (i) there are no conditions on Q, R or S, so by remark 4.12 we may assume they are 0. Since $P_1 \neq 0$, the map $(x, t, u, v, w) \to (x'=x, t'=P(t)+u, u'=u, v'=v, w'=w)$ is a coordinate change on $\mathbf{R}^5$, under which the unfolding (a) becomes $x'^3+t'x'$, which is h_1'. In case (ii) we may again assume $Q=R=S=0$, by virtue of Remark 4.12, and we have $P(t)=P_2t^2+P_3t^3+P_4t^4$, with $P_2 \neq 0$; under a suitable change of the t-coordinate $t \to t'$ the polynomial P can be transformed to $\pm t'^2$, and the unfolding (a) becomes h_2' or h_3'.

In case (iii) we may assume $R_1=1$, $R_2=0$, $S=Q=0$ by virtue of Remark 4.12, and by virtue of Lemma 4.9 we may assume $P_4=0$. If we then change coordinates by setting $t'=\sqrt[3]{P_3}t$, $v'=v/\sqrt[3]{P_3}$, the other coordinates unchanged, then (a) becomes $x^3+ux+t'^3x+t'v'x$, which is h_4'. In case (iv) we may by virtue of Remark 4.12 assume $R_1=S_2=1$, $R_2=S_1=0$, $Q=0$, and by virtue of Lemma 4.9 we may assume $P_4=\pm 1$; this gives h_5' or h_6'.

By methods similar to those used in the proof of Theorem 5.2, one can easily show that no two of the unfoldings $h_1'-h_6'$ are space-equivalent We omit the details.

Again, by using our algorithm and by virtue of the observation at the beginning of the continuation of the proof, we easily find that any space-stable unfolding of $\mu_2(x)=x^4$ is space-equivalent to an unfolding of the form

$$x^4+ux^2+P(t)x^2+Q(t)ux^2+R(t)vx^2+S(t)wx^2+vx+A(t)x+B(t)ux+C(t)vx+D(t)wx, \tag{b}$$

where P, Q, R, S, A, B, C, D are polynomials in t without constant term, P and A are at most quartic, the other polynomials are at most quadratic, and where one of the following conditions is satisfied:

either: (i) $P_1 \neq 0$, $A_1 \neq 0$; or (ii) $P_1=0$, $A_1 \neq 0$, and either $2P_2 \neq A_1R_1$ or $S_1 \neq 0$; or (iii) $A_1=0$,

$P_1 \neq 0$, $A_2 \neq 0$, and $2A_2 \neq P_1 B_1$ or $D_1 \neq 0$; or (iv) $P_1 \neq 0$, $A_1 = A_2 = 0$, $A_3 \neq 0$ and $P_1 B_1 D_2 - P_1 D_1 B_2 + 3A_3 D_1 \neq 0$; or (v) $P_1 = A_1 = 0$, $P_2 \neq 0$, $A_2 \neq 0$, $P_2 D_1 \neq A_2 S_1$.

Suppose case (i) holds. By Remark 4.12 we may assume $Q = R = S = B = C = D = 0$.

It follows easily from the implicit function theorem that we can find germs α and $\beta \in \mathcal{E}(1)$, of functions of t, such that α and β are units of $\mathcal{E}(1)$ and such that for all t near 0 we have

$$\text{(c)} \qquad \begin{aligned} \alpha^4(t) &= (P_1\beta(t) + P_2 t\beta^2(t) + P_3 t^2\beta^3(t) + P_4 t^3\beta^4(t))\,\alpha^2(t) \\ &= (A_1\beta(t) + A_2 t\beta^2(t) + A_3 t^2\beta^3(t) + A_4 t^3\beta^4(t))\,\alpha(t) \end{aligned}$$

(simply define a function $F: \mathbf{R}^3 \to \mathbf{R}^2$ by setting

$$F(t, a, b) = (a^4 - (P_1 b + P_2 b^2 t + P_3 b^3 t^2 + P_4 b^4 t^3)a^2,\ a^4 - (A_1 b + A_2 b^2 t + A_3 b^3 t^2 + A_4 b^4 t^3)a);$$

we have $F(0, A_1/P_1, A_1^2/P_1^3) = (0, 0)$ and the matrix

$$\begin{pmatrix} \dfrac{\partial F_1}{\partial a} & \dfrac{\partial F_1}{\partial b} \\ \dfrac{\partial F_2}{\partial a} & \dfrac{\partial F_2}{\partial b} \end{pmatrix} (0, A_1/P_1, A_1^2/P_1^3)$$

is non-singular, so by the implicit function theorem there are germs α and β in $\mathcal{E}(1)$, with $\alpha(0) = A_1/P_1$, $\beta(0) = A_1^2/P_1^3$, such that $F(t, \alpha(t), \beta(t)) = 0$ for t near 0).

In (b) let us replace x by $\alpha(t)x$; t by $\beta(t)t$; u by $(A_1^2/P_1^2)u$; and v by $(A_1^3/P_1^3)v$; and let us divide the resulting unfolding by $\alpha^4(t)$. Since α and β are units of $\mathcal{E}(1)$ we then obtain an unfolding which is space-equivalent to (b), and because of (c) this unfolding will have the form

$$\text{(d)} \qquad x^4 + ((A_1^2/P_1^2)\,u/\alpha^2(t))\,x^2 + tx^2 + ((A_1^3/P_1^3)\,v/\alpha^3(t))\,x + tx$$

(recall that we have set Q, R, S, B, C, D to 0!) Now clearly, for suitably chosen polynomials $Q'(t)$ and $C'(t)$, at most quadratic in t and without a constant term, and for suitably chosen germs γ and δ in $\mathfrak{m}(1)^3$, we may write the unfolding (d) as $x^4 + ux^2 + tx^2 + Q'(t)ux^2 + vx + tx + C'(t)vx + \gamma(t)ux^2 + \delta(t)vx$; moreover this unfolding is space-stable since it is space-equivalent to an unfolding of the form (b) for which condition (i) holds. By virtue of Corollary 4.8, the unfolding (d) is space-equivalent to (e): $x^4 + ux^2 + tx^2 + Q'(t)ux + vx + tx + C'(t)vx$. This unfolding is again of the form (b) and such that condition (i) holds; so by virtue of Remark 4.12 we may replace Q' and C' by 0. In other words, the unfolding (e) is space-equivalent to (f): $x^4 + ux^2 + tx^2 + vx + tx$, since both (e) and (f) are of the form (b), with the same polynomials P and A, and since they are both space-stable (because both satisfy condition (i)). Finally, if in (f) we replace t by $t - v$ and u by $u + v$ we obtain the space-equivalent unfolding h_7'.

Suppose case (ii) holds. By Remark 4.12 we may assume $S_1=1$, $S_2=0$, $Q=R=B=C=D=0$, so the unfolding (b) is $x^4+ux^2+P(t)x^2+wtx^2+vx+A(t)x$. If we set $t'=A(t)$ (since $A_1\neq 0$ the map $t\to t'=A(t)$ is a change of coordinates on $\mathbf{R}$), and if we set $w'=w/A_1$, then for a suitable polynomial $P'(t')$ without constant term, of degree at most 4, and such that $P_1'=0$, and for a suitable real number S_2' and suitable germs $\beta\in\mathfrak{m}(1)^5$ and $\gamma\in\mathfrak{m}(1)^3$ we may write the unfolding above as $x^4+ux^2+P'(t')x^2+w't'x^2+S_2w't'^2x^2+vx+t'x+\beta(t')x^2+w'\gamma(t')x^2$. By Corollary 4.8, Corollary 4.10 and Remark 4.12 this is space-equivalent to $x^4+ux^2+P'(t)x^2+wtx^2+vx+tx$, which is again of the form (b) and such that case (ii) holds. By Remark 4.12 this unfolding is space-equivalent to $x^4+ux^2+P'(t)x^2+2P_2'tvx^2+3P_3't^2vx^2+wtx^2+vx+tx$; since this is space-stable no matter what the value of P_4' we may by Lemma 4.9 assume $P_4'=0$. If we now replace t by $t-v$ and u by $u+wv+P_2'v^2-2P_3'v^3$, we obtain the space-equivalent unfolding $x^4+ux^2+P'(t)x^2+wtx^2-3P_3'v^2tx^2+tx$; replacing w by $w+3P_3'v^2$ gives the space-equivalent unfolding $x^4+ux^2+P'(t)x^2+wtx^2+tx$. We shall show this is space-equivalent to h_8'.

If $\tau_0\in[0, 1]$, define a 5-dimensional unfolding K_{τ_0} of x^4 by setting $K_{\tau_0}(x, t, u, v, w, \tau)=x^4+ux^2+(\tau_0+\tau)P'(t)x^2+wtx^2+tx$ for $x, t, u, v, w, \tau\in\mathbf{R}$. Then $\partial K_{\tau_0}/\partial\tau=P'(t)x^2$. To simplify the notation in the following let us set $\delta=u+(\tau_0+\tau)P'(t)+wt$; then $K_{\tau_0}=x^4+\delta x^2+tx$. Now $4K_{\tau_0}-x\partial K_{\tau_0}/\partial x=2\delta x^2+3tx$. Hence $(4K_{\tau_0}-x\partial K_{\tau_0}/\partial x)^2=4\delta^2x^4+12\delta tx^3+9t^2x^2$. If we subtract $(\delta^2x+3\delta t)\partial K_{\tau_0}/\partial x$ from this we get $-2\delta^3x^2-7\delta^2tx-3\delta t^2+9t^2x^2$; if we add $3\delta t^2$ $(=3\delta t^2(K_{\tau_0})^0)$ to this we get $-2\delta^3x^2-7\delta^2tx+9t^2x^2$; if we add $\delta^2(4K_{\tau_0}-x\partial K_{\tau_0}/\partial x)$ (which equals $\delta^2(2\delta x^2+3tx)$) to this we get $-4\delta^2tx+9t^2x^2$; if to this we add $4\delta^2t\partial K_{\tau_0}/\partial t$ we get $9t^2x^2+4\delta^2wtx^2+8P_2'(\tau_0+\tau)\delta^2t^2x^2+12P_3'(\tau_0+\tau)\delta^2t^3x^2$ (recall $P_4'=0$); if we subtract $2\delta wt(4K_{\tau_0}-x\partial K_{\tau_0}/\partial x)$ $(=2\delta wt(2\delta x^2+3tx))$ from this we get $9t^2x^2-6\delta wt^2x+8P_2'(\tau_0+\tau)\delta^2t^2x^2+12P_3'(\tau_0+\tau)\delta^2t^3x^2$; if we add $6\delta wt^2\partial K_{\tau_0}/\partial t$ we get $9t^2x^2+6\delta w^2t^2x^2+12P_2'(\tau_0+\tau)\delta wt^3x^2+18P_3'(\tau_0+\tau)\delta wt^4x^2+8P_2'(\tau_0+\tau)\delta^2t^2x^2+12P_3'(\tau_0+\tau)\delta^2t^3x^2$; dividing this by $9+6\delta w^2+12P_2'(\tau_0+\tau)\delta wt+18P_3'(\tau_0+\tau)\delta wt^2+8P_2'(\tau_0+\tau)\delta^2+12P_3'(\tau_0+\tau)\delta^2t$ (which is a germ of a function of t, u, w, and τ and is non-zero near 0) gives t^2x^2. We have shown that t^2x^2 is an element of

$$\langle x\partial K_{\tau_0}/\partial x,\ t\partial K_{\tau_0}/\partial x\rangle_{\mathcal{E}(6)}+\langle t\partial K_{\tau_0}/\partial t\rangle_{\mathcal{E}(5)}+\langle 1,\ K_{\tau_0},\ K_{\tau_0}^2\rangle_{\mathcal{E}(5)}$$

(here $\mathcal{E}(5)$ is the ring of germs at 0 of smooth functions of t, u, v, w, and τ); consequently t^3x^2 is also an element of the same sum of $\mathcal{E}(5)$ submodules of $\mathcal{E}(6)$; hence $\partial K_{\tau_0}/\partial\tau=P_2't^2x^2+P_3't^3x^2$ is also an element of this sum of modules, and this is true for any $\tau_0\in[0, 1]$. From this it follows by Lemma 4.5 that $x^4+ux^2+P'(t)x^2+wtx^2+tx$ is space-equivalent to $x^4+ux^2+wtx^2+tx$, which is clearly space-equivalent to h_8'.

If case (iii) holds, then by arguments similar to those used for case (i), the unfolding

(b) is space-equivalent to h_9'. Similarly in case (iv) the unfolding (b) is space-equivalent to h_{10}'.

In case (v) one may by Remark 4.12 assume $D_1=1$, $D_2=0$, $Q=R=S=B=C=0$. By the implicit function theorem one can find units α and β of $\mathcal{E}(1)$ such that

$$\alpha^4(t) = \pm(P_2\beta^2(t)+P_3t\beta^3(t)+P_4t^2\beta^4(t))\,\alpha^2(t) = (A_2\beta^2(t)+A_3t\beta^3(t)+A_4t^2\beta^4(t))\,\alpha(t)$$

for t near 0, where the $\pm$ sign is taken to be $+$ if $P_2>0$ and $-$ if $P_2<0$. By continuing the argument as in case (i) one can easily show that in case (v) the unfolding (b) is space-equivalent to h_{11}' or to h_{12}'.

It can easily be shown, by arguments similar to those used in the proof of Theorem 5.2, that no two of the unfoldings $h_7'-h_{12}'$ are space-equivalent.

By using the usual algorithm and applying the observation at the beginning of the continuation of the proof of Theorem 5.3, one readily computes that an arbitrary space-stable unfolding of $\mu_3(x)=x^5$ is space-equivalent to an unfolding of the form

$$\text{(g)}\qquad x^5+ux^3+P(t)x^3+Q(t)ux^3+R(t)vx^3+S(t)wx^3+vx^2+A(t)x^2$$
$$+B(t)ux^2+C(t)vx^2+D(t)wx^2+wx+E(t)x+F(t)ux+G(t)vx+H(t)wx,$$

where P, Q, R, S, A, B, C, D, E, F, G and H are polynomials in t without constant term, P, A, and E of degree at most 4, the others at most quadratic, and where the coefficients of these polynomials satisfy the following conditions: $E_1\neq 0$ and $4P_1E_1\neq 3A_1^2$ and the determinant

$$\begin{vmatrix} P_1 & A_1 & E_1 & 2P_2 & 2A_2 & 2E_2 \\ 0 & 0 & 0 & 2P_1 & A_1 & 0 \\ 0 & 0 & 0 & -P_1 & 0 & E_1 \\ 1 & 0 & 0 & Q_1 & B_1 & F_1 \\ 0 & 1 & 0 & R_1 & C_1 & G_1 \\ 0 & 0 & 1 & S_1 & D_1 & H_1 \end{vmatrix} \neq 0.$$

Suppose first $A_1\neq 0$. By Remark 4.12 we may assume $Q=S=B=C=D=F=G=H=0$, $R_2=0$, and we may choose R_1 to have any value such that the determinant above does not vanish (note that by choosing R_1 appropriately we can in fact ensure that the determinant does not vanish). By the inverse function theorem there are units α and $\beta\in\mathcal{E}(1)$ such that

$$\alpha^5(t) = (A_1\beta(t)+A_2\beta^2(t)t+A_3\beta^3(t)t^2+A_4\beta^4(t)t^3)\,\alpha^2(t)$$
$$= (E_1\beta(t)+E_2\beta^2(t)t+E_3\beta^3(t)t^2+E_4\beta^4(t)t^3)\,\alpha(t)$$

for t near 0.

By the same argument as was used in case (i) for the unfoldings (b) of x^4, it follows that the unfolding (g) (when $A_1 \neq 0$) is space-equivalent to an unfolding of the form

(h) $$x^5 + ux^3 + P'(t)x^3 + R_1' tvx^3 + vx^2 + tx^2 + wx + tx,$$

where P' is a polynomial in t without constant term, of degree at most 4, and R_1' is a real number, and where (since this unfolding is of the form (g) and is space-stable) we have $P_1' \neq \frac{3}{4}$ and $R_1' \neq 2P_2'$.

We shall show that any space-stable unfolding of the form (h) is space-equivalent to an unfolding of the same form in which $P_2' = P_3' = P_4' = 0$. First, since (h) is stable no matter what the value of P_4', we may by Lemma 4.9 assume $P_4' = 0$. By Remark 4.12 we may assume that $R_1' \neq 0$ and that R_1' is of opposite sign to P_2' if $P_2' \neq 0$. Now suppose P_2' and P_3' are not both zero. For $\tau_0 \in [0, 1]$, define a 4-dimensional unfolding H_{τ_0} of x^5 by setting

$$H_{\tau_0}(x, t, u, v, w) = x^5 + ux^3 + P_1' tx^3 + \tau_0(P_2' t^2 + P_3' t^3)x^3 + R_1' tvx^3 + vx^2 + tx^2 + wx + tx,$$

and define a 5-dimensional unfolding K_{τ_0} of x^5 by setting

$$K_{\tau_0}(x, t, u, v, w, \tau) = H_{\tau_0+\tau}(x, t, u, v, w), \text{ for } x, t, u, v, w, \tau \in \mathbf{R}.$$

Because of the way we chose R_1', we have that H_{τ_0} is space-stable for each $\tau_0 \in [0, 1]$. Because K_{τ_0} is a 1-dimensional unfolding of H_{τ_0} it follows easily from this, using Corollary 1.7, that

$$\mathcal{E}(6) = \langle \partial K_{\tau_0}/\partial x \rangle_{\mathcal{E}(6)} + \langle \partial K_{\tau_0}/\partial t \rangle_{\mathcal{E}(5)} + \langle \partial K_{\tau_0}/\partial u, \partial K_{\tau_0}/\partial v, \partial K_{\tau_0}/\partial w \rangle_{\mathcal{E}(4)} + \bar{K}_{\tau_0}^* \mathcal{E}(6),$$

where $\mathcal{E}(6)$ is the ring of germs of functions of x, t, u, v, w, and τ, and $\mathcal{E}(5) \subseteq \mathcal{E}(6)$ consists of those germs which do not depend on x, $\mathcal{E}(4)$ consists of those germs which do not depend on x nor t, and where $\bar{K}_{\tau_0} \in \mathcal{E}(6, 6)$ is defined by setting $\bar{K}_{\tau_0}(x, t, u, v, w, \tau) = (K_{\tau_0}(x, t, u, v, w, \tau), t, u, v, w, \tau)$ for $x, t, u, v, w, \tau \in \mathbf{R}$.

In particular, we can find germs $\gamma(x, t, u, v, w, \tau) \in \mathcal{E}(6)$; $\delta(t, u, v, w, \tau) \in \mathcal{E}(5)$; $\varepsilon(u, v, w, \tau)$, $\zeta(u, v, w, \tau)$, and $\eta(u, v, w, \tau) \in \mathcal{E}(4)$; and germs $\varkappa(t, u, v, w, \tau)$, $\lambda(t, u, v, w, \tau) \in \mathcal{E}(5)$ and $\mu(a, t, u, v, w, \tau) \in \mathcal{E}(6)$ such that

(i) $$\frac{\partial K_{\tau_0}}{\partial \tau} = P_2' t^2 x^3 + P_3' t^3 x^3 = \gamma(x, t, u, v, w, \tau) \frac{\partial K_{\tau_0}}{\partial x} + \delta(t, u, v, w, \tau) \frac{\partial K_{\tau_0}}{\partial t}$$
$$+ \varepsilon(u, v, w, \tau) \frac{\partial K_{\tau_0}}{\partial u} + \zeta(u, v, w, \tau) \frac{\partial K_{\tau_0}}{\partial v} + \eta(u, v, w, \tau) \frac{\partial K_{\tau_0}}{\partial w}$$
$$+ \varkappa(t, u, v, w, \tau) + \lambda(t, u, v, w, \tau) K_{\tau_0} + \mu(K_{\tau_0}, t, u, v, w, \tau) K_{\tau_0}^2.$$

Now we claim that when x, t, u, v, and w are 0, then the germs γ, δ, ε, ζ and η vanish (no matter what the value of τ), i.e. we claim $\gamma \in \mathfrak{m}(5)\mathcal{E}(6)$, $\delta \in \mathfrak{m}(4)\mathcal{E}(5)$ and ε, ζ and η are in $\mathfrak{m}(3)\mathcal{E}(4)$.

To prove this claim, we differentiate equation (i) and evaluate the resulting equation for $x=t=u=v=w=0$ (after substituting the correct expressions for K_{τ_0} and its derivatives). If we differentiate once, twice, ..., etc., up to seven times with respect to x, and evaluate for $x=t=u=v=w=0$, we find respectively that:

(j_1) $\delta(0,\tau)+\eta(0,\tau)=0$; ($j_2$) $2\delta(0,\tau)+2\zeta(0,\tau)=0$;

(j_3) $6P_1'\delta(0,\tau)+6\varepsilon(0,\tau)=0$; ($j_4$) $120\gamma(0,\tau)=0$;

(j_5) $600\partial\gamma/\partial x(0,\tau)+120\,\lambda(0,\tau)=0$; ($j_6$) $1800\,\partial^2\gamma/\partial x^2(0,\tau)=0$; and

(j_7) $4200\,\partial^3\gamma/\partial x^3(0,\tau)=0$.

If we differentiate equation (i) once with respect to t and then once, twice, or thrice with respect to x and evaluate when $x=t=u=v=w=0$, we find respectively:

(j_8) $$\partial\gamma/\partial x(0,\tau)+2\gamma(0,\tau)+\partial\delta/\partial t(0,\tau)+\lambda(0,\tau)=0;$$

(j_9) $$6P_1'\gamma(0,\tau)+4\partial\gamma/\partial x(0,\tau)+\partial^2\gamma/\partial x^2(0,\tau)+2\partial\delta/\partial t(0,\tau)+2\lambda(0,\tau)=0;$$

(j_{10}) $$18P_1'\partial\gamma/\partial x(0,\tau)+6\partial^2\gamma/\partial x^2(0,\tau)+\partial^3\gamma/\partial x^3(0,\tau)+6P_1'\partial\delta/\partial t(0,\tau)$$
$$+12(\tau_0+\tau)P_2'\delta(0,\tau)+6R_1'\zeta(0,\tau)+6P_1'\lambda(0,\tau)=0.$$

Now by (j_4) we have $\gamma(0,\tau)=0$, i.e. $\gamma \in \mathfrak{m}(5)\mathcal{E}(6)$. By ($j_5$) we have $\lambda(0,\tau)= -5\partial\gamma/\partial x(0,\tau)$. Substituting into ($j_8$) we find $\partial\delta/\partial t(0,\tau)=4\partial\gamma/\partial x(0,\tau)$. By ($j_6$) $\partial^2\gamma/\partial x^2(0,\tau)=0$; by ($j_9$) we now find $\partial\gamma/\partial x(0,\tau)=0$, and hence also $\partial\delta/\partial t(0,\tau)=\lambda(0,\tau)=0$. Also, by ($j_7$) we have $\partial^3\gamma/\partial x^3(0,\tau)=0$. Hence ($j_{10}$) reduces to: $12(\tau_0+\tau)P_2'\delta(0,\tau)+6R_1'\zeta(0,\tau)=0$. But by ($j_2$) we have $\delta(0,\tau)+\zeta(0,\tau)=0$, and since R_1' was so chosen that for *no* $(\tau_0+\tau)\in[0,1]$ is $12(\tau_0+\tau)P_2'=6R_1'$, it follows from the last two equations that $\delta(0,\tau)=\zeta(0,\tau)=0$ (i.e. $\delta\in\mathfrak{m}(4)\mathcal{E}(5)$ and $\zeta\in\mathfrak{m}(3)\mathcal{E}(4)$); and by ($j_1$) and ($j_3$) it now follows that η and ε are in $\mathfrak{m}(3)\mathcal{E}(4)$. This proves the claim.

Since the claim above holds for any $\tau_0\in[0,1]$ it follows by Lemma 4.5 that H_1 (which is the unfolding (h) with $P_4'=0$) is space-equivalent to $H_0=x^5+ux^3+P_1'tx^3+R_1'tvx^3+vx^2+tx^2+wx+tx$. By Remark 4.12 we may assume $R_1'=1$. If we replace t by $t-v$ and w by $w+v$ and u by $u+P_1'v+v^2$, we then get the space-equivalent unfolding $x^5+ux^3+P_1'tx^3+tvx^3+tx^2+wx+tx$, which is a member of the continuous family h_{13}' of space-stable unfoldings. This takes care of the case $A_1\neq 0$.

Let us now consider the unfolding (g) when $A_1=0$. The conditions for (g) to be space-stable now reduce to the following: $E_1 \neq 0$, $P_1 \neq 0$, and $2A_2 \neq P_1 B_1 + E_1 D_1$. By Remark 4.12 we may assume $Q=R=S=B=C=F=G=H=0$, $D_2=0$, and $D_1 \neq 2A_2/E_1$.

By the inverse function theorem we can find units α and β in $\mathcal{E}(1)$ so that for t near 0

$$\alpha^5(t) = \pm(P_1\beta(t)+P_2\beta^2(t)t+P_3\beta^3(t)t^2+P_4\beta^4(t)t^3)\,\alpha^3(t)$$
$$= (E_1\beta(t)+E_2\beta^2(t)t+E_3\beta^3(t)t^2+E_4\beta^4(t)t^3)\,\alpha(t),$$

where the $\pm$ sign is taken to be the sign of P_1/E_1.

By the same argument as was used above in the case $A_1 \neq 0$, it follows that the unfolding (g) (when $A_1=0$) is space-equivalent to an unfolding of the form

$$\text{(k)} \qquad x^5+ux^3\pm tx^3+vx^2+A'(t)x^2+D_1'twx^2+wx+tx,$$

where $A'(t)$ is a polynomial in t without constant term, of degree at most 4, and with $A_1'=0$, and where D_1' is a real number and $D_1' \neq 2A_2'$. By Remark 4.12 we may assume $D_1' \neq 0$ and is of opposite sign to A_2' if $A_2' \neq 0$.

We may then argue as we did for the unfoldings of the form (h), to show that in (k) we may assume $A'=0$. By Remark 4.12 we may then set $D_1'=1$. This gives the unfoldings $x^5+ux^3\pm tx^3+vx^2+twx^2+wx+tx$, and if in these we replace t by $t-w$; v by $v+w^2$; and u by $u\pm w$ we get h_{14}' or h_{15}' (depending on the $\pm$ sign).

By the same arguments used before one can easily show that none of the unfoldings in the family h_{13}' is space-equivalent to h_{14}' or h_{15}' ,that h_{14}' and h_{15}' are not space-equivalent to each other, and that *no two of the unfoldings* h_{13_c}', *for different values of c, are space-equivalent to each other* (so x^5 has infinitely many non-equivalent space-stable unfoldings!). In other words the space-stable unfoldings of x^5 given in the list are all essentially different.

We claim that $\mu_4(x)=x^6$ has no space-stable unfoldings. This can easily be seen by a dimension argument as follows: By virtue of Theorem 4.11 and Theorem 2.26, if x^6 has a space-stable unfolding, then it has a space-stable unfolding f of the form

$$\text{(l)} \quad f(x,t,u,v,w) = x^6+P(t,u,v,w)x^4+Q(t,u,v,w)x^3+R(t,u,v,w)x^2+S(t,u,v,w)x$$

where P, Q, R, and S are suitable polynomials in t, u, v, and w, without constant terms.

Moreover, since f is space-stable, we must have, by virtue of Corollary 4.13, that

$$\text{(m)} \qquad \mathcal{E}(2) = \langle \partial f/\partial x \,|\, \mathbf{R}^2\rangle_{\mathcal{E}(2)} + \langle \partial f/\partial t \,|\, \mathbf{R}^2\rangle_{\mathcal{E}(1)} + \langle \partial f/\partial u \,|\, \mathbf{R}^2, \partial f/\partial v \,|\, \mathbf{R}^2, \partial f/\partial w \,|\, \mathbf{R}^2\rangle_{\mathbf{R}}$$
$$+\langle 1, f\,|\,\mathbf{R}^2, f^2\,|\,\mathbf{R}^2, f^3\,|\,\mathbf{R}^2\rangle_{\mathcal{E}(1)} + \langle t^4\rangle_{\mathcal{E}(2)}$$

(where $\mathbf{R}^2$ denotes the x, t-plane (i.e. $u=v=w=0$), $\mathcal{E}(2)$ is the ring of germs of smooth functions of x and t, and $\mathcal{E}(1)$ is the ring of germs of smooth functions of t).

Now let $C = \mathcal{E}(2)/(\langle \partial f/\partial x | \mathbf{R}^2\rangle_{\mathcal{E}(2)} + \langle t^4\rangle_{\mathcal{E}(2)})$ and let $\pi: \mathcal{E}(2) \to C$ be the projection. Because f has the form (l) it is clear that the 20 elements $\pi(t^i x^j)$, $0 \leqslant i \leqslant 3$, $0 \leqslant j \leqslant 4$, form a basis of C over $\mathbf{R}$, so $\dim_{\mathbf{R}} C = 20$. Let us consider the germs $f^j | \mathbf{R}^2$, $j = 0, 1, 2, 3$. Clearly $(6f - x\partial f/\partial x) | \mathbf{R}^2 = (2Px^4 + 3Qx^3 + 4Rx^2 + 5Sx) | \mathbf{R}^2$ is divisible by t, so $\pi(f | R^2)$ can be written as a sum of monomials divisible by t. It follows that $\pi(f^2 | \mathbf{R}^2)$ is a sum of monomials divisible by t^2 and $\pi(f^3 | \mathbf{R}^2)$ is a sum of monomials divisible by t^3. But since any germ divisible by t^4 is in the kernel of π, this implies $\dim_{\mathbf{R}} \pi(\langle f^j | \mathbf{R}^2\rangle_{\mathcal{E}(1)}) \leqslant 4 - j$, for $0 \leqslant j \leqslant 3$. Hence $\dim_{\mathbf{R}} \pi(\langle 1, f | \mathbf{R}^2, f^2 | \mathbf{R}^2, f^3 | \mathbf{R}^2\rangle_{\mathcal{E}(1)}) \leqslant 4 + 3 + 2 + 1 = 10$. Moreover $\dim_{\mathbf{R}} \pi(\langle \partial f/\partial t | \mathbf{R}^2\rangle_{\mathcal{E}(1)}) \leqslant 4$ and $\dim_{\mathbf{R}} \pi(\partial f/\partial u | \mathbf{R}^2, \partial f/\partial v | \mathbf{R}^2, \partial f/\partial w | \mathbf{R}^2\rangle_{\mathbf{R}}) \leqslant 3$. So if (m) held we would have $\dim_{\mathbf{R}} C \leqslant 10 + 4 + 3 = 17 < 20$, which is impossible. Therefore f cannot be space-stable and x^6 has no space-stable unfoldings.

We shall show that $\mu_5(x, y) = x^3 + y^3$ has no space-stable unfoldings. For suppose μ_5 has a space-stable unfolding $f(x, y, t, u, v, w)$, and let $f_0 = f|_{u=v=w=0}$. By virtue of Theorem 4.11 and the observation at the beginning of the continuation of the proof of Theorem 5.3, we may clearly choose f such that f_0 has the form

$$f_0(x, y, t) = x^3 + y^3 + P(t)xy + A(t)x + E(t)y$$

for suitable polynomials P, A, and E without constant term and of degree at most 4. Let $D = \mathcal{E}(3)/(\langle t^3\rangle_{\mathcal{E}(3)} + \mathfrak{m}(3)^{12} + \langle 1, t, t^2\rangle_{\mathbf{R}})$ and let $\pi_1: \mathcal{E}(3) \to D$ be the projection. Let M be the subspace of D generated over $\mathbf{R}$ by the elements $\pi_1(xt^i)$, $\pi_1(yt^i)$ and $\pi_1(xyt^i)$, for $i = 0, 1, 2$. We claim $\pi_1(\langle \partial f_0/\partial x, \partial f_0/\partial y\rangle_{\mathcal{E}(3)}) \cap M = \{0\}$.

For let α and β be in $\mathcal{E}(3)$ and suppose $\pi_1(\alpha\partial f_0/\partial x + \beta\partial f_0/\partial y) \in M$. Since $\pi_1(\mathfrak{m}(3)^{12} + \langle t^3\rangle_{\mathcal{E}(3)}) = 0$, we may assume α and β are polynomials in x, y, and t, and we may write

$$\alpha(x, y, t) = \sum_{i,j \in \mathbf{Z}} \alpha_{ij}(t) x^i y^j; \quad \beta(x, y, t) = \sum_{i,j \in \mathbf{Z}} \beta_{ij}(t) x^i y^j,$$

where α_{ij} and β_{ij} are suitable polynomials in t and all but finitely many of them are 0, and where, of course, we set $\alpha_{ij} = \beta_{ij} = 0$ if either i or j is negative. Let us denote by $f_{ij}(t)$ the coefficient of $x^i y^j$ in $\alpha\partial f_0/\partial x + \beta\partial f_0/\partial y$ (so f_{ij} is a polynomial in t). Then we have, for all i, j, that

(n) $$f_{ij} = 3\alpha_{i-2,j} + P\alpha_{i,j-1} + A\alpha_{i,j} + 3\beta_{i,j-2} + P\beta_{i-1,j} + E\beta_{i,j}.$$

Since $\pi_1(\alpha\partial f_0/\partial x + \beta\partial f_0/\partial y) \in M$ we know f_{ij} is divisible by t^3 whenever $i \geqslant 2$ or $j \geqslant 2$, and $i + j < 12$. To show $\pi_1(\alpha\partial f_0/\partial x + \beta\partial f_0/\partial y) = 0$, we must show $f_{0,1}$, $f_{1,0}$ and $f_{1,1}$ are divisible by t^3.

Now when $i \geqslant 2$ or $j \geqslant 2$, $i+j<12$, we may solve equation (n) to find that

(o) $\alpha_{i-2,j}+\beta_{i,j-2} = -\frac{1}{3}(P\alpha_{i,j-1}+A\alpha_{i,j}+P\beta_{i-1,j}+E\beta_{i,j})$ modulo t^3 ($i \geqslant 2$ or $j \geqslant 2$, $i+j<12$).

By repeatedly substituting from equations (o) into the equations (n) for $f_{0,1}$, $f_{1,0}$ and $f_{1,1}$, one easily finds, because P, A and E are divisible by t, that $f_{0,1}$, $f_{1,0}$, and $f_{1,1}$ are divisible by t^3. In other words, $\pi_1(\alpha\partial f_0/\partial x+\beta\partial f_0/\partial y)=0$, and our claim has been proved.

Now let $D'=D/\pi_1(\langle\partial f_0/\partial x, \partial f_0/\partial y\rangle_{\mathcal{E}(3)})$ and let π_2: $\mathcal{E}(3)\to D'$ be the projection. From the form of f_0 and because of what we have just proved above it is clear that the 9 elements $\pi_2(xt^i)$, $\pi_2(yt^i)$ and $\pi_2(xyt^i)$, $i=0, 1, 2$, form an $\mathbf{R}$-basis of D'.

Corollary 4.13 clearly implies that for f to be space-stable it is necessary that D' be generated over $\mathbf{R}$ by the following 9 elements: the 3 elements $\pi_2(t^i\partial f_0/\partial t)$, for $i=0, 1, 2$; the three elements $\pi_2(\partial f/\partial u\,|_{u=v=w=0})$, $\pi_2(\partial f/\partial v\,|_{u=v=w=0})$, and $\pi_2(\partial f/\partial w\,|_{u=v=w=0})$; the two elements $\pi_2(f_0)$ and $\pi_2(f_0 t)$ (note that $3f_0-x\partial f_0/\partial x-y\partial f_0/\partial y=Pxy+2Ax+2Ey$, so $\pi_2(f_0)$ is a linear combination of monomials divisible by t; hence $\pi_2(f_0t^2)=0$); and finally the element $\pi_2(f_0^2)$ (note that since $\pi_2(f_0)$ is a sum of terms divisible by t, it follows readily that $\pi_2(f_0^2)$ is a linear combination of terms divisible by t^2, so $\pi_2(f_0^2t)=0$).

But if we denote by P_1, A_1, and E_1 the coefficient of t in P, A, and E respectively, then in terms of the basis of D' given by $\pi_2(xt^i)$, $\pi_2(yt^i)$ and $\pi_2(xyt^i)$, $i=0, 1, 2$, we have:

$$\pi_2(t^2\partial f_0/\partial t) = P_1xyt^2+A_1xt^2+E_1yt^2,$$

$$\pi_2(3f_0t) = P_1xyt^2+2A_1xt^2+2E_1yt^2,$$

and as is easily seen:

$$\pi_2(9f_0^2) = 4A_1E_1xyt^2,$$

and as these three elements of D' can clearly not be linearly independent, no matter what P, A, and E are, it follows that the 9 elements listed in the preceding paragraph cannot generate D' (since $\dim_{\mathbf{R}} D'=9$), so f is not space-stable. Hence x^3+y^3 has no space-stable unfoldings.

A similar argument shows that $\mu_6(x, y)=x^3-xy^2$ has no space-stable unfoldings.

Finally, $\mu_7(x, y)=x^2y+y^4$ has no space-stable unfoldings. This can be seen most easily as follows: Suppose f is a representative, defined near $0\in\mathbf{R}^6$, of a space-stable unfolding of μ_7. Let g be a representative of the 4-stable unfolding g_7 of μ_7 given in the list of Theorem 2.26 (recall $g_7(x, y, t, u, v, w)=x^2y+y^4+ux^2+vy^2+wx+ty$). Then for $(x, y, t, u, v, w)\in\mathbf{R}^6$ sufficiently near 0, we may consider the germs $f_{(x,y,t,u,v,w)}$ and $g_{(x,y,t,u,v,w)}\in\mathfrak{m}(6)$ (see Definition 2.3).

If $a \in \mathbf{R}$ is close enough to 0, then we have that $g_{(0,a,8a^3,-a,-6a^2,0)}(x,y,0,0,0,0)=x^2y+y^4+4ay^3$. Suppose $a<0$. Introduce new coordinates x', y' on $\mathbf{R}^2$ near 0 by setting $x'=y(4a+y)^{1/3}$ and $y'=x(-1/(4a+y))^{1/6}$. Since $a \neq 0$ this is a smooth change of coordinates on $\mathbf{R}^2$ near 0, and clearly $x^2y+y^4+4ay^3=x'^3-x'y'^2$. So for $a<0$ but near 0, we have that $g_{(0,a,8a^3,-a,-6a^2,0)}|\mathbf{R}^2$ is equivalent to $\mu_6(x,y)=x^3-xy^2$.

By assumption the germ $f_{(0,0,0,0,0,0)}$ is space-stable, hence 4-stable, and hence 4-equivalent to the unfolding $g_{(0,0,0,0,0,0)}$. This clearly implies that for any point $z \in \mathbf{R}^6$ near 0, there is a $z' \in \mathbf{R}^6$ near 0 such that the germ $g_z|\mathbf{R}^2 \in \mathfrak{m}(2)$ is equivalent to $f_{z'}|\mathbf{R}^2 \in \mathfrak{m}(2)$. In particular it follows that there are $z' \in \mathbf{R}^6$ arbitrarily close to 0 such that $f_{z'}|\mathbf{R}^2$ is equivalent to the germ $\mu_6 \in \mathfrak{m}(2)$. Now since $f_{(0,0,0,0,0,0)}$ is space-stable, equation 3.6 (c) holds for this unfolding, in fact for any q. But this equation is an algebraic condition on the q-jet of the unfolding, so clearly this equation will also hold for the unfolding f_z if $z \in \mathbf{R}^6$ is close enough to 0 (how close depends on q, of course). In any event, it follows that for any q there is a $z \in \mathbf{R}^6$ near 0 such that $f_z|\mathbf{R}^2$ is equivalent to μ_6 and such that equation 3.6 (c) holds for f_z. If q was chosen large enough, it follows f_z is space-stable, contradicting the fact that μ_6 has no space-stable unfolding. This proves that μ_7 cannot have a space-stable unfolding.

This completes the proof of Theorems 5.2 and 5.3.

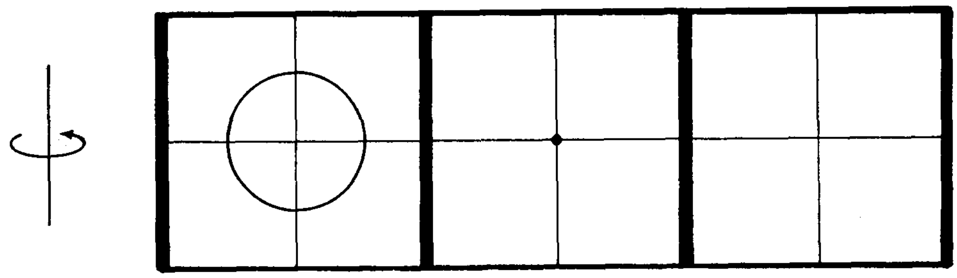
Remark. The family h'_{18} of space-stable unfoldings of x^5 in the list of Theorem 5.3 shows that for suitable r and s it is possible for a germ to have infinitely many (in fact, continuously many) nonequivalent (r, s)-stable unfoldings.

§ 6. Pictures of the time-stable unfoldings

In this section we present drawings of the unfoldings in the list of Theorem 5.2.

The drawings show the *bifurcation set* of the unfoldings, which is defined as follows: Suppose $f \in \mathfrak{m}(n+r)$ is an r-dimensional unfolding. Let $\pi: \mathbf{R}^{n+r} \to \mathbf{R}^r$ be the projection, and let $S \subseteq \mathbf{R}^{n+r}$ be the germ at 0 of the set $\{(x,u) \in \mathbf{R}^{n+r}: f|\mathbf{R}^n \times \{u\}$ has a degenerate singularity at $(x,u)\}$. Then the bifurcation set of f is the set-germ $\pi(S)$, which is the germ at 0 of a subset of $\mathbf{R}^r$.

Remark. In the language of catastrophe theory (see the beginning of § 5), if the unfolding f locally represents a gradient model for some natural process, then the catastrophe set of the gradient model (which describes what an observer would actually see in nature) will in general not be the same locally as the bifurcation set of f, so the pictures we give



$t<0$ $t=0$ $t>0$

Fig. 1 Bubble collapse: the bifurcation set of the unfolding h_2.

here do not actually show what happens in nature if we use the time-stable unfoldings as models for natural processes in catastrophe theory. However, the catastrophe set of a model is not uniquely determined; the catastrophe set can be chosen in one of several ways depending on the application. Therefore we cannot actually draw the catastrophe set of the time-stable unfoldings. However, pictures of the bifurcation set are useful because they provide a great deal of information about what the catastrophe sets can look like, and hence they enable one to see which natural processes can be described locally by which unfoldings in the list. In this sense our pictures can be interpreted as showing what happens in nature during a process whose local model is a given time-stable unfolding.

The pictures we give are in the form of a simplified cine-film, each frame of which shows the spatial configuration of the bifurcation set at a fixed moment of time. A frame taken at time t of one of these films shows a two-dimensional section of the bifurcation set rather than the actual 3-dimensional spatial configuration at time t; this lends clarity and exactness to the pictures, since we have been able to plot the actual curves to scale, and the actual 3-dimensional configuration can easily be visualized from the two-dimensional sections.

Each frame shows the coordinate axes as well as the actual curves. Where it is meaningful the scale of the pictures is given next to the films.

We do not give pictures of the unfoldings h_1, h_6, h_{10}, h_{11} or h_{12}, since these unfoldings (and their bifurcation sets) are constant in time, and pictures of their bifurcation sets at any fixed moment of time are well known and have been published elsewhere (see for example Thom [7] or Woodcock and Poston [12]).

The folds h_2-h_5 have as bifurcation sets the set-germs $\{(u, v, w, t)\,|\,t\pm u^2\pm v^2\pm w^2=0\}$, where the choice of the $\pm$ signs depends on which unfolding we are considering, and corresponds to the signs of the terms u^2x, v^2x and w^2x in the unfolding in question.

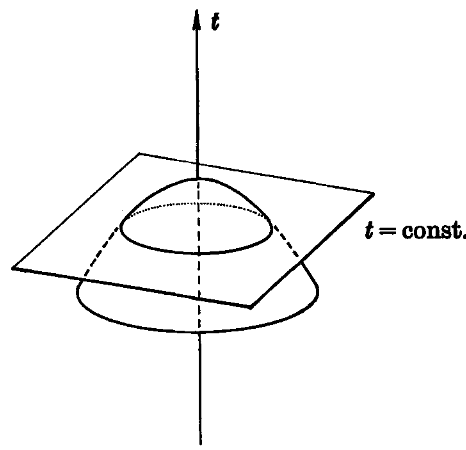


Fig. 2 The bifurcation set of the unfolding h_2.

Figure 1 is a film of the bifurcation set of h_2; each frame shows a plane section through the origin of $\mathbf{R}^3$ for a fixed value of t. The three-dimensional configuration at each fixed time can be obtained by rotating the frames about any line through the origin. When t is negative, the bifurcation set is a sphere (colloquially, a bubble). The spheres become smaller with increasing time until they are reduced to a point (at $t=0$) and vanish. For positive t the bifurcation set is empty.

The film does not show the rate of collapse of the bubbles. Instead of making a film we can draw a picture of a 3-dimensional section of the bifurcation set in which we include two spatial coordinates and the time coordinate (we set one of the spatial coordinates $=0$). Such a section is shown in figure 2: we see a paraboloid of revolution, whose axis is normal to the planes of constant time. In this picture we see how the size of the bubbles changes with time; the radius of the bubble at time t is $\sqrt{-t}$. In particular, when the bubble vanishes, at $t=0$, it is collapsing with infinite velocity.

A picture of the bifurcation set of h_5 can be obtained by running the film in figure 1 backwards at the same speed.

Figure 3 shows a film of the bifurcation set of h_3; each frame shows a section by the plane $v=0$, and the actual 3-dimensional configuration can be obtained by rotating about the vertical (w-) axis. At each time $t \neq 0$ the bifurcation set is a hyperboloid of revolution (the plane sections shown in the film are hyperbolas); at $t=0$ the bifurcation set is a cone. If, for $t<0$, we visualize the region of space enclosed by the hyperboloid as being a blob of liquid, then as time progresses this blob is pinched until, after $t=0$, it separates into two blobs which move apart; hence the name "fission". The apices of the hyperbolas shown

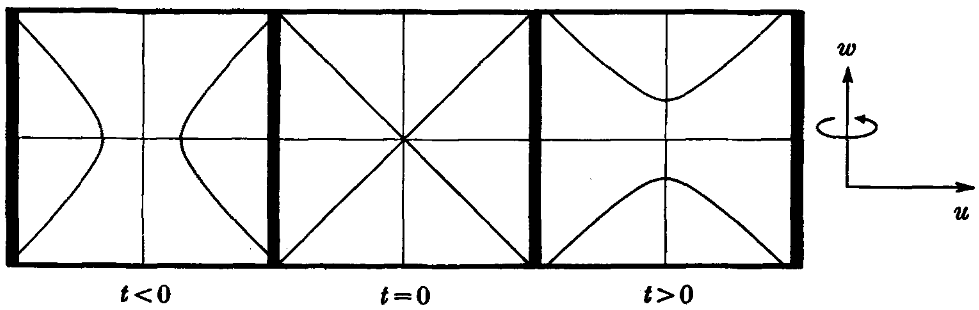


Fig. 3 Fission: the bifurcation set of the unfolding h_3.

in figure 3 for time t are at a distance $\sqrt{|t|}$ from the origin; in particular the "pinch" in the bifurcation set for negative t narrows with infinite velocity at time $t=0$, and the two components which appear when t is positive separate with infinite velocity at time $t=0$.

A film of the bifurcation set of h_4 can be obtained by rotating the frames of figure 3 about the horizontal (u-) axis rather than the vertical (w-) axis, or equivalently, as far as the bifurcation set is concerned, by running the film in figure 3 backwards at the same speed.

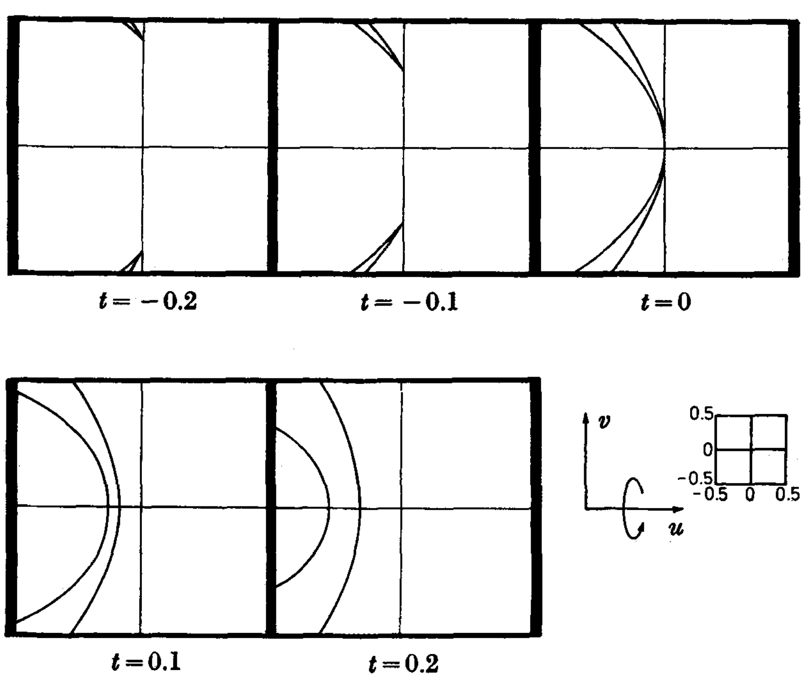


Fig. 4 Bec-à-bec: the bifurcation set of the unfolding h_7.

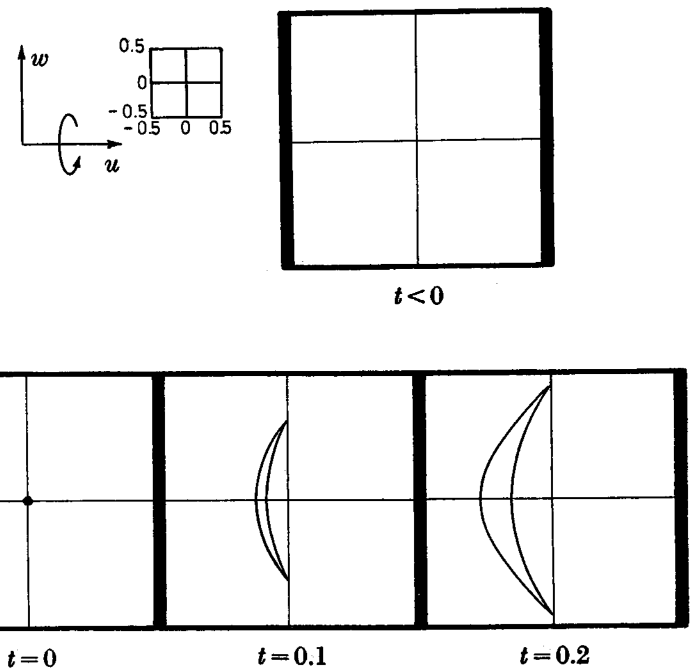


Fig. 5 The lip: the bifurcation set of the unfolding h_9.

The cusps $h_7 - h_9$ have as bifurcation sets the set-germs $\{(u, v, w, t) \mid t + (4/(3\sqrt{6}))u\sqrt{-u} + u \pm v^2 \pm w^2 = 0\}$, where the sign of the terms $\pm v^2$ and $\pm w^2$ is to be chosen to correspond to the sign of the terms v^2x and w^2x resp. in the unfolding in question.

Figure 4 shows the bifurcation set of the "bec-à-bec" unfolding h_7; each frame shows a section of the bifurcation set at some fixed time by the plane $w = 0$. When t is negative we see two cusps; as time progresses they approach, join at $t = 0$, and subsequently two smooth curves appear which move apart and away from the origin. The points of the cusps seen when $t < 0$ lie at a distance $\sqrt{-t}$ from the origin. In particular, the cusps are moving together at infinite velocity when they join. However, the two curves which form when $t > 0$ move away from the origin with finite non-zero velocity at $t = 0$ (the actual velocity is 1 in the scale we have chosen) and move away from each other with zero velocity when $t = 0$.

The above discussion refers to the plane sections of the bifurcation set shown in figure 4; the actual bifurcation set of h_7 is obtained by rotating about the horizontal (u-) axis.

Figure 5 shows the bifurcation set of the "lip" unfolding h_9; each frame shows a section of the bifurcation set at some fixed time by the plane $v = 0$. When t is negative the

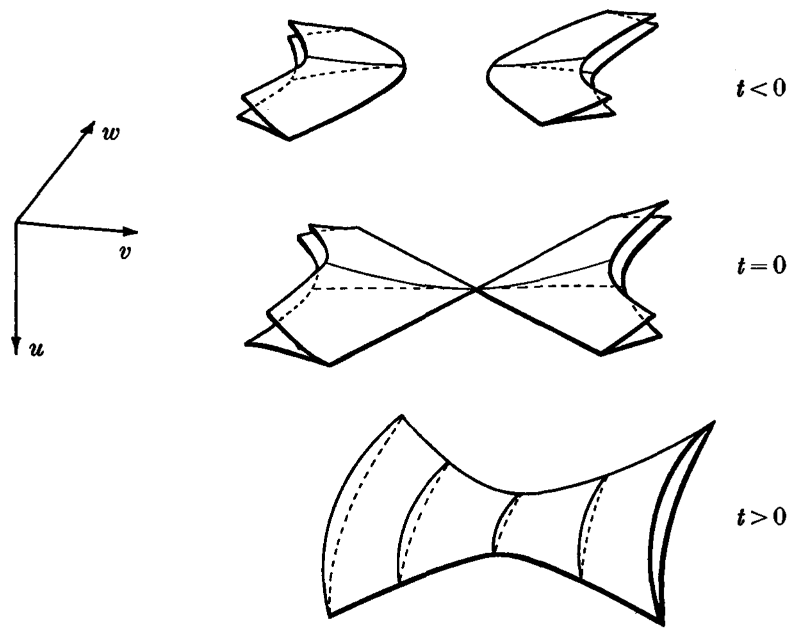


Fig. 6 Bec-à-bec to lip: the bifurcation set of the unfolding h_8.

bifurcation set is empty; at $t=0$ a point appears at the origin; subsequently a crescent appears and grows in size. The horns of the crescent are cusps whose points lie on the w-axis at a distance $\sqrt{t}$ from the origin (so the horns move apart with infinite velocity at $t=0$). The midpoints of the two curves forming the crescents move away from the origin with finite non-zero velocity ($=1$ in the scale we have used) at $t=0$, and they move away from each other with zero velocity at $t=0$.

The above discussion refers to the plane sections shown in figure 5; the actual bifurcation set of h_9 is gotten by rotating the pictures about the horizontal (u-) axis.

Figures 3, 4 and 5 together can be used to visualize the bifurcation set of the "bec-à-bec to lip" unfolding h_8. At time t, a section of the bifurcation set of h_8 by the plane $w=c$ looks like the frame of figure 4 for time $t-c^2$; a section by the plane $v=c$ looks like the frame of figure 5 for time $t+c^2$. Sections by planes $u=$constant are empty if $u>0$; if $u=0$ we see the film of figure 3 as time progresses; if $u<0$ we see simultaneously two copies of the film of figure 3 with a time-lag between them of $(8/(3\sqrt{6}))\,|u|\,\sqrt{|u|}$.

Globally we can describe the spatial configuration of the bifurcation set of h_8 as follows (for a crude sketch, see figure 6; for convenience we have changed the orientation of the coordinate axes): At negative times we see two "wedges" located symmetrically about the plane $v=0$; each wedge consists of two surfaces which meet cuspally along a branch of an hyperbola. As time progresses the two wedges approach each other and meet at the origin when $t=0$; at this time the two surfaces forming each wedge are joined along the lines $v=\pm w$. When time becomes positive the two wedges have merged to form a tube whose cross-section is a crescent (as shown in figure 5); the tube has a "seam", lying along the branches of an hyperbola, where the two surfaces forming the tube meet cuspally.

Remark. In this paper we do not give pictures of the space-stable unfoldings, those in the list of Theorem 5.3. One reason for this is that the bifurcation sets of these unfoldings can not as easily be shown pictorially as for the time-stable unfoldings. It would be meaningless to show a film of the bifurcation set, since the sequence of spatial configurations at fixed moments of time is not invariant under space-equivalence. The most reasonable sort of drawing for the space-stable unfoldings would probably be one showing the regions of space in which different sequences of events occur in time. If we denote the bifurcation set of some space-stable unfolding by $A \subseteq \mathbf{R}^4$, and if we let $\pi: \mathbf{R}^4 \to \mathbf{R}^3$ be the projection $(t, u, v, w) \mapsto (u, v, w)$, such a drawing would show where in u, v, w-space the fibres of $\pi|A$ have different topological types, where the topological type of the fibre changes, and above which points (u, v, w) two or more "branches" of A come together or branches appear or disappear. Such pictures would seem not to have as immediate an interpretation as the films of the time-stable unfoldings.

Notes added in proof: N. Baas in [13] studies a global stability notion (stability of composed mappings) which is closely related to the (local) notion of (r, s)-stability treated in the present papers; Baas and I wish to point out that these notions are not, however, exactly the same (primarily because of important (though technical) differences between the local and global cases).

Baas's preprint [13] also contains a proof, due to Mather in an unpublished manuscript, of my Corollary 1.7 to the Malgrange Preparation Theorem in a somewhat more general form. It should also be mentioned that for the case $k=2$ a version of this corollary was proved by F. Latour in [14, p. 1333].

To correct an oversight in the text it should be mentioned that the classification of germs of low codimension by Mather and Siersma referred to on page 95 has, as is well-known, been extended considerably by V. I. Arnol'd in [Функц. Анализ, 9 вып. 1 (1975), 49–50; Успехи Мат. Наук, 30 вып. 5 (1975), 3–65, and earlier papers cited there].

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