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Algebraic K-theory of spaces, with bounded control

by

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This paper is concerned with a boundedly controlled version of the algebraic K-theory of spaces functor $X \mapsto A(X)$. The word *boundedly controlled* refers to the following situation. Every object is equipped with a reference map to a metric space. In particular it makes sense to talk about *boundedness* of maps, homotopies etc.

Controlled algebraic K-theory should be related to *bounded* stable concordance theory in the same way as algebraic K-theory of spaces is related to (ordinary) stable concordance theory.

Parts of such a theory have been studied variously.

As a first example consider h-cobordisms W with a reference map $p: W \to B$ to a metric space B. One may ask when W has a *bounded product structure*. In [AH] Anderson and Hsiang have been studying such cobordisms in the special case where the lower boundary of W is of the form $M \times \mathbf{R}^k$, where M is a compact manifold and the metric space is $\mathbf{R}^k$. The answer to this question is provided by the *bounded s-cobordism theorem*. It turns out that there is a naturally defined algebraic K-theory invariant whose vanishing guarantees the existence of a bounded product structure on W. The group in which these invariants live is called the *controlled Whitehead group*.

As another example, this time on the K_0-level, we mention the *controlled finiteness obstruction*, which has been treated for example in [C]. The problem here is to decide when a space which is finitely dominated in the bounded sense is actually boundedly homotopy equivalent to a locally finite space.

In [PW1, PW2] Pedersen and Weibel have been studying a version of controlled algebraic K-theory of rings. They define the category $\mathfrak{C}_n(F_R)$ of locally finite families of free R-modules parametrized by $\mathbf{Z}^n$. They show that its K-theory is in fact an n-fold (non-connective) de-looping of the K-theory of the ring R. In [PW2] this result is generalized: To any metric space B there is associated a category $\mathfrak{C}_B(F_R)$ of locally finite families of free R-modules parametrized by the metric space B. Now assume that the metric space arises in the following way. Let X be a finite PL-subcomplex of $\mathbf{R}^\infty$, and let $o(X)$ denote the *open cone* on X with the induced metric. It is the main result of [PW2] that the functor defined by $X \mapsto K(\mathfrak{C}_{o(X)}(F_R))$ defines a *generalized homology theory*. It is also shown that one may replace the category F_R of free R-modules by any exact category in which all short exact sequences are assumed to split.

These examples show that one has been considering so far only *linear* controlled algebraic K-theory in contrast to the *non-linear* K-theory of spaces. On the other hand, from the potential applications to bounded stable concordance theory it clearly seems desirable to have available a controlled K-theory of *spaces* functor. This paper is addressed to the construction of such a functor.

Controlled algebraic K-theory of spaces is built from the category of locally finite spaces over X parametrized by a metric space B and their bounded homotopy equivalences. We obtain a functor of two variables $(X, B) \mapsto A(X; B)$. Our main result concerns the dependence of this functor on the control space B. It parallels the corresponding result of Pedersen and Weibel for the controlled K-theory of rings. The precise statement is as follows.

THEOREM. *Let K denote a PL-subcomplex of $\mathbf{R}^\infty$, and let $o(K)$ denote the* open cone *on K with metric induced from $\mathbf{R}^\infty$. Then the functor*

$$K \mapsto A(X; o(K))$$

is a generalized homology theory. Its coefficients are given by the algebraic K-theory of X with a dimension shift by one.

This result may also be viewed as giving a geometric interpretation of the homology theory associated to the spectrum $A(X)$. In fact one obtains a *non-connective* spectrum.

THEOREM. *There are homotopy equivalences $\Omega^n A(X; \mathbf{R}^n) \simeq A(X)$ for each n. Furthermore, $\pi_i A(X; \mathbf{R}^n) = K_{i-n}(\mathbf{Z}[\pi_1 X])$ for $i \leq n$.*

From a technical point of view the main ingredient in the proof of these theorems is the *decomposition technique* employed in [W1] to analyze the algebraic K-theory of generalized free products of rings. Actually we use a translation of this technique into the general framework of *categories with cofibrations and weak equivalences*, cf. [W2]. In the paper [V], which is a sequel to the present one, we explain the relationship between our definition of controlled K-theory and the linear version of Pedersen and Weibel.

§1

In this section we define the concept of a controlled space and the category $\mathfrak{R}_f(X;B)$ of boundedly finite controlled spaces over X with control in B, together with several modifications of it. We define the concept of bounded homotopy equivalence of controlled spaces. We show that the category $\mathfrak{R}_f(X;B)$ becomes a category with cofibrations and weak equivalences in the sense of [W2]. Given such a category we can define the associated K-theory. This leads to the definition of controlled K-theory. We consider various finiteness conditions in the categories of controlled spaces and discuss their effect on K-theory. In particular we state and prove a cofinality theorem for algebraic K-theory.

Let B denote a metric space with metric denoted by ϱ. Assume that B is *proper*, i.e. every closed ball in B is compact. An equivalent condition to ask is that the map $\varrho(-,b)\colon B\to\mathbf{R}_+$ is a proper map for each $b\in B$. Let c denote a positive real number and let A be a subset of B. We say that A has *diameter* $\leq c$ if A is contained in some closed ball of diameter c in B. Our basic object of study in this paper will be spaces equipped with a reference map to B which is used to measure distances. Define a *controlled space* (over B) to be a pair (Y,p), where $p\colon Y\to B$ is called the controlling map. A morphism $f\colon(Y,p)\to(Z,q)$ of controlled spaces is by definition a map $f\colon Y\to Z$ such that $qf=p$. Let $\mathfrak{Top}/B$ denote the category of controlled spaces over B. Let c denote a positive real number. A morphism $f\colon(Y,p)\to(Z,q)$ is called a *c-bounded homotopy equivalence* if there exists a map $g\colon Z\to Y$ and homotopies $\alpha\colon fg\simeq\mathrm{id}_Z$, $\beta\colon gf\simeq\mathrm{id}_Y$, which are c-bounded in the sense that the path of each point under these homotopies has diameter $\leq c$ when measured in B. The map f is called a *bounded homotopy equivalence* if it is a c-bounded homotopy equivalence for some c. Let (Y,p) and (Z,q) denote controlled spaces over B, and let $f\colon Y\to Z$ be any map of spaces. The diagram

$$\begin{tikzcd} Y \arrow[rr, "f"] \arrow[dr, "p"'] & & Z \arrow[dl, "q"] \\ & B & \end{tikzcd}$$

is *c-commutative* if $\varrho(p(y),qf(y))\leq c$, for all $y\in Y$. It is said to *commute up to bounded distance* if it is c-commutative for some c. In particular, if f is a morphism in $\mathfrak{Top}/B$, then f is c-commutative for all c. With this terminology we can say that a morphism $f\colon(Y,p)\to(Z,q)$ is a bounded homotopy equivalence if there exists a map g, and homotopies α and β as above such that the following diagrams commute up to bounded distance:

$$\begin{array}{ccc} Y\times I & \xrightarrow{\alpha} & Y \\ {\scriptstyle p\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle p} \\ B\times I & \xrightarrow{\mathrm{pr}} & B \end{array} \qquad \begin{array}{ccc} Z\times I & \xrightarrow{\beta} & Z \\ {\scriptstyle q\times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle q} \\ B\times I & \xrightarrow{\mathrm{pr}} & B \end{array}$$

The homotopy inverse $g: Z \to Y$ ned not be a morphism in $\mathfrak{Top}/B$ but it makes the diagram

$$\begin{tikzcd} Z \arrow[rr, "g"] \arrow[dr, "q"'] & & Y \arrow[dl, "p"] \\ & B & \end{tikzcd}$$

commutative up to bounded distance.

We now give the definition of the category of controlled spaces which will be used in defining controlled K-theory.

Let B be a metric space and let X be a topological space. Denote by $\mathfrak{R}(X;B)$ the following category of retractive spaces over $X\times B$: An object is a triple (Y, r, s), where $r: Y \to X\times B$, $s: X\times B \to Y$, such that $rs = \mathrm{id}$. A morphism $f: (Y, r, s) \to (Y', r', s')$ is by definition a map $f: Y \to Y'$ satisfying that $r'f = r$, $fs = s'$. We shall call the morphism f a *bounded homotopy equivalence* if it is a bounded homotopy equivalence in $\mathfrak{Top}/B$. Let $b\mathfrak{R}(X;B)$ denote the subcategory of $\mathfrak{R}(X;B)$ with the same objects and morphisms the bounded homotopy equivalences. The categories defined so far are much too general to be very useful. More specifically, we will need a notion generalizing the concept of a (finite) CW-complex. First let us define the notion of a cell. Fix a positive real number c. A *bounded n-cell (of diameter c)* is a pair $(J\times D^n, q)$, where J is a discrete index set, D^n is the n-ball, and $q: J\times D^n \to X\times B$ is a map satisfying that

(i) $\mathrm{pr}_B\, q(\{j\}\times D^n)$ has diameter at most c for each $j \in J$.

(ii) for every compact subset K of B the set $\{j\in J \mid \mathrm{pr}_B\, q(\{j\}\times D^n) \cap K \neq \emptyset\}$ is finite.

Let (Y, r, s) denote an object of $\mathfrak{R}(X;B)$, and let $(J\times D^n, q)$ denote a bounded cell. We say that (Y', r', s') is obtained from (Y, r, s) by *attaching the bounded n-cell* $(J\times D^n, q)$ if there is a map $f: J\times D^n \to Y$ such that Y' is isomorphic to the pushout of

$$Y \xleftarrow{f} J\times \partial D^n \xrightarrow{i} J\times D^n$$

(where i is the natural inclusion), and $r'|J\times D^n=q$, $r'|Y=r$. We say that an object (Y,r,s) of $\mathfrak{R}(X;B)$ has a *bounded CW-structure* if it can be obtained from $X\times B$ by attaching of bounded cells (of any diameter) in order of increasing dimension and if there is a global bound for the diameter of the cells. We call the CW-structure *(boundedly) finite* if it consists of finitely many bounded cells. Let us denote by $\mathfrak{R}_{\mathrm{f}}(X;B)$ the subcategory of $\mathfrak{R}(X;B)$ consisting of the objects with a finite bounded CW-structure and their cellular maps. Let us call an object of $\mathfrak{R}(X;B)$ *(boundedly) finite up to homotopy* if it is in the same connected component of $\mathrm{b}\mathfrak{R}(X;B)$ as an object of $\mathfrak{R}_{\mathrm{f}}(X;B)$; in other words, if there is a chain of bounded homotopy equivalences going either way between (Y,r,s) and some object with a finite bounded CW-structure. Let $\mathfrak{R}_{\mathrm{hf}}(X;B)$ denote the full subcategory of $\mathfrak{R}(X;B)$ consisting of those objects that are boundedly finite up to homotopy. Further let

$$\mathrm{b}\mathfrak{R}_{\mathrm{f}}(X;B)=\mathfrak{R}_{\mathrm{f}}(X;B)\cap\mathrm{b}\mathfrak{R}(X;B)\quad\text{and}\quad\mathrm{b}\mathfrak{R}_{\mathrm{hf}}(X;B)=\mathfrak{R}_{\mathrm{hf}}(X;B)\cap\mathrm{b}\mathfrak{R}(X;B).$$

A morphism $(Y,r,s)\to(Y',r',s')$ in $\mathfrak{R}_{\mathrm{f}}(X;B)$ is called a *cofibration* if it is isomorphic to a cellular inclusion. Define a map $i:(Y,r,s)\to(Y',r',s')$ to be a *cofibration* in $\mathfrak{R}_{\mathrm{hf}}(X;B)$ if it has the *bounded homotopy extension property*, i.e. every bounded homotopy $Y\times I\cup Y'\times 0\to Z$ may be extended to a bounded homotopy $Y'\times I\to Z$. Here we are considering Y, Y', and Z as objects of $\mathfrak{Top}/B$, i.e. we are ignoring the retractions to X.

We shall also need weaker finiteness conditions. Namely we want to admit objects, which are finitely dominated in the bounded sense. Let us call an object (Y,r,s) of $\mathfrak{R}(X;B)$ *finitely dominated* if Y has a *finite dimensional bounded CW-structure* and if there exists an object (Y',r',s') with a *finite bounded CW-structure* and a *bounded domination*, i.e. a morphism $d:Y'\to Y$ in $\mathfrak{R}(X;B)$ which is a retraction up to bounded homotopy. Denote by $\mathfrak{R}_{\mathrm{fd}}(X;B)$ the category of finitely dominated objects of $\mathfrak{R}(X;B)$ and their cellular maps. There is also a homotopy version of this. Let us call an object of $\mathfrak{R}(X;B)$ *finitely dominated up to bounded homotopy* if it is a retract of some object of $\mathfrak{R}_{\mathrm{hf}}(X;B)$. The full subcategory of $\mathfrak{R}(X;B)$ of these objects is denoted $\mathfrak{R}_{\mathrm{hd}}(X;B)$. Observe that $\mathfrak{R}_{\mathrm{fd}}(X;B)$ is actually a subcategory of $\mathfrak{R}_{\mathrm{hd}}(X;B)$. In fact if $Y'\to Y$ is a domination map with Y' a finite bounded CW-complex we may consider the mapping cylinder $T(Y\to Y')$ of a section of the domination map. It is finite up to bounded homotopy (since it is homotopy equivalent to Y'), and it contains Y as a retract. We define the notion of cofibration in $\mathfrak{R}_{\mathrm{fd}}(X;B)$ (resp. $\mathfrak{R}_{\mathrm{hd}}(X;B)$) considered before. With these definitions one observes:

LEMMA 1.1. *The categories* $\mathfrak{R}_{\mathrm{f}}(X;B)$, $\mathfrak{R}_{\mathrm{hf}}(X;B)$, $\mathfrak{R}_{\mathrm{fd}}(X;B)$, *and* $\mathfrak{R}_{\mathrm{hd}}(X;B)$ *are* *categories with cofibrations and weak equivalences, where the category of weak equivalences is given in each case by the bounded homotopy equivalences.* □

Remark. It is a technical point of some importance that the morphisms in the category $\mathfrak{R}(X;B)$ are maps which are *strictly* compatible with the reference map to B, not just up to bounded distance. For otherwise it would not be clear how to define quotients, or more generally cobase changes in this category. This differs from the corresponding definition of morphisms in the category $\mathfrak{C}_n(F_R)$ (resp. $\mathfrak{C}_B(F_R)$) of [PW1] (resp. [PW2]). In our definition the boundedness (which is of course essential) is encoded in the definition of bounded homotopy equivalences.

Let (B,ϱ) and (B',ϱ') denote metric spaces. Recall that a map $f: B \to B'$ is *proper* if the inverse image under f of a compact set is compact again. We say that f is *Lipschitz* if there exists a positive real number k such that $\varrho'(f(x),f(y)) \leq k\varrho(x,y)$ for all $x,y \in B$. A proper Lipschitz map is also called *controlled.* A Lipschitz map $f: B \to B'$ is called a *Lipschitz homotopy equivalence* if there exists a Lipschitz map $g: B' \to B$ and homotopies $gf \simeq \mathrm{id}_B$, $fg \simeq \mathrm{id}_{B'}$ which are Lipschitz maps. Finally let us call the map f a *controlled homotopy equivalence* if it is a Lipschitz homotopy equivalence and in addition the homotopies and the map g are proper.

LEMMA 1.2. *The categories $\mathfrak{R}_{\mathrm{f}}(X;B)$ (resp. $\mathfrak{R}_{\mathrm{hf}}(X;B)$, $\mathfrak{R}_{\mathrm{fd}}(X;B)$, $\mathfrak{R}_{\mathrm{hd}}(X;B)$) define a covariant functor from the category of metric spaces and controlled maps to categories with cofibrations and weak equivalences, where the weak equivalences are given by the bounded homotopy equivalences.*

Proof. A proper Lipschitz map $f: B \to B'$ of metric spaces induces an exact functor $f_*: \mathfrak{R}_{\mathrm{f}}(X;B) \to \mathfrak{R}_{\mathrm{f}}(X;B')$ by $(Y,r,s) \mapsto (Y \cup_{X\times B} X \times B', \ldots)$. The Lipschitz condition on f ensures that bounded homotopy equivalences are mapped to bounded homotopy equivalences. Since f is proper, boundedly finite objects are mapped to boundedly finite objects. Similarly the properties of homotopy finiteness and of finite domination are being preserved by f_*. □

LEMMA 1.3. *If $f: B \to B'$ is a controlled homotopy equivalence then the induced functor $f_*: \mathrm{b}S.\mathfrak{R}_{\mathrm{f}}(X;B) \to \mathrm{b}S.\mathfrak{R}_{\mathrm{f}}(X;B')$ is a weak homotopy equivalence. The same is true in the case of the categories $\mathfrak{R}_{\mathrm{hf}}(X;B)$, $\mathfrak{R}_{\mathrm{fd}}(X;B)$ and $\mathfrak{R}_{\mathrm{hd}}(X;B)$.*

Proof. The proof is literally the same in all four cases. So we just treat the first one. It suffices to show that the projection map $\pi: B \times I \to B$ induces a weak homotopy equivalence. There is a section of π induced by the inclusion $i: B\times 0 \to B\times I$. We shall show that for each n the endofunctor $i_*\pi_*$ of $\mathrm{b}S_n\mathfrak{R}_{\mathrm{f}}(X;B\times I)$ is homotopic to the identity. Define another endofunctor f of $\mathrm{b}S_n\mathfrak{R}_{\mathrm{f}}(X;B\times I)$ as follows. Let $h: B\times I\times I \to B\times I$ be a homotopy between the identity map and $i\pi$. We first define f on objects of $\mathrm{b}\mathfrak{R}_{\mathrm{f}}(X;B\times I)$ by $(Y,r,s)\mapsto(Y\times I, \mathrm{id}\times(h(r\times\mathrm{id})), (s,0))$ and on the morphisms of that category similarly. The definition clearly extends to filtered objects of $\mathrm{b}\mathfrak{R}_{\mathrm{f}}(X;B\times I)$. There are natural transformations of endofunctors of $\mathrm{b}S_n\mathfrak{R}_{\mathrm{f}}(X;B\times I)$

$$\mathrm{id}\to f\leftarrow i_*\pi_*$$

which are given by the obvious inclusion maps. This shows that $i_*\pi_*$ is homotopic to the identity. Also it is easy to see that for varying n these natural transformations assemble to a simplicial natural transformation of endofunctors of the simplicial category $\mathrm{b}S.\mathfrak{R}_{\mathrm{f}}(X;B\times I)$. This proves the lemma. □

We now want to compare the various finiteness conditions. Recall that the *approximation theorem* (Theorem 1.6.7 of [W2]) gives a sufficient condition for an exact functor of categories with cofibrations and weak equivalences to induce a homotopy equivalence on K-theory.

PROPOSITION 1.4. *The approximation theorem applies to the inclusion functors*

$$\mathfrak{R}_{\mathrm{f}}(X;B)\to\mathfrak{R}_{\mathrm{hf}}(X;B) \quad \textit{and} \quad \mathfrak{R}_{\mathrm{fd}}(X;B)\to\mathfrak{R}_{\mathrm{hd}}(X;B).$$

Proof. Let us first treat the case of finite vs. homotopy finite objects. We have to verify the following property:

Given an object (Y,r,s) in $\mathfrak{R}_{\mathrm{f}}(X;B)$ together with a morphism $f:(Y,r,s)\to(Y',r',s')$ in $\mathfrak{R}_{\mathrm{hf}}(X;B)$ there exists an object (Y_1,r_1,s_1) in $\mathfrak{R}_{\mathrm{f}}(X;B)$ and a factorization $(Y,r,s)\to(Y_1,r_1,s_1)\to(Y',r',s')$ of f, where the first map is a cofibration and the second is a bounded homotopy equivalence.

It is sufficient to find a factorization

$$(Y,s)\to(Y_1,s_1)\to(Y',s'),$$

i.e. ignoring the retractions. For we may then *define* the retraction r_1 as the composite of $Y_1\to Y'$ with $r': Y'\to X\times B$.

Since Y' is boundedly finite up to homotopy we can find a finite Y_2 together with bounded homotopy equivalences $(Y',s')\to(Y_2,s_2)$ and $(Y_2,s_2)\to(Y',s')$. By the bounded cellular approximation theorem, cf. [AM, Corollary II.10.3], we may choose a cellular map $(Y, s) \to (Y_2, s_2)$ which is boundedly homotopic to the composition $(Y, s) \to (Y', s') \to (Y_2, s_2)$. Define Y_1 to be the mapping cylinder of this map. Then there is a bounded homotopy equivalence $(Y_1, s_1) \to (Y', s')$ extending the map $(Y, s) \to (Y', s')$. This proves the first case of the proposition.

Now let (Y, r, s) be an object of $\mathfrak{R}_{\mathrm{hd}}(X; B)$. By definition there is a retraction $Y' \to Y$ where Y' is in $\mathfrak{R}_{\mathrm{hf}}(X; B)$. Choose a bounded homotopy equivalence $Y'' \to Y'$ with Y' finite. We obtain a map $Y'' \to Y$ which is a retraction up to bounded homotopy. Consequently there is a self-map α of Y'' which is *idempotent* up to bounded homotopy. By bounded cellular approximation we may assume that α and the homotopy are cellular. Therefore the *mapping telescope* of α defines an object of $\mathfrak{R}_{\mathrm{fd}}(X; B)$. Furthermore there is a bounded homotopy equivalence $\mathrm{Tel}(\alpha) \to Y$. (Refer to the proof of Proposition 2.7 below for a more detailed discussion of the mapping telescope.) Now we may copy the previous argument to verify the hypothesis of the approximation theorem. This completes the proof of the proposition. □

To describe the relationship between the categories of finite and finitely dominated objects we introduce the following terminology.

Let $\mathfrak{C}$ denote a category with cofibrations and weak equivalences, and let $\mathfrak{D}$ be a subcategory with cofibrations and weak equivalences. We say that $\mathfrak{D}$ is *cofinal* in $\mathfrak{C}$ if for each object C in $\mathfrak{C}$ there exists C' in $\mathfrak{C}$ such that $C \vee C'$ is isomorphic to an object of $\mathfrak{D}$. Here '$\vee$' denotes the sum in $\mathfrak{C}$. Assume now that $\mathfrak{C}$ and $\mathfrak{D}$ have a *cylinder functor*, and consequently also a suspension functor, cf. [W2]. Let us call $\mathfrak{D}$ a *weakly cofinal subcategory* of $\mathfrak{C}$ if for each object C in $\mathfrak{C}$ there exists C' in $\mathfrak{C}$ such that $\Sigma^k C \vee C'$ is isomorphic to an object of $\mathfrak{D}$ for some k. With this terminology we have

PROPOSITION 1.5. *The category* $\mathfrak{R}_{\mathrm{hf}}(X; B)$ *is a weakly cofinal subcategory of* $\mathfrak{R}_{\mathrm{hd}}(X; B)$.

Proof. This is almost obvious. In fact, let (Y, p, s) be finitely dominated up to bounded homotopy, and let $d: (Z, q, t) \to (Y, p, s)$ be a domination of Y with Z boundedly finite up to homotopy, and let further $e: (Y, p, s) \to (Z, q, t)$ be a section of d. Define Y' to be the mapping cone of the map Σe, and let $f: \Sigma Z \to Y'$ denote the canoncial map. Clearly Y' is an object of $\mathfrak{R}_{\mathrm{hd}}(X; B)$. Then $\Sigma d \vee f: \Sigma Z \to \Sigma Y \vee Y'$ is a bounded homotopy equivalence. This proves the proposition. □

Let $\mathfrak{C}$ be a category with cofibrations and weak equivalences and let $w\mathfrak{C}$ denote the subcategory of weak equivalences. Recall from [W2] that the algebraic K-theory of $\mathfrak{C}$ is defined to be

$$K(\mathfrak{C}, \mathrm{w}\mathfrak{C}) = \Omega|\mathrm{w}S.\,\mathfrak{C}|.$$

Here $\mathrm{w}S.\,\mathfrak{C}$ is a certain simplicial category constructed from the cofibrations and the weak equivalences in $\mathfrak{C}$.

We are now going to give the definition of the algebraic K-theory of controlled spaces. Let X be a topological space, and let B be a proper metric space. There are two definitions corresponding to the different finiteness conditions.

DEFINITION.
$$A(X;B) = \Omega|\mathrm{b}S.\,\mathfrak{R}_{\mathrm{hd}}(X;B)|$$
$$A'(X;B) = \Omega|\mathrm{b}S.\,\mathfrak{R}_{\mathrm{hf}}(X;B)|.$$

We call $A(X;B)$ (resp. $A'(X;B)$) the *algebraic K-theory of the space X with control in B*.

Remark. Proposition 1.4. tells us that there is no point in giving a third and fourth definition by replacing the category $\mathfrak{R}_{\mathrm{hf}}(X;B)$ with $\mathfrak{R}_{\mathrm{f}}(X;B)$ and $\mathfrak{R}_{\mathrm{hd}}(X;B)$ with $\mathfrak{R}_{\mathrm{fd}}(X;B)$: we would have ended up with the same functors up to homotopy. □

Let us now examine the relationship between the functors $A(X;B)$ and $A'(X;B)$. The answer is contained in the following *cofinality theorem* for algebraic K-theory. It is due to Thomason [T, Theorem 1.10.1 and Exercise 1.10.2]. For the convenience of the reader we shall explain this result here.

Let $\mathfrak{C}$ be a category with cofibrations and weak equivalences, and let $\mathfrak{D}$ be a subcategory with cofibrations and weak equivalences of $\mathfrak{C}$. Assume that $\mathfrak{C}$ has a cylinder functor and that the weak equivalences in $\mathfrak{C}$ satisfy the cylinder axiom. Let $G=\mathrm{coker}(K_0\mathfrak{D}\to K_0\mathfrak{C})$, and let $N.\,\mathfrak{G}$ denote the nerve of G considered as a discrete simplicial category, i.e. $N_k\mathfrak{G}$ $(\approx G^k)$ is viewed as a category with only identity morphisms for each k. Clearly, the geometric realization of $N.\,\mathfrak{G}$ is homotopy equivalent to BG, the classifying space of G.

THEOREM 1.6. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be as above. Suppose in addition that the following conditions are satisfied.*

(i) $\mathfrak{D}$ *is* weakly cofinal *in* $\mathfrak{C}$.

(ii) $\mathfrak{D}$ *is* saturated, *i.e. every object which is weakly equivalent to an object in* $\mathfrak{D}$ *actually belongs to* $\mathfrak{D}$.

(iii) $\mathfrak{D}$ *is* closed under quotients, *i.e. if* $C_1 \to C_2 \to C_3$ *is a cofibration sequence in* $\mathfrak{C}$, *and* C_1, C_2 *are actually objects of* $\mathfrak{D}$, *then* C_3 *is also in* $\mathfrak{D}$.

Then there is a fibration up to homotopy

$$wS.\mathfrak{D} \to wS.\mathfrak{C} \to N.\mathfrak{G}.$$

Consequently, there is an isomorphism $K_i(\mathfrak{D}, w\mathfrak{D}) \xrightarrow{\sim} K_i(\mathfrak{C}, w\mathfrak{C})$ *for* $i \geq 1$.

Proof. Let $w\mathfrak{C}$ (resp. $w\mathfrak{D}$) denote the category of weak equivalences in $\mathfrak{C}$ (resp. $\mathfrak{D}$). Define a map $f: C \to C'$ to be a v-*equivalence* iff the mapping cone $C(f)$ of f represents zero in the group G. This defines a class of weak equivalences in $\mathfrak{C}$, and of course $w\mathfrak{C} \subset v\mathfrak{C}$. By the *generic fibration theorem* (Theorem 1.6.4 of [W2]) one obtains a fibration up to homotopy

$$wS.\mathfrak{C}^v \to wS.\mathfrak{C} \to vS.\mathfrak{C}.$$

Here $\mathfrak{C}^v$ denotes the subcategory of $\mathfrak{C}$ consisting of those objects which are *acyclic* with respect to the coarse notion of weak equivalence. We are left to identify the terms in this fibration with those in the assertion of the theorem. First observe that we have a map of simplicial categories

$$p.: vS.\mathfrak{C} \to N.\mathfrak{G}.$$

It is given in degree k by

$$(* \rightarrowtail C_1 \rightarrowtail C_2 \rightarrowtail \ldots \rightarrowtail C_k) \mapsto ([C_1], [C_2]-[C_1], \ldots, [C_k]-[C_{k-1}]).$$

We claim that p_k is homotopy equivalence for each k. Observe that $p_k^{-1}(0, \ldots, 0)$ is a contractible subcategory of $vS_k\mathfrak{C}$. In fact $(* \rightarrowtail * \rightarrowtail \ldots \rightarrowtail *)$ is an initial object since $[C]=0$ in G implies that $* \to C$ is a v-equivalence. Next consider $p_k^{-1}(x_1, \ldots, x_k)$ for any $(x_1, \ldots, x_k)$ in G^k. This category is not empty: Let C_i be such that $x_i = [C_i]$. Then

$$p_k(C_1 \rightarrowtail C_1 \vee C_2 \rightarrowtail C_1 \vee C_2 \vee C_3 \rightarrowtail \ldots \rightarrowtail C_1 \vee \ldots \vee C_k) = (x_1, \ldots, x_k).$$

Since the category $\mathfrak{C}$ has a cylinder functor, and consequently also a suspension functor Σ, it follows from the additivity theorem (Theorem 1.3.2. of [W2]) that Σ acts as (-1) on the group G. We can therefore define a functor

$$\varphi: p_k^{-1}(x_1, \ldots, x_k) \to p_k^{-1}(0, \ldots, 0)$$

by

$$(B_1 \rightarrowtail \dots \rightarrowtail B_k) \mapsto (B_1 \vee \Sigma C_1 \rightarrowtail B_2 \vee \Sigma C_1 \vee \Sigma C_2 \rightarrowtail \dots \rightarrowtail B_k \vee \Sigma C_1 \vee \dots \vee \Sigma C_k).$$

There is also a functor in the other direction given by

$$\psi: p_k^{-1}(0, \dots, 0) \to p_k^{-1}(x_1, \dots, x_k)$$

$$(A_1 \rightarrowtail \dots \rightarrowtail A_k) \mapsto (A_1 \vee C_1 \rightarrowtail A_2 \vee C_1 \vee C_2 \rightarrowtail \dots \rightarrowtail A_k \vee C_1 \vee \dots \vee C_k).$$

There are obvious natural transformations $\mathrm{id} \dot{\to} \psi\varphi$ and $\mathrm{id} \dot{\to} \varphi\psi$. This implies that all fibres $p_k^{-1}(x_1, \dots, x_k)$ are homotopy equivalent to $p_k^{-1}(0, \dots, 0)$ and hence are contractible. By Quillen's Theorem A, [Q], therefore p_k is a homotopy equivalence for each k, and hence $p.$ is a homotopy equivalence by the realization lemma, cf. [W1, Lemma 5.1]. The assertion of the theorem will follow if we can show that $\mathfrak{C}^{\vee}$ is equivalent to $\mathfrak{D}$. First observe that $\Sigma^k(C \vee \Sigma C)$ is in $\mathfrak{D}$ for every C. This is true since $\mathrm{cone}(C) \simeq *$ implies that $\mathrm{cone}(C)$ is in $\mathfrak{D}$, since $\mathfrak{D}$ is saturated. Now choose C' so that $C' \vee \Sigma^k C$ is in $\mathfrak{D}$. Then from the cofibration sequence $C \to \mathrm{cone}(C) \to \Sigma C$ we deduce another cofibration sequence

$$\Sigma^k C \vee C' \to \mathrm{cone}(\Sigma^k C) \vee C' \vee \Sigma^k C \to \Sigma^k C \vee \Sigma^{k+1} C$$

where the first and second term are objects of $\mathfrak{D}$. Hence by property (ii) $\Sigma^k C \vee \Sigma^{k+1} C = \Sigma^k(C \vee \Sigma C)$ is in $\mathfrak{D}$. Let $C_1 \to C_2 \to C_3$ be a cofibration sequence in $\mathfrak{C}$. From this we obtain a cofibration sequence

$$C_1 \vee \Sigma C_1 \to C_1 \vee C_3 \vee \Sigma C_2 \to C_3 \vee \Sigma C_3$$

which is weakly equivalent to another sequence

$$C_3 \vee \Sigma C_3 \to \Sigma(C_1 \vee \Sigma C_1) \to \Sigma(C_1 \vee C_3 \vee \Sigma C_2).$$

From the preceding observation we conclude that the k-fold suspension of the first and second term are objects of $\mathfrak{D}$. Therefore we conclude that $\Sigma^{k+1}(C_1 \vee C_3 \vee \Sigma C_2)$ is in $\mathfrak{D}$. Let $K_0'\mathfrak{C}$ denote the class group of $\mathfrak{C}$ formed by introducing a relation only for *split* cofibration sequences, and let $G' = \mathrm{coker}(K_0\mathfrak{D} \to K_0'\mathfrak{C})$. Then G is generated by isomorphism classes of objects of $\mathfrak{C}$ modulo the following relations:

(i) $[D] = 0$ for $D \in \mathfrak{D}$

(ii) $[C_1] + [C_3] = [C_2]$ for each cofibration sequence $C_1 \to C_2 \to C_3$.

The group G' has the same generators and the following relations.

(i) $[D]=0$ for $D \in \mathfrak{D}$

(ii) $[C_1]+[C_3]=[C_1 \vee C_3]$ for C_1, C_3 in $\mathfrak{C}$.

Since we have shown that $\Sigma^{k+1}(C_1 \vee C_3 \vee \Sigma C_2)$ is in $\mathfrak{D}$ for every cofibration sequence $C_1 \to C_2 \to C_3$ we have that $[C_1 \vee C_3 \vee \Sigma C_2]=0$, which implies that $[C_1]+[C_3]=[C_2]$. Therefore the relations are in fact the same and $G=G'$. Now suppose that $C \in \mathfrak{C}^{\vee}$. This means by definition that $[C]=0$ in G, and hence also $[C]=0$ in G', i.e. there exists D in $\mathfrak{D}$ such that $[D]=[C]$ in $K_0'\mathfrak{C}$. But by a well known argument this holds if and only if C and D are stably equivalent, i.e. there exists C_0 such that $D \vee C_0 \simeq C \vee C_0$, cf. Lemma 1.1. of [*M*]. Choosing C_0' such that $C_0 \vee C_0'$ is in $\mathfrak{D}$ we obtain that $C \vee C_0 \vee C_0'$ is in $\mathfrak{D}$, and hence, by property (ii) again, also C is an object of $\mathfrak{D}$. Therefore the categories $\mathfrak{C}^{\vee}$ and $\mathfrak{D}$ are the same. This ends the proof of the theorem. □

We have the following application:

COROLLARY 1.7. *Let* $G=\operatorname{coker}(\pi_0 A'(X;B) \to \pi_0 A(X;B))$. *Then there is a fibration up to homotopy*

$$bS.\,\mathfrak{R}_{\mathrm{hf}}(X;B) \to bS.\,\mathfrak{R}_{\mathrm{hd}}(X;B) \to N.\,\mathfrak{G}$$

where $N.\,\mathfrak{G}$ *denotes the nerve of* G *considered as a discrete simplicial category.* □

§2.

In the second section we consider a special type of control spaces. Namely we assume that the control space is given as the *open cone* on a finite space. More precisely, we have the following definition:

Let K denote a finite PL subcomplex of S^n for some n. The *open cone* on K is the metric space defined as

$$o(K)=\{t \cdot x \mid x \in K, t \in \mathbf{R}_+\}$$

with metric induced from the embedding into $\mathbf{R}^{n+1}$. If $K \to L$ is a map of finite PL subcomplexes of S^n then there is an induced map $o(K) \to o(L)$ of the open cones and it is easy to see that this is a proper Lipschitz map. In fact $o(-)$ is a functor from finite subcomplexes of the n-sphere to the category of metric spaces and proper Lipschitz maps. Note however that it is *not* true in general that a homotopy equivalence $K \xrightarrow{\sim} L$ induces a controlled homotopy equivalence $o(K) \to o(L)$. With these notations we are now ready to state the main theorem of this section more precisely:

The functor given by $K \mapsto A(X; o(K))$ *is a generalized homology theory.*

Recall that a homotopy functor F: (spaces)$\to$(spaces) is called a *generalized reduced homology theory* if it is *pointed* (i.e. evaluated on a point it gives a contractible space) and *excisive* (i.e. it maps a cocartesian square to a cartesian square). It is well known that this implies that the homotopy groups $\pi_* F(-)$ satisfy the Eilenberg–Steenrod axioms with the possible exception of the dimension axiom. We shall prove these properties separately. First we show that the functor is pointed. In fact our statement is slightly more general.

Let B denote a metric space. Consider the product $B\times\mathbf{R}_+$ equipped with the product metric. We have

THEOREM 2.1. *$A(X;B\times\mathbf{R}_+)$ is contractible for all X and B.*

Proof. This is a version of the so-called *Eilenberg swindle*. Namely there is an endofunctor T of $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B\times\mathbf{R}_+)$ which is given on objects by

$$(Y,p,s)\mapsto(Y,\sigma p,\,s\sigma^{-1})$$

where $\sigma\colon X\times B\times\mathbf{R}_+\to X\times B\times\mathbf{R}_+$ is the shift map defined by $(x,b,t)\mapsto(x,b,t+1)$. On morphisms the functor is defined by $T(f)=f$. Since the map σ does not change distances, it takes c-bounded homotopy equivalences to c-bounded homotopy equivalences. We construct a two step homotopy from T to the identity. The intermediate step is provided by the endofunctor of $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B\times\mathbf{R}_+)$ given by

$$(Y,p,s)\mapsto(Y\times I,\bar{p},\bar{s})$$

where $\bar{p}\colon Y\times I\to X\times B\times\mathbf{R}_+$ is defined by $(y,t)\mapsto(\mathrm{pr}_{X\times B}(p(y)),\mathrm{pr}_{\mathbf{R}_+}(p(y))+t)$, and $\bar{s}$ similarly. There is a natural transformation from the identity to this functor (front inclusion) and another one from T to this functor (back inclusion). The category $\mathfrak{R}(X;B\times\mathbf{R}_+)$ has a composition law given by wedge sum (over $X\times B\times\mathbf{R}_+$). Denoting this composition law by '$\vee$' we can form the functor

$$T_k^\infty=\bigvee_k^\infty T^i$$

where T^i denotes the ith iterate of T. In fact, T_k^∞ defines an endofunctor of $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B\times\mathbf{R}_+)$. To see this, one observes that over every compact subset of $B\times\mathbf{R}_+$ the infinite wedge actually reduces to a finite one. Since the functor T^i does not change distances, i.e. it takes c-bounded maps to c-bounded maps again, the functor T_k^∞ really takes bounded homotopy equivalences to bounded homotopy equivalences. Further, since T is homotopic to the identity, we have that $T_1^\infty=T\circ T_0^\infty$ is homotopic to

$\mathrm{id} \circ T_0^\infty = T_0^\infty$. But obviously we also have a natural isomorphism $\mathrm{id} \vee T_1^\infty \approx T_0^\infty$ which implies a homotopy $\mathrm{id} \vee T_1^\infty \simeq T_1^\infty$.

The functor T can be defined for filtered objects in much the same way thus giving an endofunctor of $\mathrm{b}S_n \mathfrak{R}_{\mathrm{fd}}(X; B\times\mathbf{R}_+)$. Continuing to denote this functor by T we also obtain a homotopy as above for each n. These homotopies assemble to a similar simplicial homotopy of the simplicial category $\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X; B\times\mathbf{R}_+)$. But this space is an H-space with a homotopy inverse. So we may cancel one term, and we finally obtain a homotopy between the identity map and the trivial map. This can happen only for a contractible space. □

COROLLARY 2.2. *Let K be a finite subcomplex of S^n, and let $\mathrm{c}K$ denote the (ordinary) cone on K considered as a subcomplex of S^{n+1}. Then $A(X; o(\mathrm{c}K))$ is contractible.*

Proof. This follows from the following observation in [PW2]: If '$*$' denotes the join, then there is a controlled isomorphism $o(K*L)\approx o(K)\times o(L)$. This implies that $o(\mathrm{c}K)=o(K*(\text{point}))\approx o(K)\times o(\text{point})=o(K)\times\mathbf{R}_+$ and hence proves the corollary. □

We now turn to the proof of the excision property of the functor $A(X; o(K))$. This is shown by means of a decomposition technique which is closely analogous to that used in [W1] to analyze the algebraic K-theory of generalized free products of rings. The use of the general framework of categories with cofibrations and weak equivalences makes this approach much more perspicuous than the corresponding technique in [W1].

Consider the following situation. The control space B is given together with a decomposition $B=B_1 \cup_{B_0} B_2$. Here we assume that the B_i are also metric spaces with the induced metric. We want to look at spaces over $X\times B$ of the type considered before but which are decomposed according to the decomposition of the control space. To do this we add several technical conditions. First of all let us assume that B_0 is *bicollared* in B, i.e. there are neighborhoods $B_0\times[0,1]$ in B_2 and $B_0\times[-1,0]$ in B_1. Further let us suppose the maps $B_0\to B_1$ and $B_0\to B_2$ are cofibrations, and finally we ask that these maps are *controlled*, i.e. proper and Lipschitz. A *decomposed space* is given by definition as a pushout diagram $Y=Y_1\cup_{Y_0} Y_2$ in $\mathfrak{R}_{\mathrm{fd}}(X;B)$, where Y_i is induced from an object of $\mathfrak{R}_{\mathrm{fd}}(X;B_i)$, $i=0,1,2$, and such that the maps $Y_0\to Y_1$ and $Y_0\to Y_2$ are cofibrations. Let us denote by $\mathfrak{R}_{\mathrm{fd}}(X; B_0, B_1, B_2)$ the category of spaces which are decomposed in this way. This is a category with cofibrations in the same way as $\mathfrak{R}_{\mathrm{fd}}(X;B)$. There are *two* natural choices of weak equivalences in $\mathfrak{R}_{\mathrm{fd}}(X; B_0, B_1, B_2)$. Namely we can ask that a bounded homotopy equivalence respects the decomposition, i.e. it is a bounded homotopy equivalence on each of Y_0, Y_1, Y_2, or alternatively that it only is a bounded homotopy equivalence on the total space Y. Let $\mathrm{v}\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2)$ denote the subcategory of those bounded homotopy equivalences which respect the decomposition.

Given a category with cofibrations and *two* notions of weak equivalences there is a general theorem which describes the relationship between the K-theories associated to these, the *generic fibration theorem* of [W2]. In our context the assertion is as follows.

THEOREM 2.3. *There is a homotopy cartesian square of simplicial categories*

$$\begin{array}{ccc} \mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B_0,B_1,B_2) & \longrightarrow & \mathrm{v}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2) \\ \downarrow & & \downarrow \\ \mathrm{b}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B_0,B_1,B_2) & \longrightarrow & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2) \end{array}$$

with lower left term contractible. (Here the superscript 'b' denotes the subcategory of b-acyclic objects, *i.e. those spaces for which the structural retraction $Y \to X \times B$ is a b-equivalence.)* □

In the following we have to identify the three non-trivial terms in this cartesian square.

The first observation is

PROPOSITION 2.4. *The forgetful functor* $\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2) \to \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B)$ *is a weak homotopy equivalence.*

Proof. The idea of the proof is to use the collar of B_0 to allow for more flexibility.

The first step is to replace the decomposition of the control space $B=B_1 \cup_{B_0} B_2$ by another one $B=B'_1 \cup_{B'_0} B'_2$ where

$$B'_0=B_0\times[-1,1], \quad B'_1=B_1\cup_{B_0}B_0\times[-1,0], \quad B'_2=B_2\cup_{B_0}B_0\times[0,1].$$

Using a deformation retraction of the collar of B_0 one can easily show that the inclusion $\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2) \subset \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B'_0,B'_1,B'_2)$ is a homotopy equivalence. Also it is no loss of generality to assume that an object of (Y, Y_0, Y_1, Y_2) of the latter category satisfies the condition that $r^{-1}(X\times B_0\times[-\varepsilon,\varepsilon]) \subset Y_0$ (and hence also $r^{-1}(X\times(B_1\cup_{B_0}B_0\times[-\varepsilon,0])) \subset Y_1$ and $r^{-1}(X\times(B_2\cup_{B_0}B_0\times[0,\varepsilon])) \subset Y_2$ for some ε. Namely just replace the object Y by $Y_1\cup_{Y_0}Y_0\times[-1,1]\cup_{Y_0}Y_2$. This defines a map from $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2)$ to the subcategory of $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B'_0,B'_1,B'_2)$ of objects satisfying that condition. A deformation retraction of the collar again may be used to show that this map is a homotopy equivalence.

Denoting this subcategory of $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}(X;B_0',B_1',B_2')$ by $\mathrm{b}\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')$ we are left to show that the forgetful map

$$\mathrm{b}S.\,\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')\to \mathrm{b}S.\,\mathfrak{R}_{\mathrm{fd}}(X;B)$$

is a homotopy equivalence. This will follow from an application of the *approximation theorem* of [W2]. We have to verify the following

Assertion. Given a map $f\colon Y\to Z$, where Y is an object of $\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')$ and Z is in $\mathfrak{R}_{\mathrm{fd}}(X;B)$, there exists an object Y' in $\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')$, a cofibration $Y\rightarrowtail Y'$, and a bounded homotopy equivalence $f'\colon Y'\to Z$ extending the map f.

We shall assume without further proof the following related property of the category $\mathfrak{R}_{\mathrm{fd}}(X;B)$: Given a map $f\colon Y\to Z$ of (homotopy) finitely dominated objects over $X\times B$, there exists Y' which can be obtained from Y by attaching of cells, and a bounded homotopy equivalence $f'\colon Y'\to Z$ extending f.

Using this property we can use a cell-by-cell argument to verify the assertion above. Namely let $Y^*=Y\cup_{\partial D^n}D^n$ for some n-cell D^n. We have to show that Y^* may be decomposed in the required way to define an object of $\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')$. Let $r\colon D^n\to X\times B$ denote the restriction of the structural retraction of Y^* to D^n. Choose $\varepsilon'<\varepsilon$. By using a suitable subdivision one may find a decomposition $D_1\cup_{D_0}D_2$ of D^n such that $r^{-1}(X\times B_0\times[-\varepsilon',\varepsilon'])\subset D_0\subset r^{-1}(X\times B_0\times[-\varepsilon,\varepsilon])$ and similarly with D_1 and D_2. With this decomposition of D^n we may define $Y_1^*=Y_1\cup_{\partial D^n\cap D_1}D_1$, and similarly with Y_0^* and Y_1^*. One checks that this decomposition actually defines an object in the category $\mathfrak{R}_{\mathrm{fd}}'(X;B_0',B_1',B_2')$. (As an object may have infinitely many controlled cells, one has to be careful to choose the sequence of real numbers ε' in such a way that their limit is greater than zero.) Hence we have verified the assertion above. This finishes the proof of the proposition. □

Notation. If the context is sufficiently clear we shall denote the category of decomposed objects over $X\times B$ simply by $\mathfrak{R}_{\mathrm{fd}}(X;B)$ instead of $\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2)$. The preceding proposition says that this does not make a difference if the category of weak equivalences is the category of bounded homotopy equivalences. The concept of v-equivalence is only defined for decomposed objects anyway. So this should cause no confusion. □

It is also rather easy to analyze the fine notion of weak equivalence, i.e. the category $\mathrm{v}\mathfrak{R}_{\mathrm{fd}}(X;B)$. (Recall that $\mathrm{v}\mathfrak{R}_{\mathrm{fd}}(X;B)$ is shorthand for $\mathrm{v}\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2)$.)

PROPOSITION 2.5. *The forgetful functor*

$$vS.\mathfrak{R}_{fd}(X;B)\to bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_1)\times bS.\mathfrak{R}_{fd}(X;B_2)$$

induced by

$$Y=Y_1\cup_{Y_0}Y_2\mapsto(Y_0,Y_1,Y_2)$$

is a weak homotopy equivalence.

Proof. The category $\mathfrak{R}_{fd}(X;B)$ may be identified with a certain category of cofibrations in $\mathfrak{R}_{fd}(X;B)$ with subobject in $\mathfrak{R}_{fd}(X;B_0)$ and quotient object in the category $\mathfrak{R}_{fd}(X;B_0,B_1)\times\mathfrak{R}_{fd}(X;B_0,B_2)$. Namely, to an object of $\mathfrak{R}_{fd}(X;B_0,B_1,B_2)$ there is associated the following functorial cofibration sequence

$$Y_0\to Y_1\cup_{Y_0}Y_2\to(Y_1\cup_{Y_0}X\times B_0)\cup_{X\times B_0}(Y_2\cup_{Y_0}X\times B_0).$$

Here the notation $\mathfrak{R}_{fd}(X;B_0,B_1)$ is an abbreviation of $\mathfrak{R}_{fd}(X;B_0,B_1,B_0)$ and similarly with the other category. Actually the quotient of this cofibration lives in a certain subcategory of $\mathfrak{R}_{fd}(X;B_0,B_1)\times\mathfrak{R}_{fd}(X;B_0,B_2)$ whose objects are defined by the condition that the restriction to $X\times B_0$ is trivial, i.e. equal to $X\times B_0$ itself. Let us for the moment denote this subcategory by $\mathfrak{R}_{fd}(X;B_0,B_1)^{\#}\times\mathfrak{R}_{fd}(X;B_0,B_2)^{\#}$. This category is a subcategory of $\mathfrak{R}_{fd}(X;B)$ in a natural way. An application of the additivity theorem of [W2] now yields that the map given by

$$Y\mapsto(Y_0,Y_1/Y_0,Y_2/Y_0)$$

induces a weak homotopy equivalence

$$vS.\mathfrak{R}_{fd}(X;B)\to bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_0,B_1)^{\#}\times bS.\mathfrak{R}_{fd}(X;B_0,B_2)^{\#}.$$

(The symbol '/' denotes the quotient in the appropriate category.)

Using the collar of B_0 together with Lemma 1.3. it is easy to see that the functors

$$bS.\mathfrak{R}_{fd}(X;B_0,B_1)^{\#}\to bS.\mathfrak{R}_{fd}(X;B_1)\quad\text{and}\quad bS.\mathfrak{R}_{fd}(X;B_0,B_2)^{\#}\to bS.\mathfrak{R}_{fd}(X;B_2)$$

induced by inclusion are weak homotopy equivalences. It remains to show that this map is homotopic to the map of the proposition. Using the existence of an inverse on the H-space $vS.\mathfrak{R}_{fd}(X;B)$ it follows that the map described by

$$Y\mapsto(Y_0,Y_1/Y_0\vee Y_0,Y_2/Y_0\vee Y_0)$$

also is a weak homotopy equivalence (where '$\vee$' denotes the sum in the appropriate category). Finally another application of the additivity theorem shows that the maps

$$Y \mapsto Y_1 \quad \text{and} \quad Y \mapsto Y_1/Y_0 \vee Y_0$$

are homotopic, and similarly with Y_1 replaced by Y_2. This proves the proposition. $\square$

The main problem will be to analyze the category of bounded acyclic objects together with the fine notion of weak equivalence. There are exact functors defined as follows:

$$\alpha\colon \mathfrak{R}^b_{\mathrm{fd}}(X;B) \to \mathfrak{R}_{\mathrm{fd}}(X;B_1) \times \mathfrak{R}_{\mathrm{fd}}(X;B_2)$$

$$Y \mapsto (Y_1/Y_0;\, Y_2/Y_0)$$

$$\beta\colon \mathfrak{R}^b_{\mathrm{fd}}(X;B) \to \mathfrak{R}_{\mathrm{fd}}(X;B_0)$$

$$Y \mapsto Y_0$$

$$\gamma\colon \mathfrak{R}_{\mathrm{fd}}(X;B_1) \times \mathfrak{R}_{\mathrm{fd}}(X;B_2) \to \mathfrak{R}_{\mathrm{fd}}(X;B)$$

$$(Y,Z) \mapsto Y \cup_{X\times B} Z.$$

PROPOSITION 2.6. *These functors induce the following diagram which commutes up to homotopy. Furthermore the diagram is homotopy cartesian in a way specified below.*

$$\begin{tikzcd}
\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B) \arrow[r, "\alpha"] \arrow[d, "\beta"'] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_1) \times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_2) \arrow[d, "\gamma"] \\
\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[r] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B)
\end{tikzcd}$$

The bottom map is the natural inclusion followed by suspension.

Proof. Consider the following diagram:

$$\begin{tikzcd}
\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B) \arrow[r] \arrow[d] & \mathrm{v}S.\mathfrak{R}_{\mathrm{fd}}(X;B) \arrow[r] \arrow[d] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B) \arrow[d, "="] \\
\mathrm{b}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B) \times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[r] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B) \times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[r] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B)
\end{tikzcd}$$

The left and middle vertical map are given by $Y \mapsto (Y, Y_0)$. The lower right hand map is the projection. The diagram is clearly commutative. Both rows are fibrations up to homotopy, the upper row by Theorem 2.3, the lower one for trivial reasons, since $\mathrm{b}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B)$ is contractible. Hence the left hand square is homotopy cartesian. Now consider the diagram

$$\begin{tikzcd}
\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B) \arrow[r] \arrow[d] & \mathrm{v}S.\mathfrak{R}_{\mathrm{fd}}(X;B) \arrow[r, "a"] \arrow[d, "b"'] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_1)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_2)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[d, "d"] \\
\mathrm{b}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[r] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0) \arrow[r, "c"] & \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)
\end{tikzcd}$$

where the left hand square is identical with that of the preceding diagram, a is the map of Proposition 2.5, b is given by $Y \mapsto (Y, Y_0)$, c by $(Y,Z) \mapsto (Y \vee \Sigma Z, Z)$ and d by $(Y,Z,W) \mapsto (Y/Y_0 \vee Z/Z_0, W)$. The right horizontal maps are homotopy equivalences. This is true for a by Proposition 2.5 and for c since it is a shearing map, using the H-space structure of the spaces involved and the fact that Σ represents a homotopy inverse. The right hand square commutes up to homotopy, by an application of the additivity theorem. Namely we have the following functorial cofibration sequences in $\mathfrak{R}_{\mathrm{fd}}(X;B)$

$$Y_0 \rightarrowtail Y \twoheadrightarrow Y_1/Y_0 \vee Y_2/Y_0.$$

This gives a homotopy between the functors $Y \mapsto Y \vee \Sigma Y_0$ and $Y \mapsto Y_1/Y_0 \vee Y_2/Y_0$. Choose a specific homotopy. Then the right hand square is clearly homotopy cartesian. Since a and c are homotopy equivalences the choice of the homotopy does not matter. Hence the outer square of this diagram is homotopy cartesian in a well defined way. Composing the right horizontal maps with the projection maps away from $\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)$, and using that $\mathrm{b}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B)$ is contractible, finally gives the assertion of the proposition. □

To proceed further we introduce the following terminology. Given an object (Y,r,s) of $\mathfrak{R}_{\mathrm{fd}}(X;B)$ we define the *support* of Y to be the set

$$\operatorname{supp}(Y) = \{b \in B \mid p^{-1}(X \times \{b\}) \neq X\}.$$

We say that Y has *bounded support with respect* to B_0 if there exists a positive real number c such that $\operatorname{supp}(Y)$ has distance at most c from B_0, i.e. for each $b \in \operatorname{supp} Y$ there exists $b' \in B_0$ such that $\varrho(b,b') \leq c$. Denote by $\mathfrak{R}^b_{\mathrm{fd}}(X;B_0)_{\mathrm{b.s.}}$ the subcategory of those objects of $\mathfrak{R}^b_{\mathrm{fd}}(X;B)$ which have bounded support with respect to B_0. We claim that any bounded acyclic object is in fact v-equivalent to an object with bounded support with respect to B_0.

PROPOSITION 2.7. *The inclusion* $\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B_0)_{\mathrm{b.s.}}\to\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B)$ *is a weak homotopy equivalence.*

Proof. We deduce this from the *approximation theorem*. It suffices to verify the following property: Given a map $f\colon(Y,Y_0,Y_1,Y_2)\to(Z,Z_0,Z_1,Z_2)$, where $(Y,\ldots)$ is an object of $\mathfrak{R}^b_{\mathrm{fd}}(X;B)_{\mathrm{b.s.}}$ and $(Z,\ldots)$ is in $\mathfrak{R}^b_{\mathrm{fd}}(X;B)$, there exists an object (Y',Y_0',Y_1',Y_2') of $\mathfrak{R}^b_{\mathrm{fd}}(X;B)_{\mathrm{b.s.}}$ together with a cofibration $(Y,\ldots)\to(Y',\ldots)$ and a v-equivalence $g\colon(Y',\ldots)\to(Z,\ldots)$ such that the following triangle is commutative:

$$\begin{tikzcd}
(Y,Y_0,Y_1,Y_2) \arrow[r, "f"] \arrow[d] & (Z,Z_0,Z_1,Z_2) \\
(Y',Y_0',Y_1',Y_2') \arrow[ur, "g"'] &
\end{tikzcd}$$

Replacing Z by the mapping cylinder of f shows that it is no restriction to assume that f is a cofibration. Also it is no loss of generality to assume that

$$r^{-1}(X\times B_0\times[0,\varepsilon])=Z_0\times[0,\varepsilon]$$

(cf. the proof of Proposition 2.4). Choose a bounded contraction $h\colon Z\times I\to Z$ of Z which exists by definition of the objects in $\mathfrak{R}^b_{\mathrm{fd}}(X;B)$.

Since h is bounded we may find a subcomplex Z_1' of $Z_1-Z_0\times[0,\varepsilon]$ such that $\mathrm{supp}(Z_1')$ avoids a bounded neighborhood of B_0 and such that $h(Z_1'\times I)\subset Z_1$. Namely let $\alpha\colon Z_1\to\mathbf{R}_+$ denote the distance from B_0, i.e. $f(z)=\varrho(p(z);B_0)$. For each $d\in\mathbf{R}_+$ let $Z_1^d=\alpha^{-1}([d,\infty))\cup X\times B$. Since h is bounded there exists a number c such that $h(Z_1^d\times I)\subset Z_1^{d-c}$, if d is sufficiently large. By definition the objects of $\mathfrak{R}_{\mathrm{fd}}(X;B)$ are finite dimensional and also there is a global bound for the diameter of the cells. Therefore there exists a number e (independent of d) such that the smallest subcomplex of Z_1 generated by Z_1^d is contained in Z_1^{d-e}. If d is large enough we may therefore define Z_1' to be the smallest subcomplex containing Z_1^d. (Of course it may happen that $\mathrm{supp}(Z_1')$ is empty.) Since Y_1 has bounded support with respect to B_0 it is also no restriction to assume that $\mathrm{supp}(Z_1')$ is disjoint from $\mathrm{supp}(Y_1)$. Now define a bounded map $h_1\colon Z_1\times I\to Z_1$ as follows.

(i) $h_1|Z_1\times 0=\text{identity}$

(ii) $h_1|Z_0\times[0,\varepsilon]\times I=\text{projection to } I$

(iii) $h_1|Z_1'\times I=h$

(iv) On the remaining part of $Z_1 \times I$ choose some extension (which exists by bounded homotopy extension).

Since $h_1((Z_1 - Z_1') \times I)$ has bounded support one can find another subcomplex Z_0' of Z_1 with support in a (potentially bigger) bounded neighborhood of B_0 such that $h_1((Z_1 - Z_1') \times I) \subset Z_0'$. The end map of the homotopy h_1 maps Z_0' to itself. (In fact if $z \in Z_1'$ then $h_1(z, 1) = h(z, 1) = r(z) \in Z_0'$ since all spaces are considered as spaces over $X \times B$.) Let p_1 denote the restriction of h_1 to $Z_0' \times 1$. We claim that p_1 is an *idempotent* up to homotopy. Namely we have the following commutative diagram

$$\begin{tikzcd}
Z_0' \times I \arrow[r] \arrow[d, dashed] & Z_1 \times I \arrow[r, "(h_1 | Z_1 \times 1) \times \mathrm{id}"] & Z_1 \times I \arrow[d, "h_1"] \\
Z_0' \arrow[rr] & & Z_1
\end{tikzcd}$$

which defines the broken arrow. In fact, if $z \in Z_1' \cap Z_0'$ we have that $h_1(h_1(z, 1), t) = h_1(r(z), t) = r(z)$, and for z in $Z_1 - Z_1'$ the map h_1 takes values in Z_0' anyway by construction of the latter. Hence the left vertical arrow defines a homotopy from p_1 to p_1^2. Now form the *mapping telescope* of p_1. It is defined as the following (iterated) pushout:

$$\mathrm{Tel}(p_1) = \mathrm{pushout}(Z_0' \times [0,1] \leftarrow Z_0' \times 1 \xrightarrow{p_1} Z_0' \times [1,2] \leftarrow Z_0' \times 2 \xrightarrow{p_1} Z_0' \times [2,3] \leftarrow \ldots \xrightarrow{p_1} \ldots).$$

There is a canonical map $\mathrm{Tel}(p_1) \to Z_0'$ defined as the colimit of the following diagram, where g denotes a homotopy $p_1 \simeq p_1^2$.

$$\begin{tikzcd}
Z_0' \times [0,1] \arrow[d, "g"] & Z_0' \times 1 \arrow[l] \arrow[r] \arrow[d, "p_1^2"] & Z_0' \times [1,2] \arrow[d, "g"] & \ldots \arrow[l] \arrow[r] & \ldots \\
Z_0' \arrow[r, "="] & Z_0' \arrow[r, "="] & Z_0' & \ldots
\end{tikzcd}$$

$\mathrm{Tel}(p_1)$ is considered as a space over $X \times B$ via the composition $\mathrm{Tel}(p_1) \to Z_0' \to X \times B$. The bottom inclusion defines a map $Z_0' \to \mathrm{Tel}(p_1)$ in the other direction. The composition $\mathrm{Tel}(p_1) \to Z_0' \to \mathrm{Tel}(p_1)$ is homotopic to the identity. This implies that $\mathrm{Tel}(p_1)$ is a retract up to (bounded) homotopy of Z_0' and hence is itself finitely dominated. Now consider the following diagram:

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[d, equal] & Z_0' \arrow[r] \arrow[d, "p_1"] & Z_1 \arrow[d, "h_1 | Z_1 \times 1"] \arrow[dl] \\
Y_1 \arrow[r] & Z_0' \arrow[r] & Z_1
\end{tikzcd}$$

The map $h_1|Z_1\times 1$ obviously factors over the subcomplex Z_0' which gives the oblique arrow. Also $p_1|Y_1$=identity by construction. Since Tel(–) is a functor we obtain from this diagram a map

$$\mathrm{Tel}(p_1)\to \mathrm{Tel}(h_1|Z_1\times 1).$$

This map is actually a bounded homotopy equivalence. A homotopy inverse is provided by the map of telescopes induced by the oblique arrow. Moreover $h_1|Z_1\times 1$ is clearly a bounded homotopy equivalence (since it is homotopic to the identity). Therefore the map $\mathrm{Tel}(h_1|Z_1\times 1)\to Z_1$ is a bounded homotopy equivalence. Altogether we obtain the following commutative diagram:

$$\begin{tikzcd} Y_1 \arrow[r] \arrow[d, tail] & Z_1 \arrow[ddl, "\simeq"] \\ Z_0' \arrow[d] & \\ \mathrm{Tel}(p_1) & \end{tikzcd}$$

A similar construction may be performed for the Z_2-part of Z leading to an idempotent p_2 and a corresponding telescope. Now define

$$Y' = \mathrm{Tel}(p_2)\cup_{Z_0\times[-\varepsilon,\varepsilon]}\mathrm{Tel}(p_1).$$

By construction $Y\rightarrowtail Y'$ and Y' maps to Z by a v-equivalence. This verifies the approximation hypothesis and hence proves the proposition. □

The next step will be to analyze the category of bounded acyclic objects with bounded support with respect to B_0. Let us begin with the simplest case where B is just the product of B_0 with an interval. So suppose $B=B_0\times[-1,1]$, $B_1=B_0\times[0,1]$, $B_2=B_0\times[-1,0]$. There is an exact functor

$$\varphi\colon \mathfrak{R}_{\mathrm{fd}}(X;B_0)\times\mathfrak{R}_{\mathrm{fd}}(X;B_0)\to\mathfrak{R}^b_{\mathrm{fd}}(X;B_0,B_1,B_2)$$

given by

$$(Y,Z)\mapsto((X\times B_0\cup_Y Y\times[-1,0])\cup_{X\times B_0}(Z\times[0,1]\cup_Z X\times B_0);$$

$$Y\cup_{X\times B_0}Z;(X\times B_0\cup_Y Y\times[-1,0])\cup_{X\times B_0}Z;\ Y\cup_{X\times B_0}(Z\times[0,1]\cup_Z B_0)).$$

In other words, associated to a pair (Y,Z) there is an acyclic decomposed object over

$X \times B$, which has Y (resp. Z) sitting over $X \times B_1$ (resp. $X \times B_2$) and which is determined more or less by this condition. Let us call an object in the image of φ a *standard acyclic object*.

PROPOSITION 2.8. *If the control space B is of the form described above the functor φ induces a weak homotopy equivalence*

$$\varphi\colon \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\to \mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B).$$

Proof. Consider the composite

$$\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\xrightarrow{\varphi} \mathrm{v}S.\mathfrak{R}^d_{\mathrm{fd}}(X;B)\xrightarrow{\alpha} \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_1)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_2)$$

where α is the map $Y \mapsto (Y_1/Y_0;\, Y_2/Y_0)$. The composite $\alpha\varphi$ is given by

$$(Y,Z)\mapsto(\Sigma_{B_1}Y;\Sigma_{B_2}Z).$$

Since in the case at hand B_1 and B_2 are both controlled homotopy equivalent to B_0, and suspension induces a homotopy equivalence on the category $\mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)$, the composite map is therefore a weak homotopy equivalence. Hence φ is a homotopy quivalence if and only if α is one. The map α is just the top map of the diagram of Proposition 2.6. Since $B_0 \xrightarrow{\simeq} B$, the bottom map of that diagram is a homotopy equivalence and hence so is α. This was to be shown. □

We now wish to extend this result to a more general context. The first observation is the following

LEMMA 2.9. *Let $(B,B_0,B_1,B_2)\to(B',B_0',B_1',B_2')$ denote a v-equivalence, i.e. all the maps $B_i\to B_i'$ $(i=0,1,2)$ are controlled homotopy equivalences. Then the induced map*

$$\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B_0,B_1,B_2)\to \mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B_0',B_1',B_2')$$

is a weak homotopy equivalence.

Proof. By an application of the generic fibration theorem, one reduces to proving that the map $\mathrm{v}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0,B_1,B_2)\to \mathrm{v}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0',B_1',B_2')$ is a weak homotopy equivalence. This in turn follows by an application of Lemma 1.3. together with Proposition 2.5. □

Now suppose that $B=B_1\cup_{B_0}B_2$ and $B_0\rightarrowtail B_1$ and $B_0\rightarrowtail B_2$ are both controlled homotopy equivalences. Then the conclusion of Proposition 2.7 still holds, because there is a v-equivalence $(B_0, B_0\times[0,1], B_0\times[-1,0])\to(B_0, B_1, B_2)$. Consider again the case of open cones. Let $K_0\rightarrowtail K_1, K_0\rightarrowtail K_2$ denote inclusions of finite subcomplexes of some sphere S^n, and let K denote the pushout $K_1\cup_{K_0}K_2$. Assume that K_0 is bicollared in K. Let B (resp. B_0, B_1, B_2) denote the open cone on K (resp. K_0, K_1, K_2). There is a filtration

$$B_0 = B^{(0)}\subset B^{(1)}\subset\ldots\subset B^{(i)}\subset\ldots$$

inducing filtrations $B_0^{(i)}$ (resp. $B_1^{(i)}, B_2^{(i)}$) of B_0 (resp. B_1, B_2) satisfying that

(i) $B=\bigcup B^{(i)}$, $B_k=\bigcup B_k^{(i)}$ $(k=0,1,2)$

(ii) $B_k^{(i)}\rightarrowtail B_k^{(i+1)}$ is a controlled homotopy equivalence (in fact a controlled deformation retraction).

(iii) Let $U_i(B_0)=\{b\in B\,|\,\varrho(b,B_0)\leq i\}$. Then $U_i(B_0)\subset B^{(i)}$.

In particular there are controlled homotopy equivalences $B_0\to B_1^{(i)}, B_0\to B_2^{(i)}$ for all i. The filtration is defined as follows. Let $o_\alpha(K)=K\times[\alpha,\infty)$ considered as a subset of $o(K)$ with the induced metric. Likewise let $c_\alpha(K)=K\times[0,\alpha]/K\times 0$, i.e. the closure of $o(K)-o_\alpha(K)$. Recall that $U_i(B_0)$ denotes the set of points of B with distance $\leq i$. For each i there exists a real number $\alpha(i)$ such that $U_i(B_0)\cap o_{\alpha(i)}(K)\subset o(\text{collar}(K_0))$. Using the linear structure of the collar we may deformation retract this space to its "core", i.e. $o_{\alpha(i)}(K_0)$. Furthermore, this is a controlled deformation since the points of $U_i(B_0)$ have all bounded distance from B_0. Now let $B^{(i)}=(U_i(B_0)\cap o_{\alpha(i)}(K))\cup c_{\alpha(i)}(K)$, and $B_k^{(i)}$ similarly.

In the preceding we insisted on the condition that B_0 be bicollared in B. This condition is not satisfied in the case of open cones. But we may replace the control space B by $B'=B_1\cup_{B_0\times I}B_2$. Since the map $B'\to B$ retracting the collar inserted is a controlled homotopy equivalence, it will suffice by Lemma 1.3 to prove excision for B'. Using the collar of B_0 it is possible to extend the definition of the map φ, and we claim

PROPOSITION 2.10. *With notations as above, the map*

$$\varphi\colon \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\times \mathrm{b}S.\mathfrak{R}_{\mathrm{fd}}(X;B_0)\to \mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B')$$

given by the inclusion of standard acyclic objects is a weak homotopy equivalence.

Proof. The map φ actually takes values in the simplicial subcategory $\mathrm{v}S.\mathfrak{R}^b_{\mathrm{fd}}(X;B')_{\mathrm{b.s.}}$ of objects with bounded support with respect to B_0. Since by Proposition 2.7 the inclusion of this subcategory is a weak homotopy equivalence it suffices to show that φ is a homotopy equivalence when considered as a map with range in that subcategory. The filtration of B' described above induces a corresponding filtration of the category $\mathfrak{C}=vS.\mathfrak{R}^b_{fd}(X;B')_{b.s.}$ by property (iii) above. Let $\mathfrak{C}^{(i)}$ denote this filtration. By Lemma 2.9. the inclusions $\mathfrak{C}^{(i)}\to\mathfrak{C}^{(i+1)}$ are weak homotopy equivalences (for $i\geq 1$). But the control space $(B'^{(i)}, B_0'^{(i)}, B_1'^{(i)}, B_2'^{(i)})$ is v-equivalent to $(B_0\times[-1,1], B_0\times 0, B_0\times[0,1], B_0\times[-1,0])$ (for $i\geq 1$). Hence by Lemma 2.9. again and by Proposition 2.8. the map $: bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_0)\to\mathfrak{C}^{(i)}$ is a weak homotopy equivalence if $i\geq 1$. Since $\mathfrak{C}=\bigcup\mathfrak{C}^{(i)}$ the inclusions $\mathfrak{C}^{(i)}\to\mathfrak{C}$ are also homotopy equivalences. This proves that $\varphi: bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_0)\to\mathfrak{C}$ is a weak homotopy equivalence as asserted. □

We are now ready to conclude

THEOREM 2.11. *The functor from (finite PL subcomplexes of S^∞) to (spaces) given by*

$$K\mapsto A(X;o(K))$$

is a reduced generalized homology theory.

Proof. We already know by Corollary 2.2. that the functor $K\mapsto A(X;o(K))$ is pointed. So we have to show that it has the excision property and that it is a homotopy functor, i.e. it takes a weak homotopy equivalence $K\xrightarrow{\sim}K'$ to a weak homotopy equivalence $A(X;o(K))\to A(X;o(K'))$. As to the first part, consider the excision situation as above: $K=K_1\cup_{K_0}K_2$; K_0 is bicollared in K; K_0 and K_1 are subcomplexes of K. We must show that the square

$$\begin{tikzcd}
A(X;o(K_0)) \arrow[r] \arrow[d] & A(X;o(K_1)) \arrow[d] \\
A(X;o(K_2)) \arrow[r] & A(X;o(K))
\end{tikzcd}$$

is homotopy cartesian. Consider the following diagram. (Let B_i denote $o(K_i)$ again.)

$$\begin{tikzcd}
bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_0) \arrow[r, "\varphi"] \arrow[d, "\beta\varphi"] & vS.\mathfrak{R}^b_{fd}(X;B') \arrow[r, "\alpha"] \arrow[d, "\beta"] & bS.\mathfrak{R}_{fd}(X;B_1)\times bS.\mathfrak{R}_{fd}(X;B_2) \arrow[d, "\gamma"] \\
bS.\mathfrak{R}_{fd}(X;B_0) \arrow[r, "="] & bS.\mathfrak{R}_{fd}(X;B_0) \arrow[r] & bS.\mathfrak{R}_{fd}(X;B)
\end{tikzcd}$$

The right hand square is homotopy cartesian by Proposition 2.6, the map φ is a weak homotopy equivalence by Proposition 2.10. Hence the outer square is homotopy cartesian. This implies that the following square (which is obtained from the preceding one by crossing the lower row with the identity functor on $bS.\mathfrak{R}_{fd}(X;B_2)$ and adjusting the vertical maps accordingly) is also homotopy cartesian

$$\begin{array}{ccc} bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_0) & \xrightarrow{\alpha\varphi} & bS.\mathfrak{R}_{fd}(X;B_1)\times bS.\mathfrak{R}_{fd}(X;B_2) \\ \downarrow & & \downarrow \\ bS.\mathfrak{R}_{fd}(X;B_0)\times bS.\mathfrak{R}_{fd}(X;B_2) & \longrightarrow & bS.\mathfrak{R}_{fd}(X;B)\times bS.\mathfrak{R}_{fd}(X;B_2). \end{array}$$

Using again the fact that (the geometric realizations of) the spaces in this diagram are H-spaces with an inverse, we may cancel one factor $bS.\mathfrak{R}_{fd}(X;B_0)$ in the left column and one factor $bS.\mathfrak{R}_{fd}(X;B_2)$ in the right column. We obtain a diagram which upon geometric realization and looping gives the desired excision diagram.

For the second part of the theorem observe that it is sufficient to show that the map $A(X;o(K\times I))\to A(X;o(K))$ induced by the projection is a weak homotopy equivalence. Now from the cofibration sequence $K\rightarrowtail K\times I\twoheadrightarrow cK$ and from the excision property just proved, we obtain a fibration up to homotopy

$$A(X;o(K))\to A(X;o(K\times I))\to A(X;o(cK)).$$

(Actually it is not quite correct to apply the excision property directly to this cofibration sequence because K is not bicollared in cK. Therefore to be quite precise one should replace cK by $cK\cup_K K\times I$ which is isomorphic to cK.) By Corollary 2.2. the bottom term of this fibration is contractible. Therefore the map $A(X;o(K))\to A(X;o(K\times I))$ is a weak homotopy equivalence, and hence so is the map induced by projection which is a retraction of this map. This ends the proof of the theorem. □

Remark. One checks that the argument still works for $K=\varnothing$. Then $o(K)=*$. In view of the decomposition $\mathbf{R}=\mathbf{R}_+\cup_*\mathbf{R}_-$ this implies that the square

$$\begin{array}{ccc} A(X;*) & \longrightarrow & A(X;\mathbf{R}_+) \\ \downarrow & & \downarrow \\ A(X;\mathbf{R}_-) & \longrightarrow & A(X;\mathbf{R}) \end{array}$$

is homotopy cartesian. By Theorem 2.1. the lower left and upper right terms are contractible. The upper left term is clearly the same as $A(X)$. Hence we obtain a weak homotopy equivalence $\Omega A(X;\mathbf{R})\simeq A(X)$. It is also easy to verify that the decomposition technique also works for a control space of the form $B\times\mathbf{R}$, where B is not assumed to be an open cone. By the same argument as above we obtain:

THEOREM 2.12. *For every proper metric space B there is a weak homotopy equivalence*

$$\Omega A(X;B\times\mathbf{R})\to A(X;B). \qquad \square$$

Remark. From the theorem it follows that $\Omega^n A(X;o(S^{n-1}))=\Omega^n A(X;\mathbf{R}^n)\simeq A(X)$. Furthermore it is true that $\pi_i A(X;\mathbf{R}^n)=K_{i-n}(\mathbf{Z}[\pi_1 X])$ for $i\leq n$. The proof of this statement is given in [V]. Hence $[n]\mapsto A(X;\mathbf{R}^n)$ is a *non-connective* spectrum whose connective cover is the usual spectrum of $A(X)$.

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