INVARIANTS AND FUNDAMENTAL FUNCTIONS

BY

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Introduction

Let E be a finite-dimensional vector space over $\mathbf{R}$ and G a group of linear transformations of E leaving invariant a nondegenerate quadratic form B. The action of G on E extends to an action of G on the ring of polynomials on E. The fixed points, the *G-invariants,* form a subring. The *G-harmonic* polynomials are the common solutions of the differential equations formed by the G-invariants. Under some general assumptions on G it is shown in § 1 that the ring of all polynomials on E is spanned by products ih where i is a G-invariant and h is G-harmonic, and that the G-harmonic polynomials are of two types:

1. Those which vanish identically on the algebraic variety N_G determined by the G-invariants;

2. The powers of the linear forms given by points in N_G.

The analogous situation for the exterior algebra is examined in § 2.

Section 3 is devoted to a study of the functions on the real quadric $B=1$ whose translates under the orthogonal group $\mathbf{O}(B)$ span a finite-dimensional space. The main result of the paper (Theorem 3.2) states that (if $\dim E > 2$) these functions can always be extended to polynomials on E and in fact to $\mathbf{O}(B)$-harmonic polynomials on E due to the results of § 1.

The results of this paper along with some others have been announced in a short note [9].

§ 1. Decomposition of the symmetric algebra

Let E be a finite-dimensional vector space over a field K, let E^* denote the dual of E and $S(E^*)$ the algebra of K-valued polynomial functions on E. The sym-

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metric algebra $S(E)$ will be identified with $S((E^*)^*)$ by means of the extension of the canonical isomorphism of E onto $(E^*)^*$.

Now suppose K is the field of real numbers $\mathbf{R}$, and let $C^\infty(E)$ be the set of differentiable functions on E. Each $X \in E$ gives rise (by parallel translation) to a vector field on E which we consider as a differential operator $\partial(X)$ on E. Thus, if $f \in C^\infty(E)$, $\partial(X)f$ is the function $Y \to \{(d/dt)(f(Y+tX))\}_{t=0}$ on E. The mapping $X \to \partial(X)$ extends to an isomorphism of the symmetric algebra $S(E)$ (respectively, the complex symmetric algebra $S^c(E) = \mathbf{C} \otimes S(E)$) onto the algebra of all differential operators on E with constant real (resp. complex) coefficients.

Let H be a subgroup of the general linear group $\mathbf{GL}(E)$. Let $I(E)$ denote the set of *H-invariants* in $S(E)$ and let $I_+(E)$ denote the set of H-invariants without constant term. The group H acts on E^* by

$$(h \cdot e^*)(e) = e^*(h^{-1} \cdot e), \qquad h \in H, \ e \in E, \ e^* \in E^*,$$

and we have $I_+(E^*) \subset I(E^*) \subset S(E^*)$. An element $p \in S^c(E^*)$ is called *H-harmonic* if $\partial(J)p = 0$ for all $J \in I_+(E)$. Let $H^c(E^*)$ denote the set of H-harmonic polynomial functions and put $H(E^*) = S(E^*) \cap H^c(E^*)$. Let $I^c(E)$ and $I^c(E^*)$, respectively, denote the subspaces of $S^c(E)$ and $S^c(E^*)$ generated by $I(E)$ and $I(E^*)$. Each polynomial function $p \in S^c(E^*)$ extends uniquely to a polynomial function on the complexification E^c, also denoted by p. Let N_H denote the variety in E^c defined by

$$N_H = \{X \in E^c \mid j(X) = 0 \text{ for all } j \in I_+(E^*)\}.$$

Now suppose B_0 is a nondegenerate symmetric bilinear form on $E \times E$; let B denote the unique extension of B_0 to a bilinear form on $E^c \times E^c$. If $X \in E^c$, let X^* denote the linear form $Y \to B(X, Y)$ on E. The mapping $X \to X^*$ $(X \in E)$ extends uniquely to an isomorphism μ of $S^c(E)$ onto $S^c(E^*)$. Under this isomorphism B_0 gives rise to a bilinear form on $E^* \times E^*$ which in a well-known fashion ([5]) extends to a bilinear form $\langle , \rangle$ on $S^c(E^*) \times S^c(E^*)$. The formula for $\langle , \rangle$ is

$$\langle p, q \rangle = [\partial(\mu^{-1} p) q](0), \qquad p, q \in S^c(E^*),$$

where for any operator A: $C^\infty(E) \to C^\infty(E)$, and any function $f \in C^\infty(E)$, $[Af](X)$ denotes the value of the function Af at X. The bilinear form $\langle , \rangle$ is still symmetric and nondegenerate. Moreover, if $p, q, r \in S^c(E^*)$ and $Q = \mu^{-1}(q)$, $R = \mu^{-1}(r)$, then

$$\langle p, qr \rangle = [\partial(QR)p](0) = [\partial(Q)\partial(R)p](0) = [\partial(R)\partial(Q)p](0) = \langle \partial(Q)p, r \rangle,$$

which shows that multiplication by $\mu(Q)$ is the adjoint operator to the operator $\partial(Q)$.

Now suppose H leaves B_0 invariant; then $\langle,\rangle$ is also left invariant by H and

$$\mu(I^c(E)) = I^c(E^*). \tag{1}$$

Let P be a homogeneous element in $S^c(E)$ of degree k. If n is an integer $\geqslant k$ then the relation

$$\partial(P)((X^*)^n) = n(n-1)\dots(n-k+1)\,\mu(P)(X)(X^*)^{n-k}, \qquad X \in E^c, \tag{2}$$

can be verified by a simple computation. In particular if $X \in N_H$ then $(X^*)^n$ is a harmonic polynomial function. Let $H_1(E^*)$ denote the vector space over $\mathbf{C}$ spanned by the functions $(X^*)^n$, $(n=0,1,2,\dots, X \in N_H)$ and let $H_2(E^*)$ denote the set of harmonic polynomial functions which vanish identically on N_H.

If A is a subspace of $S^c(E^*)$ and k an integer $\geqslant 0$, A_k shall denote the set of elements in A of degree k; A is called *homogeneous* if $A = \sum_{k\geqslant 0} A_k$. The spaces $I(E^*)$, $H(E^*)$, $H_1(E^*)$ and the ideal $I_+(E^*)\,S(E^*)$ are clearly homogeneous.

LEMMA 1.1. *$H_2(E^*)$ is homogeneous.*

Proof. Let $A = I_+^c(E^*)\,S^c(E^*)$. Then N_H is the variety of common zeros of elements of the ideal A. By Hilbert's Nullstellensatz (see e.g. [18], p. 164), the polynomials in $S^c(E^*)$ which vanish identically on N_H constitute the radical $\sqrt{A}$ of A, that is the set of elements in $S^c(E^*)$ of which some power lies in A. Since A is homogeneous, $\sqrt{A}$ is easily seen to be homogeneous so the lemma follows from $H_2(E^*) = H^c(E^*) \cap \sqrt{A}$.

If C and D are subspaces of an associative algebra then CD shall denote the set of all finite sums $\sum_i c_i d_i$ $(c_i \in C, d_i \in D)$.

THEOREM 1.2. *Let U be a compact group of linear transformations of a vector space W_0 over $\mathbf{R}$. Then*

$$S(W_0^*) = I(W_0^*)\,H(W_0^*). \tag{3}$$

Let B_0 be any strictly positive definite symmetric bilinear form on $W_0 \times W_0$ invariant under U (such a B_0 exists). Then $H^c(W_0^*)$ is the orthogonal direct sum,

$$H^c(W_0^*) = H_1(W_0^*) + H_2(W_0^*). \tag{4}$$

Proof. Using an orthonormal basis of W_0 it is not hard to verify that the bilinear form $\langle,\rangle$ is now strictly positive definite on $S(W_0^*) \times S(W_0^*)$. On combining this fact with the remark above about the adjoint of $\partial(Q)$ the orthogonal decomposition

$$S(W_0^*)_k = (I_+(W_0^*)\,S(W_0^*))_k + H(W_0^*)_k \tag{5}$$

is quickly established for each integer $k \geqslant 0$. Now (3) follows by iteration of (5). In order to prove (4) consider the orthogonal complement M of $(H_1(W_0^*))_k$ in $(H^c(W_0^*))_k$. Let $q \in (H^c(W_0^*))_k$, $Q = \mu^{-1}(q)$. Then $q \in M \Leftrightarrow [\partial(Q)h](0) = 0$ for all $h \in (H_1(W_0^*))_k \Leftrightarrow \partial(Q)((X^*)^k) = 0$ for all $X \in N_U$. In view of (2) this last condition amounts to q vanishing identically on N_U; consequently $M = (H_2(W_0^*))_k$. This proves the formula (4) since all terms in it are homogeneous.

Remark 1. Theorem 1.2 was proved independently by B. Kostant who has also sharpened it substantially in the case when W_0 is a compact Lie algebra and U is its adjoint group (see [11 a]).

Remark 2. In the case when U is the orthogonal group $\mathbf{O}(n)$ acting on $W_0 = \mathbf{R}^n$ then $I(W_0^*)$ consists of all polynomials in $x_1^2 + \dots + x_n^2$ and $H(W_0^*)$ consists of all polynomials $p(x_1, \dots x_n)$ which satisfy Laplace's equation. In view of (5) a harmonic polynomial $\not\equiv 0$ is not divisible by $x_1^2 + \dots + x_n^2$. Since the ideal $(x_1^2 + \dots + x_n^2)\, S^c(W_0^*)$ equals its own radical, $H_2(W_0^*) = 0$ in this case. Theorem 1.2 therefore states that each polynomial $p = p(x_1, \dots x_n)$ can be decomposed $p = \sum_k (x_1^2 + \dots + x_n^2)^k h_k$ where h_k is harmonic and that the complex polynomials $(a_1 x_1 + \dots + a_n x_n)^k$ where $a_1^2 + \dots + a_n^2 = 0$, $k = 0, 1 \dots$ span the space of all harmonic polynomials. These facts are well known (see e.g. [2], p. 285 and [13]).

THEOREM 1.3. *Let V_0 be a finite-dimensional vector space over $\mathbf{R}$ and let G_0 be a connected semisimple Lie subgroup of $\mathbf{GL}(V_0)$ leaving invariant a nondegenerate symmetric bilinear form B_0 on $V_0 \times V_0$. Then*

$$S(V_0^*) = I(V_0^*)\, H(V_0^*),$$

$$H^c(V_0^*) = H_1(V_0^*) + H_2(V_0^*), \quad (\textit{direct sum}).$$

We shall reduce this theorem to Theorem 1.2 by use of an arbitrary compact real form $\mathfrak{u}$ of the complexification $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}_0$ of G_0. Let V denote the complexification of V_0 and let B denote the unique extension of B_0 to a bilinear form on $V \times V$. The Lie algebra $\mathfrak{gl}(V_0)$ of $\mathbf{GL}(V_0)$ consists of all endomorphisms of V_0 and $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{gl}(V_0)$. Consequently the complexification $\mathfrak{g}$ is a subalgebra of the Lie algebra $\mathfrak{gl}(V)$ of all endomorphisms of V. Let U and G denote the connected Lie subgroups of $\mathbf{GL}(V)$ (considered as a real Lie group) which correspond to $\mathfrak{u}$ and $\mathfrak{g}$ respectively. The elements of G_0 extend uniquely to endomorphisms of V whereby G_0 becomes a Lie subgroup of G leaving B invariant. This implies that

$$B(T \cdot Z_1, Z_2) + B(Z_1, T \cdot Z_2) = 0, \qquad Z_1, Z_2 \in V,\ T \in \mathfrak{g}_0. \tag{6}$$

However, since $(T_1+iT_2)\cdot Z = T_1\cdot Z + iT_2\cdot Z$ for $T_1, T_2 \in \mathfrak{g}_0$, $Z\in V$ it is clear that (6) holds for all $T\in\mathfrak{g}$ so, by the connectedness of G, B is left invariant by G.

LEMMA 1.4. *There exists a real form W_0 of V on which B is strictly positive definite and which is left invariant by U.*

Proof. By the usual reduction of quadratic forms the space V_0 is an orthogonal direct sum $V_0 = V_0^- + V_0^+$ where V_0^- and V_0^+ are vector subspaces on which $-B_0$ and B_0, respectively, are strictly positive definite. Let J denote the linear transformation of V determined by

$$JZ = iZ \text{ for } Z\in V_0^-, \quad JZ = Z \text{ for } Z \in V_0^+.$$

Then the bilinear form

$$B'(Z_1, Z_2) = B(JZ_1, JZ_2) \qquad (Z_1, Z_2 \in V),$$

is strictly positive definite on V_0. Let $\mathbf{O}(B), \mathbf{O}(B') \subset \mathbf{GL}(V)$ denote the orthogonal groups of B and B' respectively and let $\mathbf{O}(B_0')$ denote the subgroup of $\mathbf{O}(B')$ which leaves V_0 invariant, i.e. $\mathbf{O}(B_0') = \mathbf{O}(B') \cap \mathbf{GL}(V_0)$. Now

$$U \subset G \subset \mathbf{O}(B) = J\mathbf{O}(B')J^{-1}.$$

On the other hand, the identity component of the group $J\mathbf{O}(B_0')J^{-1}$ is a maximal compact subgroup of the identity component of $J\mathbf{O}(B')J^{-1}$. By an elementary special case of Cartan's conjugacy theorem, (see e.g. [10] p. 218), this last group contains an element g such that $g^{-1}Ug \subset J\mathbf{O}(B_0')J^{-1}$. Then the real form $W_0 = gJV_0$ of V has the properties stated in the lemma. In fact, $U\cdot W_0 \subset W_0$ is obvious and if $X\in W_0$ then since $J^{-1}g^{-1}J \in \mathbf{O}(B')$, we have

$$B(X,X) = B'(J^{-1}X, J^{-1}X) = B'(J^{-1}g^{-1}X, J^{-1}g^{-1}X) \geqslant 0.$$

Now the bilinear form B is nondegenerate on $V_0\times V_0$, $W_0\times W_0$ and $V\times V$. As remarked before this induces the isomorphisms

$$\mu_1\colon S^c(V_0)\to S^c(V_0^*), \quad \mu_2\colon S^c(W_0)\to S^c(W_0^*), \quad \mu\colon S(V)\to S(V^*),$$

all of which are onto. By restriction of a complex-valued function on V to V_0 and to W_0 respectively we get the isomorphisms

$$\lambda_1\colon S(V^*)\to S^c(V_0^*), \quad \lambda_2\colon S(V^*)\to S^c(W_0^*),$$

both of which are onto. Since $S(V) = S((V^*)^*)$ we get by restricting complex-valued functions on V^* to V_0^* and to W_0^* respectively, the isomorphisms

$$\Lambda_1\colon S(V)\to S^c(V_0), \quad \Lambda_2\colon S(V)\to S^c(W_0).$$

Then we have the commutative diagram

$$\begin{array}{ccccc} S^c(V_0) & \xleftarrow{\Lambda_1} & S(V) & \xrightarrow{\Lambda_2} & S^c(W_0) \\ \downarrow{\scriptstyle \mu_1} & & \downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle \mu_2} \\ S^c(V_0^*) & \xleftarrow{\lambda_1} & S(V^*) & \xrightarrow{\lambda_2} & S^c(W_0^*) \end{array}$$

Corresponding to the actions of G_0 on V_0, of U on W_0 and of G on V we consider the spaces of invariants $I^c(V_0^*)$, $I^c(V_0)$, $I^c(W_0)$, $I^c(W_0^*)$ and $I(V)$, $I(V^*)$.

LEMMA 1.5. *Let* $\lambda = \lambda_2 \lambda_1^{-1}$, $\Lambda = \Lambda_2 \Lambda_1^{-1}$. *Then*

$$\lambda(I^c(V_0^*)) = I^c(W_0^*), \quad \Lambda(I^c(V_0)) = I^c(W_0).$$

Proof. Since $G_0 \subset G$ it is clear that $\lambda_1(I(V^*)) \subset I^c(V_0^*)$. On the other hand, let $p \in I^c(V_0^*)$. If $Z \in \mathfrak{g}_0$ let d_Z denote the unique derivation of $S^c(V_0^*)$ which satisfies $(d_Z \cdot v^*)(X) = v^*(Z \cdot X)$ for $v^* \in V_0^*$, $X \in V_0$. Then

$$d_Z \cdot p = 0. \tag{7}$$

Let $(X_1, \ldots, X_n)$ be a basis of V_0, $(x_1, \ldots, x_n)$ the dual basis of V_0^*, $(z_1, \ldots, z_n)$ the basis of V^* dual to $(X_1, \ldots, X_n)$ considered as a basis of V. Then (7) is an identity in $(x_1, \ldots, x_n)$ which remains valid after the substitution $x_1 \to z_1, \ldots, x_n \to z_n$. This means that

$$\delta_Z \cdot (\lambda_1^{-1} p) = 0, \tag{8}$$

where δ_Z is the derivation of $S(V^*)$ which satisfies $(\delta_Z \cdot v^*)(X) = v^*(Z \cdot X)$ for $v^* \in V^*$, $X \in V$. However δ_Z can be defined for all $Z \in \mathfrak{g}$ by this last condition and (8) remains valid for all $Z \in \mathfrak{g}$. Since G is connected, this implies $\lambda_1^{-1} p \in I(V^*)$. Thus $\lambda_1(I(V^*)) = I^c(V_0^*)$; similarly $\lambda_2(I(V^*)) = I^c(W_0^*)$ and the first statement of the lemma follows. The second statement follows from the first, taking into account (1) and the diagram above.

LEMMA 1.6. *Let* $P \in S^c(V_0)$, $q \in S^c(V_0^*)$. *Then*

$$\partial(\Lambda P)(\lambda q) = \lambda(\partial(P) q). \tag{9}$$

Proof. First suppose $P = X \in V_0$, $q = \mu_1(Y)$ $(Y \in V_0)$. In this case one verifies easily that both sides of (9) reduce to $B(X, Y)$. Next observe that the mappings $q \to \partial(\Lambda X)\lambda q$ and $q \to \lambda(\partial(X) q)$ are derivations of $S^c(V_0^*)$ which coincide on V_0^*, hence on all of $S^c(V_0^*)$. Since the mappings $P \to \partial(\Lambda P)$ and $P \to \partial(P)$ are isomorphisms, (9) follows in general.

Combining the two last lemmas we get

LEMMA 1.7. $\lambda(H^c(V_0^*)) = H^c(W_0^*)$.

Now we apply the isomorphism λ^{-1} to the relation (3) in Theorem 1.2. Using Lemmas 1.5 and 1.7 we get the first formula in Theorem 1.3. Next we note that due to Lemma 1.5 the varieties N_U and N_{G_0} coincide. Consequently $\lambda(H_i(V_0^*)) = H_i(W_0^*)$ $(i=1,2)$ so Theorem 1.3 follows.

Remark. In the case when the ideals $I_+^c(W_0^*)\,S^c(W_0^*)$ and $I_+^c(V_0^*)\,S^c(V_0^*)$ are prime ideals they are equal to their own radicals. Hence it follows from (5) (and the analogous relation for V_0^*) that $H_2(W_0^*) = H_2(V_0^*) = \{0\}$. In this case Theorems 1.2 and 1.3 are contained in the results of Maass [13], proved quite differently.

§ 2. Decomposition of the exterior algebra

Let E be a finite-dimensional vector space over $\mathbf{R}$ as in § 1 and let $\Lambda(E)$ and $\Lambda(E^*)$, respectively, denote the Grassmann algebras over E and its dual. Each $X \in E$ induces an anti-derivation $\delta(X)$ of $\Lambda(E^*)$ given by

$$\delta(X)\,(x_1 \wedge \ldots \wedge x_m) = \sum_{k=1}^{m} (-1)^{k-1} x_k(X)\,(x_1 \wedge \ldots \wedge \hat{x}_k \wedge \ldots \wedge x_m),$$

where $\hat{x}_k$ indicates omission of x_k. The mapping $X \to \delta(X)$ extends uniquely to a homomorphism of the tensor algebra $T(E)$ over E into the algebra of all endomorphisms of $\Lambda(E^*)$. Since $\delta(X \otimes X) = \delta(X)^2 = 0$ there is induced a homomorphism $P \to \delta(P)$ of $\Lambda(E)$ into the algebra of endomorphisms of $\Lambda(E^*)$. As will be noted below, this homomorphism is actually an isomorphism.

Now suppose B is any nondegenerate symmetric bilinear form on $E \times E$. The mapping $X \to X^*\,(X^*(Y) = B(X, Y))$ extends to an isomorphism μ of $\Lambda(E)$ onto $\Lambda(E^*)$. We obtain a bilinear form $\langle , \rangle$ on $\Lambda(E^*) \times \Lambda(E^*)$ by the formula

$$\langle p, q \rangle = [\delta(\mu^{-1}(p))\,q]\,(0). \tag{1}$$

If $x_1, \ldots, x_k, y_1, \ldots, y_l \in E^*$ then

$$\langle x_1 \wedge \ldots \wedge x_k,\, y_1 \wedge \ldots \wedge y_l \rangle = 0 \quad \text{or} \quad (-1)^{\frac{1}{2}k(k-1)} \det\,(B(\mu^{-1}x_i, \mu^{-1}y_j)), \tag{2}$$

depending on whether $k \neq l$ or $k = l$. It follows that $\langle , \rangle$ is a symmetric nondegenerate bilinear form. Also if $Q \in \Lambda(E)$, $q = \mu(Q)$ then the operator $p \to p \wedge q$ on $\Lambda(E^*)$ is the adjoint of the operator $\partial(Q)$. It is also easy to see from (1) and (2) that the mapping

$P \to \delta(P)$ $(P \in \Lambda(E))$ above is an isomorphism. Finally, if B is strictly positive definite the same holds for $\langle , \rangle$.

Now let G be a group of linear transformations of E. Then G acts on E^* as before and acts as a group of automorphisms of $\Lambda(E)$ and $\Lambda(E^*)$. Let $J(E)$ and $J(E^*)$ denote the set of G-invariants in $\Lambda(E)$ and $\Lambda(E^*)$ respectively; let $J_+(E)$ and $J_+(E^*)$ denote the subspaces consisting of all invariants without constant term. An element $p \in \Lambda(E^*)$ is called *G-primitive* if $\delta(Q)p = 0$ for all $Q \in J_+(E)$. Let $P(E^*)$ denote the set of all G-primitive elements.

THEOREM 2.1. *Let B be a nondegenerate, symmetric bilinear form on $E \times E$ and let G be a Lie subgroup of* $\mathbf{GL}(E)$ *leaving B invariant. Suppose that either* (i) *G is compact and B positive definite or* (ii) *G is connected and semisimple. Then*

$$\Lambda(E^*) = J(E^*)\, P(E^*). \tag{3}$$

The proof is quite analogous to that of Theorems 1.2 and 1.3. For the case (i) one first establishes the orthogonal decomposition

$$\Lambda(E^*) = \Lambda(E^*)\, J_+(E^*) + P(E^*) \tag{4}$$

in the same manner as (5) in § 1. Then (3) follows by iteration of (4). The non-compact case (ii) can be reduced to the case (i) by using Lemma 1.4. We omit the details since they are essentially a duplication of the proof of Theorem 1.3.

Example. Let V be an n-dimensional Hilbert space over $\mathbf{C}$. Considering the set V as a $2n$-dimensional vector space E over $\mathbf{R}$ the unitary group $\mathbf{U}(n)$ becomes a subgroup G of the orthogonal group $\mathbf{O}(2n)$. Let $Z_k = X_k + iY_k$ $(1 \leqslant k \leqslant n)$ be an orthonormal basis of V, $z_1, \ldots, z_n$ the dual basis of V^*, and put $x_k = \frac{1}{2}(z_k + \bar{z}_k)$, $y_k = -\frac{1}{2}i(z_k - \bar{z}_k)$, $(1 \leqslant k \leqslant n)$.

Let F denote the vector space over $\mathbf{C}$ consisting of all $\mathbf{R}$-linear mappings of V into $\mathbf{C}$. The exterior algebra $\Lambda(F)$ is the direct sum

$$\Lambda(F) = \sum_{0 \leqslant a, b} F_{a,b},$$

where $F_{a,b}$ is the subspace of $\Lambda(F)$ spanned by all multilinear forms of the type

$$(z_{\alpha_1} \wedge \ldots \wedge z_{\alpha_a}) \wedge (\bar{z}_{\beta_1} \wedge \ldots \wedge \bar{z}_{\beta_b}),$$

where $1 \leqslant \alpha_1 < \alpha_2 < \ldots < \alpha_a \leqslant n$, $1 \leqslant \beta_1 < \beta_2 < \ldots < \beta_b \leqslant n$. The G-invariants $J(E^*)$ are given by the space J of invariants of $\mathbf{U}(n)$ acting on F. It is clear that $J = \sum_{a,b} J_{a,b}$ where $J_{a,b} = J \cap F_{a,b}$. Now if $\varrho_i \in \mathbf{R}$ $(1 \leqslant i \leqslant n)$ the mapping

$$(Z_1, \ldots, Z_n) \to (e^{i\varrho_1} Z_1, \ldots, e^{i\varrho_n} Z_n)$$

is unitary. As a consequence one finds that $J_{a,b}=0$ if $a \neq b$ and that if $f \in J_{a,a}$ then

$$f = \sum A_{\alpha_1 \ldots \alpha_a} (z_{\alpha_1} \wedge \ldots \wedge z_{\alpha_a}) \wedge (\bar{z}_{\alpha_1} \wedge \ldots \wedge \bar{z}_{\alpha_a}).$$

Now, there always exists a unitary transformation of V mapping $Z_{\alpha_i} \to Z_i$ $(1 \leqslant i \leqslant a)$. It follows that $A_{1 \ldots a} = A_{\alpha_1 \ldots \alpha_a}$ so f is a constant multiple of $(\sum_{\alpha=1}^n z_\alpha \wedge \bar{z}_\alpha)^a$. Since $z_\alpha \wedge \bar{z}_\alpha = -2i(x_\alpha \wedge y_\alpha)$ it is clear that $J(E^*)$ is the algebra generated by $u = \sum_{\alpha=1}^n x_\alpha \wedge y_\alpha$. In view of Theorem 2.1 each $q \in \Lambda(E^*)$ can be written

$$q = \sum_k u^k \wedge p_k, \tag{5}$$

where each p_k satisfies $\delta(u) p_k = 0$. This result is of course well known (Hodge), even for all Kähler manifolds (compare [17], Théorème 3, p. 26).

§ 3. Fundamental functions on quadrics

Let G be a topological group, H a closed subgroup, and G/H the set of left cosets gH with the usual topology. If f is a function on G/H and $x \in G$ then f^x denotes the function on G/H given by $f^x(gH) = f(xgH)$.

Definition. A complex-valued continuous function f on G/H is called *fundamental* if the vector space V_f over $\mathbf{C}$ spanned by the functions f^x $(x \in G)$ is finite-dimensional.

Fundamental functions arise of course in a natural fashion in the theory of finite-dimensional representations of topological groups. First we remark that if π denotes the natural mapping of G onto G/H then f is fundamental on G/H if and only if $f \circ \pi$ is fundamental on G (viewed as $G/\{e\}$). But the fundamental functions on G are just the linear combinations of matrix coefficients of finite-dimensional representations of G (see e.g. [11], Prop. 2.1, p. 497). Considering Kronecker products of representations, the fundamental functions on G (and also those on G/H, by the remark above) are seen to form an algebra.

Let G be a topological transformation group of a topological space E. A G-equivariant imbedding of E into a finite-dimensional vector space V is a one-to-one continuous mapping i of E into V and a representation α of G on V such that $\alpha(g)\, i(e) = i(g \cdot e)$ for all $e \in E$, $g \in G$. If V is provided with a strictly positive definite quadratic form which is left invariant by all $\alpha(g)$ $(g \in G)$ then i is called an orthogonal G-equivariant imbedding.

It is known ([14], [16]) that if U is a compact Lie group and K a closed subgroup then U/K has an orthogonal U-equivariant imbedding.

LEMMA 3.1. *Let U be a compact Lie group and K a closed subgroup. Let i be any orthogonal U-equivariant imbedding of U/K into a vector space V over* **R**. *Then the fundamental functions on U/K are precisely the functions $p \circ i$ where p is a complex-valued polynomial function on V.*

Proof. Putting $v_0 = i(K)$ we have

$$i(uK) = \alpha(u)\, v_0 \quad (u \in K).$$

Let $F(U)$ denote the algebra of all fundamental functions on U and let S denote the subalgebra of $F(U)$ generated by the constants and all functions on U of the form $u \to \langle \alpha(u)\, v_0, v\rangle$ where $v \in V$ and $\langle , \rangle$ denotes the inner product on V. If φ is a continuous function on U and $x \in U$ we define the left and right translate of φ by $\varphi^{L(x)}(y) = \varphi(x^{-1}y)$, $\varphi^{R(x)}(y) = \varphi(yx^{-1})$, $y \in U$. Let us verify that

$$K = \{x \in U \mid \varphi^{R(x)} = \varphi \quad \text{for all } \varphi \in S\}. \tag{1}$$

It is clear that K is contained in the right hand side of (1). On the other hand, if $\varphi^{R(x)} = \varphi$ for all $\varphi \in S$ we find in particular that $\langle \alpha(x)\, v_0 - v_0, v\rangle = 0$ for all $v \in V$. Hence $\alpha(x)\, v_0 = v_0$ and, since i is one-to-one, $x \in K$. Now S is a subalgebra of $F(U)$ which contains the constants and is invariant under all left translations and the complex conjugation. From (1) and Lemma 5.3 in [11] p. 515 it follows that

$$S = \{\varphi \in F(U) \mid \varphi^{R(k)} = \varphi \quad \text{for all } k \in K\}. \tag{2}$$

Now let f be a fundamental function on U/K. Then $\varphi = f \circ \pi \in F(U)$ and by (2), $\varphi \in S$. By the definition of S there exist finitely many vectors $v_1, \ldots, v_r \in V$ such that if we put

$$s_i(u) = \langle \alpha(u)\, v_0, v_i\rangle \quad (u \in U),$$

then

$$\varphi = \sum A_{n_1 \ldots n_r} s_1^{n_1} \ldots s_r^{n_r}, \quad A_{n_1 \ldots n_r} \in \mathbf{C}. \tag{3}$$

Let l_i denote the linear function $v \to \langle v, v_i\rangle$ on V. Then (3) implies that

$$f = \sum A_{n_1 \ldots n_r} l_1^{n_1} \ldots l_r^{n_r} \circ i,$$

proving the lemma.

Remark. Lemma 3.1 is closely related to Theorem 3 in [15], stated without proof.

Consider now the quadric $C_{p,q} \subset \mathbf{R}^{p+q+1}$ given by the equation

$$Q(X) \equiv x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q+1}^2 = -1 \quad (p \geqslant 0,\ q \geqslant 0). \tag{4}$$

The orthogonal group $\mathbf{O}(Q)=\mathbf{O}(p,q+1)$ acts transitively on $C_{p,q}$; the subgroup leaving the point $(0,\ldots,0,1)$ on $C_{p,q}$ fixed is isomorphic to $\mathbf{O}(p,q)$ so we make the identification

$$C_{p,q}=\mathbf{O}(p,q+1)/\mathbf{O}(p,q). \tag{5}$$

It is clear that the restriction of a polynomial on $\mathbf{R}^{p+q+1}$ to $C_{p,q}$ is a fundamental function.

THEOREM 3.2. *Let f be a fundamental function on $\mathbf{C}_{p,q}$. Assume $(p,q)\neq(1,0)$. Then there exists a polynomial $P=P(x_1,\ldots,x_{p+q+1})$ on $\mathbf{R}^{p+q+1}$ such that*

$$f=P \quad \text{on } C_{p,q}.$$

If $p=0$ then this theorem is an immediate consequence of Lemma 3.1. The general case requires some preparation.

Let U be a topological group and K a closed subgroup. A representation α of U on a Hilbert space $\mathfrak{H}$ is said to be of *class 1* (with respect to K) if it is irreducible and unitary and if there exists a vector $\mathbf{e}\neq 0$ in $\mathfrak{H}$ which is left fixed by each $\alpha(k)$, $k\in K$.

LEMMA 3.3. *The representations of the group $\mathbf{SO}(n)$ of class 1 (with respect to $\mathbf{SO}(n-1)$) are (up to equivalence) precisely the natural representations of $\mathbf{SO}(n)$ on the eigenspaces of the Laplacian Δ on the unit sphere $\mathbf{S}^{n-1}$.*

This lemma is essentially known ([1]), but we shall indicate a proof. Let α be a representation of $\mathbf{SO}(n)$ of class 1. If φ is the spherical function on $\mathbf{S}^{n-1}=\mathbf{SO}(n)/\mathbf{SO}(n-1)$ corresponding to α, i.e., $\varphi(u\mathbf{SO}(n-1))=\langle \mathbf{e},\alpha(u)\,\mathbf{e}\rangle$, then α is equivalent to the natural representation of $\mathbf{SO}(n)$ on the space V_φ spanned by the translates φ^x, $(x\in\mathbf{SO}(n))$. (See, for example, Theorem 4.8, Ch. X, in [10]). The elements of V_φ are all eigenfunctions of Δ for the same eigenvalue. The space V_φ must exhaust the eigenspace of Δ for this eigenvalue because otherwise there would exist two linearly independent eigenfunctions of Δ invariant under $\mathbf{SO}(n-1)$ corresponding to the same eigenvalue. This is impossible as one sees by expressing Δ in geodesic polar coordinates. All eigenspaces of Δ are obtained in this way.

Each eigenfunction of Δ on $\mathbf{S}^{n-1}$ is a fundamental function on $\mathbf{O}(n)/\mathbf{O}(n-1)$, hence the restriction of a polynomial which by Theorem 1.2 can be assumed harmonic. On the other hand, let $P\neq 0$ be a homogeneous harmonic polynomial on $\mathbf{R}^n$ of degree m. (By Remark 2 following Theorem 1.2 these exist for each integer $m\geqslant 0$.) Using the expression of the Laplacian $\tilde{\Delta}$ on $\mathbf{R}^n$ in polar coordinates,

$$\tilde{\Delta} = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta,$$

one finds that the restriction $\bar{P}$ of P to $\mathbf{S}^{n-1}$ satisfies

$$\Delta \bar{P} = -m(m+n-2)\,\bar{P}.$$

This shows, as is well known, that the eigenvalues of Δ are $-m(m+n-2)$, where m is a non-negative integer.

LEMMA 3.4. *Let (U, π) denote the universal covering group of $\mathbf{SO}(n)$ $(n \geqslant 3)$ and let K denote the identity component of $\pi^{-1}(\mathbf{SO}(n-1))$. Let α be a representation of U of class 1 (with respect to K). Then there exists a representation α_0 of $\mathbf{SO}(n)$ such that $\alpha_0 = \alpha \circ \pi$.*

Proof. The mapping $\psi: uK \to \pi(u)\,\mathbf{SO}(n-1)$ is a covering map of U/K onto $\mathbf{SO}(n)/\mathbf{SO}(n-1) = \mathbf{S}^{n-1}$ which is already simply connected. Hence ψ is one-to-one so $K = \pi^{-1}(\mathbf{SO}(n-1))$. Let $\mathbf{e} \neq 0$ be a common fixed vector for all $\alpha(k)$, $k \in K$. Then in particular $\alpha(z)\,\mathbf{e} = \mathbf{e}$ for all z in the kernel of π. By Schur's lemma $\alpha(z)$ is a scalar multiple of the identity I; hence $\alpha(z) = I$ for all z in the kernel of π and the lemma follows.

Now we need more notation. Let $\mathfrak{o}(r, s)$ denote the Lie algebra of the orthogonal group $\mathbf{O}(r, s)$, put $\mathfrak{o}(r) = \mathfrak{o}(r, 0) = \mathfrak{o}(0, r)$ and let $\mathfrak{o}(n, \mathbf{C})$ denote the Lie algebra of the complex orthogonal group $\mathbf{O}(n, \mathbf{C})$. Consider now the following diagram of Lie groups and their Lie algebras:

$$\begin{array}{ccc} & G & \\ & \nearrow \ \nwarrow & \\ G_0 & \uparrow & U_0 = \mathbf{SO}(p+q+1) \\ & H & \\ \uparrow & \nearrow \ \nwarrow & \uparrow \\ H_0 & & K_0 = \mathbf{SO}(p+q) \end{array} \qquad \begin{array}{ccc} & \mathfrak{g} = \mathfrak{o}(p+q+1, \mathbf{C}) & \\ & \nearrow \ \nwarrow & \\ \mathfrak{g}_0 = \mathfrak{o}(p, q+1) & \uparrow & \mathfrak{u} = \mathfrak{o}(p+q+1) \\ \uparrow & \mathfrak{h} = \mathfrak{o}(p+q, \mathbf{C}) & \uparrow \\ & \nearrow \ \nwarrow & \\ \mathfrak{h}_0 = \mathfrak{o}(p, q) & & \mathfrak{k} = \mathfrak{o}(p+q) \end{array}$$

In the diagram on the right the arrows denote imbeddings. The imbedding of $\mathfrak{o}(p, q)$ into $\mathfrak{o}(p, q+1)$ is the one which corresponds to the inclusion (5) and the imbeddings of $\mathfrak{o}(p+q)$ in $\mathfrak{o}(p+q+1)$ and of $\mathfrak{o}(p+q, \mathbf{C})$ in $\mathfrak{o}(p+q+1, \mathbf{C})$ are to be understood similarly. In the diagram on the left are Lie groups corresponding to the Lie algebras on the right; here the arrows mean inclusions among the identity components. G_0 and H_0 respectively stand for the groups $\mathbf{O}(p, q+1)$ and $\mathbf{O}(p, q)$ in (5). Let G, U_0, H, K_0 denote the analytic subgroups of $\mathbf{GL}(p+q+1, \mathbf{C})$ corresponding to the subalgebras $\mathfrak{g}$, $\mathfrak{u}$, $\mathfrak{h}$, $\mathfrak{k}$ in the right hand diagram.

For the proof of Theorem 3.2 we have to consider four cases:

I $p=0$; II $q=0$; III $p=1, q=1$; IV p, q arbitrary.

Case I is contained in Lemma 3.1, G_0 being compact. The proof in Case II will be based on the compactness of H_0. In Case III we shall use the fact that the identity component of $\mathbf{O}(1,2)$ is a well-imbedded linear Lie group in the sense of [4], p. 327. Finally, Case IV is reduced to the three previous cases by a suitable method of descent.

The case $p=0$ being settled, suppose $q=0$. Consider the representation ϱ of G_0 on V_f given by $\varrho(x^{-1})F=F^x$ $(F\in V_f)$. This representation is completely reducible because G_0 is semisimple (since $q=0$, p is >1) and has finitely many components ([3], Théorème 3 b, p. 85). We may therefore assume ϱ irreducible. Since G_0 is transitive on $C_{p,0}$ we can suppose $f(0, \dots, 0, 1)\neq 0$. Moreover, since the subgroup H_0 is now compact we may, by replacing f with the average $\int_{H_0} f^h dh$ assume that $f^h=f$ for each $h\in H_0$. Now there is induced a representation $d\varrho$ of $\mathfrak{g}_0$ onto V_f by

$$[d\varrho(X)F](m)=\left\{\frac{d}{dt}(F(\exp(-tX)\cdot m))\right\}_{t=0} \tag{6}$$

for $F\in V_f$, $X\in\mathfrak{g}_0$, $m\in C_{p,0}$. Next $d\varrho$ extends to a representation $d\varrho^c$ of the complex Lie algebra $\mathfrak{g}$ on V_f and finally $d\varrho^c$ extends to a representation (also denoted $d\varrho^c$) on V_f of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$.

Let Γ denote the Casimir element in $U(\mathfrak{g})$. Since Γ lies in the center of $U(\mathfrak{g})$ and since ϱ is irreducible it follows by Schur's lemma that $d\varrho^c(\Gamma)=\gamma I$ where $\gamma\in\mathbf{C}$. Consider now the representation $\tilde{\varrho}$ of G_0 on the space of C^∞-functions on G_0/H_0 given by $\tilde{\varrho}(x^{-1})F=F^x$. Although infinite-dimensional this representation extends (as by (6)) to a representation $d\tilde{\varrho}^c$ of $U(\mathfrak{g})$ and thereby $d\tilde{\varrho}^c(\Gamma)$ is a second order differential operator on G_0/H_0, annihilating the constants and invariant under the action of G_0. It follows without difficulty that $d\tilde{\varrho}^c(\Gamma)$ is the Laplace-Beltrami operator corresponding to the invariant Riemannian structure on G_0/H_0 which is induced by the Killing form of $\mathfrak{g}_0$. According to [8] this Riemannian structure is $2(p-1)$ times the Riemannian structure of $C_{p,0}$ induced by the quadratic form $x_1^2+\dots+x_p^2-x_{p+1}^2$ on $\mathbf{R}^{p+1}$. The corresponding Laplace-Beltrami operators are proportianal by the reciprocal proportionality factor. Now, since f is necessarily differentiable, ϱ is the restriction of $\tilde{\varrho}$ to V_f. Putting together these facts we conclude that each function in V_f is an eigenfunction of the Laplacian Δ' on $C_{p,0}$ with eigenvalue $2(p-1)\gamma$.

On the other hand, the Lie algebra $\mathfrak{u}$ of $\mathbf{SO}(p+1)$ is a compact real form of $\mathfrak{g}$.

By restriction ϱ induces a representation of this Lie algebra on V_f. This representation extends to a representation (also denoted ϱ) on V_f of the universal covering group U of $\mathbf{SO}(p+1)$. This representation is of class 1 with respect to the connected Lie subgroup of U with Lie algebra $\mathfrak{u} \cap \mathfrak{g}_0$, the function f being the fixed vector. By Lemma 3.4 ϱ induces a representation of $\mathbf{SO}(p+1)$ of class 1 (with respect to $\mathbf{SO}(p)$), which then can be described by Lemma 3.3. Consider now the representation ϱ^* of $\mathbf{SO}(p+1)$ on $C^\infty(S^p)$ given by $\varrho^*(x^{-1})F = F^x$. Under this representation $(d\varrho^*)^c(\Gamma) = -(2(p-1))^{-1}\Delta$; the minus sign is due to the fact that the negative Killing form of $\mathfrak{u}$ induces a positive definite Riemannian structure on $\mathbf{SO}(p+1)/\mathbf{SO}(p)$. Now it follows that $-2(p-1)\gamma$ is an eigenvalue of the Laplacian Δ on S^p, so $-2(p-1)\gamma = -m(m+p-1)$, where m is a non-negative integer.

Now let P be a homogeneous polynomial of degree m on $\mathbf{R}^{p+1}$ satisfying

$$\frac{\partial^2 P}{\partial x_1^2} + \dots + \frac{\partial^2 P}{\partial x_{p+1}^2} = 0.$$

We can select P such that $P(0, \dots, 0, 1) \neq 0$ and by integrating over the isotropy group of $(0, \dots, 0, 1)$, such that

$$P(x_1, \dots, x_{p+1}) \equiv P((x_1^2 + \dots + x_p^2)^{\frac{1}{2}}, 0, \dots, 0, x_{p+1}).$$

If we substitute $x_{p+1} \to i x_{p+1}$ in $P(x_1, \dots, x_{p+1})$ we obtain a homogeneous polynomial $Q(x_1, \dots, x_{p+1})$ of degree m satisfying

$$\Delta^* Q \equiv \frac{\partial^2 Q}{\partial x_1^2} + \frac{\partial^2 Q}{\partial x_2^2} + \dots + \frac{\partial^2 Q}{\partial x_p^2} - \frac{\partial^2 Q}{\partial x_{p+1}^2} = 0,$$

$$Q(x_1, \dots, x_{p+1}) \equiv Q((x_1^2 + \dots + x_p^2)^{\frac{1}{2}}, 0, \dots, 0, x_{p+1}), \quad Q(0, \dots, 0, 1) \neq 0.$$

Now the operator Δ^* can be expressed in terms of the coordinates on $C_{p,0}$ and the "distance" $r = (-x_1^2 - \dots - x_p^2 + x_{p+1}^2)^{\frac{1}{2}}$. One finds (compare Lemma 21, p. 278, in [7]) that in these coordinates

$$\Delta^* = -\frac{\partial^2}{\partial r^2} - \frac{p}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta',$$

where Δ' is the Laplacian on $C_{p,0}$. Now $Q = r^m \bar{Q}$ where $\bar{Q}$ is the restriction of Q to $C_{p,0}$ so we obtain for $r = 1$

$$\Delta' \bar{Q} = m(m+p-1)\bar{Q}.$$

Thus the functions f and $\bar{Q}$ have the same eigenvalue. Both are invariant under the isotropy group of $(0, \dots, 0, 1)$ and neither vanishes at that point. According to Cor. 3.3, Ch. X i [10], f and $\bar{Q}$ are proportional so the proof is finished in the case $q = 0$.

Now we come to Case III: $p=q=1$. We shall use the diagram following the proof of Lemma 3.4. Again let f be a fundamental function on $C_{1,1}$ and let V_f denote the vector space over $\mathbf{C}$ spanned by all translates f^x, $x \in G_0$. Consider the representation ϱ of G_0 on V_f defined by $\varrho(x^{-1}) F = F^x$. For the same reason as in Case II we may assume ϱ irreducible and $f(0,0,1) \neq 0$. As before consider the representations $d\varrho, d\varrho^c$. Since the identity component of G_0 is a well-imbedded linear Lie group there exists a representation ϱ^c of G on V_f whose differential is the previous $d\varrho^c$ ([4], p. 329). Let α denote the restriction of ϱ^c to U_0.

LEMMA 3.5. *α is of class 1 (with respect to K_0).*

Proof. Let $X \in \mathfrak{h}_0$ and put $p_0 = (0,0,1)$. Then for each $F \in V_f$

$$[d\varrho(X) F](p_0) = \left\{\frac{d}{dt}(F(\exp(-tX)\cdot p_0))\right\}_{t=0} = 0,$$

and by induction
$$[(d\varrho(X))^m F](p_0) = 0 \quad (m \geqslant 1). \tag{7}$$

Since $d\varrho^c(iX) = i\, d\varrho^c(X)$, (7) implies

$$[(d\alpha(X))^m F](p_0) = 0 \quad (X \in \mathfrak{k},\ F \in V_f). \tag{8}$$

Now, since $K_0 = \mathbf{SO}(2)$ is abelian, V_f is a direct sum of one-dimensional subspaces, $V_f = \sum_{i=1}^r V_i$, each of which is invariant under $\alpha(K_0)$. Let $d\alpha(X)_i$ denote the restriction of $d\alpha(X)$ to V_i, and let χ_i denote the homomorphism of K_0 into $\mathbf{C}$ determined by $\chi_i(\exp X) = \exp(d\alpha(X)_i)$. Then by (8) $\chi_i(\exp X) F_i(p_0) = F_i(p_0)$, $F_i \in V_i$, so if $k \in K_0$, $f = \sum F_i$,

$$[\alpha(k) f](p_0) = \sum_{i=1}^r \chi_i(k) F_i(p_0) = \sum_{i=1}^r F_i(p_0) = f(p_0).$$

Thus the vector
$$f^* = \int_{K_0} (\alpha(k) f)\, dk$$

in V_f is $\neq 0$ and invariant under K_0. This proves the lemma.

LEMMA 3.6. *The vector $f^* \in V_f$ is invariant under $\varrho(h)$ for each h in the identity component of H_0.*

In fact, $\alpha(k) f^* = f^*$ $(k \in K_0)$ so $d\alpha(X) f^* = 0$ $(X \in \mathfrak{k})$; hence $d\varrho^c(X) f^* = 0$ for all X in the complexification $\mathfrak{h}$ of $\mathfrak{k}$. The lemma now follows.

Since ϱ is irreducible we have $V_f = V_{f^*}$. Thus it suffices to prove Theorem 3.2 for the function f^*. By a procedure similar to that in Case II it is found that

$$\Delta' f^* = m(m+1) f^*, \tag{9}$$

where Δ' is the Laplace-Beltrami operator on $C_{1,1}$ corresponding to the pseudo-Riemannian structure on $C_{1,1}$ induced by $x_1^2-x_2^2-x_3^2$, and m is a non-negative integer.

On the other hand, let P be a homogeneous polynomial of degree m on $\mathbf{R}^3$ satisfying

$$\frac{\partial^2 P}{\partial x_1^2}+\frac{\partial^2 P}{\partial x_2^2}+\frac{\partial^2 P}{\partial x_3^2}=0 \quad (P(0,0,1)\neq 0);$$

$$P(x_1,x_2,x_3)=\sum_k A_k(x_1^2+x_2^2)^k x_3^{m-2k} \quad (A_k\in \mathbf{C}).$$

If we substitute $x_2\to ix_2$, $x_3\to ix_3$ in $h(x_1,x_2,x_3)$ we obtain a homogeneous polynomial $Q(x_1,x_2,x_3)$ of degree m satisfying

$$\frac{\partial^2 Q}{\partial x_1^2}-\frac{\partial^2 Q}{\partial x_2^2}-\frac{\partial^2 Q}{\partial x_3^2}=0 \quad (Q(0,0,1)\neq 0);$$

$$Q(x_1,x_2,x_3)=\sum_k B_k(x_1^2-x_2^2)^k x_3^{m-2k} \quad (B_k\in \mathbf{C}).$$

As in Case II it follows that the restriction $\bar{Q}$ of Q to $C_{1,1}$ satisfies the equation (9). Also $\bar{Q}^h=\bar{Q}$ for each $h\in H_0$.

LEMMA 3.7. *The functions f^* and $\bar{Q}$ are proportional.*

Proof. In the Lorentzian manifold $C_{1,1}$ we consider the retrograde cone D with vertex $(0,0,1)$ ([7], p. 287). In geodesic polarcoordinates on D let $(\Delta')_r$ denote the restriction of Δ' to functions depending on the radiusvector r alone. Then by Lemma 25 in [7]

$$(\Delta')_r=\frac{d^2}{dr^2}+2\coth r\frac{d}{dr} \qquad (r>0).$$

Since
$$\frac{d^2g}{dr^2}+2\coth r\frac{dg}{dr}=\frac{1}{\sinh r}\left(\frac{d^2}{dr^2}-1\right)(g(r)\sinh r),$$

it follows that the solutions of (9) in D which depend on r alone are given by

$$g(r)\sinh r=A\sinh(\lambda r)+B\cosh(\lambda r), \qquad \lambda^2=m(m+1)+1,\ \lambda>0$$

where $A,B\in \mathbf{C}$. Now both functions f^* and $\bar{Q}$ satisfy this equation in D but since they are bounded in a neighborhood of $(0,0,1)$ it is clear that $B=0$ so f^* and $\bar{Q}$ are proportional on D. But these functions are analytic on the connected manifold $C_{1,1}$, so, being proportional on the open subset D, are proportional everywhere. This proves Theorem 3.2 in Case III.

Finally, we consider Case IV and assume $p \geq 1$, $q \geq 1$. Let f be a fundamental function on $C_{p,q}$. Again we consider the representation ϱ of G_0 on V_f given by $\varrho(x^{-1}) F = F^x$, and assume as we may that ϱ is irreducible and that $f(0, \ldots, 0, 1) \neq 0$. Since the subgroup $H^* = \mathbf{O}(p) \times \mathbf{O}(q)$ of H_0 is compact we can also assume that $\varrho(h) f = f$ for all $h \in H^*$. It follows that on $C_{p,q}$

$$f(x_1, \ldots, x_{p+q+1}) = f((x_1^2 + \ldots + x_p^2)^{\frac{1}{2}}, 0, \ldots, 0, (x_{p+1}^2 + \ldots + x_{p+q}^2)^{\frac{1}{2}}, x_{p+q+1}). \tag{10}$$

On the quadric $y_1^2 - y_2^2 - y_3^2 = -1$ we consider now the function

$$f^*(y_1, y_2, y_3) = f(y_1, 0, \ldots, 0, y_2, y_3).$$

This function f^* is well defined since $(y_1, 0, \ldots, 0, y_2, y_3) \in C_{p,q}$ and is a fundamental function on the quadric $C_{1,1}$. As shown above there exists a polynomial $P^*(y_1, y_2, y_3)$ such that

$$f^*(y_1, y_2, y_3) = P^*(y_1, y_2, y_3) \quad \text{for } y_1^2 - y_2^2 - y_3^2 = -1.$$

By (10) f^* is even in the first two variables so P^* can be assumed to contain y_1 and y_2 in even powers alone. Combining the equations above we find that

$$f(x_1, \ldots, x_{p+q+1}) = P^*((x_1^2 + \ldots + x_p^2)^{\frac{1}{2}}, (x_{p+1}^2 + \ldots + x_{p+q}^2)^{\frac{1}{2}}, x_{p+q+1})$$

on $C_{p,q}$. Due to the assumptions made on P^* the right-hand side of this equation is a polynomial on $\mathbf{R}^{p+q+1}$. This disposes of Case IV so Theorem 3.2 is now completely proved.

Remarks. Some special cases of Theorem 3.2 have been proved before. The case $p=0$ (for which $\mathbf{O}(p, q+1)$ is compact) was already proved by Hecke [6] (for $q=2$) and Cartan [1]. If $p=2$, $q=0$ then $C_{p,q}$ is the 2-dimensional Lobatchefsky space of constant negative curvature. In this case Theorem 3.2 was proved by Loewner [12] using special features of the Poincaré upper half plane.

The assumption that $(p, q) \neq (1, 0)$ is essential for the validity of Theorem 3.2. In fact, consider the function f on the quadric $x_1^2 - x_2^2 = -1$ defined by

$$f(x_1, x_2) = \sinh^{-1}(x_1).$$

The group $\mathbf{O}(1, 1)$ is generated by the transformations

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \to \begin{Bmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{Bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} \to \begin{Bmatrix} \pm x_1 \\ -x_2 \end{Bmatrix}.$$

It is easy prove that $\dim_C(V_f) = 2$. Thus f is fundamental but is certainly not the restriction of a polynomial on $\mathbf{R}^2$.

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