QUASI-INVARIANCE AND ANALYTICITY OF MEASURES ON COMPACT GROUPS

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1. Introduction

In this paper we study an extension to compact abelian groups of the two celebrated theorems of F. and M. Riesz [15] concerning analytic measures on the circle group. The content of these theorems is as follows:

Let μ be a Borel measure on the circle satisfying

$$\int_{-\pi}^{+\pi} e^{in\theta} d\mu(\theta) = 0 \quad (n = 1, 2, 3, \dots).$$

Then

A. *μ is absolutely continuous with respect to Lebesgue measure.*

B. *If μ vanishes identically*(1) *on a set of positive Lebesgue measure, then μ must be the zero measure.*

It is not hard to see that A and B together are equivalent to the following:

The collection of Borel sets on which μ vanishes identically is invariant under rotation.

This is the assertion concerning analytic measures that we extend to compact abelian groups. Before stating this extension we make the basic definitions.

In all that follows G is a compact abelian group and $\hat{G}$ its discrete dual. An "ordering" of $\hat{G}$ is given by a fixed non-trivial(2) homomorphism ψ of $\hat{G}$ into the group R of real numbers. The mapping $\psi: \hat{G} \to R$ is a continuous homomorphism and thus induces a continuous

(1) *μ vanishes identically* on E if $\mu(F) = 0$ for all Borel subsets F of E.

(2) We assume that $\psi(\hat{G}) \neq \{0\}$.

homomorphism φ: $R \to G$ of the associated dual groups; φ is the unique mapping of R into G satisfying

$$\sigma(\varphi(t)) = e^{i\psi(\sigma)t} \quad (t \in R,\ \sigma \in \hat{G}).$$

By *measure* on G we shall always mean finite complex regular Borel measure. If μ is a measure on G, we denote by $|\mu|$ the associated total variation measure. μ is said to *vanish identically* on a Borel subset E of G if $|\mu|(E)=0$. μ is called *quasi-invariant* under φ if the collection of Borel subsets of G on which μ vanishes identically is invariant under translation by elements of $\varphi(R)$.

A measure or function on G is called *φ-analytic* if its Fourier transform vanishes on the "negative half" $\{\sigma: \psi(\sigma)<0\}$ of $\hat{G}$.

MAIN THEOREM. *Let μ be a φ-analytic measure on G. Then μ is quasi-invariant under φ.*

This result is established in sections 6 and 7. The method is roughly as follows. In section 6 a covering group $R \times K$ for G is constructed in which the one parameter subgroup $\varphi(R)$ is "unwound", and the Main Theorem for G is reduced to the corresponding result for the covering group. In section 7 this result for analytic measures on $R \times K$ is established by decomposing such measures into analytic measures concentrated on fibers $R \times \{u\}$ and applying the one-dimensional F. and M. Riesz Theorems to these measures.

The earlier sections of the paper are devoted to consequences of the Main Theorem. In section 3 we study extensions of the first F. and M. Riesz theorem stated above. Although a φ-analytic measure need not be absolutely continuous with respect to Haar measure on G, we obtain a class of Borel sets on which φ-analytic measures must necessarily vanish. These are the Borel subsets of G that intersect each coset of $\varphi(R)$ in a set of linear measure zero. We also obtain an extension of the theorem of Bochner [2] that states that a measure on the n-torus whose Fourier transform vanishes off the positive octant must be absolutely continuous.

In section 4 we establish analogues of the second of the F. and M. Riesz theorems. We obtain a theorem that has as consequences the facts that, if $\varphi(R)$ is dense in G, an absolutely continuous φ-analytic measure cannot vanish identically on a set of positive Haar measure (due to Helson, Lowdenslager and Malliavin [10]), and an arbitrary φ-analytic measure cannot vanish identically on an open subset of G.

Section 5 is devoted to further consequences of the Main Theorem. The results are in part refinements of theorems of Helson-Lowdenslager [9] and Bochner [2].

In what follows we shall use without comment the basic results of abstract harmonic analysis, for which see [13]. We denote Fourier transform by $\hat{}$ and convolution by $*$.

We use the fact (see [5] or see the recent paper of Heble and Rosenblatt, *Proc. Amer. Math. Soc.*, 14 (1963), 177–184) that if μ and λ are measures on G, and E is a Borel subset of G, then $x \to \lambda(-x+E)$ is a Borel function on G and

$$\mu * \lambda(E) = \int_G \lambda(-x+E)\,d\mu(x).$$

For measures μ and λ, $\mu << \lambda$ will mean that μ is absolutely continuous with respect to λ. If μ is a measure on G, we shall simply say that μ is *absolutely continuous* or that μ is *singular* if it is absolutely continuous or singular with respect to Haar measure on G.

The results of this paper have been announced in [6].

2. Properties of quasi-invariant measures

The Main Theorem, proved in sections 6 and 7, asserts the quasi-invariance under φ of φ-analytic measures on G. In this section we study group and measure theoretic consequences of quasi-invariance for arbitrary measures. The results obtained are applied, together with the Main Theorem, in the following three sections to establish properties of φ-analytic measures.

It is not hard to show that a measure μ on the circle group is quasi-invariant under rotation if and only if it is absolutely continuous and does not vanish identically on any set of positive Lebesgue measure; equivalently, if and only if $|\mu|$ and $\lambda * |\mu|$ are mutually absolutely continuous, for λ Haar measure on the circle. The main theorem of this section, Proposition 2.3, is the analogue of this result in this context we are considering. Before proving this theorem we must make the appropriate definitions and prove two lemmas.

A Borel subset E of G will be called *null in the direction of* φ if for each x in E the coset $x+\varphi(R)$ intersects E in a set of linear measure zero, and will be called *thick in the direction of* φ if for each x in E the coset $x+\varphi(R)$ intersects E in a set of positive linear measure. More precisely, E is null (resp., thick) in the direction of φ if for each x in E,

$$\{t\colon t \in R, x+\varphi(t) \in E\}$$

has zero Lebesgue measure (resp., has positive Lebesgue measure).

We shall denote by ϱ the image on G of the measure $(1+x^2)^{-1}dx$ on R under the mapping $\varphi\colon R \to G$. Then, for any bounded Borel measurable function f on G, we have

$$\int_G f\,d\varrho = \int_{-\infty}^{+\infty} f(\varphi(t))\frac{1}{1+t^2}\,dt.$$

The main property of the measure ϱ that we use in what follows is that for a Borel subset E of G, $\{x\colon x \in G, \varrho(-x+E) > 0\}$ is precisely the set of those x that are such that $x+\varphi(R)$

intersects E in a set of positive linear measure. In particular, this set is invariant under translation by elements of $\varphi(R)$.

It will be convenient for us to rephrase the null and thickness conditions in terms of the measure ϱ. Because of the formula

$$\varrho(-x+E)=\int_G \chi_{-x+E}\,d\varrho=\int_{-\infty}^{+\infty}\chi_E(x+\varphi(t))\frac{1}{1+t^2}\,dt,$$

a Borel subset E of G is null in the direction of φ if and only if $\varrho(-x+E)=0$, all x in E, and is thick in the direction of φ if and only if $\varrho(-x+E)>0$, all x in E.

We shall call a measure μ on G *absolutely continuous in the direction of* φ if $\mu(E)=0$ for all Borel sets E that are null in the direction of φ.

LEMMA 2.1. *Let μ be a measure on G. Then the following are equivalent:*

1° *μ is absolutely continuous in the direction of φ;*

2° $|\mu| << \varrho * |\mu|$.

Proof. (2° implies 1°) Let E be a Borel subset of G that is null in the direction of φ, so for each x in G, $\varrho(-x+E)=0$. Then

$$\varrho*|\mu|(E)=\int_G \varrho(-x+E)\,d|\mu|(x)=0,$$

so, by 2°, $|\mu|(E)=0$.

(1° implies 2°) Let E be a Borel subset of G with $\varrho*|\mu|(E)=0$. We must prove $|\mu|(E)=0$. Define the $\varphi(R)$-invariant subsets G_0 and G_1 of G by

$$G_0=\{x\colon x\in G,\ \varrho(-x+E)=0\},$$
$$G_1=\{x\colon x\in G,\ \varrho(-x+E)>0\}.$$

Let $E_0=E\cap G_0$ and $E_1=E\cap G_1$. If x is in E_0, then

$$0\leqslant \varrho(-x+E_0)\leqslant \varrho(-x+E)=0,$$

so E_0 is null in the direction of φ. And since μ is assumed absolutely continuous in the direction of φ, we have $|\mu|(E_0)=0$. Furthermore, by hypothesis

$$\int_G \varrho(-x+E)\,d|\mu|(x)=\varrho*|\mu|(E)=0,$$

and $\varrho(-x+E)>0$ for x in E_1, so $|\mu|(E_1)=0$. This proves that

$$|\mu|(E)=|\mu|(E_0)+|\mu|(E_1)=0.$$

We shall call a measure μ on G *non-vanishing in the direction of* φ if $|\mu|(E)>0$ for each Borel subset E of G that is thick in the direction of φ and for which[1] $|\mu|(E+\varphi(R))>0$.

LEMMA 2.2. *Let μ be a measure on G. Then the following are equivalent.*

1° *μ is non-vanishing in the direction of φ;*

2° $\varrho * |\mu| << |\mu|$.

Proof. (2° implies 1°) Assume that 2° is true and 1° false. Then there is a Borel set E, thick in the direction of φ, with $|\mu|(E)=0$ and $|\mu|(E+\varphi(R))>0$. Because of 2°,

$$0=\varrho * |\mu|(E)=\int_G \varrho(-x+E)\,d|\mu|(x).$$

Since E is thick in the direction of φ, $E+\varphi(R)$ is precisely the set of x where $\varrho(-x+E)>0$, and this contradicts $|\mu|(E+\varphi(R))>0$.

(1° implies 2°) Let E be a Borel subset of G with $|\mu|(E)=0$. We must prove $\varrho * |\mu|(E)=0$. Define the Borel subsets G_0, G_1, E_0 and E_1 of G as in the proof of Lemma 2.1. As in that proof, E_0 is null in the direction of φ, so

$$\varrho * |\mu|(E_0)=\int_G \varrho(-x+E_0)\,d|\mu|(x)=0.$$

E_1 is thick in the direction of φ, for x is in E_1 if and only if $(x+\varphi(R))\cap E$ has positive linear measure, in which case

$$(x+\varphi(R))\cap E=(x+\varphi(R))\cap E_1.$$

Since $0\leqslant |\mu|(E_1)\leqslant |\mu|(E)=0$ and μ is non-vanishing in the direction of φ, we have $|\mu|(E_1+\varphi(R))=0$. Thus

$$\varrho * |\mu|(E_1)=\int_G \varrho(-x+E_1)\,d|\mu|(x)=0,$$

since the integrand is positive precisely for those x in $E_1+\varphi(R)$. This completes the proof that

$$\varrho * |\mu|(E)=\varrho * |\mu|(E_0)+\varrho * |\mu|(E_1)=0.$$

We are now able to state the main result of this section.

PROPOSITION 2.3. *Let μ be a measure on G. Then the following are equivalent:*

1° *μ is quasi-invariant under φ;*

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[1] If E is thick in the direction of φ, $E+\varphi(R)=\{x:\varrho(-x+E)>0\}$, and thus $E+\varphi(R)$ is Borel.

2° *μ is absolutely continuous and non-vanishing in the direction of φ;*

3° *$|\mu|$ and $\varrho * |\mu|$ are mutually absolutely continuous.*

Proof. (3° implies 1°) For any measure μ, $\varrho * |\mu|$ is quasi-invariant under φ because of

$$\varrho * |\mu|(E) = \int_G \varrho(-x+E)\,d|\mu|(x),$$

$$\varrho * |\mu|(\varphi(t)+E) = \int_G \varrho(-x+\varphi(t)+E)\,d|\mu|(x),$$

and the fact that, for each t in R,

$$\{x\colon x \in G,\ \varrho(-x+E)>0\} = \{x\colon x\in G,\ \varrho(-x+\varphi(t)+E)>0\}.$$

It follows that μ is quasi-invariant under φ if $|\mu|$ is mutually absolutely continuous with respect $\varrho * |\mu|$.

(1° implies 3°) Assume μ is quasi-invariant under φ. For any Borel set E,

$$\varrho * |\mu|(E) = \int_G |\mu|(-x+E)\,d\varrho(x) = \int_{-\infty}^{+\infty} |\mu|(-\varphi(t)+E)\frac{1}{1+t^2}\,dt.$$

Thus if $|\mu|(E)=0$, so that $|\mu|(-\varphi(t)+E)=0$, all t, we have $\varrho * |\mu|(E)=0$. And if $\varrho * |\mu|(E)=0$, then $|\mu|(-\varphi(t)+E)=0$ for some t, and thus $|\mu|(E)=0$. This shows $|\mu|$ and $\varrho * |\mu|$ mutually absolutely continuous.

This completes the proof of Proposition 2.3, as the equivalence of 2° and 3° is a consequence of the preceding two lemmas.

Before stating the last result of this section, which is the analogue in our context of a result of Plessner [14] for the circle group, one further definition is necessary. We shall say that a measure μ on G *translates continuously in the direction of* φ if

$$\lim_{t\to 0} \|\mu_t - \mu\| = 0,$$

where $\|\cdot\|$ is the total variation norm, and for each t in R the translated measure μ_t is defined by

$$\mu_t(E) = \mu(\varphi(t)+E).$$

PROPOSITION 2.4. *Let μ be a measure on G. Then the following are equivalent.*

1° *μ is absolutely continuous in the direction of φ;*

2° *μ translates continuously in the direction of φ.*

Proof. (2° implies 1°) Let E be a Borel subset of G null in the direction of φ. We must prove $\mu(E)=0$. For each positive integer n denote by ν_n the measure on G that is the image under the mapping $\varphi: R \to G$ of the uniform measure of total mass 1 on the interval $[-1/n, +1/n]$. Because μ translates continuously in the direction of φ,

$$\lim_{n\to\infty} \|\nu_n * \mu - \mu\| = 0,$$

and in particular

$$\mu(E) = \lim_{n\to\infty} \nu_n * \mu(E).$$

But for each n,

$$\nu_n * \mu(E) = \int_G \nu_n(-x+E)\, d\mu(x) = \int_G \left(\frac{n}{2}\int_{-1/n}^{+1/n} \chi_E(x+\varphi(t))\, dt\right) d\mu(x) = 0,$$

since E is null in the direction of φ. This proves $\mu(E)=0$.

(1° implies 2°). We shall denote by $M(G)$ the Banach space of measures on G supplied with the total variation norm $\|\cdot\|$, and by $M_c(G)$ the subset of $M(G)$ consisting of those measures that translate continuously in the direction of φ. It is easy to check that $M_c(G)$ is a closed linear subspace of $M(G)$ and contains $\nu_n * \mu$, for ν_n defined in the first part of the proof and μ any measure on G. Suppose now that μ is a measure on G absolutely continuous in the direction of φ. We shall prove that $\nu_n * \mu$ converges to μ *weakly* in the Banach space $M(G)$. This will complete the proof that μ is in $M_c(G)$, for each $\nu_n * \mu$ is in the strongly closed linear subspace $M_c(G)$ of $M(G)$, and a strongly closed linearly subspace must be weakly closed (see [7], theorem 13, p. 422). To show that $\nu_n * \mu$ converges weakly to μ it suffices to demonstrate (see [7], theorem 5, p. 308)

$$\lim_{n\to\infty} \nu_n * \mu(E) = \mu(E)$$

for an arbitrary Borel subset E of G. Denote by F the subset of G consisting of those x for which

$$\lim_{n\to\infty} \frac{n}{2}\int_{-1/n}^{+1/n} \chi_E(x+\varphi(t))\, dt = \chi_E(x).$$

F is a Borel set, and by the metric density theorem (see for example [8], p. 211) its complement $G\backslash F$ is a set null in the direction of φ. Thus, since μ is absolutely continuous in the direction of φ, $|\mu|(G\backslash F)=0$; equivalently

$$\lim_{n\to\infty} \frac{n}{2}\int_{-1/n}^{+1/n} \chi_E(x+\varphi(t))dt = \chi_E(x)$$

almost everywhere with respect to $|\mu|$. So as a consequence of the Lebesgue bounded convergence theorem we have

$$\begin{aligned}\lim_{n\to\infty} \nu_n * \mu(E) &= \lim_{n\to\infty} \int_G \nu_n(-x+E)\, d\mu(x)\\ &= \lim_{n\to\infty} \int_G \left(\frac{n}{2}\int_{-1/n}^{+1/n} \chi_E(x+\varphi(t))\, dt\right) d\mu(x)\\ &= \int_G \chi_E(x)\, d\mu(x) = \mu(E).\end{aligned}$$

This completes the proof.

3. Extensions of the first Riesz theorem

In this section we establish, as consequences of the Main Theorem and the results of the preceeding section, absolute continuity properties of analytic measures. The results given here are extensions of the first of the F. and M. Riesz theorems.

THEOREM 3.1. *Let μ be a φ-analytic measure on G. Then μ is absolutely continuous in the direction of φ. Equivalently, μ translates continuously in the direction of φ.*

Proof. Immediate from the Main Theorem, Proposition 2.3 and Proposition 2.4.

If μ is a measure on G, the *ψ-conjugate* of μ is defined to be that measure μ_ψ (if such exists) whose Fourier transform satisfies

$$\hat{\mu}_\psi(\sigma) = \begin{cases} \hat{\mu}(\sigma), & \psi(\sigma) > 0\\ 0, & \psi(\sigma) = 0\\ -\hat{\mu}(\sigma), & \psi(\sigma) < 0.\end{cases}$$

Suppose that μ is a measure on G that has a ψ-conjugate μ_ψ. Since $\mu = \frac{1}{2}((\mu+\mu_\psi)+(\mu-\mu_\psi))$, Theorem 3.1 applied to $\mu+\mu_\psi$ and $\mu-\mu_\psi$ yields the following, which is equivalent to Theorem 3.1.

THEOREM 3.2. *Let μ be a measure on G having a ψ-conjugate. Then μ is absolutely continuous in the direction of φ and translates continuously in the direction of φ.*

The following theorem is the extension to our context of a result for the circle group due to Rudin [16] and Carleson [4].

THEOREM 3.3. *Let E be a closed subset of G. Then the following are equivalent:*

1° *E is null in the direction of φ;*

2° *For each continuous function h on E there is a continuous φ-analytic function f on G that agrees with h on E.*

Proof. (1° implies 2°) Let μ be any measure on G that is orthogonal to all continuous φ-analytic functions on G. Then in particular, $\int_G \sigma d\mu = 0$ for each σ in $\hat{G}$ with $\psi(\sigma) \geqslant 0$. Equivalently, $\hat{\mu}$ vanishes on $\{\sigma: \sigma \in G, \psi(\sigma) \leqslant 0\}$. So μ is analytic and by Theorem 3.1, $\mu(E_1) = 0$ for each Borel $E_1 \subset E$, or $|\mu|(E) = 0$. Since $|\mu|(E) = 0$ for any measure μ that is orthogonal to all continuous φ-analytic functions, 2° is a consequence of Theorem 1 of [1].

(2° implies 1°) Assume that 1° is false and 2° true. By translating E if necessary we may assume that $\varphi(\mathrm{R}) \cap E$ has positive linear measure. Let $K = \varphi^{-1}(E)$, so K is a closed subset of R having positive Lebesgue measure. We have assumed that ψ is a nontrivial homomorphism and thus φ is a non-trivial homomorphism. Consequently, it is possible to find an interval I of R and a point k of K so that $I \cap K$ has positive Lebesgue measure and $\varphi(k)$ is not in $\varphi(I)$. By the Tietze extension theorem there is a continuous function h on E equal to 0 on $E \cap \varphi(I)$ and 1 at $\varphi(k)$. By the assumption of 2° there is a continuous φ-analytic function f on G agreeing with h on E. The function $f \circ \varphi$ on R has an analytic extension to the upper half-plane since f is φ-analytic, and it is 0 on $I \cap K$ and 1 at k. But this is impossible since a bounded analytic function in a half-plane cannot have boundary valuez zero on a set of positive Lebesgue measure without vanishing identically. The completes the proof of Theorem 3.3.

If H is the n-torus, its dual $\hat{H}$ is the group of lattice points in real n-space. Bochner's extension of the first F. and M. Riesz theorem (see [2]) states that any measure on the n-torus whose Fourier transform vanishes off the positive octant of the lattice points must be absolutely continuous. The following consequence of Theorem 3.1 is an extension of the Bochner theorem. To obtain Bochner's theorem from this result, define ψ_j, for $j = 1, \dots, n$, by $\psi_j(m_1, \dots, m_n) = m_j$. Then if μ is a measure on the n-torus whose Fourier transform vanishes off the positive octant, $\mu_{\psi_j} = \mu$, $j = 1, \dots, n$.

THEOREM 3.4. *Let H be the n-torus, μ a measure on H and $\{\psi_1, \dots, \psi_n\}$ a linearly independent*(1) *set of homomorphisms of $\hat{H}$ into R. Assume that for $j = 1, \dots, n$ the conjugate measure μ_{ψ_j} exists. Then μ must be absolutely continuous.*

Proof. Denote by φ_j: $R \to H$ the homomorphism dual to ψ_j: $\hat{H} \to R$. Let $M(G)$ be the Banach space of measures on G supplied with the total variation norm. For x in G, denote by $T_x\mu$ the measure on G defined by $T_x\mu(E) = \mu(x + E)$. By Theorem 3.2, for $j = 1, \dots, n$, the mapping $t \to T_{\varphi_j(t)}\mu$ of R into $M(G)$ is continuous. Since the ψ_j are linearly independent, for t small enough, the mapping

(1) We assume there are no constants $c_1, \dots, c_n$ so that $c_1\psi_1 + \dots + c_n\psi_n$ is the zero mapping from $\hat{H}$ to R.

$$(t_1, \ldots, t_n) \to \varphi_1(t_1) \ldots \varphi_n(t_n)$$

is a homeomorphism of a neighborhood of the origin in n-space with a neighborhood of the identity element of H. Thus, by the continuity of $t \to T_{\varphi_j(t)}\mu$ for $j = 1, \ldots, n$, $x \to T_x \mu$ is a continuous mapping of G into $M(G)$. So by the generalized Plessner theorem (see for example, Theorem, p. 230, [17]) μ must be absolutely continuous with respect to Haar measure of H.

4. Extensions of the second Riesz theorem

In this section we establish, as consequences of the Main Theorem and the results of section 2, non-vanishing properties of analytic measures. The results given here are extensions of the second of the F. and M. Riesz theorems.

THEOREM 4.1. *Let μ be a φ-analytic measure on G. Then μ is nonvanishing in the direction of φ.*

Proof. Immediate from the Main Theorem and Proposition 2.3.

Because of Theorem 4.1, a φ-analytic measure vanishing identically on an open subset U of G must also vanish identically on the set $\varphi(R) + U$. But we cannot conclude from this that μ is the zero measure unless $\varphi(R)$ is dense in G; or equivalently, that $\psi: \hat{G} \to R$ is one-one. Thus we make this assumption below in order to be able to draw the stronger conclusion.

Part 2° of the following is due to Helson, Lowdenslager and Malliavin in [10]. We have not been able to obtain a proof of part 1° by their methods.

THEOREM 4.2. *Assume $\varphi(R)$ dense in G. Let μ be a φ-analytic measure on G that either*

1° *vanishes identically on an open subset of G or*

2° *is absolutely continuous and vanishes identically on a Borel set of positive Haar measure.*

Then μ is the zero measure.

Proof. (1°) Suppose μ vanishes identically on the open subset U of T. U is thick in the direction of φ, so by Theorem 4.1, μ vanishes identically on $\varphi(R) + U$. But $\varphi(R)$ is dense in G, so $\varphi(R) + U = G$.

(2°) We shall denote normalized Haar measure on G by m_G. Let E be a Borel subset of G with $m_G(E) > 0$ and $|\mu|(E) = 0$. We must show μ is the zero measure. By Theorem 4.1 and Lemma 2.2, $\varrho * |\mu|(E) = 0$. Thus

$$0 = \varrho * |\mu|(E) = \int_G (-x + E) d|\mu|(x),$$

so $|\mu|(G_1)=0$, if G_1 is defined to be

$$\{x: x\in G, \varrho(-x+E)>0\},$$

a Borel set invariant under translations by elements of $\varphi(R)$. Since $\varrho * m_G = m_G$,

$$\int_G (-x+E)\,dm_G(x) = m_G(E) > 0,$$

so that $m_G(G_1)>0$. Consequently the characteristic function χ_{G_1} of G_1 is a non-zero element of $L_1(G)$ fixed under translations by elements of $\varphi(R)$; and since elements of $L_1(G)$ translate continuously, χ_{G_1} is left fixed by translation by any element of $\varphi(R)^- = G$. Evidently then $m_G(G_1)=1$, which proves μ is the zero measure since $m_G(G_1)=1$, $|\mu|(G_1)=0$, and $|\mu| << m_G$.

5. Further properties of analytic measures

In this section we establish some further properties of measures on G that are absolutely continuous in the direction of φ. As a consequence of Theorem 3.1, these results are valid for all φ-analytic measures on G. For such measures, the results that we state are in part extensions of results of Bochner [2] and Helson-Lowdenslager [10].(1)

Throughout this section, for μ a measure on G, $\mu=\mu_a+\mu_s$ is its decomposition into parts absolutely continuous and singular with respect to Haar measure on G. It is clear from the definition of absolute continuity in the direction of φ that μ_a and μ_s will be absolutely continuous in the direction of φ if μ is.

PROPOSITION 5.1. *Let μ be a measure absolutely continuous in the direction of φ. Let K be a closed subset of R. If $\hat{\mu}$ vanishes off $\psi^{-1}(K)$, then $\hat{\mu}_a$ and $\hat{\mu}_s$ also vanish off K.*

For μ a φ-analytic measure and $K=\{t: t\leqslant 0\}$, this result is essentially Theorem 7 of [9].

It is convenient to establish a lemma before the proof of Proposition 5.1.

LEMMA 5.2. *Let μ be a singular measure on G absolutely continuous in the direction of φ. Let λ be the image under φ: $R\to G$ of some measure ν on R. Then $\lambda*\mu$ is also singular.*

Proof. Since μ is singular we can find a Borel subset B of G having Haar measure 1 and $|\mu|(B)=0$. Let B_0 be the intersection of

$$\{\varphi(t)+B: t \text{ rational}\},$$

(1) Bochner has informed us that he is able to obtain the conclusions of Proposition 5.1, 5.3 and 5.4 for φ-analytic measures using the results of [9]. Frank Forelli has obtained a special case of Lemma 5.2 for φ-analytic measures using quite different methods.

so B_0 has Haar measure 1 and $|\mu|(-\varphi(t)+B_0)=0$ for rational t. $|\mu|$ is absolutely continuous in the direction of φ, so by Proposition 2.4, $t\to|\mu|(-\varphi(t)+B_0)$ is continuous and thus $|\mu|(-\varphi(t)+B_0)=0$ for all $t\in R$. Let D be an arbitrary Borel subset of B_0. Then $\mu(-\varphi(t)+D)=0$ for all $t\in R$, so

$$\lambda * \mu(D) = \int_G \mu(-x+D)d\lambda(x) = \int_{-\infty}^{+\infty} \mu(-\varphi(t)+D)d\nu(t) = 0.$$

This proves $|\lambda*\mu|(B_0)=0$, and since B_0 has Haar measure 1, $\lambda*\mu$ must be singular.

We now proceed to the proof of Proposition 5.1. Suppose the proposition false. Then there is an element σ_0 of $\hat{G}$ not in $\psi^{-1}(K)$ with $\hat{\mu}_a(\sigma_0)\neq 0$. Since K is a closed subset of R and $\psi(\sigma_0)$ is not in K it is possible to find a measure ν on R with $\hat{\nu}\equiv 0$ on K and $\hat{\nu}(\psi(\sigma_0))\neq 0$. Let λ be the image of ν under φ: $R\to G$. Then $\hat{\lambda}=\hat{\nu}\circ\psi$, and as a consequence $\lambda*\mu=0$, since $(\lambda*\mu)\hat{}=\hat{\lambda}\hat{\mu}$, $\hat{\mu}$ vanishes off $\psi^{-1}(K)$ and $\hat{\lambda}$ vanishes on $\psi^{-1}(K)$. So we have $\lambda*\mu_a=-\lambda*\mu_s$. $\lambda*\mu_a$ is absolutely continuous, and by Lemma 5.2 applied to μ_s, $\lambda*\mu_s$ is singular. Thus $\lambda*\mu_a=\lambda*\mu_s=0$. But

$$\widehat{\lambda*\mu_a}(\sigma_0)=\hat{\lambda}(\sigma_0)\hat{\mu}_a(\sigma_0)=\hat{\nu}(\psi(\sigma_0))\hat{\mu}_a(\sigma_0)\neq 0,$$

so we have a contradiction, completing the proof of Proposition 5.1.

In the following two results, we assume for simplicity of statement that $\hat{G}$ is R with the discrete topology and ψ: $\hat{G}\to R$ the identity mapping, so that G is the Bohr compactification of the reals. For the validity of the results it would have sufficed to assume ψ one-one.

PROPOSITION 5.3. *Let μ be a singular measure on G that is absolutely continuous in the direction of φ. Then $\{\sigma: \sigma\in R, \hat{\mu}(\sigma)\neq 0\}$ is a subset of R containing no isolated points.*(1)

Proof. Suppose σ_0 were an isolated point of $\{\sigma: \sigma\in R, \hat{\mu}(\sigma)\neq 0\}$. Let ν be a measure on R with $\hat{\nu}(\sigma_0)\neq 0$ and $\hat{\nu}=0$ at all other points where $\hat{\mu}$ is non-zero. Let λ be the image of ν under the mapping φ: $R\to G$, so that $\hat{\lambda}=\hat{\nu}$, since we have identified $\hat{G}$ with R under ψ. Then

$$\widehat{\lambda*\mu}(\sigma)=\hat{\lambda}(\sigma)\hat{\mu}(\sigma)=\hat{\nu}(\sigma)\hat{\mu}(\sigma),$$

which is zero if $\sigma\neq\sigma_0$ and non-zero for $\sigma=\sigma_0$. So $\lambda*\mu$ must be a non-zero multiple of a character times Haar measure on G. But by Lemma 5.2, $\lambda*\mu$ is singular, so we have our contradiction.

(1) We assume it contains no isolated points considered as a subset of R with the usual topology.

PROPOSITION 5.4. *Let K be a countable closed subset of R. Let μ be a measure on G that is absolutely continuous in the direction of φ and whose Fourier transform vanishes off K. Then μ is absolutely continuous.*

Proof. By Proposition 5.1, $\hat{\mu}_s$ also vanishes off K. And by Proposition 5.3, $\{\sigma: \sigma \in R, \hat{\mu}_s(\sigma) \neq 0\}$ can have no isolated points. But the only subset of K having no isolated points is the empty set since K is countable and closed. Thus $\mu_s = 0$.

The final result of this section is a refinement of Lemma 5.2, which is valid for measures that are quasi-invariant under φ, and thus by the Main Theorem, for all φ-analytic measures on G.

PROPOSITION 5.5. *Let μ be a measure on G that is quasi-invariant under φ. Let λ be the image under $\varphi: R \to G$ of some measure ν on R. Then $\lambda * \mu << \mu$.*

Proof. Let E be a Borel subset of G with $|\mu|(E) = 0$. Then for any Borel subset F of E,

$$\lambda * \mu(F) = \int_G \mu(-x + F) d\lambda(x) = \int_{-\infty}^{+\infty} \mu(-\varphi(t) + F) d\nu(t) = 0,$$

since the last integrand is identically zero. This proves that $\lambda * \mu << \mu$.

Theorem 5.4 leads easily to an analogue of Bochner's Theorem for the (countably) infinite dimensional torus T^∞. Indeed, since we may identify $T^{\infty\wedge}$ with the (weak) direct sum of countably many replicas of the integers, let the positive "octant" in $T^{\infty\wedge}$ consist of those sequences of integers $n = (n_1, n_2, \ldots)$ with $n_i \geqslant 0$ (only finitely many n_i are non-zero). Then if μ in $M(T^\infty)$ has its Fourier transform $\hat{\mu}$ vanishing off the octant, μ is absolutely continuous with respect to Haar measure; in fact $|\mu|$ is equivalent to Haar measure if $\mu \neq 0$.

For let $a_1, a_2, \ldots$ be an increasing sequence of independent positive real numbers tending to ∞. Then the isomorphism ψ of $T^{\infty\wedge}$ into R defined by

$$\psi(n) = \sum n_i a_i,$$

maps the octant onto a closed countable subset of the non-negative reals, as is easily seen. Thus with φ the dual map of R into T^∞, μ is clearly φ-analytic, so that by Theorems 3.1 and 5.4, μ is absolutely continuous. Moreover, since ψ is one-one, $\varphi(R)$ is dense in T^∞, and so Theorem 4.2 applies, and $|\mu|$ cannot vanish on a set of positive Haar measure if $\mu \neq 0$.

6. Reduction to the covering group

In this section we construct a covering group for G and show that the Main Theorem is a consequence of an analogous result for the covering group. This result for the covering group is established in the following section, completing the proof of the Main Theorem.

In order to clarify matters we shall outline our procedure in this section. We find a closed subgroup K of G so that the homomorphism γ: $R \times K \to G$ defined by

$$\gamma(t,u) = \varphi(t) + u \quad (t \in R,\ u \in K),$$

is a covering mapping for G; that is, γ is onto, has as kernel a discrete subgroup, and is a local homeomorphism. (This construction was suggested to us by [11], p. 451.) A measure λ on $R \times K$ is called *analytic* if its Fourier transform $\hat{\lambda}$, defined on the dual group $R \times \hat{K}$, satisfies $\hat{\lambda}(s,\sigma) = 0$ for $s < 0$. And λ is called *quasi-invariant* if the collection of all Borel sets on which λ vanishes identically is invariant under translation by the subgroup $R \times \{e\}$ of $R \times K$, where e is the identity element of K. We define a mapping $\mu \to \lambda_\mu$ of measures on G into measures on $R \times K$ which satisfies 1°. *If μ is φ-analytic, then λ_μ is analytic;* 2°. *If λ_μ is quasi-invariant, then μ is quasi-invariant under φ.* Thus the Main Theorem is reduced to the assertion that each analytic measure on $R \times K$ is quasi-invariant. And this result is Theorem 7.1 of the following section.

We first construct the subgroup K of G. Choose any element χ_0 of $\hat{G}$ with $\psi(\chi_0) > 0$. By changing the scale of ψ if necessary we may assume $\psi(\chi_0) = 1$. K is defined to be the annihilator in G of the subgroup of $\hat{G}$ generated by χ_0; that is,

$$K = \{x\colon x \in G,\ \chi_0(x) = 1\}.$$

K is a closed subgroup of G and thus a compact abelian group. We shall denote by T the locally compact abelian group $R \times K$ and by γ: $T \to G$ the mapping defined by

$$\gamma(t,u) = \varphi(t) + u \quad (t \in R,\ u \in K).$$

γ is clearly a continuous homomorphism; its kernel will be denoted by D. Since $\chi_0(\varphi(t)) = e^{i\psi(\chi_0)t} = e^{it}$ for all $t \in R$, $\varphi(t)$ is in K if and only if t is an integer multiple of 2π. Thus

$$D = \{(2\pi n,\ -\varphi(2\pi n))\colon n = 0,\ \pm 1, \ldots\}.$$

In particular D is a discrete subgroup of T.

Throughout this section we shall denote by I the interval $\{t\colon -\pi < t \leqslant +\pi\}$ in R. The next lemma shows that γ: $R \times K \to G$ is a covering mapping and has $I \times K$ as fundamental domain. The result is essentially Theorem 2.3 of [11].

LEMMA 6.1. *γ maps $I\times K$ one-one onto G and is a homeomorphism on the interior of $I\times K$.*

Proof. Let $x\in G$. Suppose that $\chi_0(x)=e^{it}$, where t is chosen in I. Let $u=x-\varphi(t)$, so $\chi_0(u)=e^{it}e^{-it}=1$, and $u\in K$. Then $x=\gamma(t,u)$, proving the onto part of the assertion. For the one-one part, suppose that (t_1,u_1) and (t_2,u_2) in $I\times K$ have the same image under γ. Then $\varphi(t_1)+u_1=\varphi(t_2)+u_2$ and as a consequence $\varphi(t_2-t_1)=u_2-u_1\in K$, so t_2-t_1 must be an integral multiple of 2π. But t_1 and t_2 are both in I, so $t_1=t_2$, which implies $u_1=u_2$.

To see γ is a homeomorphism on the interior of $I\times K$, suppose $\varphi(t_\delta)+u_\delta\to\varphi(t)+u$, $-\pi<t$, $t_\delta<\pi$. Then any cluster point of $\{\varphi(t_\delta-t)\}=\{\varphi(t_\delta)-\varphi(t)\}$ is a cluster point of $\{u-u_\delta\}$, hence in K. So if t' is a cluster point of $\{t_\delta\}$ then $\varphi(t'-t)\in K$, whence $t'-t$ is an integral multiple of 2π; since $-\pi\leqslant t'\leqslant\pi$ and $-\pi<t<\pi$, we must have $t'-t=0$. Thus $t_\delta\to t$, so that $\varphi(t_\delta)\to\varphi(t)$, whence $u_\delta\to u$ and $(t_\delta,u_\delta)\to(t,u)$.

We shall denote by $\beta: G\to I\times K$ the mapping inverse to the one-one correspondence $\gamma: I\times K\to G$. Haar measure on G, D and T will be denoted by m_G, m_D and m_T respectively. We assume m_G normalized to have total mass 1, and m_D normalized to give each point of D mass 1. Lemma 6.1 shows that G is naturally isomorphic to the quotient group T/D. Thus by p. 131 of [13], it is possible to normalize m_T so that(¹)

$$\int_T f\,dm_T=\int_G((m_D * f)\circ\beta)\,dm_G \tag{6.1}$$

for each continuous f on T having compact support.

We proceed next to the construction of the mapping $\mu\to\lambda_\mu$ from measures on G to measures on T. We denote by $C(G)$ and $C_0(T)$ the Banach spaces of continuous functions on G and continuous functions on T zero at infinity. Their duals are the spaces $M(G)$ and $M(T)$ of measures on G and T. We denote by $C_c(T)$ the space of continuous functions on T having compact support.

To define the mapping $\mu\to\lambda_\mu$ we shall make use of a fixed function g in $C_0(T)$. In order to construct the mapping $\mu\to\lambda_\mu$ we need only assume that g satisfies

$$g(u_1,t)=g(u_2,t)\quad(u_1,\ u_2\in K,\ t\in R),$$

and

$$g(u,t)=O(t^{-2})\quad\text{as}\quad t\to\pm\infty.$$

Later in order to establish properties of $\mu\to\lambda_\mu$, we shall put further conditions on g.

(¹) $m_D * f(u)=\sum_{d\in D}f(u+d)$.

If μ is absolutely continuous, so $\mu = fm_G$ for some m_G integrable function f on G, we want the associated measure λ_μ to be $(f \circ \gamma) g m_T$. So one way of proceeding would be by extending the mapping $fm_G \to (f \circ \gamma) g m_T$ by continuity to all of $M(G)$. Instead we use a somewhat more indirect method which is more convenient.

We define a mapping L: $C_0(T) \to C(G)$ as follows. For each f in $C_0(T)$, Lf is the function $(m_D \ast (fg)) \circ \beta$ on G. It is simple to check that, because of the conditions assumed on g, L is a continuous linear transformation of $C_0(T)$ into $C(G)$. The adjoint mapping L^*: $M(G) \to M(T)$, defined by

$$\int_T f d(L^* \mu) = \int_G (Lf)\, d\mu \quad (f \in C_0(T)),$$

is continuous in both the norm and weak* topologies of $M(G)$ and $M(T)$.

For each μ in $M(G)$ we define λ_μ to be $L^* \mu$.

We first show, for a large class of absolutly continuous μ, that λ_μ is the measure we wanted.

LEMMA 6.2. *Let h be a function in $C(G)$. Then*

$$L^*(hm_G) = (h \circ \gamma) g m_T.$$

Proof. It suffices to show, that for each function b in $C_c(T)$,

$$\int_T bd(L^*(hm_G)) = \int_T (h \circ \gamma) bg dm_T. \tag{6.2}$$

By the definition of L and L^*

$$\begin{aligned}\int_T bd(L^*(hm_G)) &= \int_G (Lb) h dm_G \\ &= \int_G ((m_D \ast (bg)) \circ \beta) h dm_G = \int_G ((m_D \ast (bg)) \circ \beta)((h \circ \gamma) \circ \beta) dm_G \\ &= \int_G ((h \circ \gamma)(m_D \ast (bg)) \circ \beta dm_G = \int_G (m_D \ast ((h \circ \gamma) bg)) \circ \beta dm_G.\end{aligned} \tag{6.3}$$

Formulas (6.1) is valid for $f = (h \circ \gamma) bg$, so

$$\int_G (m_D \ast ((h \circ \gamma) bg)) \circ \beta dm_G = \int_T (h \circ \gamma) bg dm_T. \tag{6.4}$$

(6.3) together with (6.4) implies (6.2).

We define a measure λ in T to be *quasi-invariant* if the collection of Borel sets on which λ vanishes identically is invariant under translation by elements of $R \times \{e\}$.

A further condition on g is necessary in order to relate quasi-invariance on G and T; we shall assume that the function g never vanishes on T.

LEMMA 6.3. *Let μ be a measure on G, U a Baire subset of G, $V=\beta(U)$ and d an element of D. Then*

$$\int_{d+V} (1/g) d\lambda_\mu = \mu(U). \tag{6.5}$$

Proof. By the definition of λ_μ, for any f in $C_0(T)$,

$$\int_T f d\lambda_\mu = \int_G (m_D * (fg)) \circ \beta d\mu. \tag{6.6}$$

Thus for any b in $C_c(T)$, taking $f=b/g$ in (6.6), we get

$$\int_T (b/g) d\lambda_\mu = \int_G (m_D * b) \circ \beta d\mu. \tag{6.7}$$

For any bounded open interval J of R, denote by B_J the class of all bounded Baire functions b on T, vanishing off $J \times K$, that satisfy (6.7). B_J contains all b in $C_c(T)$ with support in $J \times K$, and is clearly closed under monotone convergence of bounded sequences. Thus B_J contains all bounded Baire functions on T vanishing off $J \times K$. This shows in particular that (6.7) holds for b the characteristic function of $d+V$. But in this case, $m_D * b$ is the characteristic function of $\gamma^{-1}(U)$, so $(m_D * b) \circ \beta$ is the characteristic function of U and the equality (6.7) becomes (6.5).

COROLLARY 6.4. *Let μ be a measure on G, E a Baire subset of G. Then μ vanishes identically on E if and only if λ_μ vanishes identically on $\gamma^{-1}(E)$.*

Proof. Suppose μ does not vanish identically on E. Then there is a Baire subset U of E with $\mu(U) \neq 0$. By Lemma 6.3,

$$\int_V (1/g) d\lambda_\mu = \mu(U) \neq 0,$$

where $V=\beta(U)$ is a subset of $\gamma^{-1}(E)$, so λ_μ cannot vanish identically on $\gamma^{-1}(E)$. For the converse, suppose that λ_μ does not vanish identically on $\gamma^{-1}(E)$. Let

$$F=\beta(E)=\gamma^{-1}(E) \cap (I \times K).$$

Then, since $\gamma^{-1}(E)=\bigcup_{d \in D}(d+F)$, there is some d_0 is D so that λ_μ does not vanish identically on d_0+F. Let V be a Baire subset of F such that $\int_{d_0+V}(1/g)\,d\lambda_\mu \neq 0$, and let $U=\gamma(V)$. Then, by Lemma 6.3,

$$\mu(U)=\int_{d_0+V}(1/g)\,d\lambda_\mu \neq 0,$$

so μ cannot vanish identically on E.

PROPOSITION 6.5. *Let μ be a measure on G. If λ_μ is quasi-invariant, then μ is quasi-invariant under φ.*

Proof. Let E be a Baire subset of G on which μ vanishes identically, t an element of R. By Corollary 6.4, λ_μ vanishes identically on $\gamma^{-1}(E)$. λ_μ is quasi-invariant, so λ_μ vanishes identically on $(t,e)+\gamma^{-1}(E)$, which is $\gamma^{-1}(\varphi(t)+E))$. Again using Corollary 6.4, μ vanishes identically on $\varphi(t)+E$. This proves that the collection of Baire subsets of G on which μ vanishes identically is invariant under translation by elements of $\varphi(R)$. As a consequence, since μ is regular, the collection of Borel subsets of G on which μ vanishes identically is invariant under the translation by elements of $\varphi(R)$; that is, μ is quasi-invariant under φ.

The same sort of proof establishes the converse of Proposition 6.5. However we shall not need this result.

The dual group $\hat{T}$ of $T=R\times K$ can be identified with $R\times\hat{K}$; we associate with the element (s,σ) of $R\times\hat{K}$ the character χ of $R\times K$ defined by

$$\chi(t,u)=e^{ist}\sigma(u) \quad (t\in R,\ u\in K).$$

A function or measure on $R\times K$ will be called *analytic* if its Fourier transform vanishes on the "negative half" $\{(s,\sigma): s<0\}$ of $R\times\hat{K}$.

We require one final condition on the function g in order to relate analyticity on G and T; we assume the function g to be analytic.

We first show that functions satisfying all of the conditions that we have imposed on g actually exist. Let h be any continuous function on R, nowhere vanishing, satisfying $|h(t)|\leqslant 1/(1+t^2)$, all t, and $\hat{h}(s)=0$ if $s<0$. Then we can define g on $R\times K$ by $g(t,u)=h(t)$. It is simple to check that this function satisfies the conditions.

We now proceed to the proof that φ-analyticity of μ implies analyticity of λ_μ. We establish first two lemmas. We shall denote by $M_a(G)$ and $M_a(T)$ the subsets of $M(G)$ and $M(T)$ consisting of measures that are φ-analytic and analytic respectively.

LEMMA 6.6. *$M_a(T)$ is a weak* closed linear subspace of $M(T)$.*

Proof. Let h be a Lebesgue integrable continuous function vanishing at infinity on R satisfying $\hat{h}(s)=0$ if $s>0$ and $\hat{h}(s)\neq 0$ if $s<0$. Define the function f on $R\times K$ by $f(t,u)=$

$h(t)$. Then $f \in C_0(T)$, f is integrable with respect to m_T, and its Fourier transform $\hat{f}$, defined on $R \times \hat{K}$, satisfies $\hat{f}(s,\sigma)=0$ if $s>0$ and $\hat{f}(s,\sigma) \neq 0$ if $s<0$. Let λ be a measure in $M(T)$. Then $\widehat{\lambda * f} = \hat{\lambda}\hat{f}$ will be the zero function on $R \times \hat{K}$ precisely when $\hat{\lambda}$ vanishes on the subset $\{(s,\sigma): s<0\}$ of $R \times K$; that is, when λ is analytic. This shows that $M_a(T)$ consists of those λ in $M(T)$ satisfying $\lambda * f = 0$. But $\lambda * f = 0$ if and only if λ is orthogonal to all translates of f, and f is in $C_0(T)$, which proves that $M_a(T)$ is weak* closed. It is clear that $M_a(T)$ is a linear subspace of $M(T)$, so the proof is complete.

LEMMA 6.7. *Let χ in $\hat{G}$ satisfy $\psi(\chi) \geq 0$. Then $L^*(\chi m_G)$ is an analytic measure on T.*

Proof. By Lemma 6.2, $L^*(\chi m_G) = (\chi \circ \gamma) g m_T$. So it suffices to show that the function $(\chi \circ \gamma) g$ is analytic. Denote by χ_1 the restriction of the character χ of G to the subgroup K, so $\chi_1 \in \hat{K}$. Then for (s,σ) in $R \times K$,

$$\begin{aligned}\widehat{(\chi \circ \gamma) g}(s, \sigma) &= \int_T e^{-ist}\,\overline{\sigma(u)}[(\chi \circ \gamma) g](t, u)\,dm_T(t, u)\\ &= \int_T e^{-ist}\,\overline{\sigma(u)} e^{i\psi(\chi)t}\,\chi(u) g(t, u)\,dm_T(t, u)\\ &= \hat{g}(s-\psi(\chi), \sigma\bar{\chi}_1).\end{aligned} \tag{6.8}$$

Since we have assumed $\psi(\chi) \geq 0$, if s is negative, (6.8) together with the analyticity of g shows that $\widehat{(\chi \circ \gamma) g}(s,\sigma)=0$. This proves $(\chi \circ \gamma) g$ analytic.

PROPOSITION 6.8. *Let μ be a measure on G. If μ is φ-analytic, then λ_μ is analytic.*

Proof. We must prove that L^* maps $M_a(G)$ into $M_a(T)$. Let $\{\theta_\alpha: \alpha \in J\}$ be an approximate identity on G consisting of trigonometric polynomials;[1] that is, a net of trigonometric polynomials satisfying $\theta_\alpha * f \to f$ in $C(G)$ for each f in $C(G)$. Choose μ in $M_a(G)$. Then each $\mu * \theta_\alpha$ is a linear combination of characters χ with $\psi(\chi) \geq 0$, so by Lemma 6.7, each $L^*((\mu * \theta_\alpha) m_G)$ is in $M_a(T)$. Since $\{\theta_\alpha: \alpha \in J\}$ is an approximate identity, $(\mu * \theta_\alpha) m_G \to \mu$ weak*. L^* is continuous in the weak* topologies, so $L^*\mu$ is the weak* limit of $\{L^*((\mu * \theta_\alpha) m_G): \alpha \in J\}$, and thus by Lemma 6.6, is in $M_a(T)$.

7. Completion of proof of Main Theorem

As a consequence of Proposition 6.5 and 6.8, the Main Theorem will be established once we have proved the following.

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(1) A trigonometric polynomial is a linear combination of characters.

THEOREM 7.1. *Let μ be a measure on T. If μ is analytic, then μ is quasi-invariant.*

This section is devoted to the proof of Theorem 7.1. The method is roughly as follows. By Lemma 6.6, the space $M_a(T)$ of analytic measures in T is a weak* closed linear subspace of the space $M(T)$ of all measures on T. Thus, by the Alaoglu theorem, its unit ball

$$B = \{\mu: \mu \in M_a(T), \|\mu\| \leqslant 1\}$$

is compact in the weak* topology. We denote by B_e the set of extreme points of B and by $\bar{B}_e$ the weak* closure of B_e. It is a consequence (see Proposition 7, p. 87 of [3]) of the Krein-Milman theorem that each measure μ in B will have a representation of the form [1]

$$\mu = \int_{\bar{B}_e} \lambda d\nu(\lambda) \tag{7.1}$$

for some non-negative measure ν of total mass 1. The measures in $\bar{B}_e$ are shown to be concentrated on lines $R \times \{u\}$ in $R \times K$; and the one-dimensional F. and M. Riesz theorems applied to these measures demonstrates that they are quasi-invariant. The quasi-invariance of each measure in $M_a(T)$ is then obtained as a consequence of the quasi-invariance of the measures in $\bar{B}_e$ and the existence of the representation (7.1).

Our first aim is the establishment of the quasi-invariance of the measures in $\bar{B}_e$. This will be achieved by a sequence of lemmas.

LEMMA 7.2. *Let ν be a measure on R whose Fourier transform vanishes on $\{s: s<0\}$. Then*

$$\{D: D \text{ Borel}, |\nu|(D) = 0\} \tag{7.2}$$

is invariant under translation.

Proof. We can assume ν is not the zero measure. By the one-dimensional F. and M. Riesz theorems for R, ν must be absolutely continuous and not vanish identically on any set of positive Lebesgue measure. Thus (7.2) is simply the collection of Borel subsets of R having Lebesgue measure zero.

LEMMA 7.3. *Let μ be a measure in $M_a(T)$ with carrier contained in $R \times \{u\}$. Then μ is quasi-invariant.*

Proof. Define the measure ν on R by $\nu(D) = \mu(D \times \{u\})$ for each Borel subset D of R. Then, for any s in R,

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[1] The meaning of this integral is discussed after Proposition 7.7.

$$\hat{\nu}(s) = \int_{-\infty}^{+\infty} e^{-ist}\, d\nu(t) = \int_{R\times K} e^{-ist}\, d\mu(t,\, v)$$

$$= \int_{R\times K} e^{-ist}\, \chi_1(v) d\mu(t,\, v) = \hat{\mu}(s,\, \chi_1), \tag{7.3}$$

where χ_1 is the character of K identically equal to 1. μ is analytic, and thus $\hat{\mu}(s,\chi_1)=0$ if $s<0$. So, by (7.3), $\hat{\nu}(s)=0$ if $s<0$, and the conclusion of Lemma 7.2 applies to ν. Now let E be any Borel subset of T with $|\mu|(E)=0$, t_0 an element of R. We want to show $|\mu|((t_0,e)+E)=0$. Let $D=\{t\colon (t,u)\in E\}$, so $|\nu|(D)=|\mu|(E)=0$. The collection of null sets of ν is translation invariant; thus $|\mu|((t_0,e)+E)=|\nu|(t_0+D)=0$.

Lemma 7.4. *Let μ be a measure in $M_a(T)$. If h is a bounded continuous function on T satisfying*

$$h(t_1,u) = h(t_2,u) \quad (t_1,\, t_2 \in R,\, u\in K) \tag{7.4}$$

then $h\mu$ is also in $M_a(T)$.

Proof. Let $\mathcal{H}$ be the collection of all continuous functions h on T satisfying (7.4) which are such that $h\mu$ is in $M_a(T)$. If χ is a character of K and h on T is defined by $h(t,u)=\chi(u)$, then h is in $\mathcal{H}$. For, if $s<0$, $\sigma\in\hat{K}$, then

$$\widehat{h\mu}(s,\, \sigma) = \int_{R\times K} e^{-ist}\, \overline{\sigma(u)} h(t,\, u) d\mu(t,\, u)$$

$$= \int_{R\times K} e^{-ist}\, \overline{\sigma(u)} \chi(u) d\mu(t,\, u) = \hat{\mu}(s,\, \sigma\bar{\chi}) = 0$$

by the analyticity of μ. $\mathcal{H}$ is clearly a linear space of functions and closed under uniform convergence. Thus, since each continuous function on K can be approximated uniformly by linear combinations of characters, $\mathcal{H}$ consists of all continuous functions h on T satisfying (7.4).

Lemma 7.5. *Let μ be a measure in B_e. Then there is some u in K for which μ has carrier contained in $R\times\{u\}$.*

Proof. Since μ is an extreme point of B, $\|\mu\|=1$. Assume that there are points (t_0,u_0) and (t_1,u_1) in the carrier of μ with $u_0 \neq u_1$. We must show that this contradicts the assumption that μ is in B_e. Let f be a continuous function on K, $0\leqslant f\leqslant 1$, $f=0$ near u_0 and $f=1$ near u_1. Define the function h on T by $h(t,u)=f(u)$. Then h satisfies (7.4), so by Lemma 7.4, the measures $h\mu$ and $(1-h)\mu$ are in $M_a(T)$. These measures are non-zero and not multiples of each other. For u_0 is in the carrier of $(1-h)\mu$ and not the carrier of $h\mu$; and u_1 is in the carrier of $h\mu$ and not the carrier of $(1-h)\mu$. We have

$$\mu = \|h\mu\| \left(\frac{1}{\|h\mu\|} h\mu\right) + \|(1-h)\mu\| \left(\frac{1}{\|(1-h)\mu\|}(1-h)\mu\right),$$

and

$$\|h\mu\| + \|(1-h)\mu\| = \int_T |h|\, d|\mu| + \int_T |1-h|\, d|\mu|$$

$$= \int_T h d|\mu| + \int_T (1-h) d|\mu| = |\mu|(T) = 1,$$

which shows that μ could not be an extreme point of B.

LEMMA 7.6. *Let μ be a measure in $\bar{B}_e$. Then there is some u in K so that μ has carrier contained in $R \times \{u\}$.*

Proof. Let $\{\mu_\alpha\colon \alpha \in J\}$ be a net in B_e converging weak* to μ. By Lemma 7.5, there is for each $\alpha \in J$ a point u_α in K so that μ_α has carrier contained in $R \times \{u_\alpha\}$. By the compactness of K, the net $\{u_\alpha\colon \alpha \in J\}$ has at least one cluster point. Let u be such a cluster point. We will prove the carrier of μ is contained in $R \times \{u\}$. Let f be any continuous function on T having compact support contained in the complement of $R \times \{u\}$. Then the net

$$\left\{\int_T f d\mu_\alpha\colon \alpha \in J\right\}$$

of real numbers has a cofinal subnet consisting of zeros. But

$$\int_T f d\mu = \lim_{\alpha \in J} \int_T f d\mu_\alpha$$

so $\int_T f d\mu = 0$. This proves the carrier of μ contained in $R \times \{u\}$.

We can now assert the quasi-invariance of the measures in $\bar{B}_e$.

PROPOSITION 7.7. *Let μ be a measure in $\bar{B}_e$. Then μ is quasi-invariant.*

Proof. Immediate from Lemmas 7.3 and 7.6.

We now proceed to the second part of the proof of Theorem 7.1, which reduces the question of the quasi-invariance of an arbitrary measure in $M_a(T)$ to that of measures in $\bar{B}_e$, which has been settled.

Let μ be an arbitrary measure in $M_a(T)$ with $\|\mu\| = 1$, so μ is in B. To complete the proof of Theorem 7.1 we must show μ quasi-invariant. By Lemma 6.6 and the Alaoglu theorem, B is compact in the weak$*$ topology of $M(T)$. It is a consequence (see Proposition 7, p. 87 of [3]) of the Krein-Milman theorem that there will be a non-negative measure ν of total mass 1 on the compact space $\bar{B}_e$ so that

$$\mu = \int_{\bar{B}_e} \lambda d\nu(\lambda). \tag{7.5}$$

The integral in (7.5) is to be interpreted in the following sense;(1) for each f in $C_0(T)$,

$$\mu(f) = \int_{\bar{B}_e} \lambda(f) d\nu(\lambda).$$

One further lemma is needed before we are able to complete the proof of Theorem 7.1. This lemma states roughly that, in the situation we are considering, the formula

$$|\mu| = \int_{\bar{B}_e} |\lambda| \, d\nu(\lambda)$$

is meaningful and valid.

LEMMA 7.8. *Let E be a Baire subset of T. Then*

$$\lambda \to |\lambda|(E)$$

is a Borel function on B and

$$|\mu|(E) = \int_{\bar{B}_e} |\lambda|(E) d\nu(\lambda).$$

Let us assume the lemma true and complete the proof of Theorem 7.1; the lemma will be proved last.

Since μ is regular, to establish the quasi-invariance of μ it suffices to show that the collection of Baire subsets E of T with $|\mu|(E)=0$ is invariant under translation by elements of $R \times \{e\}$. So suppose E is Baire, $|\mu|(E)=0$ and $t \in R$. We must show $|\mu|((t,e)+E)=0$. By Lemma 7.8, since $|\mu|(E)=0$,

$$\{\lambda \colon \lambda \in \bar{B}_e, |\lambda|(E) \neq 0\} \tag{7.6}$$

is Borel and has ν measure 0. By Proposition 7.7, the set

$$\{\lambda \colon \lambda \in \bar{B}_e, |\lambda|((t,e)+E) \neq 0\}$$

(1) We now use the notation $\lambda(f) = \int f d\lambda$, $\mu(f) = \int f d\mu$.

is identical with (7.6), and thus in particular has ν measure 0. As a consequence, again using Lemma 7.8,

$$|\mu|((t,e)+E)=\int_T |\lambda|((t,e)+E)d\nu(\lambda)=0.$$

So except for the proof of Lemma 7.8, we have completed the proof of Theorem 7.1, and thus of the Main Theorem. We now prove this lemma.

We continue to use $\|\cdot\|$ to denote the total variation norm for measures in $M(T)$ and will use $\|\cdot\|_\infty$ for the supremum norm for $C_0(T)$. Let f be any bounded continuous function on T with $f\geqslant 0$. Then, for any measure η on T,

$$|\eta|(f)=\int_T fd|\eta|=\|f\eta\|=\sup\left\{\left|\int_T fhd\eta\right|: h\in C_0(T),\ \|h\|_\infty\leqslant 1\right\}. \tag{7.7}$$

For each h in $C_0(T)$ with $\|h\|_\infty \leqslant 1$ define the function L_h on B by $L_h(\eta)=|\eta(fh)|$. Each L_h is continuous on B. And (7.7) shows that the function

$$\eta\to|\eta|(f) \quad (\eta\in B),$$

is the pointwise supremum of the family $\{L_h\}$ of continuous functions, and is thus a Borel function on B. In particular, it is possible to form the integral

$$\int_{\bar{B}_e}|\lambda|(f)d\nu(\lambda).$$

We have, using the representation (7.5) and the validity of (7.7) for $\eta=\mu$,

$$|\mu|(f)=\sup_{\|h\|_\infty\leqslant 1}\left|\int_T fhd\mu\right|=\sup_{\|h\|_\infty\leqslant 1}\left|\int_{\bar{B}_e}\lambda(fh)d\nu(\lambda)\right|$$
$$\leqslant \sup_{\|h\|_\infty\leqslant 1}\int_{\bar{B}_e}|\lambda|(f)d\nu(\lambda)=\int_{\bar{B}_e}|\lambda|(f)d\nu(\lambda).$$

this proves that

$$|\mu|(f)\leqslant\int_{\bar{B}_e}|\lambda|(f)d\nu(\lambda) \tag{7.8}$$

for all bounded continuous f on T with $f\geqslant 0$. Suppose that the inequality (7.8) were strict for some f_0. We may assume $0\leqslant f_0\leqslant 1$. Then

$$1=\|\mu\|=\int_T 1d|\mu|=\int_T f_0d|\mu|+\int_T(1-f_0)d|\mu|. \tag{7.9}$$

We have assumed

$$\int_T f_0 d|\mu| < \int_{\bar{B}_e} |\lambda|(f_0)d\nu(\lambda); \tag{7.10}$$

and

$$\int_T (1-f_0)d|\mu| \leqslant \int_{\bar{B}_e} |\lambda|(1-f_0)d\nu(\lambda) \tag{7.11}$$

because of (7.8) applied to the non-negative function $1-f_0$. (7.9), (7.10) and (7.11) lead to the contradiction

$$\begin{aligned} 1 &< \int_{\bar{B}_e} |\lambda|(f_0)d\nu(\lambda) + \int_{\bar{B}_e} |\lambda|(1-f_0)d\nu(\lambda) \\ &= \int_{\bar{B}_e} |\lambda|(1)d\nu(\lambda) = \int_{\bar{B}_e} 1d\nu(\lambda) = 1. \end{aligned}$$

This shows that we have the equality

$$|\mu|(f) = \int_{\bar{B}_e} |\lambda|(f)d\nu(\lambda) \tag{7.12}$$

for all non-negative bounded continuous f on T, and thus for all bounded continuous f on T. Let us denote by $\mathcal{B}$ the collection of all bounded Baire functions f on T for which the mapping

$$\lambda \to |\lambda|(f) \quad (\lambda \in \mathcal{B}),$$

is Borel and the equality (7.12) holds. We have shown that $\mathcal{B}$ contains all bounded continuous functions on T; and because of the bounded convergence theorem, $\mathcal{B}$ is closed under pointwise convergence of bounded sequences. Thus $\mathcal{B}$ consists of all bounded Baire functions on T. Since the characteristic function of a Baire subset of T is a Baire function, the lemma is established.

8. Some further questions

There are several questions connected with the above results which deserve mention. First, most of our deductions from the Main Theorem are valid in the context of one-parameter groups of homeomorphisms of compact topological spaces. In particular, the results of sections 2 and 5 can be established in this generality. It is conceivable that a version of the Main Theorem itself is also valid in the context. Here is a possible generalization. Let X be a compact space and $\{T_t\}$ a one-parameter group of homeomorphisms of X. Call a measure μ on X $\{T_t\}$-*analytic* if the vector valued integral

$$\int_{-\infty}^{+\infty} h(t)T_t\mu dt$$

is zero for all h in $L^1(R)$ whose Fourier transforms vanish for $t \leqslant 0$. (In the case that $X = G$ and T_t is translation by $\varphi(t)$, this agrees with our previous definition of analyticity.) Then a generalization of our Main Theorem would be the assertion that a $\{T_t\}$-analytic measure μ is quasi-invariant under $\{T_t\}$; that is, the collection of $|\mu|$-null sets is $\{T_t\}$ invariant. Indeed, with this definition of analyticity (and T_t right translation by $\varphi(t)$), the Main Theorem continues to hold even when G is non-commutative.(1)

Another possible extension of some of our results is to the context of Dirichlet algebras (for the relevant definitions, see [12]). The collection of φ-analytic continuous functions on G is a Dirichlet algebra on G. Theorem 3.1 says precisely that each Borel subset of G that is of measure zero for all of the *representing measures* for the algebra must be of measure zero for all of the *annihilating measures* for the algebra. It is conceivable that a corresponding result holds for a wider class of Dirichlet algebras.

9. Locally compact abelian groups

We have neglected the (seemingly) more general situation in which G is a locally compact abelian group simply because the main results in this case follow easily from the compact case. Here we shall only note how the Main Theorem and the analogues of A and B (Theorems 3.1 and 4.1) are obtained.

Suppose that $\varphi: R \to G$ is a continuous homomorphism into the locally compact abelian group G, and μ is a φ-analytic measure on G (where "φ-analytic" is defined as earlier in terms of the dual of φ). Let $\tau: G \to G^a$ be the $(1-1)$ continuous homomorphism of G into its Bohr compactification G^a. The induced map of measures on G into measures on G^a, which we may as well again call τ, takes μ into a measure $\tau\mu$ which is $\tau\varphi$-analytic on G^a, as is easily seen. Thus $\tau\mu$ is quasi-variant under $\tau\varphi$; since τ maps Borel subsets of G onto Borel subsets of G^a, while $\tau\nu(\tau E) = \nu(E)$ for any measure ν on G and Borel $E \subset G$, it is trivial to conclude that μ is quasi-invariant under φ, so that the Main Theorem holds.

Similarly Theorem 3.1 (which might be obtained by showing its derivation from the Main Theorem involves no essential use of compactness) follows once we note that a Borel

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(1) Sketch of proof. Let $H = \varphi(R)^-$, a compact abelian group, and let B be the $\{T_t\}$-analytic measures of norm $\leqslant 1$, with B_e, B_e^- defined as before in terms of B. For $f \in C(G)$ constant on (left) cosets mod H, $T_t f\mu = f T_t \mu$, so $\int h(t)\, T_t f\, d\mu(t) = f \int h(t)\, T_t\, d\mu(t)$, and $f\mu$ is $\{T_t\}$-analytic if μ is. Using such f one can now argue as in 7.5 that $\mu \in B_e$ has carrier contained in a coset of H; and as in 7.6 the same applies to $\mu \in B_e^-$. But $\{T_t\}$-analyticity for a μ carried by H coincides with φ-analyticity for the map $\varphi: R \to H$. Thus $\mu \in B_e^-$ has a left translate which is φ-analytic, hence quasi-invariant, and μ is quasi-invariant. The final argument of § 7 now completes the proof.

set $E \subset G$ null in the direction of φ has τE null in the direction of $\tau\varphi$, whence $0 = |\tau\mu|(\tau E) = \tau|\mu|(\tau E) = |\mu|(E)$. And Theorem 4.1 follows since, if E is thick in the direction of φ and(1) $|\mu|(E+\varphi(R)) > 0$, we have the same statements for τE and $\tau\varphi$, so that $|\mu|(E) = \tau|\mu|(\tau E) = |\tau\mu|(\tau E) > 0$.

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(1) Since $\tau E + \tau\varphi(R)$ is a Borel set in G^a, $E + \varphi(R) = \tau^{-1}(\tau E + \tau\varphi(R))$ is locally Borel, hence a Borel subset of G since it is contained in a τ-compact set.