REMARK ON A PROBLEM OF LUSIN

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1. In 1915, (see [2] for an edition with added commentary), Lusin asked whether, for every measurable function f on $[0, 2\pi]$, finite or infinite, there is a trigonometric series, with coefficients converging to zero, which converges almost everywhere to f.

The problem was solved in the affirmative by Menchoff, [3], [4] (also, see [1]), for the case where f is finite almost everywhere. Bari, ([2], p. 527), also solved the problem for the finite case, with Haar functions instead of trigonometric functions; an interesting but easier bit of mathematics.

By substituting convergence in measure for almost everywhere convergence, Menchoff, [5], then answered Lusin's question. He showed that for every measurable f on $[0, 2\pi]$, finite or infinite, there is a trigonometric series, with coefficients converging to zero, which converges in measure to f. This work of Menchoff is difficult to understand. Fortunately, Talalyan has given a brilliant and lucid treatment of this problem, summarized in [7], where he proves Menchoff's theorem for every normal Schauder basis in $L_p[a, b]$, $p > 1$.

The original Lusin problem remains unanswered, not only for the trigonometric functions but for any Schauder basis in any L_p, $p > 1$. It is not even known whether any such series converges almost everywhere to $+\infty$; in particular, this is not known for the Haar functions.

Schauder, [6], originally introduced the idea of basis for the space $C[0, 1]$ as well as for the L_p spaces. It is natural to ask whether Lusin's problem has an affirmative answer using this system of functions. It is our purpose here to show that it does. The problem for this case is, of course, of a much lower order of difficulty than for the original trigonometric functions, or even for the Haar functions. Nevertheless, it turns out to be of technical interest in its own right.

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2. In a separable Banach space X, a *Schauder basis* is a (countable) set $\{x_n\}$ in X such that, for every $x \in X$, there is a unique series $\sum_{n=1}^{\infty} a_n x_n$ which converges to x in the norm of X.

For $X = C[0, 1]$, for convenience, we define a Schauder basis in a slightly different form from that given originally by Schauder. Let

$$x_{-1}^{(1)}(t) = t, \quad t \in [0,1], \qquad x_0^{(1)}(t) = \begin{cases} 2t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ 2-2t, & \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$
$$x_{-1}^{(2)}(t) = 1-t, \quad t \in [0,1],$$

$$x_1^{(1)}(t) = \begin{cases} 4t, & 0 \leqslant t \leqslant \frac{1}{4}, \\ 2-4t, & \frac{1}{4} \leqslant t \leqslant \frac{1}{2}, \\ 0, & \frac{1}{2} \leqslant t \leqslant 1, \end{cases} \qquad x_1^{(2)}(t) = \begin{cases} 4t-2, & \frac{1}{2} \leqslant t \leqslant \frac{3}{4}, \\ 4-4t, & \frac{3}{4} \leqslant t \leqslant 1, \\ 0, & 0 \leqslant t \leqslant \frac{1}{2}, \end{cases}$$

. . . .

$$x_m^{(k)}(t) = \begin{cases} 2^{m+1}t - 2(k-1), & \dfrac{k-1}{2^m} \leqslant t \leqslant \dfrac{2k-1}{2^{m+1}}, \\ 2k - 2^{m+1}t, & \dfrac{2k-1}{2^{m+1}} \leqslant t \leqslant \dfrac{k}{2^m}, \\ 0, & \text{elsewhere}, \end{cases}$$
$$k = 1, \ldots, 2^m; \quad m = 2, 3, 4, \ldots.$$

It is an easy matter to show that this countable set of functions ordered by

$$x_{-1}^{(1)},\ x_{-1}^{(2)},\ x_0^{(1)},\ x_1^{(1)},\ x_1^{(2)}, \ldots,\ x_m^{(1)}, \ldots,\ x_m^{(2m)}, \ldots$$

is a Schauder basis for $C[0, 1]$.

We shall need the fact, also easy to show, that if f is a continuous function which is zero at all the points $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, (2^{m+1}-1)/2^{m+1}$, then if its expansion is

$$a_{-1}^{(1)} x_{-1}^{(1)} + a_{-1}^{(2)} x_{-1}^{(2)} + \ldots$$

it follows that
$$a_{-1}^{(1)} = a_{-1}^{(2)} = \ldots = a_m^{(1)} = \ldots = a_m^{(2m)} = 0.$$

Moreover, if $f \in C[0, 1]$ and $\varepsilon > 0$, there is a $g \in C[0, 1]$ such that

$$\|g\| \leqslant \|f\|, \quad m[x : f(x) + g(x)] < \varepsilon$$

and g vanishes at $0, 1, \frac{1}{2}, \frac{1}{4}, \frac{3}{4}, \ldots, (2^{m+1}-1)/2^{m+1}$, where $\|f\| = \max\{|f(x)| : x \in [0, 1]\}$.

A point $k/2^m$, $m = 0, 1, 2, \ldots$; $k = 0, 1, 2, \ldots$ will be called a *dyadic point*. An interval will be called *dyadic* if its end points are dyadic points. The *rank* of a dyadic interval of length $1/2^m$ is the number m. The *rank* of a dyadic point $k/2^m$ (k odd) is the number m.

3. Let $E \subset [0,1]$ be measurable and let f be the function which is $+\infty$ on E and 0 on the complement, $\complement E$, of E. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that

$$\lim_{n\to\infty} \varepsilon_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \varepsilon_n = +\infty.$$

Let $\{\eta_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \eta_n < +\infty$.

For each k, let J_k be a finite set of dyadic intervals such that

$$m[(E - J_k) \cup (J_k - E)] < \tfrac{1}{2}\eta_k.$$

Let m_1 be the highest rank of end points of the intervals complementary to J_1. Let g_1 be a non-negative continuous function which vanishes on the complement of J_1 and at all dyadic points of rank not exceeding $m_1 + 1$ and which is equal to ε_1 on a subset $I_1 \subset J_1$ with $m(I_1) > m(J_1) - \frac{1}{2}\eta_1$. The Schauder expansion of g_1 has a partial sum

$$a_{m_1}^{(1)} x_{m_1}^{(1)} + \dots + a_{n_1}^{(2^{n_1})} x_{n_1}^{(2^{n_1})}$$

with positive coefficients not exceeding ε_1 and

$$\sum_{i=m_1}^{n_1} \sum_{j=1}^{2^i} a_i^{(j)} x_i^{(j)} > \tfrac{1}{2}\varepsilon_1 \text{ on } I_1$$
$$= 0 \text{ on } \complement J_1.$$

Let m_2 be the highest rank of end points of the intervals complementary to J_2. Let g_2 be a non-negative continuous function which vanishes on the complement of J_2 and at all dyadic points of rank not greater than $\max(n_1, m_2) + 1$ and which is equal to ε_2 on a subset $I_2 \subset J_2$ with $m(I_2) > m(J_2) - \frac{1}{2}\eta_2$. The Schauder expansion of g_2 has a partial sum

$$a_{n_1+1}^{(1)} x_{n_1+1}^{(1)} + \dots + a_{n_2}^{(2^{n_2})} x_{n_2}^{(2^{n_2})}$$

with positive coefficients not exceeding ε_2 and

$$\sum_{i=n_1+1}^{n_2} \sum_{j=1}^{2^i} a_i^{(j)} x_i^{(j)} > \tfrac{1}{2}\varepsilon_2 \quad \text{on} \quad I_2$$
$$= 0 \quad \text{on} \quad \complement J_2.$$

By continuing in this way, we obtain a sequence $n_1 < n_2 < \dots < n_k < \dots$ such that, for every k, there is a series

$$a_{n_k+1}^{(1)} x_{n_k+1}^{(1)} + \dots + a_{n_{k+1}}^{(2^{n_k+1})} x_{n_{k+1}}^{(2^{n_k+1})}$$

with positive coefficients not exceeding ε_{k+1} and

$$\sum_{i=n_k+1}^{n_{k+1}} \sum_{j=1}^{2^i} a_i^{(j)} x_i^{(j)} > \tfrac{1}{2} \varepsilon_{k+1} \quad \text{on} \quad I_{k+1}$$
$$= 0 \qquad \text{on} \quad \complement J_{k+1}.$$

We then obtain a series expansion in the Schauder functions

$$a_{-1}^{(1)} x_{-1}^{(1)} + \ldots + a_n^{(1)} x_n^{(1)} + \ldots + a_n^{(2^n)} x_n^{(2^n)} + \ldots .$$

Let
$$R = \bigcup_{n=1}^{\infty} \bigcap_{s=n}^{\infty} I_s \quad \text{and} \quad T = \bigcup_{n=1}^{\infty} \bigcap_{s=n}^{\infty} \complement J_s.$$

Our series converges to $+\infty$ on R and to a finite function on T.

Now, for each n,
$$\bigcap_{s=n}^{\infty} I_s \subset E \cup Z_n,$$

where $m(Z_n) = 0$ and
$$m(E - \bigcap_{s=n}^{\infty} I_s) < \sum_{s=n}^{\infty} \eta_s.$$

It follows that
$$m[(R - E) \cup (E - R)] = 0.$$

Moreover, for each n,
$$\bigcap_{s=n}^{\infty} \complement J_s \subset \complement E \cup Y_n,$$

where $m(Y_n) = 0$ and
$$m(\complement E - \bigcap_{s=n}^{\infty} \complement J_s) < \sum_{s=n}^{\infty} \eta_s.$$

It follows that
$$m[(T - \complement E) \cup (\complement E - T)] = 0.$$

We have thus proved the

LEMMA 1. *If* $E \subset [0, 1]$ *is measurable, there is a sequence*

$$a_{-1}^{(1)},\ a_{-1}^{(2)},\ a_0^{(1)}, \ldots, a_n^{(1)}, \ldots, a^{(2^n)}, \ldots$$

of non-negative numbers, converging to zero, such that the series

$$a_{-1}^{(1)} x_{-1}^{(1)} + \ldots + a_n^{(1)} x_n^{(1)} + \ldots + a_n^{(2^n)} x_n^{(2^n)} + \ldots$$

converges to $+\infty$ *almost everywhere on* E *and to a finite function almost everywhere on* $\complement E$.

4. We turn now to a consideration of the finite case which rests on the following two remarks.

Remark 1. Let $I_1, \ldots, I_r;\ J_1, \ldots, J_s$ be a partition of $[0, 1]$ into dyadic intervals such that the ranks of the J_i are all the same number m, and the ranks of the I_j

are all smaller than m. Let f be a continuous function which vanishes on each I_j, and at the end points of each J_i, and is either non-negative or non-positive on each J_i. It follows readily from the definition of the functions $x_{-1}^{(1)}, x_{-1}^{(2)}, x_0^{(1)}, \ldots$ that, in the series expansion of f, $a_i^{(j)} = 0$ whenever $i < m$, so that f has an expansion

$$f = a_m^{(1)} x_m^{(1)} + \ldots + a_m^{(2^m)} x_m^{(2^m)} + \ldots.$$

Moreover, every partial sum of this series has norm not exceeding the norm of f, vanishes on each I_j, is non-negative on those J_i for which f is non-negative, and is non-positive on those J_i for which f is non-positive

Remark 2. If f is a continuous function on $[0, 1]$, $\varepsilon > 0$, and n is given, there is an $m > n$, a partition $I_1, \ldots, I_r$; $J_1, \ldots, J_s$ of $[0, 1]$, and a continuous function g such that

a) $m[x : f(x) \neq g(x)] < \varepsilon$,
b) the intervals J_i are of rank m and the intervals I_j are of rank smaller than m,
c) g vanishes on each I_j, at the end points of each J_i, and is non-negative or non-positive on each J_i,
d) $\|g\| \leqslant \|f\|$.

In order to prove this, let $$F = [x : f(x) \neq 0].$$

Then F is the union of pairwise disjoint open intervals $K_1, K_2, \ldots$. Let $K_1, \ldots, K_t$ be such that $m(\bigcup_{i=1}^{t} K_i) > m(F) - \frac{1}{3}\varepsilon$. Shrink and partition each K_i, $i = 1, \ldots, t$, so that it is composed of dyadic intervals. Then partition the complementary intervals so that they are dyadic and comprise $I_1, \ldots, I_r$. Then further partition the subintervals of the K_i so that they have the desired rank and comprise $J_1, \ldots, J_s$. The above shrinking should be of an amount less than $\frac{1}{3}\varepsilon$. Now alter f on a set of measure less than ε so that the resulting function g has properties c) and d).

We are now ready to prove the

LEMMA 2. *If f is continuous, $\|f\| = k$, and $E = [x : f(x) = 0]$, then for every $\varepsilon > 0$ and m there is a series*

$$a_m^{(1)} x_m^{(1)} + \ldots + a_n^{(2^n)} x_n^{(2^n)}, \quad n > m,$$

such that none of the coefficients are greater than k in absolute value,

$$\|a_m^{(1)} x_m^{(1)} + \ldots + a_i^{(j)} x_i^{(j)}\| \leqslant k, \quad i = m, \ldots, n, \quad j = 1, \ldots, 2^i,$$

the functions $$a_m^{(1)} x_m^{(1)} + \dots + a_i^{(j)} x_i^{(j)}$$

all vanish on a set F *with* $m(E-F) < \varepsilon$, *and*

$$|f(x) - (a_m^{(1)} x_m^{(1)} + \dots + a_n^{(2^n)} x_n^{(2^n)})| < \varepsilon$$

on a set of measure greater than $1-\varepsilon$.

Proof. Let g be the function of Remark 2 corresponding to f, m, and ε. By Remark 1, the Schauder series for g has a finite subseries with the desired properties.

LEMMA 3. *If* f *is continuous and* $E = [x : f(x) = 0]$, *then for every* $\eta > 0$ *and* m *there is a series*

$$a_m^{(1)} x_m^{(1)} + \dots + a_n^{(2^n)} x_n^{(2^n)}$$

such that all the coefficients are no greater than η *in absolute value,*

$$|f(x) - (a_m^{(1)} x_m^{(1)} + \dots + a_n^{(2^n)} x_n^{(2^n)})| < \eta$$

on a set of measure greater than $1-\eta$, *and all functions*

$$a_m^{(1)} x_m^{(1)} + \dots + a_i^{(j)} x_i^{(j)}, \quad i = m, \dots, n, \quad j = 1, \dots, 2^i,$$

vanish on a set F, *where* $$m(E-F) < \eta.$$

Proof. There are continuous functions $f_1, \dots, f_r$, all vanishing on E, such that

$$\|f_i\| \leqslant \eta, \ i = 1, \dots, r, \text{ and } f = f_1 + \dots + f_r.$$

Apply Lemma 2 to f_1, with $\varepsilon = \eta/r$ and $m = m_1$. Obtain a finite sum

$$a_{m_1}^{(1)} x_{m_1}^{(1)} + \dots + a_n^{2^{n_1}} x_{n_1}^{2^{n_1}}.$$

Apply Lemma 2 to f_2, with $\varepsilon = \eta/r$ and $m = n_1 + 1$. Continue in this way. The series

$$a_{m_1}^{(1)} x_{m_1}^{(1)} + \dots + a_{m_r}^{(1)} x_{m_r}^{(1)} + \dots + a_{n_r}^{2^{n_r}} x_{n_r}^{2^{n_r}}$$

has the desired properties.

LEMMA 4. *If* f *is measurable and finite almost everywhere,* $\varepsilon > 0$, *and* m *is a positive integer, then if* $E = [x : f(x) \leqslant \varepsilon]$ *and* $\eta > 0$, *there is an expansion*

$$a_m^{(1)} x_m^{(1)} + \dots + a_n^{(2^n)} x_n^{(2^n)}$$

such that all coefficients do not exceed $\varepsilon + \eta$ *in absolute value,*

$$|a_m^{(1)} x_{m_*}^{(1)}(t) + \ldots + a_i^{(j)} x_j^{(i)}(t)| \leqslant \varepsilon, \quad i = m, \ldots, n, \quad j = 1, \ldots, 2^i,$$

for every $t \in D$, *where* $D \subset E$ *and* $m(D) > m(E) - \eta$, *and*

$$|f(t) - (a_m^{(1)} x_m^{(1)}(t) + \ldots + a_n^{(2^n)} x_n^{(2^n)}(t)| \leqslant \eta,$$

for every $t \in H$, *where* $m(H) > 1 - \eta$.

Proof. Let $f_1 = \chi_E \cdot f$ and $f_2 = f - f_1$, where χ_E is the characteristic function of E. There are continuous functions g_1 and g_2 such that

$$m[t : f_1(t) \neq g_1(t)] < \tfrac{1}{4}\eta, \quad m[t : f_2(t) \neq g_2(t)] < \tfrac{1}{4}\eta, \quad \text{and} \quad \|g_1\| \leqslant \varepsilon.$$

By Lemma 2, there is a series

$$b_m^{(1)} x_m^{(1)} + \ldots + b_n^{(2^n)} x_n^{(2^n)},$$

whose coefficients are less than ε in magnitude, such that

$$|b_m^{(1)} x_m^{(1)}(t) + \ldots + b_i^{(j)} x_i^{(j)}(t)| \leqslant \varepsilon, \quad i = m, \ldots, n, \quad j = 1, \ldots, 2^i,$$

for all $t \in D \subset E$, where $m(D) > m(E) - \eta$, and

$$|g_1(t) - (b_m^{(1)} x_m^{(1)}(t) + \ldots + b_n^{(2^n)}(t))| \leqslant \tfrac{1}{2}\eta$$

for every $t \in K$, where $m(K) > 1 - \tfrac{1}{2}\eta$.

By Lemma 3, there is a series

$$c_m^{(1)} x_m^{(1)} + \ldots + c_n^{2^n} x_n^{2^n}$$

whose coefficients are less than η in magnitude, where the same n can be taken for both cases by allowing enough coefficients to vanish, such that

$$|g(t) - (c_m^{(1)} x_m^{(1)}(t) + \ldots + c_n^{(2^n)} x_n^{(2^n)}(t)| \leqslant \tfrac{1}{2}\eta$$

for every $t \in G$, where $m(G) > 1 - \tfrac{1}{2}\eta$.

The series $$(b_m^{(1)} + c_m^{(1)}) x_m^{(1)} + \ldots + (b_n^{(2^n)} + c_n^{(2^n)}) x_n^{(2^n)}$$

has the desired properties with $H = K \cap G$.

We may now prove the

THEOREM 1. *If f is a measurable function, finite almost everywhere, there is a series*

$$a_{-1}^{(1)} x_{-1}^{(1)} + a_{-1}^{(2)} x_{-1}^{(2)} + a_0^{(1)} x_0^{(1)} + \dots,$$

whose coefficients converge to zero, which converges to f almost everywhere.

Proof. Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that $\sum_{n=1}^{\infty} \varepsilon_n < +\infty$. There is a series

$$a_{-1}^{(1)} x_{-1}^{(1)} + a_{-1}^{(2)} x_{-1}^{(2)} + \dots + a_{m_1}^{(1)} x_{m_1}^{(1)} + \dots + a_{m_1}^{(2^{m_1})} x_{m_1}^{(2^{m_1})}$$

such that $$|f(t) - (a_{-1}^{(1)} x_{-1}^{(1)}(t) + \dots + a_{m_1}^{(2^{m_1})} x_{m_1}^{(2^{m_1})}(t))| < \varepsilon_1$$

for $t \in E_1$, where $m(D_1) < \varepsilon_1$ with $D_1 = \complement E_1$.

Let $$f_1 = f - (a_{-1}^{(1)} x_{-1}^{(1)} + \dots + a_{m_1}^{(2^{m_1})} x_{m_1}^{(2^{m_1})}).$$

By Lemma 4, there is a series

$$a_{m_1+1}^{(1)} x_{m_1+1}^{(1)} + \dots + a_{m_2}^{(2^{m_2})} x_{m_2}^{(2^{m_2})}$$

such that $$|a_i^{(j)}| \leqslant \varepsilon_1 + \varepsilon_2$$

and $$|a_{m_1+1}^{(1)} x_{m_1+1}^{(1)}(t) + \dots + a_i^{(j)} x_i^{(j)}(t)| \leqslant \varepsilon_1, \quad i = m_1 + 1, \dots, m_2, \quad j = 1, \dots, 2^i,$$

for every $t \in H_1$, where $m(H_1) > 1 - 2\varepsilon_1$. Moreover,

$$|f_1(t) - (a_{m_1+1}^{(1)} x_{m_1+1}^{(1)}(t) + \dots + a_{m_2}^{(2^{m_2})} x_{m_2}^{(2^{m_2})}(t)| \leqslant \varepsilon_2$$

for all $t \in E_2$, where $m(D_2) < \varepsilon_2$ with $D_2 = \complement E_2$.

Let $$f_2 = f_1 - (a_{m_1+1}^{(1)} x_{m_1+1}^{(1)} + \dots + a_{m_2}^{(2^{m_2})} x_{m_2}^{(2^{m_2})}).$$

By Lemma 4, there is a series

$$a_{m_2+1}^{(1)} x_{m_2+1}^{(1)} + \dots + a_{m_3}^{(2^{m_3})} x_{m_3}^{(2^{m_3})}$$

such that $$|a_i^{(j)}| \leqslant \varepsilon_2 + \varepsilon_3$$

and $$|a_{m_2+1}^{(1)} x_{m_2+1}^{(1)}(t) + \dots + a_i^{(j)} x_i^{(j)}(t)| \leqslant \varepsilon_2, \quad i = m_2 + 1, \dots, m_3, \quad j = 1, \dots, 2^i,$$

for every $t \in H_2$, where $m(H_2) > 1 - 2\varepsilon_2$. Moreover,

$$|f_2(t) - (a_{m_2+1}^{(1)} x_{m_2+1}^{(1)}(t) + \dots + a_{m_3}^{(2^{m_3})} x_{m_3}^{(2^{m_3})}(t)| \leqslant \varepsilon_3$$

for all $t \in E_3$, where $m(D_3) < \varepsilon_3$ with $D_3 = \complement E_3$.

In this way, we obtain an increasing sequence $\{m_k\}$ and sequences $\{H_k\}$, $\{E_k\}$ of sets such that, for every k, there is a series

$$a_{m_k+1}^{(1)}\, x_{m_k+1}^{(1)} + \ldots + a_{m_{k+1}}^{(2^{m_{k+1}})}\, x_{m_{k+1}}^{(2^{m_{k+1}})}$$

with
$$|a_i^{(j)}| \leqslant \varepsilon_k + \varepsilon_{k+1}.$$

$$|a_{m_k+1}^{(1)}\, x_{m_k+1}^{(1)}(t) + \ldots + a_i^{(j)}\, x_i^{(j)}(t)| \leqslant \varepsilon_k, \quad i = m_k+1, \ldots, m_{k+1}, \quad j = 1, \ldots, 2^i,$$

for all $t \in H_k$, where $m(H_k) > 1 - 2\varepsilon_k$. Moreover,

$$|f_k(t) - (a_{m_k+1}^{(1)}\, x_{m_k+1}^{(1)} + \ldots + a_{m_{k+1}}^{(2^{m_{k+1}})}\, x_{m_{k+1}}^{(2^{m_{k+1}})}(t)| \leqslant \varepsilon_{k+1}$$

for all $t \in E_{k+1}$. where $m(D_{k+1}) < \varepsilon_{k+1}$, where $D_{k+1} = \complement E_{k+1}$ and

$$f_k = f_{k-1} - (a_{m_{k-1}+1}^{(1)}\, x_{m_{k-1}+1}^{(1)} + \ldots + a_{m_k}^{(2^{m_k})}\, x_{m_k}^{(2^{m_k})}).$$

We now show that the series

$$a_{-1}^{(1)}\, x_{-1}^{(1)} + a_{-1}^{(2)}\, x_{-1}^{(2)} + a_0^{(1)}\, x_0^{(1)} + \ldots$$

converges almost everywhere to f.

We first observe that, for every k,

$$f = f_k + a_{-1}^{(1)}\, x_{-1}^{(1)} + \ldots + a_{m_k}^{(2^{m_k})}\, x_{m_k}^{(2^{m_k})}.$$

For every $r > k+1$ and $i = 1, \ldots, 2^r$, we have

$$|f(t) - (a_{-1}^{(1)}(t) + \ldots + a_{m_r}^{(i)}\, x_{m_r}^{(i)}(t))| = |f_k(t) - (a_{m_{k+1}}^{(1)}\, x_{m_{k+1}}^{(1)}(t) + \ldots + a_{m_r}^{(i)}\, x_{m_r}^{(i)}(t))|$$
$$\leqslant \varepsilon_{k+1} + \varepsilon_{k+1} + \ldots + \varepsilon_r,$$

for every $t \in E_{k+1} \cap (\bigcap_{s=k+1}^{\infty} H_s)$.

But the measure of this set exceeds $1 - 3\sum_{s=k+1}^{\infty} \varepsilon_s$. The almost everywhere convergence of the series to f then follows since $\lim_{k\to\infty} \sum_{s=k}^{\infty} \varepsilon_s = 0$.

That the coefficients converge to 0 is obvious.

5. In conclusion, we may state

THEOREM 2. *If f is a measurable function, finite or infinite, on* $[0, 1]$, *there is a Schauder series, with coefficients converging to zero, which converges almost everywhere to f.*

Proof. First $f = f_1 + f_2 + f_3$, where f_3 is finite, f_1 is $+\infty$ on a set E_1 and 0 on $\complement E_1$, and f_2 is $-\infty$ on a set E_2 and 0 on $\complement E_2$. By Lemma 1, there are Schauder series

$$a_{-1}^{(1)}\, x_{-1}^{(1)} + a_{-1}^{(2)}\, x_{-1}^{(2)} + a_0^{(1)}\, x_0^{(1)} + \ldots \text{ and } b_{-1}^{(1)}\, x_{-1}^{(1)} + b_{-1}^{(2)}\, x_{-1}^{(2)} + b_0^{(1)}\, x_0^{(1)} + \ldots,$$

the first of which converges almost everywhere to a function g_1 which is $+\infty$ on E_1 and finite on $\complement E_1$, the second of which converges almost everywhere to a function g_2 which is $-\infty$ on E_2 and finite on $\complement E_2$, and are such that the coefficients converge to 0 for both series. Let

$$g = f_3 - g_1 \chi_{\complement E_1} - g_2 \chi_{\complement E_2}.$$

Then g is finite and measurable so that, by Theorem 1, there is a Schauder series

$$c_{-1}^{(1)} x_{-1}^{(1)} + c_{-1}^{(2)} x_{-1}^{(2)} + c_0^{(1)} x_0^{(1)} + \ldots,$$

with coefficients converging to 0, which converges almost everywhere to g. The Schauder series

$$(a_{-1}^{(1)} + b_{-1}^{(1)} + c_{-1}^{(1)}) x_{-1}^{(1)} + (a_{-1}^{(2)} + b_{-1}^{(2)} + c_{-1}^{(2)}) x_{-1}^{(2)} + \ldots$$

has coefficients which converge to 0 and converges almost everywhere to f.

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