CUTS

BY

E. MICHAEL

University of Washington, Seattle, Wash., U.S.A.(1)

1. Introduction

What does it mean to cut a topological space X along a subset A? Consider two examples:

(1) X is the plane, and A is a triod (i.e. a Y).

(2) X is a Möbius band, and A is the equator.

Note that both sets A have empty interior, or, in the terminology of [20], are *thin*; this is necessary if "cutting" is to make much sense. Now in both examples, it is intuitively clear what happens when X is cut along A: The space X is replaced by a space $\mathbf{X}$, and if (2) $p: \mathbf{X} \twoheadrightarrow X$ is the function which maps each point of $\mathbf{X}$ to the point of X where it came from before cutting, while $\mathbf{A}$ denotes $p^{-1}(A)$, then p maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$. In (1), $\mathbf{X}$ is the plane with a (topologically) circular hole, and $\mathbf{A}$ is the boundary of the hole. In (2), where cutting is occasionally performed as a parlor trick, $\mathbf{X}$ is a cylinder, $\mathbf{A}$ is a circle which is one of the two components of the boundary of $\mathbf{X}$, and $p|\mathbf{A}$ is a double covering. Let us try to identify those common properties of p and $\mathbf{A} \subset \mathbf{X}$ which will lead to a general concept of cutting. (3)

First of all, p is continuous and closed, and—as observed above—maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$. Moreover, in both examples $p|\mathbf{A}$ is finite-to-one, but in general this requirement must be somewhat relaxed, as the following example shows:

(3) In the plane, $\mathbf{X}$ consists of the intervals joining $(0,0)$ to $(x,1)$ for all x in $S=\{1,\frac{1}{2},\frac{1}{3},\ldots,0\}$, and $A=\{(0,0)\}$.

---

(1) Supported in part by a National Science Foundation grant.

(2) We use $\twoheadrightarrow$ to denote an *onto* map.

(3) It should be noted that a somewhat different method of cutting was implicitly considered by R. H. Fox in [7]. In many important cases (including Examples (1), (2), and (3)), Fox's cuts agree with ours; in general, however, Fox's map p_F is a restriction of our map p, and the range of p_F (unlike the range of p) need not be all of X. The exact relation between these two ways of cutting will be established in section 16.

In this example, it is intuitively clear (remembering that $\mathbf{X}-\mathbf{A}$ must be homeomorphic to $X-A$) that $\mathbf{X}$ consists of the intervals joining $(x,0)$ to $(x,1)$ for x in S, while A consists of the points $(x,0)$ with x in S, and p is obvious. Note that here $p|\mathbf{A}$ is not finite-to-one (since $p^{-1}(0,0)=S\times\{0\}$), but it is (compact, totally disconnected)-to-one, and that will always be the case.

To isolate the essential property of $\mathbf{A}\subset\mathbf{X}$, consider what would happen if we cut $\mathbf{X}$ along $\mathbf{A}$. If the original cut of X along A was performed properly, the answer must be: Nothing happens. More precisely, if $\mathbf{p}$: $\tilde{\mathbf{X}}\twoheadrightarrow\mathbf{X}$ is the map obtained by cutting $\mathbf{X}$ along $\mathbf{A}$, then $\mathbf{p}$ is a homeomorphism. Now the crucial property of $\mathbf{A}\subset\mathbf{X}$ which causes this behavior is that $\mathbf{A}$ *nowhere cuts* $\mathbf{X}$ in the following sense: (1) $\mathbf{A}$ is thin in $\mathbf{X}$, and whenever $x\in\mathbf{A}$ and U is a neighborhood of x in $\mathbf{X}$, then $U-\mathbf{A}$ does not split into two disjoint open sets both having x in their closure.

This completes our discussion of what to expect from cuts, and brings us to our principal theorem, which asserts that, for Tychonoff spaces, such cuts are always possible, and are *essentially unique* in the following sense: If p: $\mathbf{X}\twoheadrightarrow X$ and p_1: $\mathbf{X}_1\twoheadrightarrow X$ are both consequences, satisfying our conditions, of cutting X along A, then p and p_1 are *equivalent*; that is, there exists a homeomorphism k: $\mathbf{X}\twoheadrightarrow\mathbf{X}_1$ such that $p=p_1\circ k$.

THEOREM 1.1. *Let X be a Tychonoff space and A a thin subset.* (2) *Then there exist—essentially uniquely—a Tychonoff* (3) *space $\mathbf{X}$ with nowhere cutting subset $\mathbf{A}$, and a continuous closed p: $\mathbf{X}\twoheadrightarrow X$ which maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$ and maps $\mathbf{A}$ (compact, totally disconnected)-to-one onto A.*

In the situation described in Theorem 1.1, we will say that $(\mathbf{X},\mathbf{A},p)$ is an (X,A)-*cut*.

After some preliminary results in section 2 on proper maps (i.e. closed maps with inverse images of points compact) and monotone-light factorizations, and in section 3 on nowhere cutting sets, the proof of Theorem 1.1 will be given in section 4. It is not long, and depends only on Propositions 2.3, 3.5, 3.11 and 3.12.

Since the map p in Theorem 1.1 is proper, it follows [12; Theorem 2.2] that if X is compact, so is $\mathbf{X}$, and if A is compact, so is $\mathbf{A}$. The same is true for paracompactness and local compactness [12; Theorem 2.2]. Unfortunately, however, metrizability of X need not be inherited by $\mathbf{X}$, as the following example shows:

(1) This concept (but not this terminology) was introduced by J. de Groot [11]. (See also Freudenthal [7].) For compact spaces, it has recently received considerable attention from Sklyarenko (see, for instance, [17]), who calls Y a *perfect* compactification of X if $Y-X$ nowhere cuts Y. While some of our work is similar in spirit to Sklyarenko's, there is little overlap.

(2) Note that we need not assume that A is closed.

(3) Propositions 2.4 and 2.5 imply that essential uniqueness actually extends to Hausdorff spaces $\mathbf{X}$ (since, subject to the other assumptions, they are automatically Tychonoff spaces).

(4) X consists of 0 and all n^{-1} for $n=1,2,\ldots$, and A consists of 0 alone.

Here geometric intuition is little help in determining $\mathbf{X}$ and p: $\mathbf{X}\to X$. (Note that $\mathbf{X}\neq X$, since $X-A$ splits into $\{n^{-1}\mid n \text{ even}\}$ and $\{n^{-1}\mid n \text{ odd}\}$, both having 0 in their closure). As a matter of fact, $\mathbf{X}=\beta(X-A)$ (that is, the Stone-Čech compactification of the countable discrete space $X-A$), and p is the unique extension of the identity map on $X-A$.[1] This follows from Theorem 1.1, once it is known that $\beta(X-A)$ is totally disconnected [10; p. 90] and that, if E is a Tychonoff space, $\beta E-E$ nowhere cuts βE (Proposition 3.5). This example is rather typical; whenever one's geometric intuition fails to produce $\mathbf{X}$ for metrizable X, one may expect $\mathbf{X}$ to be non-metrizable. All the same, it would be nice to have a somewhat more precise criterion for metrizability, and fortunately—at least for separable metric X—a very satisfactory one is available. Let us say that $A\subset X$ *nowhere scatters* X if, whenever $a\in A$ and U is a neighborhood of a in X, then any disjoint open covering of $U-A$ is locally finite at a. This concept is discussed in section 6. Now it is not hard to show that $\mathbf{A}$ nowhere scatters $\mathbf{X}$ if and only if A nowhere scatters X (Proposition 7.2), and that, in a first countable[2] space, nowhere cutting sets are nowhere scattering (Proposition 6.2). Hence if $\mathbf{X}$ is to be metrizable, or even first countable, A must nowhere scatter X. (This implies that, in Example (4) above, $\mathbf{X}$ cannot be metrizable). That establishes the easier half of the following fundamental result, proved in section 7.

THEOREM 1.2. *If* $(\mathbf{X},\mathbf{A},p)$ *is an* (X,A)*-cut, and* X *is separable*[3] *metric, then the following are equivalent.*

(a) $\mathbf{X}$ *is separable metric.*

(b) $\mathbf{X}$ *is first countable*

(c) A *nowhere scatters* $\mathbf{X}$.

For compact spaces, Theorem 1.2—together with Theorem 1.1—has some bearing on the Freudenthal compactification [8], which is discussed at the end of section 7.

While in general we have only an existence theorem to guarantee that cuts are possible, there are three situations where we can describe them more explicitly. Firstly, if X is metrizable, A nowhere scatters X, and $X-A$ is locally connected, then $\mathbf{X}$ is simply the completion (or part of the completion if X is not complete) of $X-A$ remetrized in a natural manner (see section 8). Secondly (see section 9), if X is a locally finite simplicial complex and A a subcomplex, then $\mathbf{X}$ and $\mathbf{A}$ are also easily described complexes, and p is piecewise

(1) More generally, $\mathbf{X}=\beta(X-A)$ whenever X is compact metric and $X-A$ is 0-dimensional.

(2) X is *first countable* if, for $x\in X$, there exists a countable base for the neighborhoods of x.

(3) I do not know whether this result, or something like it, remains true for arbitrary metric spaces. (See, however, the special cases treated in Theorem 8.1 and Proposition 11.1.)

linear. Finally (section 10), if X is a mapping cylinder with base A, then $\mathbf{X}$ is a mapping cylinder with base $\mathbf{A}$.

Perhaps the most important example of nowhere cutting sets are sets which are collared in the sense of M. Brown [5] (see Example 3.6). If $\mathbf{A}$ is collared in $\mathbf{X}$, we say that A is *multicollared* in X, and such sets are studied in section 11.

In section 12 we show that if a thin subset has certain properties locally, then it also has them globally. A typical result, whose proof depends on a theorem of M. Brown [5; Theorem 1], asserts that a locally multicollared subset of a metric space is multicollared.

If $X-A$ has only finitely many components, then these are in a natural one-to-one correspondence with the components of $\mathbf{X}$, and, under certain additional hypotheses (which are always satisfied if X is E^n and A is non-empty, closed, and connected), with the components of $\mathbf{A}$. This is proved in section 13.

In section 14 we generalize the process of cutting to a process for "completing a spread", and generalize Theorem 1.1 accordingly. In section 15, we compare this completion process to a somewhat different one due to R. H. Fox. In section 16, finally, we compare the cutting process which grows out of Fox's completion to the one studied in this paper.

To conclude, the author gratefully acknowledges many helpful conversations with Morton Brown and John Isbell.

2. Proper maps, monotone-light factorizations, and spreads

According to Bourbaki [4; § 10, Definition 1 and Theorem 1], a continuous map $f\colon E\to F$ is *proper* if it is closed, and inverse images of points are compact. (Such maps are called compact by Vaĭnstein [21], fitting by Henriksen and Isbell [12], and perfect by Aleksandrov [1].) Proper maps are compact in the sense of Whyburn [24; 8] (i.e. inverse images of compact sets are compact); the converse is true in metric spaces [24; 8.2], but not in general.

The following result is known, but I have not found an explicit statement in print.

PROPOSITION 2.1. *Let $f\colon E\to F$ be proper. If E is dense in a Hausdorff space E_1, and $F\subset F_1$, then any continuous extension $f_1\colon E_1\to F_1$ of f maps E_1-E into F_1-F.*

Proof. Suppose not. Then there is an $x\in E_1-E$ with $f_1(x)\in F$. Let $y\in f_1(x)$ and $A=f^{-1}(y)$. Now A is compact and $x\notin A$, so there exist disjoint open U and V in E_1 with $x\in U$ and $A\subset V$. Then $x\in (U\cap E)^-\subset (E-V)^-$, so

$$y=f_1(x)\in F\cap (f_1(E-V))^-=F\cap (f(E-V))^-=f(E-V),$$

which is false. That completes the proof.

Proposition 2.1 shows that the requirements of Theorem 1.1 are somewhat redundant. In fact, if p: $\mathbf{X} \twoheadrightarrow X$ is a proper map which maps $\mathbf{X}-\mathbf{A}$ homeomorphically onto $X-A$, then p automatically maps $\mathbf{A}$ onto A.

The following result was proved by Bourbaki [4; § 10, Proposition 5], and will be used later.

PROPOSITION 2.2. *If h: $E \to G$ and g: $G \to F$ are continuous, if G is Hausdorff, and if $f = g \circ h$, then f is proper if and only if g and h are both proper.*

Recall that a continuous f: $E \to F$ is *monotone* (resp. light) if $f^{-1}(y)$ is connected (resp. totally disconnected) for every $y \in F$. If f: $E \to F$ is continuous, then $f = g \circ h$ is a *monotone-light* factorization of f if, for some G, the map h: $E \twoheadrightarrow G$ is monotone and g: $G \to F$ is light. Moreover, we also require that h be a *quotient* (=quasi-compact) map, meaning that U is open in G whenever $h^{-1}(U)$ is open in E. This last requirement is not standard, but it permits the conclusion, following Whyburn [23; Theorem 2], that the monotone-light factorization—if it exists—is *essentially unique*. In fact, G must then—essentially—be the space of all components of all $f^{-1}(y)$ $(y \in F)$ with the quotient topology, and h: $E \twoheadrightarrow G$ must be the quotient map. There is then a unique g: $G \twoheadrightarrow F$ such that $g \circ h = f$, and this g is continuous. If F is a T_1-space, it can be shown that g must be light, so that then the monotone-light factorization of f exists. While G is a T_1-space whenever F is, G need not be Hausdorff even when E and F are both metrizable. If f is proper, however, then G is Hausdorff whenever E and F are Hausdorff, so that g and h are then also proper by Proposition 2.2. This was asserted by Ponomarev [16] for *compact* E and F (which is all we need in the proof of Proposition 2.3 below), but his proof is valid in general. Since these facts are neither well known nor very accessible, we take time out to outline their proofs.

To show that g light if F is T_1, let $y \in F$, and suppose $C \subset g^{-1}(y)$ is connected. Then $\bar{C}$ is connected, and $\bar{C} \subset g^{-1}(y)$ since $g^{-1}(y)$ is closed. Since the inverse image, under a monotone quotient map, of a *closed* connected set is connected (this is easily verified), $h^{-1}(\bar{C})$ must be connected, and hence is contained in a component of $f^{-1}(y)$. Hence $\bar{C}$—and thus C—contains at most one point.

To show that G is Hausdorff whenever E and F are Hausdorff and f is proper, let $y_1, y_2 \in G$, and let us separate them by disjoint open sets U_1 and U_2. If $g(y_1) \neq g(y_2)$, this is trivial. If $g(y_1) = g(y_2) = z$, then $h^{-1}(y_1)$ and $h^{-1}(y_2)$ are different components of the compact Hausdorff space $f^{-1}(z)$, and hence [3; p. 233] there exists a clopen C_1 in $f^{-1}(z)$ containing $h^{-1}(y_1)$ and not $h^{-1}(y_2)$. Then C_1 and $C_2 = f^{-1}(z) - C_1$ can be separated in E by disjoint open sets W_1 and W_2. Now $W = W_1 \cup W_2$ is an open set containing $f^{-1}(z)$, so, since f is closed

z has an open neighborhood V in F with $f^{-1}(V) \subset W$. Now if $W_i' = W_i \cap f^{-1}(V)$, then W_i' is an open inverse set under h, so we may let $U_i = h(W_i')$.

The result just proved, together with Proposition 2.2, implies that G inherits any property possessed by both E and F which implies Hausdorff and is preserved by images (or inverse images) of proper maps. Among such properties are metrizability [19], paracompactness [12], and normality [23; Theorem 9], but *not* the property of being a Tychonoff space [12; 4.2]. Nevertheless, we have the following result, which is all we shall really need concerning monotone-light factorizations.

PROPOSITION 2.3. *If f: $E \to F$ is proper, and if E and F are Tychonoff spaces, then so is the middle space G of the* (*essentially unique*) *monotone-light factorization of f, and the factors are proper.*

Proof. Let $\bar{f}$: $\beta E \to \beta F$ be the continuous extension of f. If $\bar{f} = \bar{h} \circ \bar{g}$ is the monotone-light factorization of $\bar{f}$, then the middle space $\bar{G}$ is (compact) Hausdorff by the result of Ponomarev [16] considered earlier. Since f is proper, $E = \bar{f}^{-1}(F)$ by Proposition 2.1. Let $G = g^{-1}(F)$ (so that $E = \bar{h}^{-1}(G)$), and let $g = \bar{g} | G$ and $h = \bar{h} | E$. Then g and h are proper, and $f = g \circ h$ is a monotone-light factorization of f, with middle space G. That completes the proof.

Note that Proposition 2.2 was not needed to show that the factors in Proposition 2.3 are proper.

We conclude this section with a look at spreads in the sense of R. H. Fox [7]. According to Fox, a continuous f: $X \to Y$ is a *spread* if X and Y are T_1 and if the clopen subsets of all $f^{-1}(U)$, with U open in Y, are a base for the open subsets of X. Every spread is clearly light, but a simple example in [7; p. 255, footnote] shows that the converse is false. Nevertheless, we have the following result, a special case of which appears in [7; p. 255, footnote]. (We use the fact that a totally disconnected compact Hausdorff space is 0-*dimensional*, in the sense of having a clopen (=closed and open) base [14; p. 20, A].)

PROPOSITION 2.4. *If p: $E \to F$ is light and proper, and if E is Hausdorff, then p is a spread.*

Proof. Let $x \in E$, and let U be a neighborhood of x in E. Let $Z = p^{-1}(p(x))$. Since Z is 0-dimensional, there exists a clopen C_1 in Z with $x \in C_1 \subset (U \cap Z)$. Let $C_2 = Z - C_1$. Then C_1 and C_2 are disjoint compact subsets of E, so can be separated by disjoint open sets W_1 and W_2, and we may suppose $W_1 \subset U$. Let $W = W_1 \cup W_2$; then W is an open neighborhood of Z in E, so $V = F - p(E - W)$ is an open neighborhood of $p(x)$ in F, with $p^{-1}(V) \subset W$. But now $W_1 \cap p^{-1}(V)$ is a clopen subset of $p^{-1}(V)$, and

$$x \in (W_1 \cap p^{-1}(V)) \subset U,$$

which completes the proof.

Further results on the relation of spreads to proper maps are found in sections 14 and 15.

The following proposition is due to J. R. Isbell; the analogous result for regular spaces was proved by Fox in [7].

PROPOSITION 2.5. (Isbell). *If p: $E \to F$ is a spread, and if F is a Tychonoff space, then so is E.*

Proof. Since p is a spread, E is by assumption T_1. Now let $x \in E$, and let V be a basic open neighborhood of x, so that V is clopen in $p^{-1}(W)$ for some open neighborhood W of $p(x)$. To define a continuous f: $E \to I$ with $f(x) = 0$ and $f(E - V) = 1$, first pick a continuous g: $F \to I$ with $g(p(x)) = 0$ and $g(F - W) = 1$, and then let f be $g \circ p$ on V and 1 elsewhere. This f is continuous, since it is $g \circ p$ on $\bar{V}$ and 1 on $E - V$, and $\bar{V}$ and $E - V$ are closed sets whose union is E. That completes the proof.

3. Nowhere cutting sets

If A is a subset of a topological space X, then A *nowhere cuts* X if A is thin in X (i.e. has empty interior) and if, whenever $a \in A$ and U is a neighborhood of a in X, then $U - A$ does not split into two disjoint open sets both having a in their closure. (This requirement clearly need only be satisfied by all U in some local base at a.) Our first three lemmas follow directly from the definition.

LEMMA 3.1. *The following properties of a thin $A \subset X$ are equivalent.*

(a) *A nowhere cuts X.*

(b) *If U is open in X, and $\{U_1, U_2\}$ is a disjoint open covering of $U - A$, then $\{\bar{U}_1, \bar{U}_2\}$ is a disjoint open covering of U (where $\bar{U}_i$ is the closure of U_i in U).*

The following corollary shows how "nowhere cutting" is related to some similar concepts. The implication (b) $\Rightarrow$ (c) follows from Lemma 3.1, and the others are clear.

COROLLARY 3.2. *Suppose $A \subset X$ is thin. Then always (a) $\Rightarrow$ (b) $\Rightarrow$ (c); if X is locally connected, then (a), (b), and (c) are equivalent.*

(a) *$X - A$ is locally connected at A (i.e. if $a \in A$, every neighborhood U of a in X contains a neighborhood V of a with $V - A$ connected).*

(b) *A nowhere cuts X.*

(c) *If $U \subset X$ is open and connected, then $U - A$ is connected.*

Note that, if X is not locally connected, the conditions of Corollary 3.2 need not be equivalent. For instance, let Q be the rationals, and $X = Q \times I$; then $A = Q \times \{0\}$ satisfies (b) (see Proposition 3.4) but not (a), while $A = Q \times \{\frac{1}{2}\}$ satisfies (c) but not (b).

LEMMA 3.3. *Let $A \subset X$, and let $\mathcal{U}$ be a collection of open subsets of X which covers A. Then A nowhere cuts X if and only if $A \cap U$ nowhere cuts U for every $U \in \mathcal{U}$.*

We now prove two important propositions.

PROPOSITION 3.4. *If E is a topological space, $E \times \{0\}$ nowhere cuts $E \times I$.*

Proof. Clearly $E \times \{0\}$ is thin. Now let $(x, 0) \in E \times \{0\}$. It suffices to show that, if $U = V \times [0,t)$ is a *rectangular* neighborhood of $(x,0)$ in $E \times \mathrm{I}$, and if $\{U_1, U_2\}$ is a disjoint open covering of $U - A$, then U_1 and U_2 cannot both have $(x,0)$ in their closure. Now for each $x' \in V$, $\{x'\} \times (0,t)$ is connected, and thus entirely in U_1 or entirely in U_2. Hence $U_i = V_i \times (0,t)$, where $\{V_1, V_2\}$ is a disjoint open covering of V. But then $(x,0) \in U_i$ only if $x \in V_i$, and that completes the proof.

Note that the proof of Proposition 3.4 actually proves the following stronger result: *If $B \subset Y$ is thin, and $Y - B$ is locally connected at B (see Corollary 3.2), then $E \times B$ nowhere cuts $E \times Y$.*

Our next result was proved in the proof—although not explicitly stated in the statement—of [13; Lemma 4]. (See also [17; Corollary to Theorem 1].) For completeness, we reproduce the proof.

PROPOSITION 3.5. [13] [17]. *If E is a Tychonoff space, $\beta E - E$ nowhere cuts βE.*

Proof. Let $x \in \beta E - E$, let U be an open neighborhood of x in βE, and let $U \cap E$ split into two disjoint open sets V_1 and V_2. Since E is dense in βE, the set $(U \cap E)^- = \overline{V}_1 \cup \overline{V}_2$ contains U, where the closures are taken in βE. We must show that $x \notin \overline{V}_1 \cap \overline{V}_2$.

Suppose $x \in \overline{V}_1 \cap \overline{V}_2$. Pick a continuous φ: $\beta E \to I$ such that $\varphi(x) = 0$ and $\varphi(\beta E - U) = 1$. Define f: $E \to I$ by

$$f(x) = \tfrac{1}{2} \quad \text{if} \quad x \in V_2 \quad \text{and} \quad \varphi(x) < \tfrac{1}{2},$$
$$f(x) = \varphi(x) \quad \text{otherwise.}$$

Then f is continuous on each of the open sets V_2 and $E - (\{x \in E \mid \varphi(x) \leq \frac{1}{2}\} \cap V_2)$, and since these sets cover E, f is continuous on E. The continuous extension $\bar{f}$ of f over βE coincides with φ on $\overline{V}_1$, so $\bar{f}(x) = 0$, and is $\geq \frac{1}{2}$ on $\overline{V}_2$, so $\bar{f}(x) \geq \frac{1}{2}$. This contradiction shows that $x \notin \overline{V}_1 \cap \overline{V}_2$, and the proof is complete.

We now come to our list of examples of nowhere cutting sets $A \subset X$. For Example 3.6, recall that A is *collared* in X in the sense of [5] if there exists a homeomorphism h from $A \times [0,1)$ onto an open neighborhood of A in X such that $h(a,0) = a$ for every $a \in A$.

EXAMPLE 3.6. *A is collared in X.* (This formally strengthens Proposition 3.4, and follows from Proposition 3.4 and Lemma 3.3.)

EXAMPLE 3.7. *A is thin, and $X-A$ is locally connected at A.* (See Corollary 3.2.)

EXAMPLE 3.8. *X is an n-manifold, and* dim $A \leqslant n-2$. (This follows from Corollary 3.2 and the fact [14; p. 48, Corollary 1] that a connected n-manifold remains connected after removal of a subset of dimension $\leqslant n-2$.)

EXAMPLE 3.9. *X is a metric space, A is thin, and $X-A$ is uniformly locally connected.* (1) (This follows from Example 3.7.)

EXAMPLE 3.10. *E is a Tychonoff space, $E \subset X \subset \beta E$, and $A = X-E$.* (This follows from Proposition 3.5 and the trivial fact that,if B nowhere cuts Y and if $(Y-B) \subset X \subset Y$, then $B \cap X$ nowhere cuts X.)

We now prove some results which will be needed elsewhere in this paper.

PROPOSITION 3.11. *Let $f: X \twoheadrightarrow Y$ be a quotient map. Let $A \subset Y$, with $f^{-1}(y)$ connected for all $y \in A$, and suppose $f^{-1}(A)$ nowhere cuts X. Then A nowhere cuts Y.*

Proof. We use Lemma 3.1. Clearly A is thin. Suppose U is open in Y, and $\{V_1, V_2\}$ is a disjoint open covering of $U-A$. Let $U' = f^{-1}(U)$, and $V_i' = f^{-1}(V_i)$ for $i = 1,2$. Then U' is open in X, and $\{V_1', V_2'\}$ is a disjoint open cover of $U' - f^{-1}(A)$. If $\overline{V}_i'$ is the closure of V_i in U', then, by Lemma 3.1, $\{\overline{V}_1', \overline{V}_2'\}$ is a disjoint open covering of U'. Since $f^{-1}(z)$ is connected for all $z \in A$, $\overline{V}_i' = f^{-1}(f(\overline{V}_i'))$ for $i = 1,2$, so that $\{f(\overline{V}_1'), f(\overline{V}_2')\}$ is a disjoint open covering of U. By Lemma 3.1, that completes the proof.

The following result was proved by E. Sklyarenko [17; Lemma 2].

PROPOSITION 3.12 [17]. *Let X be Hausdorff, $p: X \twoheadrightarrow Y$ proper, $A \subset Y$ nowhere cutting, and $p|p^{-1}(Y-A)$ one-to-one. Then p is monotone.*

Proof. If $y \in Y-A$, then $p^{-1}(y)$ consists of a single point. So suppose $y \in A$. If $p^{-1}(y)$ is not connected, then $p^{-1}(y)$ is the union of two disjoint closed—hence compact—subsets C_1 and C_2, which can be separated by disjoint open sets W_1 and W_2. If $W = W_1 \cup W_2$, then W is a neighborhood of $p^{-1}(y)$, so $p(W)$ is a neighborhood of y in Y. Let $V_i = W_i - p^{-1}(A)$; since p maps $p^{-1}(Y-A)$ homeomorphically onto $Y-A$, $\{p(V_1), p(V_2)\}$ is a disjoint open cover of $p(W)-A$. Since $\overline{V}_1$ and $\overline{V}_2$ intersect $p^{-1}(y)$, both $(p(V_1))^-$ and $(p(V_2))^-$ contain y. That contradicts the assumption that A nowhere cuts Y, and the proof is complete.

If X is normal in Proposition 3.12, then—as the proof shows—it suffices if p is open or closed (rather than proper).

(1) A metric space M is *uniformly locally connected* if to every $\varepsilon > 0$ corresponds a $\delta > 0$ such that any two points x, y of M, with $\varrho(x,y) < \delta$, lie in a connected subset of M of diameter $< \varepsilon$.

PROPOSITION 3.13. *If $Z \subset Y \subset X$, and if $Y - Z$ nowhere cuts Y and $X - Y$ nowhere cuts X, then $X - Z$ nowhere cuts X.*

Proof. We use Lemma 3.1. Clearly $X - Z$ is thin in X. Let $U \subset X$ be open, and $\{U_1, U_2\}$ a disjoint open cover of $U - (X - Z) = U \cap Z$. Let $V = U \cap Y$, $V_1 = U_1 \cap Y$, and $V_2 = U_2 \cap Y$. If W_i is the closure of V_i in V $(i = 1,2)$, then $\{W_1, W_2\}$ is a disjoint open covering of V (Lemma 3.1), and hence, if R_i is the closure of W_i in U (=the closure of V_i in U), then $\{R_1, R_2\}$ is a disjoint open cover of U. By Lemma 3.1, that completes the proof.

Recall that $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut if the conditions of Theorem 1.1 are satisfied.

LEMMA 3.14. *If $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut, and if U is open in X, then $(p^{-1}(U), p^{-1}(U \cap A), p|p^{-1}(U))$ is a $(U, U \cap A)$-cut.*

Proof. Clearly $U \cap A$ is thin in U, and $p^{-1}(U \cap A)$ nowhere cuts $p^{-1}(U)$ by Lemma 3.3. The other requirements for a $(U, U \cap A)$-cut in Theorem 1.1 are also clearly satisfied, and that is all we need.

4. Proof of Theorem 1.1

Let $\bar{j}: \beta(X - A) \to \beta X$ be the continuous extension of the injection map $j: X - A \to X$. Let $\hat{X} = \bar{j}^{-1}(X)$, and $f = \bar{j}|\hat{X}$. Then $f: \hat{X} \twoheadrightarrow X$ is proper. We will show that, if $p: \mathbf{X} \twoheadrightarrow X$ and $\mathbf{A} = p^{-1}(A)$, then $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut if and only if p is the light factor of a monotone-light factorization of f. Since such factorizations exist (with the middle space Tychonoff and both factors proper) and are essentially unique (Proposition 2.3), that will prove our theorem.

Suppose $\hat{X} \xrightarrow{h} \mathbf{X} \xrightarrow{p} X$ is the monotone-light factorization of f, with h and p proper, and let $\mathbf{A} = p^{-1}(A)$. Now $f^{-1}(X - A) = X - A$ by Proposition 2.1, so f maps $f^{-1}(X - A)$ homeomorphically onto $X - A$, and hence p maps $p^{-1}(X - A) = \mathbf{X} - \mathbf{A}$ homeomorphically onto $X - A$. By assumption, p is light and proper. Finally, $\mathbf{A}$ nowhere cuts $\mathbf{X}$ by Proposition 3.11, because h is closed and $h^{-1}(\mathbf{A}) = f^{-1}(A) = \hat{X} - (X - A)$, which nowhere cuts $\hat{X}$ by Example 3.10.

Suppose, conversely, that $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut. Let $k = p^{-1}|(X - A)$; then $k: (X - A) \to \mathbf{X}$ is a homeomorphism into. Let $\bar{k}: \beta(X - A) \to \beta\mathbf{X}$ and $\bar{p}: \beta\mathbf{X} \to \beta X$ be the continuous extensions of k and p, respectively. Then $\bar{p} \circ \bar{k}$ agrees with $\bar{j}$ on the dense set $X - A$, and hence on all of $\beta(X - A)$. Now $\bar{p}^{-1}(X) = \mathbf{X}$ by Proposition 2.1, so $\bar{k}^{-1}(\mathbf{X}) = \bar{j}^{-1}(X) = \hat{X}$. Let $g = \bar{k}|\hat{X}$. Then $f = p \circ g$, p and g are proper, and p is light, so we need only check that g is monotone. Note that $g^{-1}(\mathbf{X} - \mathbf{A}) = X - A$ by Proposition 2.1, and g is one-to-one on $X - A$; hence g is monotone by Proposition 3.12, and the proof is complete.

5. Lifting mappings

In this section we prove a theorem (Theorem 5.3) which generalizes the uniqueness part of Theorem 1.1. The theorem is preceded by some preliminary results on extending maps defined on dense sets; these results have some independent interest, and will also be used to prove Theorem 14.2.

Our first result is a very general extension theorem for continuous functions defined on sets with nowhere cutting complement. Recall that a set-valued function φ from E to the space 2^F of non-empty subsets of F is *upper semi-continuous* if $\{x \in E \mid \varphi(x) \subset V\}$ is open in E for every open $V \subset F$, and that f: $E \to F$ is a *selection* for φ if it is continuous and $f(x) \in \varphi(x)$ for every $x \in E$.

PROPOSITION 5.1. *Let φ: $E \to 2^F$ be upper semi-continuous, with F regular and every $\varphi(x)$ compact and totally disconnected, and let $B \subset E$ be nowhere cutting. Then any selection g for $\varphi \mid (E-B)$ extends uniquely to a selection f for φ.*

Proof. Uniqueness of f follows from the denseness of $E-B$. To prove existence, it suffices [4; § 8, Theorem 1] to show that, if $x_0 \in B$ and $E_0 = (E-B) \cup \{x_0\}$, then g can be extended to a continuous f: $E_0 \to F$ with $f(x_0) \in \varphi(x_0)$. Let us do that.

For each neighborhood U of x_0 in E_0, let

$$S(U) = (g(U - \{x_0\}))^- \cap \varphi(x_0).$$

Since φ is upper semi-continuous, $S(U)$ is not empty. Hence if

$$\mathcal{S} = \{S(U) \mid U \text{ a neighborhood of } x_0 \text{ in } E_0\},$$

$\mathcal{S}$ is a collection of closed subsets of $\varphi(x_0)$ with the finite intersection property, and so has a non-empty intersection S.

Now pick any $y_0 \in S$, and define f: $E_0 \to F$ by

$$f(x) = g(x) \quad \text{if} \quad x \in E - B,$$
$$f(x_0) = y_0.$$

We will show that f is continuous (which will imply that y_0 is actually the only point in S).

We need only show continuity at x_0. So let V be an open neighborhood of y_0 in F, and let C_1 be clopen in $\varphi(x_0)$ with $y_0 \in C_1 \subset (V \cap \varphi(x_0))$. Let $C_2 = \varphi(x_0) - C_1$. Then C_1 and C_2 are disjoint compact subsets of F, so can be separated by open sets W_1 and W_2, and we may suppose $W_1 \subset V$. Let $W = W_1 \cup W_2$, and $U = g^{-1}(W)$; then $\{x_0\} \cup U$ is a neighborhood of x_0 in E_0 since φ is upper semi-continuous. Let $U_i = g^{-1}(W_i)$ for $i = 1, 2$. Now $\{x_0\}$ nowhere cuts E_0 (since B nowhere cuts E), so x_0 is not in the closure of both U_1 and U_2 and hence $\{x_0\} \cup U_1$ or $\{x_0\} \cup U_2$ is a neighborhood of x_0. But if $\{x_0\} \cup U_2$ were a neighborhood of x_0, then

$$y_0 \in S \subset S(\{x_0\} \cup U_2) \subset (g(U_2))^- \subset \overline{W}_2,$$

which contradicts $y_0 \in W_1$. Hence $\{x_0\} \cup U_1$ is a neighborhood of x_0. But

$$f(\{x_0\} \cup U_1) = g(U_1) \cup \{y_0\} \subset V,$$

so f is continuous at x_0, and the proof is complete.

COROLLARY 5.2. *Let* π: $\mathbf{X} \to Y$ *and* q: $\mathbf{Y} \to Y$ *be continuous, with* $\mathbf{Y}$ *regular and* q *light and proper. Let* $\mathbf{A} \subset \mathbf{X}$ *be nowhere cutting. Then any continuous* g: $(\mathbf{X} - \mathbf{A}) \to \mathbf{Y}$, *such that* $\pi | (\mathbf{X} - \mathbf{A}) = q \circ g$, *can be extended uniquely to a continuous* $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ *such that* $\pi = q \circ \mathbf{f}$.

Proof. For each $x \in \mathbf{X}$, let $\varphi(x) = q^{-1}(\pi(x)) = (q^{-1} \circ \pi)(x)$. Since q is closed, the set-valued map q^{-1} is upper semi-continuous, and hence so is φ. Also, if $x \in X - A$, then $\varphi(x)$ is compact and totally disconnected, and $g(x) \in \varphi(x)$. Hence, by Proposition 5.1 there exists an $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$ satisfying our requirements, which completes the proof.

THEOREM 5.3. *Let* $(\mathbf{X}, \mathbf{A}, p)$ *be an* (X, A)*-cut, and* $(\mathbf{Y}, \mathbf{B}, q)$ *a* (Y, B)*-cut. Suppose* f: $X \to Y$ *is continuous with* $A \supset f^{-1}(B)$. *Then there exists a unique map* $\mathbf{f}$: $\mathbf{X} \to \mathbf{Y}$, *with* $\mathbf{A} \supset \mathbf{f}^{-1}(\mathbf{B})$, *making the following diagram commutative.*

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\ \mathbf{f}\ } & \mathbf{Y} \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow{\ f\ } & Y \end{array}$$

If f *is a homeomorphism onto, so is* $\mathbf{f}$.

Proof. Define g: $\mathbf{X} - \mathbf{A} \to \mathbf{Y}$ by $g(x) = q^{-1}(f(p(x)))$, and note that g is single-valued and continuous. Let $\pi = f \circ p$. The first assertion now follows from Corollary 5.2.

If f is a homeomorphism onto, we similarly obtain a continuous $\mathbf{h}$: $\mathbf{Y} \to \mathbf{X}$ such that $p \circ \mathbf{h} = f^{-1} \circ q$. But $\mathbf{h} \circ \mathbf{f}$ is then the identity on the dense set $\mathbf{X} - \mathbf{A}$, and hence on all of $\mathbf{X}$, and similarly $\mathbf{f} \circ \mathbf{h}$ is the identity on $\mathbf{Y}$. Hence $\mathbf{f}$ and $\mathbf{h}$ are inverse homeomorphisms onto, and the proof is complete.

Note that Corollary 5.4, with f a homeomorphism onto, yields a new proof—not depending on section 2—of essential uniqueness in Theorem 1.1.

6. Nowhere scattering sets

Recall from section 1 that a subset A of X *nowhere scatters* X if, whenever U is a neighborhood of $a \in A$, then every disjoint open covering $\mathfrak{V}$ of $U-A$ is locally finite at a (i.e. a has a neighborhood W which intersects only finitely many $V \in \mathfrak{V}$).

It is easily checked that, for $A \subset X$ to nowhere scatter X, it is sufficient if each neighborhood U of $a \in A$ contains a neighborhood V of a such that $V-A$ has only finitely many components. (This occurs, for example, if A is a subcomplex of a locally finite simplicial complex.) That this condition is not necessary, however, is shown by Example (3) of the introduction.

If the definition of nowhere scattering is altered by requiring $\mathfrak{V}$ to be finite and W to intersect only one $V \in \mathfrak{V}$, then one obtains a characterization of nowhere cutting. Hence if the definition of nowhere scattering is merely changed by requiring $\mathfrak{V}$ to be finite, one obtains a characterization of sets A which are both nowhere cutting and nowhere scattering. This yields the following analogue of Lemma 3.1.

LEMMA 6.1. *The following properties of a thin $A \subset X$ are equivalent.*

(a) *A nowhere cuts and nowhere scatters X.*

(b) *If U is open in X, and $\{V_\alpha\}_\alpha$ is a disjoint open covering of $U-A$, then $\{\overline{V}_\alpha\}_\alpha$ is a disjoint open covering of U (where $\overline{V}_\alpha$ denotes the closure of V_α in U).*

It is now easily checked that all results in section 3 after Lemma 3.1, with the exception of Proposition 3.5 and Example 3.10, remain true if "nowhere cutting" is replaced by "nowhere cutting and nowhere scattering".

Simple examples, like a point on a line, show that a nowhere scattering subset need not be nowhere cutting. The converse implication is, in general, also false (consider Example 3.10, with E the integers), but we can prove:

PROPOSITION 6.2. *In first countable spaces, nowhere cutting sets are nowhere scattering.*

Proof. Let X be such a space and A a nowhere cutting subset. Let $x \in A$, let U be a neighborhood of x in X, and $\mathfrak{V}$ a disjoint open covering of $U-A$. We suppose that no neighborhood of x intersects only finitely many $V \in \mathfrak{V}$, and obtain a contradiction.

First, let us show that no $V \in \mathfrak{V}$ has x in its closure. Suppose $x \in \overline{V}_0$ for some $V_0 \in \mathfrak{V}$; then V_0 and $(U-A)-V_0$ form a disjoint open covering of $U-A$, so x is not in the closure of $(U-A)-V_0$. Hence some neighborhood of x intersects only V_0, contradicting our assumption.

Now let $B_1, B_2, \ldots$ be a base for the neighborhoods of x in X. By induction, we can pick elements V_i and W_i of $\mathfrak{V}$ $(i=1,2,\ldots)$, all distinct, such that $V_i \cap B_i \neq \emptyset$ and $W_i \cap B_i \neq \emptyset$

for all i; this is easily done, since $x \in (U-A)^-$ (because A is thin). Now let $V = \bigcup_{i=1}^{\infty} V_i$, and $W = (U-A) - V$. Then V and W form a disjoint open covering of $U-A$, and both have x in their closure, which is a contradiction. This completes the proof.

In the remainder of this section, we characterize nowhere scattering sets under special circumstances. These results will not be needed for the proof of Theorem 1.2.

PROPOSITION 6.3. *If X is a locally compact* [1] *metric space and $A \subset X$ is 0-dimensional, then A nowhere scatters X if, and only if, every disjoint open covering of $X-A$ is locally finite at every point of A.*

Proof. The condition is clearly necessary. To prove sufficiency, let $a \in A$, let U be an open neighborhood of a in X, and $\mathcal{V}$ a disjoint open covering of $U-A$. We must show that $\mathcal{V}$ is locally finite at a.

Let R be open, with $a \in R \subset \bar{R} \subset U$ and $\bar{R}$ compact. Let C be a clopen subset of A such that $a \in C \subset (R \cap A)$, and let

$$N = \{x \in R \mid d(x, C) < d(x, A-C)\}.$$

Then N is open in X, and $N \cap A = C$. Also $B = \bar{N} - N$ does not intersect A and is compact. Now let $\mathcal{W} = \{V \cap \bar{N} \mid V \in \mathcal{V}\}$; it suffices to show that $\mathcal{W}$ is locally finite at a. Now $\mathcal{W} = \mathcal{W}_1 \cup \mathcal{W}_2$, where

$$\mathcal{W}_1 = \{W \in \mathcal{W} \mid W \cap B \neq \emptyset\}, \text{ and } \mathcal{W}_2 = \{W \in \mathcal{W} \mid W \subset N\}.$$

Since $\mathcal{W}_1$ is a disjoint, open covering of B, and B is compact, $\mathcal{W}_1$ is finite. Now the elements of $\mathcal{W}_2$, as well as their union W_2, are clopen in $\bar{N} - A$ and in $N - A$, and hence in $X - A$. Hence $\mathcal{W}_2 \cup \{(X-A) - W_2\}$ is a disjoint open covering of $X - A$, and is therefore —along with $\mathcal{W}_2$—locally finite at a. Hence so is $\mathcal{W}$, and the proof is complete.

In the proof of Proposition 6.5 below, we need the following unpublished result of H. H. Corson.

LEMMA 6.4 (H. H. Corson). *A compact Hausdorff space, which is the continuous image of a separable metric space, is metrizable.*

Proof. If E is compact Hausdorff, then $E \times E$ is regular and the diagonal Δ is closed in $E \times E$. Hence $(E \times E) - \Delta$ can be covered by open sets whose closures in $E \times E$ miss Δ; since $(E \times E) - \Delta$ is also the continuous image of a separable metric space, it has the Lindelöf property, so $(E \times E) - \Delta$ is covered by countably many such sets. Hence Δ is a

[1] It suffices if X is locally compact at every point of A.

G_δ in $E \times E$, which implies that E is metrizable. (*Proof:* Let $\Delta = \bigcap_{n=1}^{\infty} U_n$, with U_n open, and pick open $V_{n,i}$ in E ($i = 1, \ldots, k(n)$) such that

$$\Delta \subset \bigcup_{i=1}^{k(n)} (V_{n,i} \times V_{n,i}) \subset U_n \text{ for all } n;$$

then the set of all such $V_{n,i}$ is a countable base for E.) That completes the proof.

Recall that, if E is a topological space and $x \in E$, then the *quasi-component* of E containing x is

$$\bigcap \{C \subset E \mid C \text{ clopen}, \quad x \in C\}.$$

The space $Q(E)$ of quasi-components of E is topologized by taking all sets of the form

$$\{C \in Q(E) \mid C \subset U\} \quad U \text{ clopen in } E$$

as a base for the open sets. This topology makes the natural projection π: $E \twoheadrightarrow Q(E)$ continuous, but not, in general, a quotient map.

PROPOSITION 6.5. *If X is a compact metric space, the following properties of a 0-dimensional subset A are equivalent:*

(a) *A nowhere scatters X.*
(b) *Every disjoint open covering of $X - A$ is finite.*
(c) *The space Q of quasi-components of $X - A$ is compact metric.*
(d) *$X - A$ has only countably many clopen* (in $X - A$) *subsets.*

Proof. (a) ⇒ (*b*). Let $\mathcal{U}$ be a disjoint open covering of $X - A$. By assumption, each $x \in A$ has an open neighborhood intersecting only finitely many $U \in \mathcal{U}$. Surely every point of $X - A$ has such a neighborhood, so X is covered by such neighborhoods, and hence, being compact, by finitely many of them. Hence $\mathcal{U}$ is finite.

(b) ⇒ (a). This follows from Proposition 6.3.

(b) ⇒ (c). Compactness of Q follows from (b), and metrizability from Lemma 6.4.

(c) ⇒ (d). (After Freudenthal [9; 2.4].) By (c), Q has a countable, clopen base $\mathcal{B}$. Thus each clopen set $S \subset Q$ is the union of sets in $\mathcal{B}$, and, since S is compact, of finitely many. Hence Q, and thus also $X - A$, has only countably many clopen sets.

(d) ⇒ (b). If $\mathcal{U}$ were an infinite disjoint covering of $X - A$, each of the uncountably many subcollections of $\mathcal{U}$ would have a different clopen union, contradicting (d).

Note that, under the assumptions of Proposition 6.5, the property of A nowhere scattering X was characterized entirely in terms of properties of $X - A$ alone.

7. Proof of Theorem 1.2; application to Freudenthal compactification

Throughout this section, let $(\mathbf{X}, \mathbf{A}, p)$ be an (X, A)-cut.

PROPOSITION 7.1. *A nowhere scatters X if, and only if, $\mathbf{A}$ nowhere scatters $\mathbf{X}$.*

Proof. Suppose $\mathbf{A}$ nowhere scatters $\mathbf{X}$. Let $a \in A$, let U be a neighborhood of a in X, and $\mathcal{V}$ a disjoint open covering of $U - A$. Now for each $x \in p^{-1}(a)$, the set $p^{-1}(U)$ is a neighborhood of x in $\mathbf{X}$, and

$$p^{-1}(\mathcal{V}) = \{p^{-1}(V) \mid V \in \mathcal{V}\}$$

is a disjoint open covering of $p^{-1}(U) - \mathbf{A}$. Hence x has a neighborhood N_x with the property of intersecting only finitely many elements of $p^{-1}(\mathcal{V})$. Since $p^{-1}(a)$ is compact, it is covered by finitely many open (in $\mathbf{X}$) sets with this property, and hence by a single one (their union), say N. Then $p(N)$ is a neighborhood of a (since p is closed) intersecting only finitely many $V \in \mathcal{V}$.

Suppose A nowhere scatters X. Let $x \in \mathbf{A}$, let U be a neighborhood of x in $\mathbf{X}$, and $\mathcal{V}$ a disjoint open covering of $U - \mathbf{A}$. Since p is a spread (Proposition 2.4), there exists a neighborhood M of $p(x)$ in X, and a clopen subset N of $p^{-1}(M)$, such that $x \in N \subset U$. Then $\{p(V \cap N) \mid V \in \mathcal{V}\}$ together with $(M - A) - p(V)$ form a disjoint open covering of $M - A$, and hence $p(x)$ has a neighborhood W intersecting only finitely many elements of this covering. But then $p^{-1}(W) \cap N$ intersects only finitely many elements of $\mathcal{V}$, and the proof is complete.

Proposition 6.2 and half of Proposition 7.1 imply that (b) $\Rightarrow$ (c) in Theorem 1.2. To prove the harder result (c) $\Rightarrow$ (a), we need some preliminary results about bases in X and $\mathbf{X}$.

First, some notation: A base $\mathcal{B}$ for $X - A$ is called *full* if, whenever $a \in A$, U a neighborhood of a in X, and C clopen in $U - A$, then $(V \cap C) \in \mathcal{B}$ for some neighborhood V of a in X. If G is open in $\mathbf{X} - \mathbf{A}$, let

$$G^* = \bigcup \{U \subset \mathbf{X} \mid U \text{ open in } \mathbf{X},\ U \cap (\mathbf{X} - \mathbf{A}) = G\}.$$

Note that, since $\mathbf{A}$ is thin in $\mathbf{X}$, $G^* \subset \bar{G}$.

LEMMA 7.2. *If $\mathcal{B}$ is a full base for $X - A$, then $\{(p^{-1}(B))^* \mid B \in \mathcal{B}\}$ is a base for $\mathbf{X}$.*

Proof. Suppose $\mathcal{B}$ is a full base for $X - A$. Let $x \in \mathbf{X}$, and V a neighborhood of x. We must find a $B \in \mathcal{B}$ such that $x \in p^{-1}(B)^* \subset V$. Pick an open U in $\mathbf{X}$ such that $x \in U \subset \bar{U} \subset V$. If $x \in \mathbf{X} - \mathbf{A}$, we now merely pick $B \in \mathcal{B}$ such that $p(x) \in B \subset p(U - \mathbf{A})$. So suppose $x \in \mathbf{A}$. Since p is a spread (Proposition 2.4), there exists a neighborhood M of $p(x)$ in X, and a clopen subset N of $p^{-1}(M)$, such that $x \in N \subset U$. By assumption, there exists a neighborhood W of $p(x)$ in X such that $B = W \cap p(N - \mathbf{A})$ is in $\mathfrak{B}$. Now

$$p^{-1}(B) = p^{-1}(W) \cap (N - \mathbf{A}) = (p^{-1}(W) \cap N) \cap (\mathbf{X} - \mathbf{A}),$$

so

$$x \in (p^{-1}(W) \cap N) \subset p^{-1}(B)^* \subset p^{-1}(B)^- \subset \bar{N} \subset \bar{U} \subset V,$$

which completes the proof.

Lemma 7.3. *A Hausdorff space E, which is a continuous image of a separable metric space, has a countable collection $\mathfrak{U}$ open subsets such that, if $x \neq y$ are in E, then there exist disjoint U_x, U_y in $\mathfrak{U}$ with $x \in U_x$ and $y \in U_y$. If E is 0-dimensional, the $U \in \mathfrak{U}$ may be chosen clopen.*

Proof. Since E is Hausdorff, the diagonal Δ is closed in $E \times E$. Hence $(E \times E) - \Delta$ can be covered by open rectangles $V \times W$; if E is 0-dimensional, V and W may be chosen clopen. Since $(E \times E) - \Delta$ has the Lindelöf property, this covering has a countable subcovering $\{V_i \times W_i\}_{i=1}^{\infty}$, and we let $\mathfrak{U} = \{V_i\}_{i=1}^{\infty} \cup \{W_i\}_{i=1}^{\infty}$.

Lemma 7.4. *If X is separable metric, and A nowhere scatters X, then $X - A$ has a countable full base.*

Proof. Let $\{B_n\}_{n=1}^{\infty}$ be a countable base for X. For each n, let E_n be the space of quasi-components of $B_n - A$, and let f_n: $B_n - A \to E_n$ be the natural map. Topologize E_n by taking as base for the open sets all sets U such that $f_n^{-1}(U)$ is clopen in $B_n - A$. Clearly E_n is Hausdorff and 0-dimensional, and f_n is continuous, so we can apply Lemma 7.3. Let $\mathfrak{U}_n$ be a collection of clopen subsets of E_n with the property guaranteed by Lemma 7.4, and, for convenience, let us suppose that it is closed under finite intersections. Let $\mathfrak{W}_n = \{f_n^{-1}(U) \mid U \in \mathfrak{U}_n\}$, and let $\mathfrak{F}$ be the collection of finite unions of sets of the form $B_m \cap W$ ($W \in \mathfrak{W}_n$, $m, n = 1, 2, \ldots$). Let us show that $\mathfrak{F}$, which is clearly a countable base for $X - A$, is full.

Let $a \in A$, U a neighborhood of a in X, and C a clopen subset of $U - A$. We must find a neighborhood V of a such that $(V \cap C) \in \mathfrak{F}$. Since $\{B_n\}_{n=1}^{\infty}$, is a base for X, we may assume that $U = B_k$ for some k. Denote $B_k - A$ by B_k'.

First, let us show that, if $x \in C$, then there exists a $W_x \in \mathfrak{W}_k$ containing x, and a neighborhood N_x of a, such that $(N_x \cap W_x) \subset C$. Let Q be the quasi-component of B_k' which contains x; clearly $Q \subset C$. Using our assumptions about $\mathfrak{U}_k$, we can now pick a decreasing sequence $\{S_j\}_{j=1}^{\infty}$ from $\mathfrak{W}_k$ whose intersection is Q. Let $S_0 = B_k'$. Let $\mathfrak{D}$ be the disjoint open covering of B_k' consisting of C and all sets of the form $(B_k' - C) \cap (S_j - S_{j+1})$ for $j = 0, 1, \ldots$. Since A nowhere scatters X, there exist a neighborhood N_x of a which intersects only finitely many elements of $\mathfrak{D}$. Hence, for some j_0, the set N_x does not intersect

$(B'_k - C) \cap (S_j - S_{j+1})$ for any $j \geq j_0$. Let $W_x = S_{j_0}$. Then N_x does not intersect $(B'_k - C) \cap W_x$, so $N_x \cap W_x$ does not intersect $B'_k - C$, and thus, since $W_x \subset B'_k$, we have $(N_x \cap W_x) \subset C$.

Now $\{W_x | x \in C\}$ is a subcollection of the countable collection $\mathcal{W}_k$, and hence can be re-indexed as $\{W_j\}_{j=1}^{\infty}$. For each j, let N_j be a neighborhood of a such that $(N_j \cap W_j) \subset C$. Note that $C \subset \bigcup_{j=1}^{\infty} W_j$. Now let $\mathcal{U}$ be the disjoint open covering of B'_k consisting of $B'_k - C$ and all sets $C \cap (W_j - \bigcup_{i=1}^{j-1} W_i)$ for $j = 1, 2, \ldots$. Since A nowhere scatters X, there exists a neighborhood N of a which intersects only finitely many elements of $\mathcal{U}$, and hence misses $C \cap (W_j - \bigcup_{i=1}^{j-1} W_i)$ for all j greater than some index r. Hence $(N \cap C) \subset \bigcup_{j=1}^{r} W_j$. Now pick m so that $a \in B_m \subset (N \cap (\bigcap_{j=1}^{r} N_j))$. Then

$$(B_m \cap C) = (B_m \cap (N \cap C)) \subset \bigcup_{j=1}^{r} (B_m \cap W_j) = \bigcup_{j=1}^{r} (B_m \cap (N_j \cap W_j)) \subset (B_m \cap C).$$

Hence

$$(B_m \cap C) = (\bigcup_{j=1}^{r} (B_m \cap W_j)) \in \mathfrak{J},$$

which completes the proof.

Proof of Theorem 1.2. That (a) $\Rightarrow$ (b) is obvious. As observed earlier, (b) $\Rightarrow$ (c) follows from Propositions 6.2 and 7.1. Lastly, (c) $\Rightarrow$ (a) follows from Lemmas 7.2 and 7.4 and Urysohn's metrization theorem, which completes the proof of Theorem 1.2.

We conclude this section by using Theorem 1.2 to relate cuts to the Freudenthal compactification. If D is a separable metric space, then a metric compactification X of D is called a *Freudenthal compactification* of D if $X - D$ is 0-dimensional and nowhere cuts X. If it exists, the Freudenthal compactification is essentially unique [8]. We now prove

PROPOSITION 7.5. *Let A be a thin, 0-dimensional subset of a compact metric space X, and suppose that $X - A$ satisfies one of the equivalent properties of Proposition* 6.5. *If $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut, then $\mathbf{X}$ is a Freudenthal compactification of $\mathbf{X} - \mathbf{A}$.*

Proof. By Proposition 6.5, A nowhere scatters X, so $\mathbf{X}$ is metrizable by Theorem 1.2. Since X is compact and p is proper, $\mathbf{X}$ must be compact [12]. Since p is a spread (Proposition 2.4) and A is 0-dimensional, so is $\mathbf{A} = p^{-1}(A)$. Finally, $\mathbf{A}$ always nowhere cuts $\mathbf{X}$, and that completes the proof.

The following corollary was proved (differently) by Freudenthal [8] [9].

COROLLARY 7.6. *The following properties of a separable metric space D are equivalent:*

(a) *D has a Freudenthal compactification.*

(b) *D satisfies the conditions of Corollary* 6.5, *and has a metric compactification X with $X - A$ 0-dimensional.*

Proof. That (a) $\Rightarrow$ (b) follows from Proposition 6.2 and Corollary 6.5, while (b) $\Rightarrow$ (c) follows from Proposition 7.5.

8. Locally connected metric spaces

If (Y,d) is a locally connected metric space, one can always obtain a metric d^* on Y such that

(a) d^* and d generate the same topology,

(b) d^* is uniformly locally connected, [1]

(c) $d^* \geqslant d$,

(d) $d^*(x,y) \leqslant a$ if x and y lie in a connected subset of d-diameter $\leqslant a$.

In fact, if Y is connected, such a d^* is obtained [22] by letting

$$d^*(x,y) = \inf \{\operatorname{diam} A \mid A \subset X,\ A \text{ connected, } x \text{ and } y \text{ in } A\}.$$

If Y is not connected, one can still obtain such a d^* as follows: In each of the (open) components Y_α of Y, pick a point p_α. If x and y are in the same component of Y, define $d^*(x,y)$ as above; if they are in different components, say $x \in Y_\alpha$ and $y \in Y_\beta$, let

$$d^*(x,y) = d^*(x,p_\alpha) + d(p_\alpha,p_\beta) + d^*(p_\beta,y) + 1.$$

It is easy to chek that this works.

For simplicity, the following result is stated for complete spaces, but, as we shall see, an analogous result is true in general. The hypothesis that A nowhere scatters X is essential, however, for otherwise X could not be metrizable (Propositions 6.2 and 7.1).

Our proof will be purely metric, making no use of the general existence theorem for cuts.

THEOREM 8.1. *Let X be a complete metric space, $A \subset X$ thin and nowhere scattering, and $X - A$ locally connected. Let (Y,d) be $X - A$ with the induced metric, let $\mathbf{X}$ be the completion of (Y,d^*) (where d^* is as above), let p: $\mathbf{X} \to X$ be the uniformly continuous extension of the injection i: $(Y,d^*) \to X$, and let $\mathbf{A} = \mathbf{X} - Y$. Then $(\mathbf{X},\mathbf{A},p)$ is an (X,A)-cut.*

Before proving Theorem 8.1, we need a lemma, whose notation is that of the theorem.

LEMMA 8.2. *If $a \in A$, $x_n \in \mathbf{X} - \mathbf{A}$, and $p(x_n) \to a$, then some subsequence x_{n_k} of x_n converges in $\mathbf{X}$ (to an x with $p(x) = a$).*

Proof. We will pick a decreasing sequence of infinite sets S_k $(k \geqslant 1)$ of positive integers such that, if $m,\ n \in S_k$, and $k > 1$, then $d^*(x_n,x_m) \leqslant k^{-1}$. This will suffice, for we can then inductively pick n_k so that $n_k \in S_k$ and $n_{k+1} > n_k$.

---

[1] See footnote (1) on page 9

Begin by letting S_1 be the set of all positive integers. Suppose that $S_1, \ldots, S_k$ have been found, and let us pick S_{k+1}. Let U be the open $\frac{1}{2}(k+1)^{-1}$-sphere about a in X, and let $\mathfrak{V}$ be the set of components of $U-A$; since $X-A$ is locally connected, each $V \in \mathfrak{V}$ is open. Since A nowhere scatters X, there exists a neighborhood W of a which intersects only finitely many $V \in \mathfrak{V}$. Hence S_k has an infinite subset S_{k+1} such that all $p(x_n)$ with $n \in S_{k+1}$ are in one element of $\mathfrak{V}$, say V_0. But V_0 is connected with d-diameter $\leqslant (k+1)^{-1}$, hence with d^*-diameter $\leqslant (k+1)^{-1}$, and thus S_{k+1} satisfies all our requirements. This completes the proof of Lemma 8.2.

Proof of Theorem 8.1. We must verify that $\mathbf{X}$, $\mathbf{A}$, and p satisfy all the requirements of Theorem 1.1.

(a) *p is closed:* Let E be closed in $\mathbf{X}$, and let $y \in X$ be in $(p(E))^-$, so that $p(x_n) \to y$ for suitable $x_n \in E$. We must show that $y \in p(E)$.

Pick $z_n \in X - A$ whose distance from x_n is less than n^{-1}. Then $d(p(x_n), p(z_n)) < n^{-1}$, so $p(z_n) \to y$. If $y \in X-A$, then $z_n \to p^{-1}(y)$, so $x_n \to p^{-1}(y)$, hence $p^{-1}(y) \in E$ and $y \in p(E)$. If $y \in A$, then, by Lemma 8.2, z_n has a subsequence z_{n_k} converging to some $z \in \mathbf{X}$ with $p(z) = y$. But then $x_{n_k} \to z$, and hence $z \in E$ and $y \in p(E)$.

(b) *p is onto:* This follows from (a).

(c) *p maps $\mathbf{X} - \mathbf{A}$ homeomorphically onto $X - A$:* This follows from the definitions.

(d) $p(\mathbf{A}) \subset A$: This follows from (c) and Proposition 2.1.

(e) *$p^{-1}(x)$ is compact for $x \in A$:* Since $\mathbf{X}$ and X are metrizable, and p is closed, $p^{-1}(x)$ has compact boundary [19] [21]. But $p^{-1}(x)$ is a subset of the thin set $\mathbf{A}$, and thus coincides with its boundary.

(f) $\mathbf{A}$ *nowhere cuts* $\mathbf{X}$: $\mathbf{A}$ is thin in $\mathbf{X}$ by definition, and $\mathbf{X} - \mathbf{A}$ is uniformly locally connected. Hence $\mathbf{A}$ nowhere cuts $\mathbf{X}$ by Example 3.9.

(g) *p is light:* We will actually prove that p is a spread (see section 2). Let U be a neighborhood of $x \in \mathbf{X}$, and pick r so that S, the d^*-sphere about x of radius r, is contained in U. Let W be the d-sphere about $p(x)$ of radius $\frac{1}{4}r$. Let $\mathfrak{V}$ be the collection of components of $p^{-1}(W) - A$. Since $\mathbf{X} - \mathbf{A}$ is uniformly locally connected, $\mathbf{A}$ nowhere cuts and nowhere scatters $\mathbf{X}$ (see Example 3.9 and the remark following Lemma 6.1), and hence (Proposition 6.1) $\{\overline{V} \mid V \in \mathfrak{V}\}$ is a disjoint open covering of $p^{-1}(W)$, with $\overline{V}$ denoting the closure of V in $p^{-1}(W)$. Pick $V_0 \in \mathfrak{V}$ such that $x \in \overline{V}_0$. It remains to show that $\overline{V}_0 \subset U$.

If $y, z \in V_0$, then $p(y)$, $p(z)$ are in the connected subset $p(V_0)$ of $X - A$, and since $p(V_0) \subset W$, we have $d^*(y,z) \leqslant \frac{1}{2}r$. Since $x \in \overline{V}_0$, it follows that $d^*(x,y) \leqslant \frac{1}{2}r$ for every $y \in \overline{V}_0$, so $\overline{V}_0 \subset S \subset U$. That completes the proof.

We conclude this section with a remark about what happens to Theorem 8.1 if X is not complete. In that case, $\mathbf{X}$ is not the whole completion of (Y,d^*); it is the subset of this completion consisting of all points which are limits of sequences $u_n \in Y$ such that $i(u_n)$ converges in X. The map p: $\mathbf{X} \to X$ and the set $\mathbf{A}$ are defined as before. The above proof of Theorem 8.1 goes through unchanged in this more general situation.

9. Simplicial complexes

Let K be a locally finite simplicial complex, and let $|K|$ carry the usual (metrizable) topology for which $U \subset |K|$ is open provided its intersection with each simplex of K is relatively open. Call a subcomplex L of K *thin* if each simplex of L is a face of at least one simplex of K which is not in L. It is clear that L is thin in K if and only if $|L|$ is thin in $|K|$ in the sense of having empty interior.

Suppose now that L is thin in K. Let K^* be the disjoint union of the maximal simplices of K, and let π: $K^* \twoheadrightarrow K$ be the natural projection. If s and t are simplices of K^*, with faces s' and t' such that $\pi(s') = \pi(t') \notin L$, then we write $x \sim y$ whenever $x \in s'$, $y \in t'$, and $\pi(x) = \pi(y)$. We now define the equivalence relation $\equiv$ on $|K^*|$ by letting $x \equiv y$ whenever there exists a sequence $x = x_0, \ldots, x_n = y$ such that $x_i \sim x_{i+1}$ for $i = 0, \ldots, n-1$. Let $E = |K^*|/\equiv$ be the quotient space, and q: $|K^*| \twoheadrightarrow E$ the quotient map. Since $x \equiv y$ implies $\pi(x) = \pi(y)$, there exists a p: $E \twoheadrightarrow |K|$ such that $\pi = p \circ q$. Now let $\mathbf{K}$ be the set of all $q(s)$, with s a simplex of K^*. In general, $\mathbf{K}$ need not be a complex, because the intersection of two simplices of $\mathbf{K}$ need not be a simplex of $\mathbf{K}$, but only a finite union of such simplices. However, under these circumstances $\mathbf{K}'$, the barycentric subdivision of $\mathbf{K}$, *is* a complex, and p: $\mathbf{K}' \twoheadrightarrow K'$ (where K' is the barycentric subdivision of K) is simplicial. Note that $|\mathbf{K}| = |\mathbf{K}'| = E$. Finally, let $\mathbf{L} = p^{-1}(L)$.

THEOREM 9.1. *If L is a thin subcomplex of a locally finite simplicial complex K, and if $\mathbf{K}$, $\mathbf{L}$, and p: $\mathbf{K} \twoheadrightarrow K$ are as above, then $(|\mathbf{K}|, |\mathbf{L}|, p)$ is a $(|K|, |L|)$-cut.*

Proof. It follows from the definitions that p maps $|\mathbf{K}| - |\mathbf{L}|$ homeomorphically onto $K - L$, and maps $|\mathbf{L}|$ onto $|L|$. Let us show that p is closed. If K is a finite complex, this is clear, since then $|\mathbf{K}|$ is compact. In general, let $A \subset |\mathbf{K}|$ be closed, and let us show that $p(A)$ is closed. By definition, we must show that $p(A) \cap S$ is closed for every simplex s of K. But $p^{-1}(s)$ is a *finite* subcomplex of $\mathbf{K}$, so that $A \cap p^{-1}(s)$ is compact, and hence $p(A) \cap s = p(A \cap p^{-1}(s))$ is also compact. So p is closed; since K is locally finite, each $p^{-1}(x)$ is finite, so p is proper and light. It remains to show that $|\mathbf{L}|$ nowhere cuts $|\mathbf{K}|$. We do this by showing that $|\mathbf{L}|$ satisfies the conditions of Example 3.7.

First of all, $\mathbf{L}$ is clearly thin in $\mathbf{K}$, so $|\mathbf{L}|$ is thin in $|\mathbf{K}|$. Now let $x \in |\mathbf{L}|$, and let U be a neighborhood of x in $|\mathbf{K}|$; we must find a neighborhood V of x in $|\mathbf{K}|$ such that $V - |\mathbf{L}|$

is connected. Let S be the star of x in $\mathbf{K}$. Now the definitions imply that, if s and t are simplices in S, then there exist simplices $s=s_0,\dots,s_n=t$ in S such that $s_i \cap s_{i+1} \notin \mathbf{L}$ for $i=0,\dots,n-1$. But this implies that, if V is a barycentric neighborhood of x in $|S|$, then $V-|\mathbf{L}|$ is polygonally connected, and that completes the proof.

Now that we have a combinatorial description of cuts, what can be said about L in case K is an $(n+1)$-manifold, and L is dimensionally homogeneous? If $\dim L<n$, then $(\mathbf{K},\mathbf{L})=(K,L)$. If $\dim L=n+1$, then L is not thin. So suppose $\dim L=n$. The following example shows what can happen:

EXAMPLE 9.2. *Let $|K|=E^3$, and let $|L|$ consist of two tangent spheres. Then $|\mathbf{L}|$ has three components: Two of them are spheres, and the third consists of two tangent spheres.*

The above example shows that the components of $|\mathbf{L}|$ need not be manifolds, but the following theorem asserts that they cannot get much worse than two tangent spheres.

THEOREM 9.3. *Let K be a triangulated $(n+1)$-manifold (not necessarily combinatorial), and L a homogeneously n-dimensional subcomplex. Then $\mathbf{L}$ is homogeneously n-dimensional, and each $(n-1)$-simplex of $\mathbf{L}$ is the face of exactly two n-simplices of $\mathbf{L}$.*

Proof. First we must show that $|\mathbf{L}|$ is the union of n-simplices. Let $x\in|\mathbf{L}|$, and let s be an $(n+1)$-simplex of $\mathbf{K}$ containing x. Then $p(s)$ is an $(n+1)$-simplex of K containing $p(x)$. Since K is a manifold, we can pick $(n+1)$-simplices $t_0,\dots,t_k$ in K, all containing $p(x)$, such that $t_0=p(s)$, $(t_i \cap t_{i+1})\notin L$ for $i=0,\dots,k-1$, and t_k has an n-dimensional face u which contains $p(x)$ and is in L. Then the definitions imply that $x\in p^{-1}(u)\cap s_k$, where s_k is the unique $(n+1)$-simplex of $\mathbf{K}$ such that $p(s_k)=t_k$. But $p^{-1}(u)\cap s_k$ is an n-simplex of $\mathbf{L}$, so $\mathbf{L}$ is indeed the union of n-simplices.

Next, let s^{n-1} be an $(n-1)$-simplex of $\mathbf{L}$; we must show that S^n, the collection of n-simplices of $\mathbf{L}$ which contain s^{n-1}, has exactly two elements.

Let s^{n+1} be any element of S^{n+1}, the collection of all $(n+1)$-simplices of $\mathbf{K}$ which contain s^{n-1}. The definition of $\mathbf{K}$ implies that S^{n+1} is obtained by starting with s^{n+1}, and then successively adding—in both directions—simplices which intersect some element of S^{n+1} already obtained in an n-simplex not contained in $\mathbf{L}$. Now the definition of $\mathbf{K}$ implies that every element of S^n is a face of exactly one element of S^{n+1}. Hence if the above process of adding simplices ends in both directions with an $(n+1)$-simplex having a face in S^n, then S^n has exactly two elements and we are through. If not, the two arms of the process would meet, and each element of S^{n+1} would intersect two others in n-simplices of $\mathbf{K}$ not lying in $\mathbf{L}$. But then each element of $T^{n+1}=\{p(s)\,|\,s\in S^{n+1}\}$ would intersect two others in

n-simplices of K not lying in L; since K is a manifold, T^{n+1} would then consist of *all* $(n+1)$-simplices of K containing $p(s^{n-1})$, and so $p(s^{n-1})$ would not lie in any n-simplex of L, which contradicts our assumptions. That completes the proof.

10. Mapping cylinders

If $f: E \to F$ is a continuous function, then the mapping cylinder C_f is obtained by taking the disjoint union $(E \times I) \cup F$, and identifying $(x, 0)$ with $f(x)$ for all $x \in E$. Let $\pi_f: (E \times I) \cup F \twoheadrightarrow C_f$ be the quotient map. Then π_f maps F homeomorphically onto $\pi_f(F)$ (called the *base* of C_f), and this is used to identify $\pi_f(F)$ with F.

LEMMA 10.1. *If f is closed, then so is π_f.*

Proof. Let $A \subset (E \times I) \cup F$ be closed, and write $A \cap (E \times \{0\}) = B \times \{0\}$ and $A \cap F = C$. Then

$$\pi_f^{-1}(\pi_f(A)) \cap (E \times I) = A \cup (f^{-1}(f(B) \cup C) \times \{0\}),$$

and
$$\pi_f^{-1}(\pi_f(A)) \cap F = f(B) \cup C,$$

so $\pi_f^{-1}(\pi_f(A))$ is closed in $(E \times I) \cup F$, and therefore $\pi_f(A)$ is closed in C_f.

LEMMA 10.2. *If E and F are Tychonoff spaces, and $f: E \twoheadrightarrow F$ is proper, then C_f is Tychonoff.*

Proof. Let $\bar{f}: \beta E \to \beta F$ be the continuous extension of f, and write π for $\pi_{\bar{f}}$. Let $X = \pi((E \times I) \cup F)$. Since f is proper, $\bar{f}^{-1}(F) = E$ by Proposition 2.1, so $\pi^{-1}(X) = (E \times I) \cup F$. Hence $\pi | (E \times I) \cup F$ is a closed map—and hence a quotient map—from $(E \times I) \cup F$ onto X, so that the Tychonoff space X is homeomorphic to C_f. This completes the proof.

The requirement in Lemma 10.2 that f is proper seems to be essential. In fact, if E is the Tychonoff plank $(=(\{x < \Omega\} \times \{\alpha < \omega\}) - (\Omega, \omega))$ and F is a point, then $f: E \twoheadrightarrow F$ is continuous and closed, but C_f not a Tychonoff space.

PROPOSITION 10.3. *If E and F are Tychonoff spaces, $f: E \twoheadrightarrow F$ light and proper, and if $p = \pi_f | E \times I$, then $(E \times I, E \times \{0\}, p)$ is a (C_f, F)-cut.*

Proof. Since f is onto, F is thin in C_f. By Lemma 10.2, C_f is Tychonoff. By Lemma 10.1, p is closed. And $E \times \{0\}$ nowhere cuts $E \times I$ by Proposition 3.4. All other requirements are obviously satisfied, and that completes the proof.

LEMMA 10.4. *If $f: E \twoheadrightarrow F$ is a monotone quotient map, then F nowhere cuts C_f. (Hence, if C_f is Tychonoff, (C_f, F) is a (C_f, F)-cut.)*

Proof. Since f is a quotient map, so is $\pi_f | E \times I$. The lemma now follows from Proposition 3.11.

J. Isbell asked what happens when the requirements on f are relaxed, and conjectured the answers given below.

Suppose h: $E \to G$ and g: $G \twoheadrightarrow F$ are continuous, and $f = h \circ g$. Then we can define a continuous g^*: $C_h \twoheadrightarrow C_f$ as follows: If k: $(E \times I) \cup G \twoheadrightarrow (E \times I) \cup F$ is the map which is the identity on $E \times I$ and is g on G, then $g^* = \pi_f \circ k \circ \pi_h^{-1}$. This g^* is single valued, and is continuous because $g^* \circ \pi_h = \pi_f \circ k$, which is continuous. Note that, if g is closed, then so is k, and hence also g^*. Similarly, if g is proper or light, so is g^*.

PROPOSITION 10.5. *Let E and F be Tychonoff spaces, and f: $E \twoheadrightarrow F$ proper. Let $E \overset{h}{\twoheadrightarrow} G \overset{g}{\twoheadrightarrow} F$ be the monotone-light factorization of f (see Proposition 2.3). Then (C_h, G, g^*) is a (C_f, F)-cut.*

Proof. By Proposition 2.5, h and g are proper. By Lemma 10.2, C_h and C_f are both Tychonoff. Since g is proper and light, so is g^*. By Lemma 10.4, G nowhere cuts C_h. All other requirements are clearly satisfied, and that completes the proof.

THEOREM 10.6. *Let f: $E \to F$ be continuous, with $f(E)$ dense in F, and C_f Tychonoff. Then f has a factorization $E \overset{h}{\to} G \overset{g}{\twoheadrightarrow} F$, with $h(E)$ dense in G, C_h Tychonoff, and g light and proper, such that (C_h, G, g^*) is a (C_f, F)-cut.*

Proof. Let $\bar{\pi}$: $\beta(E \times I) \cup \beta F \to \beta C_f$ be the continuous extension of π_f: $(E \times I) \cup F \to C_f$. Since $\pi_f(E \times I)$ is dense in C_f, we have $\bar{\pi}(\beta(E \times I) = \beta C_f$. Let $R = \bar{\pi}^{-1}(C_f) \cap \beta(E \times I)$, and let $\hat{\pi} = \bar{\pi} | R$. Then $\hat{\pi}$: $R \twoheadrightarrow C_f$ is proper, and hence (Proposition 2.3) has a monotone-light factorization $R \overset{q}{\twoheadrightarrow} S \overset{p}{\twoheadrightarrow} C_f$, with p and q proper. Let $G = p^{-1}(F)$, and let us show that (S, G, p) is a (C_f, F)-cut. We know that p is light and proper. Now π_f maps $E \times (0,1]$ homeomorphically onto $C_f - F$, so (Proposition 2.1) $E \times (0,1] = \pi^{-1}(C_f - F)$, and hence $E \times (0,1] = q^{-1}(S - G)$; this implies that p maps $S - G$ homeomorphically onto $C_f - F$. It remains to show that G nowhere cuts S. Since q is closed and monotone, it suffices to check—by Proposition 3.11—that $q^{-1}(G)$ nowhere cuts R. But $(E \times I) - (E \times (0,1]) = E \times \{0\}$ nowhere cuts $E \times I$ (Proposition 3.4), and $R - (E \times I)$ nowhere cuts R (Example 3.10), so $q^{-1}(G) = R - (E \times (0,1])$ nowhere cuts R (Proposition 3.13).

Define h: $E \to G$ by $h(x) = q(x,0)$, and g: $G \twoheadrightarrow F$ by $g = p | G$. Then $g(h(x)) = p(q(x,0)) = \pi_f(x,0) = f(x)$ for all $x \in E$, so $f = g \circ h$. Since p is light and proper, so is g. Let us show that (C_h, G, g^*) is a (C_f, F)-cut. Define k: $C_h \twoheadrightarrow S$ by $k = p^{-1} \circ g^*$. Then k is single valued and continuous, because $k \circ \pi_h$ is the identity on G and coincides with q on $E \times I$. Also k is one-to-one and onto, and maps G identically onto G. Now $p\circ k = g^*$, which is proper because g is proper. Hence k is a homeomorphism by Proposition 2.2. Since (S, G, p) is a (C_f, F)-cut, this implies that (C_h, G, g^*) is also a (C_f, F)-cut.

11. Multicollared sets

According to M. Brown [5], a subset B of a space Y is *collared* in Y if there exists a homeomorphism h from $B \times [0,1)$ onto an open neighborhood of B in Y such that $h(b,0) = b$ for every $b \in B$. Collared sets are always nowhere cutting (Example 3.6); if, with the notation of Theorem 1.1, $\mathbf{A}$ is collared in $\mathbf{X}$, we say that A is *multicollared* in X.

Let us illustrate this concept by answering a rather special question: If X is an n-manifold (without boundary), and $A \subset X$ is multicollared, what can $\mathbf{A}$ look like? Since $\mathbf{A}$ is collared in $\mathbf{X}$, we have $\mathbf{A} \times (\frac{1}{2}, 1)$ homeomorphic to a subset V of $\mathbf{X} - \mathbf{A}$ which is open in $\mathbf{X}$. Then $p(V)$, which is homeomorphic to V, is open in X, so $\mathbf{A} \times E^1$ is an n-manifold. Now if $n = 2$ or 3, this implies [3] that $\mathbf{A}$ is an $(n-1)$-manifold (so that each component of $\mathbf{A}$ is a line or a circle if $n = 2$), and hence, if A is closed, $\mathbf{X}$ is an n-manifold with boundary $\mathbf{A}$. If $n \geqslant 4$, however, $\mathbf{A}$ need not be a manifold: In fact, for $n \geqslant 4$ there exist non-manifolds B such that $B \times E^1 = E^n$ [2]; if we let $X = B \times E^1$ and $A = B \times \{0\}$, then $\mathbf{A}$ is the disjoint union of two copies of B, and is thus not a manifold. (Whether such examples exist with X triangulable and A a subcomplex is unknown.)

We now return to the general theory. For multicollared sets the metrizability question is easily settled:

PROPOSITION 11.1. *If $A \subset X$ is multicollared, and if X is metrizable, so is* $\mathbf{X}$.

Proof. Let h be a homeomorphism from $\mathbf{A} \times [0,1)$ onto an open neighborhood U of $\mathbf{A}$ in $\mathbf{X}$. Then $\mathbf{A}$ is homeomorphic to $h(\mathbf{A} \times \{\frac{1}{2}\}) \subset (\mathbf{X} - \mathbf{A})$, so $\mathbf{A}$ is metrizable, and hence so is $U \supset \mathbf{A}$. Also $U = p^{-1}(p(U))$, and $p(U)$ is open in X and hence an F_σ in X, so U is an F_σ in $\mathbf{X}$. Thus the paracompact space $\mathbf{X}$, being the union of the open F_σ-set U and the dense set $\mathbf{X} - \mathbf{A}$, both of which are metrizable, is itself metrizable [6; Corollary 1.2]. That completes the proof.

We now characterize multicollared sets in terms of mapping cylinders (see Section 10). Since collared sets are defined in terms of $I' = [0,1)$—rather than $I = [0,1]$—to permit the use of *open* neighborhoods, we will find it convenient to use the "half-open" mapping cylinder C'_f, which is defined just like C_f except that I is replaced by I'. All results which were proved for C_f is section 10 are equally valid for C'_f.

Recall from section 1 (just before the statement of Theorem 1.1) that two maps $f_1\colon E_1 \twoheadrightarrow F$ and $f_2\colon E_2 \twoheadrightarrow F$ are called *equivalent*, and we write $f_1 \equiv f_2$, if there exists a homeomorphism $h\colon E_1 \twoheadrightarrow E_2$ such that $f_1 = f_2 \circ h$.

PROPOSITION 11.2. *The following properties of a thin subset A of a Tychonoff space X are equivalent:*

(a) *A is multicollared.*

(b) *For some proper map f: $\hat{A} \twoheadrightarrow A$, there exists a homeomorphism k from C_f' onto an open neighborhood U of A in X such that $k(a) = a$ for all $a \in A$.*

Moreover, if f is as in (b), *and if* $(\mathbf{X}, \mathbf{A}, p)$ *is an (X, A)-cut, then* $f \equiv p \,|\, \mathbf{A}$.

Proof. (a) ⇒ (b). Let $(\mathbf{X}, \mathbf{A}, p)$ be an (X, A)-cut. Let h be a homeomorphism from $A \times I'$ (where I' denotes $[0,1)$) onto an open neighborhood $\mathbf{U}$ of $\mathbf{A}$ in $\mathbf{X}$ such that $h(x,0) = x$ for every x in $\mathbf{A}$. Let $U = p(\mathbf{U})$; then $\mathbf{U} = p^{-1}(U)$, so $p \,|\, \mathbf{U}$: $\mathbf{U} \twoheadrightarrow U$ is closed, and hence so is $p \circ h$: $A \times I' \twoheadrightarrow U$. Now let $f = p \,|\, \mathbf{A}$, and let φ: $\mathbf{A} \times I' \twoheadrightarrow C_f'$ be the projection map. Since f is closed, so is φ (Lemma 10.1). Moreover, if x_1, $x_2 \in \mathbf{A} \times I'$, then $\varphi(x_1) = \varphi(x_2)$ if and only if $p \circ h(x_1) = p \circ h(x_2)$. We can therefore define a homeomorphism k: $C_f' \twoheadrightarrow U$ by $k = (p \circ h) \circ \varphi^{-1}$, and then $k(a) = a$ for every $a \in A$.

(b) ⇒ (a). Let φ: $\hat{A} \times I' \to C_f'$ be the projection map. Then, by Proposition 10.3, $(\hat{A} \times I', \hat{A} \times \{0\}, \varphi)$ is a (C_f', A)-cut, and hence $(\hat{A} \times I', \hat{A} \times \{0\}, k \circ \varphi)$ is a (U, A)-cut. Now if $(\mathbf{X}, \mathbf{A}, p)$ is an (X, A)-cut, then, by Lemma 3.14, $(p^{-1}(U), \mathbf{A}, p \,|\, p^{-1}(U))$ is also a (U, A)-cut. Since cuts are essentially unique, and $\hat{A} \times \{0\}$ is collared in $\hat{A} \times I'$, we must have $\mathbf{A}$ collared in $p^{-1}(U)$, and hence in $\mathbf{X}$.

Finally, under these circumstances we have

$$p \,|\, \mathbf{A} \equiv k \circ \varphi \,|\, (\hat{A} \times \{0\}) = \varphi \,|\, (\hat{A} \times \{0\}) \equiv f,$$

and that completes the proof.

Local properties of multicollared sets are treated in the next section.

An interesting example of a multicollared set was pointed out to me by J. Isbell. Let X be the n-fold cartesian product of the circle, and Y the n-fold symmetric product, obtained from X by identifying points whose coordinates differ only by a permutation. Let π: $X \twoheadrightarrow Y$ be the quotient map. Let A consist of those points of X whose coordinates are not all different. Then A is a multicollared subset of X. Moreover, Y and $\mathbf{X}$ are manifolds with boundaries $\pi(A)$ and $\mathbf{A}$, respectively, and $\pi \circ p$: $\mathbf{X} \twoheadrightarrow Y$ is an $n!$-fold covering of Y. (If $n = 2$, then X is a torus, Y a Möbius band, and $\mathbf{X}$ a cylinder.)

12. Local properties

M. Brown's principal theorem about collared sets [5; Theorem 1] asserts that *locally collared subsets of a metric space are collared*, where $B \supset Y$ is called *locally collared* if B can be covered by open subsets U of Y for which $U \cap A$ is collared in U (or, equivalently, in Y). If we similarly call $A \subset X$ *locally multicollared* if A can be covered by open subsets U of X such that $U \cap A$ is multicollared in U (or, equivalently by Lemma 3.14, in X), then we can prove the following consequence and analogue of Brown's theorem.

THEOREM 12.1. *A locally multicollared subset of a metric space is multicollared.*

Proof. Let X be metric, $A \subset X$ locally multicollared. Then A is covered by open subsets U of X for which $U \cap A$ is multicollared in U. Now if $(\mathbf{X}, \mathbf{A}, p)$ is an (X,A)-cut, then

$$(p^{-1}(U), \quad p^{-1}(U \cap A), \quad p|p^{-1}(U))$$

is a $(U, U \cap A)$-cut for each U by Lemma 3.14, and hence $p^{-1}(U \cap A)$ is collared in $p^{-1}(U)$. Hence $\mathbf{A}$ is locally collared in $\mathbf{X}$, and is therefore collared in $\mathbf{X}$ (by Brown's theorem) provided $\mathbf{A}$ has a metrizable neighborhood V in $\mathbf{X}$. Now each $p^{-1}(U \cap A)$ is collared in $\mathbf{X}$, and hence (just as in the proof of Proposition 11.1) has a metrizable neighborhood V_U in X. But then V, the union of all the V_U, is a neighborhood of $\mathbf{A}$ which is locally metrizable and paracompact (by [12; Theorem 2.2], since p is proper and $V = p^{-1}(p(V))$), and hence is metrizable by the Nagata-Smirnov theorem. That completes the proof.

Recall that a function $f\colon \hat{A} \twoheadrightarrow A$ is a *double covering* of A if A can be covered by open subsets V for which $f^{-1}(V)$ splits into two disjoint open subsets, both of which f maps homeomorphically onto V; if one can take $V = A$, then f is called the *trivial double covering.* If $(\mathbf{X}, \mathbf{A}, p)$ is an (X,A)-cut, we say that A *(trivially) double-cuts* X if $p|\mathbf{A}$ is a (trivial) double covering of A; similarly, A *locally (trivially) double cuts* X if A is covered by open subsets U of X for which $A \cap U$ (trivially) double cuts U. Note that, locally, double-cutting and trivially double-cutting is the same thing, but the equator of a Möbius band shows that globally they are distinct.

The following result is proved just like Theorem 12.1 (except that Brown's theorem and Proposition 11.1 are, of course, not needed in the proof).

PROPOSITION 12.2. *A locally double-cutting subset of a Tychonoff space is double-cutting.*

According to [4], $A \subset X$ is *bi-collared* in X if there exists a homeomorphism h from $A \times (-1,1)$ onto an open neighborhood of A in X such that $h(a,0) = a$ for every $a \in A$. More generally, call A *double-collared* in X if there exists a double covering $f\colon \hat{A} \twoheadrightarrow A$, and a homeomorphism h from C_f' (see the definition before Proposition 11.2) onto an open neighborhood of A in X, such that $h(a) = a$ for all $a \in A$. Now it follows from Proposition 11.2 that *A is double-collared (resp. bi-collared) in X if and only if A is multicollared and double-cutting (resp. trivially double-cutting) in X.* Define *locally bi-collared* and *locally* *double-collared* in the obvious way. As before, bi-collared and double-collared coincide locally, but the equator of a Möbius band shows that globally they are distinct.

The following result now follows immediately from Theorem 12.1 and Proposition 12.2.

COROLLARY 12.3. *A locally bi-collared (=locally double-collared) subset of a metric space is double-collared.*

13. Components

Let $(\mathbf{X},\mathbf{A},p)$ be an (X,A)-cut, and suppose that $X-A$ has only finitely many components $C_1,\dots,C_n$. If $D_i=p^{-1}(C_i)$, then $D_1,\dots,D_n$ are the components of $\mathbf{X}-\mathbf{A}$. Clearly $\mathbf{X}=\bar{D}_1\cup\dots\cup\bar{D}_n$, where $\bar{D}_i$ is the closure of D_i in $\mathbf{X}$. Moreover, since $\mathbf{A}$ nowhere cuts $\mathbf{X}$, and the D_i are all open in $\mathbf{X}-\mathbf{A}$, the sets $\bar{D}_1,\dots,\bar{D}_n$ are disjoint. Since they are connected, they are the components of $\mathbf{X}$.

Now let $\mathbf{A}_i=\bar{D}_i\cap\mathbf{A}$ for $i=1,\dots,n$. Then the $\mathbf{A}_i$ are disjoint, and $\mathbf{A}=\mathbf{A}_1\cup\dots\cup\mathbf{A}_n$. Hence the $\mathbf{A}_i$ are the components of $\mathbf{A}$, provided they are connected and non-empty. This need not always happen even when A is connected: For example, if X is a circle, and A consists of one point, then $\mathbf{X}$ is a closed interval, and $\mathbf{A}=\mathbf{A}_1$ consists of the two end points. Another example is obtained by taking $X=E^2$ and A an open line segment in X; here $\mathbf{A}=\mathbf{A}_1$ also consists of two disjoint copies of A. However, we can prove the following result, which was first obtained by J. Jaworowski for $X=E^n$ and multicollared A.

THEOREM 13.1. *Suppose that X is paracompact, locally connected, and unicoherent,* (1) *and that $A\neq\emptyset$ is closed and connected. Then each $\mathbf{A}_i$ is connected and non-empty, so that $\mathbf{A}$ and $X-A$ have the same number of components.*

Proof. If some $\mathbf{A}_i$ were empty, then D_i would be clopen in $\mathbf{X}$. But then C_i is a clopen subset of X which is neither empty nor X, which contradicts the connectedness of X.

Let us show that $\mathbf{A}_i$ is connected. Suppose not. Then $\mathbf{A}_i=E\cup F$, where E and F are non-empty, disjoint, and closed. Since X is paracompact, and p proper, $\mathbf{X}$ is also paracompact [12; Theorem 1.2] and thus normal. Hence there exist disjoint open subsets U and V of $\mathbf{X}$ containing E and F, respectively. Let $W=U\cup V$.

Before continuing, let us show that $X-C_i$ is connected. In fact,

$$X-C_i=A\cup(\textstyle\bigcup_{j\neq i}\bar{C}_j),$$

where A and each $\bar{C}_j$ is connected, and $A\cap\bar{C}_j$ is not empty (otherwise $C_j=\bar{C}_j$ would be a non-trivial clopen subset of X). This implies that $X-C_i$ is connected.

(1) X is *unicoherent* [18] if it is connected and, whenever $X=A\cup B$ with A and B both closed and connected, then $A\cap B$ is connected. Examples include all connected, locally connected, simply connected Hausdorff spaces (in particular E^n for all n, and S^n for $n>1$), as well as projective n-space P^n for $n>1$.

Now $D_i - W = \bar{D}_i - W$ is closed in $\mathbf{X}$, so $p(D_i - W)$ is closed in X. Hence its complement $R = (X - C_i) \cup p(W)$ is open. Since $X - C_i$ is connected and R is locally connected, $X - C_i$ is contained in a connected, open subset S of R. Now $S \cup C_i = X$, X is unicoherent, and S and C_i are connected and open, so $S \cap C_i$ is connected by [18; Theorem 3]. Also $p^{-1}(S)$ is open in $\mathbf{X}$ and contains A_i, so $p^{-1}(S) \cap D_i$ intersects both U and V. Since it is also contained in $W = U \cup V$, it is disconnected. But $p^{-1}(S) \cap D_i = p^{-1}(S \cap C_i)$, which is homeomorphic to $S \cap C_i$, yielding a contradiction.

14. The proper completion of a spread

Let D and X be Tychonoff spaces, and $f: D \to X$ a spread (see section 2). Then an extension $p: \mathbf{X} \to X$ of f is a *proper completion* of f if $\mathbf{X}$ is Tychonoff, p is light and proper (hence a spread by Proposition 2.4), and $\mathbf{X} - D$ nowhere cuts $\mathbf{X}$.(1) The following theorem generalizes Theorem 1.1, and reduces to it in case f is a homeomorphism from D onto a dense subset of X.

THEOREM 14.1. *If D and X are Tychonoff spaces, every spread $f: D \to X$ has an—essentially unique*(2)*—proper completion $p: \mathbf{X} \to X$.*

Proof. Let us prove existence; uniqueness will follow from Theorem 14.2 below.

Let $\bar{f}: \beta D \to \beta X$ be the unique continuous extension of f, let $E = \bar{f}^{-1}(X)$, and let $g = \bar{f} | E$. Then $g: E \to X$ is proper, and hence (Proposition 2.3) has a monotone-light factorization $E \xrightarrow{q} \mathbf{X} \xrightarrow{p} X$, with p and q both proper and $\mathbf{X}$ Tychonoff. Let us show that q maps D homeomorphically into $\mathbf{X}$ (this follows easily from Proposition 2.1 in the important special case where f, regarded as a map from D onto $f(D)$, is proper).

To prove that q is one-to-one on D, we must show that two points $d_1 \neq d_2$ of D cannot lie in a connected subset of some $g^{-1}(x)$. In fact, if $d_1, d_2 \in (D \cap g^{-1}(x)) = f^{-1}(x)$, then some basic neighborhood U of d_1 misses d_2, where U is clopen in $f^{-1}(W)$ for some open neighborhood W of x. Let $V = g^{-1}(W)$. Then $V \cap D = f^{-1}(W)$, and since $E - D$ nowhere cuts E (Example 3.10), $\bar{U}$ (the closure of U in V) is clopen in V (Lemma 3.1). Since $g^{-1}(x) \subset V$, we see that d_1 and d_2 are separated by the clopen subset $\bar{U} \cap g^{-1}(x)$ of $g^{-1}(x)$, and hence do not lie in a connected subset of $g^{-1}(x)$.

To prove that $q | D$ is a homeomorphism, it remains to show that $q(U)$ is open in $q(D)$ for every open $U \subset D$, and it suffices to consider basic sets U, which are clopen in $f^{-1}(W)$ for some open W in X. If $V = g^{-1}(W)$, then, as we saw above, $\bar{U}$ (the closure of

(1) By Propositions 2.4 and 2.5, $\mathbf{X}$ is automatically Tychonoff if it is Hausdorff.

(2) In the sense that any two such completions $p: \mathbf{X} \to X$ and $p_1: \mathbf{X}_1 \to X$ are *equivalent* (i.e. there exists a homeomorphism $k: \mathbf{X} \rightarrowtail \mathbf{X}_1$, which keeps D pointwise fixed, such that $p = p_1 \circ k$).

U in V) is clopen in V. Now V is an inverse set under g, and hence also under q, so that U is also an inverse set under q (since q is monotone). Hence $q(\overline{U})$ is open in $\mathbf{X}$, and

$$q(U)=q(\overline{U}\cap D)=q(\overline{U})\cap q(D),$$

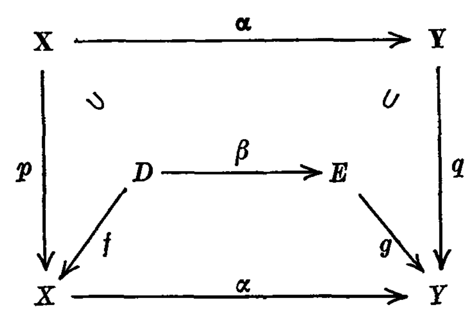
so that $q(U)$ is open in $q(D)$.

As observed above, $E-D$ nowhere cuts E (Example 3.10). Since $q|D$ is a homeomorphism, $q^{-1}(\mathbf{X}-q(D))=\mathbf{X}-D$ by Proposition 2.1; since q is monotone, this implies (Proposition 3.11) that $\mathbf{X}-q(D)$ nowhere cuts $\mathbf{X}$.

To complete the proof of existence, we now identify D with $q(D)$, so that $p|q(D)$ becomes identified with f.

To establish essential uniqueness in Theorem 14.1, we now prove the following analogue of Theorem 5.3.

THEOREM 14.2. *Let* p: $\mathbf{X}\to X$ *and* q: $\mathbf{Y}\to Y$ *be proper completions of spreads* f: $D\to X$ *and* g: $E\to Y$, *respectively. Let* α: $X\to Y$ *be continuous. Then any continuous* β: $D\to E$ *such that* $g\circ\beta=\alpha\circ f$ *can be extended to a continuous* $\boldsymbol{\alpha}$: $\mathbf{X}\to\mathbf{Y}$ *such that* $q\circ\boldsymbol{\alpha}=\alpha\circ p$.



If α *and* β *are homeomorphisms onto, so is* $\boldsymbol{\alpha}$.

Proof. Letting $\pi=\alpha\circ p$ and $\mathbf{A}=\mathbf{X}-D$, the first part of the theorem follows from Corollary 5.2 (with g replaced by β, and $\mathfrak{f}$ by $\boldsymbol{\alpha}$).

If α and β are homeomorphisms onto, then we can similarly extend β^{-1} to a continuous $\boldsymbol{\alpha}'$: $\mathbf{Y}\to\mathbf{X}$ such that $p\circ\boldsymbol{\alpha}'=\alpha^{-1}\circ q$. But $\boldsymbol{\alpha}'\circ\boldsymbol{\alpha}$ is then the identity on the dense subset D of $\mathbf{X}$, and hence on all of $\mathbf{X}$. Similarly, $\boldsymbol{\alpha}\circ\boldsymbol{\alpha}'$ is the identity on $\mathbf{Y}$. Hence $\boldsymbol{\alpha}$ and $\boldsymbol{\alpha}'$ are inverse homeomorphisms onto, and the proof is complete.

The following result is now a consequence of Propositions 2.4 and 2.5, Theorem 14.1, and the fact that a restriction of a spread is spread.

COROLLARY 14.3. *If* F *is a Tychonoff space, then a map* f: $E\to F$ *is a spread if and only if* f *can be extended to a light proper map* p: $G\to F$ *for some Tychonoff space* $G\supset E$.

15. The Fox completion of a spread

In [7], R. H. Fox presented a method of completing spreads which is, in general, different from the one considered in the previous section (and leads to a different concept of cutting). We shall refer to it as the *Fox completion*, to distinguish it from the proper completion considered in section 14. We need two preliminary concepts.

Fox calls a spread p: $E \to F$ *complete* if it satisfies the following condition: If $y \in F$, and if to every open neighborhood W of y one assigns a quasi-component (see section 6) Q_W of $p^{-1}(W)$ so that $Q_{W_1} \subset Q_{W_2}$ whenever $W_1 \subset W_2$, then $\bigcap_W Q_W$ is non-empty (and hence consists of one point).

If $A \subset X$ is thin, we say that A *nowhere separates* X if, whenever U is a neighborhood of $x \in A$, and $\mathfrak{V}$ is a clopen covering of $U - A$, then x has a neighborhood W such that $W - A$ is covered by finitely many $V \in \mathfrak{V}$. If a particular $x \in A$ has this property, we say that A *does not separate* X *at* x. (Clearly every nowhere separating set is nowhere cutting and nowhere scattering, but the converse is, in general, false (see section 16).)

We now call p: $\mathbf{X} \to X$ a *Fox completion* of a spread f: $D \to X$ if p is a complete spread which extends f, and if $\mathbf{X} - D$ nowhere separates $\mathbf{X}$. This is the definition given in [15], where it is shown that it generalizes the definition given by Fox in [7] for locally connected D. The following result was proved in [7] for locally connected D, and in [15] for arbitrary D.

THEOREM 15.1. *Every spread has an—essentially unique* (1)*—Fox completion.*

To claryfy the relation between Fox completions and proper completions, we need

PROPOSITION 15.2. *If E is Hausdorff, every light, proper map p: $E \to F$ is a complete spread.*

Proof. That p is a spread was asserted in Proposition 2.4. To show that p is complete, let $y \in Y$ and the Q_W be as in the definition of complete spread. Let us show that $Q_W \cap p^{-1}(y) \neq \varnothing$ for every W. If not, the fact that f is closed implies that there exists an open neighborhood $V \subset W$ of y such that $p^{-1}(V)$ misses Q_W. But then $Q_V \subset Q_W \cap p^{-1}(V) = \varnothing$ which is impossible.

Since every $Q_W \cap p^{-1}(y)$ is non-empty, the collection of all such sets has the finite intersection property, and hence (since $p^{-1}(y)$ is compact) has a non-empty intersection. That completes the proof.

(1) See footnote (2) on page 29.

The converse of Proposition 15.2 is false, as is shown by mapping an infinite discrete space into a point.

From Proposition 15.2, and the essential uniqueness of both proper and Fox completions, we now obtain the following corollary.

COROLLARY 15.3. *Let* $f: D \to X$ *be a spread and let* $p: \mathbf{X} \to X$ *and* $p_F: \mathbf{X}_F \to X$ *be proper and Fox completions of* f*, respectively. Then the following are equivalent:*

(a) p *and* p_F *are equivalent (i.e. proper and Fox completions of* f *coincide).*
(b) p_F *is a proper map.*
(c) $\mathbf{X} - D$ *nowhere separates* X.

In general, Fox and proper completions of a spread are different, as examples in the text will show. They are, however, always related by the following result.

THEOREM 15.4. *If* $p: \mathbf{X} \to X$ *is a proper completion of a spread* $f: D \to X$*, and if*

$$Y = D \cup \{x \in \mathbf{X} - D \mid \mathbf{X} - D \quad \text{does not separate} \quad \mathbf{X} \text{ at } x\},$$

then $g = p \mid Y$ *is a Fox completion of* f.

Proof. Clearly $Y - D$ nowhere separates Y. Since p is a spread (Proposition 2.4), so is g. It remains to show that g is complete.

Let $x \in X$, and for each open neighborhood W of x let Q_W be a quasi-component of $g^{-1}(W)$ so that $W_1 \subset W_2$ implies $Q_{W_1} \subset Q_{W_2}$. We must find a $y \in \bigcap_W Q_W$.

For each W, let Q'_W be the quasi-component of $p^{-1}(W)$ which contains Q_W. Then $Q'_{W_1} \subset Q'_{W_2}$ whenever $W_1 \subset W_2$, so (since p is a complete spread by Proposition 15.1) there is a $y \in \bigcap_W Q'_W$. Clarly $y \in p^{-1}(x)$.

Let us show that $y \in Y$. This is clear if $y \in D$, so suppose $y \notin D$. Let N be an open neighborhood of y in $\mathbf{X}$, and $\mathcal{V}$ a clopen covering of $N \cap D$. Pick an open neighborhood W of x and a clopen subset U of $p^{-1}(W)$ so that $y \in U \subset N$. Let $\mathcal{R} = \{V \cap U \mid V \in \mathcal{V}\}$; then $\mathcal{R}$ is a clopen covering of $U \cap D$. Now $Y - D$ nowhere separates Y, so $Q_W \cap D \neq \emptyset$ by [15; Lemma 4.1], and hence $Q'_W \cap D \neq \emptyset$. Since $y \in Q'_W \cap U$, we have $Q'_W \subset U$, and thus $Q'_W \cap R_0 \neq \emptyset$ for some $R_0 \in \mathcal{R}$. Now $\bar{R}_0 \cap U$ is clopen in U because $\mathbf{X} - D$ nowhere cuts $\mathbf{X}$ (Proposition 3.1), so that $Q'_W \subset \bar{R}_0 \cap U$. Hence $\bar{R}_0 \cap U$ is a neighborhood of y in $\mathbf{Z}$, and $(\bar{R}_0 \cap U) \cap D = R_0 \subset V_0$ for some $V_0 \in \mathcal{V}$. Hence $\mathbf{X} - D$ does not separate $\mathbf{X}$ at y, and therefore $y \in Y$.

Thus $y \in Y \cap Q'_W = g^{-1}(W) \cap Q'_W$, so it remains to show that $g^{-1}(W) \cap Q'_W \subset Q_W$. Since the opposite inclusion follows from the definition of Q'_W, it suffices to show that no clopen subset U of $g^{-1}(W)$ disconnects $g^{-1}(W) \cap Q'_W$. Suppose it did. Since $Y - D$ nowhere cuts Y, we would have $\bar{U} \cap p^{-1}(W)$ clopen in $p^{-1}(W)$ (by Proposition 3.1) and disconnecting Q'_W, which is impossible. That completes the proof.

16. Fox cuts and proper cuts

If A is a thin subset of a T_1-space, then applying Theorem 15.2 and Proposition 2.1 to the injection map $f: X - A \to X$ yields the following analogue of Theorem 1.1.

THEOREM 16.1. *Let A be a thin subset of a T_1-space X. Then there exist—essentially uniquely—a T_1-space $\mathbf{X}_F$ with nowhere separating subset $\mathbf{A}_F$, and a spread p_F: $\mathbf{X}_F \to X$ which maps $\mathbf{X}_F - \mathbf{A}_F$ homeomorphically onto $X - A$ and maps $\mathbf{A}_F$ into A.*

Let us call the triple $(\mathbf{X}_F, \mathbf{A}_F, p_F)$ described in Theorem 16.1 a *Fox (X,A)-cut*. If X is a Tychonoff space, then the triple $(\mathbf{X}, \mathbf{A}, p)$ which has hitherto simply been called an (X,A)-cut will now, for distinction, be called a *proper (X,A)-cut*. Applying Theorem 15.4 to the present special situation, we see that $\mathbf{X}_F$ can always be obtained as a certain subset of $\mathbf{X}$, with $p_F = p \mid \mathbf{X}_F$. In general p_F (unlike p) need not be onto X, as is shown by Example (4) of the introduction, where $\mathbf{A}_F$ is empty. (A similar example can be constructed with both X and $X - A$ locally connected).

It follows from Corollary 15.3 that, if $(\mathbf{X}, \mathbf{A}, p)$ is a proper (X,A)-cut, then it is a Fox (X,A)-cut if, and only if, $\mathbf{A}$ nowhere separates $\mathbf{X}$. To prepare for some further (and more useful) criteria in this direction, we now introduce the following concept, which is formally stronger than nowhere scattering: If $A \subset X$, then A *nowhere shatters* X if, whenever U is a neighborhood of $x \in A$ and $\mathcal{V}$ is a clopen covering of $U - A$, then x has a neighborhood W such that $W - A$ is covered by finitely many $V \in \mathcal{V}$. Clearly A nowhere separates X if and only if A nowhere cuts and nowhere shatters X.

That nowhere shatters is actually strictly stronger than nowhere scatters is seen by letting X be the space of ordinals $\leqslant \Omega$ (the first uncountable ordinal), and letting $A = \{\Omega\}$; it is not hard to check that here A nowhere scatters (and nowhere cuts) X, but A does not nowhere shatter X. I don't know whether the concepts coincide in metrizable spaces, or whether they are interchangeable in Proposition 6.2. We do, however, have the following result.

LEMMA 16.2. *"A nowhere scatters X" and "A nowhere shatters X" are equivalent under either of the following circumstances.*

(a) *X is separable metric.*

(b) *$X - A$ is locally connected.*

Proof. We need only show that, in either case, if A nowhere scatters X then A nowhere shatters X. So let U be an open neighborhood of $x \in A$, and $\mathcal{V}$ a clopen covering of $U - A$.

(a) Let $\{V_n\}_{n=1}^{\infty}$ be a countable subcovering of $\mathcal{V}$, and let $U_n = V_n - \bigcup_{i=1}^{n-1} V_i$. Then $\{U_n\}_{n=1}^{\infty}$ is a disjoint open covering of $U - A$, so x has a neighborhood W such that $W - A$ is covered by finitely many U_n, and hence by finitely many $V \in \mathcal{V}$.

(b) If $\mathcal{C}$ is the collection of components of $U - A$, then $\mathcal{C}$ is a disjoint, open refinement of $\mathcal{V}$. But then x has a neighborhood W such that $W - A$ is covered by finitely many $C \in \mathcal{C}$, and hence by finitely many $V \in \mathcal{V}$. That completes the proof.

Another useful fact about nowhere shattering sets is the following result, which is analogous to—and has the same proof as—Proposition 7.1.

LEMMA 16.3. *If* $(\mathbf{X}, \mathbf{A}, p)$ *is an* (X, A)*-cut, then* $\mathbf{A}$ *nowhere shatters* $\mathbf{X}$ *if, and only if,* A *nowhere shatters* X.

We now prove

THEOREM 16.4. *Let* $(\mathbf{X}, \mathbf{A}, p)$ *and* $(\mathbf{X}_F, \mathbf{A}_F, p_F)$ *be proper and Fox* (X, A)*-cuts, respectively. Then* (a), (b) *and* (c) *are always equivalent, and they are equivalent to* (d) *if* X *is separable metric.*

(a) p *is equivalent to* p_F (i.e. *proper and Fox* (X, A)*-cuts coincide*).

(b) p_F *is a proper map.*

(c) A *nowhere shatters* X.

(d) $\mathbf{X}$ *is separable metric.*

Proof. The equivalence of (a) and (b) is a special case of Corollary 15.3. By that corollary, they are also equivalent to $\mathbf{A}$ nowhere separating $\mathbf{X}$; since $\mathbf{A}$ surely nowhere cuts $\mathbf{X}$, this is equivalent to $\mathbf{A}$ nowhere shattering $\mathbf{X}$, which is equivalent to (c) by Lemma 16.3. If X is separable metric, finally, (c) is equivalent to (d) by Lemma 16.2 (a) and Theorem 1.2. That completes the proof.

It follows from Proposition 15.3 that Fox and proper (X, A)-cuts surely coincide if X has a base consisting of sets U such that $X - U$ has only finitely many components; in particular, this occurs if A is a subcomplex of a locally finite simplicial complex X. An example where these cuts coincide, but where no such base can be found, is given by Example (3) of the introduction.

In conclusion, consider a metrizable space X with thin subset A. Although there are many situations where $\mathbf{X}_F$ is metrizable while $\mathbf{X}$ is not (e.g. Example (4) of the introduction where $\mathbf{A}_F$ is empty), an example in [15; section 5] shows that in general $\mathbf{X}_F$ need not be metrizable. If $X - A$ is locally connected, however, Fox [7; p. 246, Lemma] showed that $\mathbf{X}_F$ must have a countable base (and hence be metrizable) whenever X does. We now prove

PROPOSITION 16.5. *If A is a thin subset of a metrizable space, and if $X - A$ is locally connected, then $\mathbf{X}_F$ is metrizable. In fact, given a metric on X one can explicitly construct the metric space $\mathbf{X}_F$ by the method of section 8.*

Proof. It suffices to observe that Theorem 8.1 (as well as the remark at the end of section 8) remains true without assuming that A nowhere scatters X, provided "(X,A)-cut" is replaced by "Fox (X,A)-cut". In fact, without assuming that A nowhere scatters X, part (g) of the proof of Theorem 8.1 explicitly proves that p is a spread, and it is easy to chek that this spread is complete. Since $\mathbf{X} - \mathbf{A}$ is uniformly locally connected, $\mathbf{A}$ nowhere separates $\mathbf{X}$, and hence $(\mathbf{X}, \mathbf{A}, p)$ satisfies all the conditions for a Fox (X,A)-cut. That completes the proof.

References

[1]. ALEKSANDROV, P. S., Some results in the theory of topological spaces obtained within the last twenty-five years. *Uspehi Mat. Nauk*, 15 (1960), 25–95; = *Russian Math. Surveys*, 15 (1960), vol. 2, 23–83.

[2]. BING, R. H., The cartesian product of a certain non-manifold and a line is E^4. *Bull. Amer. Math. Soc.*, 64 (1958), 82–84.

[3]. BORSUK, K., On the decomposition of manifolds into curves and surfaces. *Fund. Math.*, 33 (1945), 273–298.

[4]. BOURBAKI, N., *Topologie Générale*. Chapters I and II, Third Edition, Hermann, Paris, 1961.

[5]. BROWN, M., Locally flat embeddings of topological manifolds. *Ann. of Math.*, 75 (1962), 331–341.

[6]. CORSON, H. H. & MICHAEL, E., Metrizability of countable unions. To appear in *Illinois J. Math.*

[7]. FOX, R. H., Covering spaces with singularities. *Algebraic geometry and topology*. A symposium in honor of S. Lefschetz, 243–257. Princeton University Press, Princeton, N. J., 1957.

[8]. FREUDENTHAL, H., Neuaufbau der Endentheorie. *Ann. of Math.*, 43 (1942), 261–279.

[9]. —— Kompaktisierung und Bikompaktisierung. *Indag. Math.*, 14 (1951), 184–192.

[10]. GILLMAN, L. & JERISON, M., *Rings of Continuous Functions*. Van Nostrand, New York, 1960.

[11]. DE GROOT, J., *Topologische Studien, Compactificatie, Voortzetting van Afbeeldingen en Samenhang*. Thesis, University of Groningen, 1942.

[12]. HENRIKSEN, M. & ISBELL, J. R., Some properties of compactifications. *Duke Math. J.*, 25 (1958), 83–106.

[13]. —— Local connectedness in the Stone-Čech compactification. *Illinois J. Math.*, 1 (1957), 574–582.

[14]. HUREWICZ, W. & WALLMAN, H., *Dimension Theory*. Princeton University Press, 1948.

[15]. MICHAEL, E., Completing a spread (in the sense of R. H. Fox) without local connectedness. *Indag. Math.*, 25 (1963), 629–633.

[16]. PONOMAREV, V. I., On continuous decomposition of bicompacts. *Uspehi Mat. Nauk*, 12 (1957), 335–340.

[17]. SKLYARENKO, E. G., On perfect bicompact extensions. *Dokl. Akad. Nauk SSSR*, 137 (1961), 39–41; = *Soviet Math. Dokl.*, 2 (1961), 238–240.

[18]. Stone, A. H., Incidence relations in unicoherent spaces. *Trans. Amer. Math. Soc.*, 65 (1949), 427–447.
[19]. —— Metrizability of decomposition spaces. *Proc. Amer. Math. Soc.*, 7 (1956), 361–366.
[20]. —— *General Topology*. To appear.
[21]. Vainstein, E. A., On closed mappings. *Moskov. Gos. Univ. Uč. Zap.*, 155, Mat. 5 (1952), 3–53.
[22]. Whyburn, G. T., A certain transformation on metric spaces. *Amer. J. Math.*, 54 (1932), 367–376.
[23]. —— Open and closed mappings. *Duke Math. J.*, 17 (1950), 69–74.
[24]. —— Open mappings on locally compact spaces. *Mem. Amer. Math. Soc.*, 1 (1950).

Received June 4, 1963.