On the propagation of singularities for pseudo-differential operators of principal type

Nils Dencker

1. Introduction

Let P be a properly supported pseudo-differential operator of order m on a C^∞ manifold X. We shall assume that the symbol of P is a sum of terms homogeneous of degree $m, m-1, \ldots$ and we denote the principal symbol by p.

Definition 1.1. P is said to satisfy condition (P) if there is no C^∞ complex valued function q in $T^*X\setminus 0$ such that $\operatorname{Im} qp$ takes both positive and negative values on a bicharacteristic of $\operatorname{Re} qp$ where $q\neq 0$.

By a bicharacteristic of $\operatorname{Re} qp$ we mean an integral curve of the Hamilton field $\operatorname{Re} H_{qp}$ on which $\operatorname{Re} qp$ vanishes. (Some authors call this a null-bicharacteristic.) We say that P is of principal type if $dp\neq 0$ when $p=0$. For operators of principal type satisfying condition (P) and with no bicharacteristics trapped over a point, Nirenberg and Treves [5] proved local solvability when the principal symbol is analytic. Beals and Fefferman [1] extended their result to the C^∞ case. Hörmander [3] proved semi-global solvability by studying the propagation of singularities for the adjoint operator. In this paper we shall study the case which was left open in [3].

Definition 1.2. We denote by $\mathscr{C}_3$ the set of $(x, \xi)\in T^*X\setminus 0$ such that $p(x, \xi)=0$ and $\operatorname{Im} qp$ vanishes of third order at (x, ξ) for some $q\in C^\infty(T^*X\setminus 0)$ such that $q(x, \xi)\neq 0$.

Observe that $\mathscr{C}_3$ contains the set $\mathscr{C}_{13}$ defined by Hörmander [3], for which there are also global conditions. The definition implies that a bicharacteristic γ of, say, $\operatorname{Re} p$ is a one dimensional bicharacteristic of p as long as it remains in $\mathscr{C}_3$, that is, $p=0$ on γ and $H_p\neq 0$ is proportional to the tangent vector.

When studying the singularities we shall use the Sobolev spaces $H_{(s)}$ of distributions which are mapped into L^2 by any pseudo-differential operator of order s.

When u is a distribution we define the regularity function $s_u^*(x, \xi)$ for $(x, \xi)\in T^*X\setminus 0$ as the supremum of all real s such that $u\in H_{(s)}$ at (x, ξ), that is, $u=u_1+u_2$ where $u_1\in H_{(s)}$ and $(x, \xi)\notin WF(u_2)$.

Arbitrarily close to a one dimensional bicharacteristic in $\mathscr{C}_3$ there may exist bicharacteristics on which $d\operatorname{Re} p$ and $d\operatorname{Im} p$ are linearly independent. There we know that s_u^* is a superharmonic function with respect to a natural analytic structure if $Pu\in C^\infty(X)$. When approaching the one dimensional bicharacteristic the superharmonicity degenerates to the minimum principle with respect to constant functions, and we are led to the following theorem.

Theorem 1.3. *Let P be a properly supported pseudo-differential operator of order m on a C^∞ manifold X, satisfying condition* (P). *Let J be a compact interval on a one dimensional bicharacteristic in $\mathscr{C}_3$. If $u\in\mathscr{D}'(X)$, and s is a real constant such that $s\leqq s_{Pu}^*+m-1$ on J and $s\leqq s_u^*$ at ∂J, then $s\leqq s_u^*$ on J.*

Thus if $s\leqq s_{Pu}^*$ on J then $\min(s_u^*, s+m-1)$ satisfies the minimum principle with respect to constant functions on J, that is, either it is monotonic or else it rises monotonically to a maximum value and falls monotonically afterwards. Note that Theorem 1.3 gives additional information on s_u^* even in the cases of Theorems 6.1 and 6.6 in [3], which on embedded one dimensional bicharacteristics only give information on the infimum of s_u^*.

In the proof of Theorem 1.3 we shall use the Weyl calculus developed by Hörmander [4]. For definitions, notations and calculus results we refer the reader to [4]. The plan of the paper is as follows. In Section 2 we reduce the proof of Theorem 1.3 to the *a priori* estimates of Proposition 2.7. We define a metric in Section 3 which is a modification of the one used by Beals and Fefferman [1] to prove the local solvability of operators satisfying condition (P). In Section 4 we state and prove the *a priori* estimates we shall use in the proof of Proposition 2.7. When localizing these estimates we must have operators which approximately commute with P. In order to construct such operators we have to find uniformly bounded solutions to the Hamilton equations. This is done in Section 5 and the results are used in Section 6 to construct solutions with special properties. Finally we prove Proposition 2.7 in Section 7, thus finishing the proof of Theorem 1.3.

I would like to thank my teacher Professor Lars Hörmander who suggested this problem to me and whose constant encouragement and advice have been of invaluable help.

2. Reduction to a priori estimates

In this section we shall reduce the proof of Theorem 1.3 to certain *a priori* estimates (Proposition 2.7). For simplicity we do this in several steps, where we microlocalize and prepare the symbol of the operator.

By multiplying with an elliptic pseudo-differential operator of order $1-m$ we may assume that $m=1$. The symbol of P is then an asymptotic sum of homogeneous terms

$$p(x, \xi)+p_0(x, \xi)+p_{-1}(x, \xi)+\dots$$

where the principal symbol p is homogeneous of degree 1 and p_j is homogeneous of degree j in the ξ variables.

To prove Theorem 1.3 it suffices to show that if J is a compact interval on a one dimensional bicharacteristic in $\mathscr{C}_3$, and if $s<s_u^*+1/15$, $s<s_{Pu}^*$ on J and $s<s_u^*$ on ∂J, then $s\leqq s_u^*$ on J. In fact, since $s_u^*>-N$ on J for some N, we obtain Theorem 1.3 by iterating this result with $s<k/15-N$. Since conjugation by an elliptic operator of order s does not change the principal symbol, it suffices to prove the case $s=0$. Thus, Theorem 1.3 will follow if we prove the following

Proposition 2.1. *Assume that P is of order* 1 *and satisfies the condition* (P). *Assume that J is a compact interval on a one dimensional bicharacteristic in $\mathscr{C}_3$, and u is a distribution such that for some $\varepsilon>0$*

$$u\in H_{(-1/15)} \quad \text{and} \quad Pu\in H_{(\varepsilon)} \quad \text{on } J,$$
$$u\in H_{(\varepsilon)} \quad \text{at } \partial J.$$

Then it follows that $u\in H_{(0)}$ on J.

We shall now prepare the operator so that microlocally it becomes a differential operator in the x_0 variable and a pseudo-differential operator in the x' variables depending on the parameter x_0, $(x_0, x')\in\mathbf{R}^{n+1}$. We shall use the symbol classes $S(h_0^s, g_0)$ in $T^*\mathbf{R}^n$, where g_0 is the metric

$$g_0(t, \tau) = |t|^2+|\tau|^2/(1+|\xi'|^2) \quad \text{at} \quad (x', \xi')$$

and

$$h_0^2 = \sup_{t,\tau} g_0(t, \tau)/g_0^\sigma(t, \tau) = (1+|\xi'|^2)^{-1} \quad \text{at} \quad (x', \xi').$$

Assume that $\mathbf{R}\supseteqq I\ni t\to\gamma(t)\in J$ is a compact interval on a one dimensional bicharacteristic in $\mathscr{C}_3$, which does not have the radial direction. (Proposition 2.1 is empty if the direction is radial.) Then Proposition 2.5 in [3] gives that we can extend γ to a homogeneous canonical transformation χ from a neighborhood of $I\times(0, \bar\xi)\subseteqq T^*\mathbf{R}^{n+1}$, $\bar\xi=(0, \dots, 0, 1)$, such that for some q homogeneous of degree 0, the pullback $\chi^*(qp)$ is of the form $\xi_0+if(x, \xi')$ in a conical neighborhood of $I\times(0, \bar\xi)$.

Here $f\in C^\infty$ when $\xi'\neq 0$, and f is homogeneous of degree 1 in the ξ' variables. The assumptions imply that f does not change sign for fixed (x', ξ') and that f vanishes of degree 3 on $I\times(0, \bar{\xi})$. If we conjugate with a Fourier integral operator of order 0 corresponding to χ, and multiply by a suitable elliptic pseudo-differential operator, we can get the symbol equal to $\xi_0+if(x, \xi')+q_0(x, \xi)$ in a conical neighborhood of $I\times(0, \bar{\xi})$ apart from terms homogeneous of degree -1 and lower. Here q_0 is homogeneous of degree 0, and we may now assume that q_0 is independent of ξ_0 and vanishes on $I\times(0, \bar{\xi})$. In fact, by Malgrange's preparation theorem we can find e and r homogeneous of degree -1 and 0 respectively such that

$$q_0(x, \xi) = e(x, \xi)(\xi_0+if(x, \xi'))+r(x, \xi')$$

in a conical neighborhood of $I\times(0, \bar{\xi})$. Indeed, when $\xi_n=1$ we can do so locally, hence we get this decomposition in a neighborhood of $I\times(0, \bar{\xi})$ by a partition of unity, and may then extend it by homogeneity. If we multiply by a pseudo-differential operator with symbol $1-e(x, \xi)$ the term of degree 0 in the symbol becomes r. To show that we may assume that $r(x, \xi')$ vanishes on $I\times(0, \bar{\xi})$, we take $a(x, \xi')$ elliptic and of degree 0 in the ξ' variables, and conjugate the operator by a^w. We then get the symbol

$$\xi_0+if(x, \xi')+i(H_p a)(x, \xi')/a(x, \xi')+r(x, \xi')$$

apart from terms of degree -1 and lower. Here H_p is the Hamilton field of $p(x, \xi)=\xi_0+if(x, \xi')$. The term of order 0 in this symbol is equal to 0 on $I\times(0, \bar{\xi})$ if

$$i\partial_{x_0} a(x, \xi')+r(x, \xi')a(x, \xi') = 0$$

since $H_p=\partial_{x_0}$ then. This equation is satisfied by

$$a(x, \xi') = \exp\left(i\int_c^{x_0} r(t, x', \xi')\, dt\right)$$

which is elliptic of order 0 and defined in a conical neighborhood of $I\times(0, \bar{\xi})$.

It is clear that in the same way we may successively make the lower order terms independent of ξ_0 and vanishing on J. However, since it suffices to prove Proposition 2.1 when $\varepsilon\leqq 14/15$, we may ignore these terms. Thus we obtain the following

Lemma 2.2. *In the proof of Proposition* 2.1 *we may assume that*

a) $J=I\times(0, \bar{\xi})$, *where I is a compact interval on the real axis and* $\bar{\xi}=(0, \ldots, 0, 1)$,

b) *in a conical neighborhood of J the symbol of P is of the form* $\xi_0+if(x, \xi')+r(x, \xi')$, *where f and $r\in C^\infty$ are homogeneous of degree* 1 *and* 0, *respectively, when* $|\xi'|\geqq 1$,

c) *f is real and does not change sign for fixed* (x', ξ'),

d) *f and r vanish on J of degree* 3 *and* 1, *respectively.*

Having reduced the symbol microlocally, it is natural to use the spaces $H'_{(s)}\subseteq \mathcal{S}'(\mathbf{R}^{n+1})$ with the norm

$$\|u\|'_{(s)}=\left((2\pi)^{-n-1}\int|\hat{u}(\xi)|^2(1+|\xi'|^2)^s\,d\xi\right)^{1/2}<\infty$$

to measure the regularity in the x' variables. We are going to localize these spaces by using operators in the x' variables, depending on the parameter x_0. Then, if the symbol of P is of the form $\xi_0+if(x,\xi')+r(x,\xi')$, we can use the calculus in $\mathbf{R}^n$ and consider x_0 as a parameter.

Let $B^\infty(\mathbf{R}, S(1,g_0))$ be the space of bounded C^∞functions on $\mathbf{R}$ with values in $S(1,g_0)$. Thus, if $\psi\in B^\infty(\mathbf{R}, S(1,g_0))$, then $\psi\in C^\infty(\mathbf{R}^{2n+1})$ and we have the estimate

$$|D_x^\alpha D_{\xi'}^\beta\psi(x,\xi')|\leqq C_{\alpha,\beta}(1+|\xi'|)^{-|\beta|}$$

for all α, β. It is easy to see that if $\psi\in B^\infty(\mathbf{R}, S(1,g_0))$ then ψ^w is continuous in $\mathcal{S}(\mathbf{R}^{n+1})$, $\mathcal{S}'(\mathbf{R}^{n+1})$ and $H'_{(s)}$ for all s. We shall now study the connection between $H'_{(s)}$ regularity and $H_{(s)}$ regularity.

Lemma 2.3. a) *Let* $(y,\eta')\in\mathbf{R}^{2n+1}$, $\eta'\neq 0$, *and assume that*

$$\psi(x,\xi')\in B^\infty(\mathbf{R}, S(1,g_0))$$

is homogeneous of degree 0 *for large* ξ', $\psi(y,r\eta')\neq 0$ *for large* r, *and* $\psi^w u\in H'_{(s)}$, $u\in\mathcal{S}'$. *Then* $u\in H_{(s)}$ *at* (y,η_0,η') *for all* η_0.

b) *Assume that* $\xi'\neq 0$ *in* $WF(u)$, $u\in\mathcal{S}'$, *and that* $u\in H_{(s)}$ *at* (y,η_0,η') *for all* η_0. *Then it follows that* $\psi^w u\in H'_{(s)}$ *if* $\psi(x,\xi')\in B^\infty(\mathbf{R}, S(1,g_0))$ *has support in a sufficiently small conical neighborhood of* (y,η').

Proof of Lemma 2.3. a) Choose $\chi(\xi)\in C^\infty(\mathbf{R}^{n+1})$ homogeneous of degree 0 for large ξ, such that $\chi(\xi)=1$ when $|\xi_0|\leqq c|\xi'|+1$, and $\chi(\xi)=0$ when $|\xi_0|\geqq C(|\xi'|+1)$. If $\psi(x,\xi')\in B^\infty(\mathbf{R}, S(1,g_0))$, then the composition $\chi^w\psi^w$ is a pseudo-differential operator, which, since $|\xi_0|\leqq C(|\xi'|+1)$ in $\operatorname{supp}\chi$, can be computed by the standard calculus (see Sjöstrand [6, Appendix]). If $\psi^w u\in H'_{(s)}$ then it is clear that $\chi^w\psi^w u\in H_{(s)}$, since $1\leqq(1+|\xi|^2)/(1+|\xi'|^2)\leqq 1+2C^2$ in $\operatorname{supp}\chi$ and χ is independent of the x variables. Thus, we find that $u\in H_{(s)}$ at (y,η_0,η') when $|\eta_0|\leqq c|\eta'|$. Since the constant c can be chosen arbitrarily large, we obtain a).

b) After multiplication with a suitable cut-off function, we may assume that $u\in\mathcal{E}'$. Then, since $\xi'\neq 0$ in $WF(u)$, we find that $|\xi_0|\leqq C|\xi'|$ in $WF(u)$. Choose $\chi(\xi)\in C^\infty(\mathbf{R}^{n+1})$ homogeneous of degree 0 for large ξ, such that $\chi=1$ in a conical neighborhood of $WF(u)$ and $\chi(\xi)=0$ when $|\xi_0|\geqq C'(|\xi'|+1)$. If

$$\psi(x,\xi')\in B^\infty(\mathbf{R}, S(1,g_0))$$

has support in a sufficiently small conical neighborhood of (y,η'), then $\chi^w\psi^w u\in H_{(s)}$.

In fact, this follows since $u\in H_{(s)}$ at (y, η_0, η') for all η_0, and $\chi^w\psi^w$ is a pseudo-differential operator which can be computed by the standard calculus. As before, we find that $\chi^w\psi^w u\in H'_{(s)}$, because $1\leqq(1+|\xi|^2)/(1+|\xi'|^2)\leqq 1+2(C')^2$ in $\operatorname{supp}\chi$ and χ is independent of the x variables.

Now we are going to prove that $(1-\chi^w)\psi^w u\in H'_{(t)}$ for all t. Since $\chi=1$ in a conical neighborhood of $WF(u)$, we can choose $\chi_1(\xi)\in C^\infty(\mathbf{R}^{n+1})$ homogeneous of degree 0 for large ξ, such that $\chi_1=1$ in a conical neighborhood of $WF(u)$ and $\chi=1$ on $\operatorname{supp}\chi_1$. Now $(1-\chi^w)\psi^w u=(1-\chi^w)\psi^w\chi_1^w u+(1-\chi^w)\psi^w(1-\chi_1^w)u$, where, as before, the symbol of $(1-\chi^w)\psi^w\chi_1^w$ can be computed by the standard calculus. Since $\chi=1$ on $\operatorname{supp}\chi_1$ the calculus gives that $(1-\chi^w)\psi^w\chi_1^w u\in H_{(t)}\subseteqq H'_{(t)}$ for all positive t. Since $\chi_1=1$ on $WF(u)$, we find that $(1-\chi_1^w)u\in H_{(t)}\subseteqq H'_{(t)}$ for all positive t. This implies that $\psi^w(1-\chi_1^w)u\in H'_{(t)}$, and since $1-\chi(\xi)$ is bounded and independent of the x variables, that $(1-\chi^w)\psi^w(1-\chi_1^w)u\in H'_{(t)}$ for all positive t. This completes the proof of the lemma.

Definition 2.4. When $\xi'\neq 0$ in $WF(u)$, $u\in\mathcal{S}'$, we say that $u\in H'_{(s)}$ at (y, η') if $\psi^w u\in H'_{(s)}$ for some $\psi(x, \xi')\in B^\infty(\mathbf{R}, S(1, g_0))$ homogeneous of degree 0 for large ξ', such that $\psi(y, r\eta')\neq 0$ for large r.

Let $WF_{(s)}(u)=\{(x, \xi)\colon u\notin H_{(s)} \text{ at } (x, \xi)\}$ and let

$$WF'_{(s)}(u) = \{(x, \xi')\colon u\notin H'_{(s)} \text{ at } (x, \xi')\},$$

if $\xi'\neq 0$ in $WF(u)$, $u\in\mathcal{S}'$. Then Lemma 2.3 gives

$$\pi_0(WF_{(s)}(u)) = WF'_{(s)}(u)$$

where $\pi_0(x, \xi_0, \xi')=(x, \xi')$.

Proposition 2.5. *Assume that* $P'(x, D)=D_0+iF^w(x, D')+R^w(x, D')$, *where* F *and* $R\in C^\infty(\mathbf{R}^{2n+1})$ *are bounded functions of* x_0 *with values in* $S(h_0^{-1}, g_0)$ *and* $S(1, g_0)$ *respectively. Also assume that* F *is real, does not change sign for fixed* (x', ξ') *and vanishes of order* 3 *on the rays through* $J=I\times(0, \bar{\xi})$, *where* $I=[-1, 1]$ *and* $\bar{\xi}=(0, \ldots, 0, 1)$, *and that* R *vanishes on the rays trough* J. *If* $\xi'\neq 0$ *in* $WF(v)$, $v=0$ *when* $|x_0|>1$ *and for some* $\varepsilon>0$

$$P'v\in H'_{(\varepsilon)} \quad \text{on} \quad \pi_0(J), \quad v\in H'_{(-1/15)} \quad \text{and} \quad P'v\in H'_{(-1/15)},$$

then it follows that

$$v\in H'_{(0)} \quad \text{on} \quad \pi_0(J).$$

Proof that Proposition 2.5. implies Proposition 2.1. By Lemma 2.2 we may assume that the symbol of P is equal to $\xi_0+if(x, \xi')+r(x, \xi')$ in a conical neighborhood of $J=I\times(0, \bar{\xi})$ where f and $r\in C^\infty(\mathbf{R}^{2n+1})$ are bounded functions of x_0

with values in $S(h_0^{-1}, g_0)$ and $S(1, g_0)$ respectively. After a change of scale in the x_0 variable we may assume that $I=[-1, 1]$. Then Lemma 2.2 gives that

$$P'(x, D) = D_0 + if^w(x, D') + r^w(x, D') \tag{2.1}$$

fulfills the requirements in Proposition 2.5. We shall prove that Proposition 2.1 follows from Proposition 2.5 by microlocalizing in a conical neighborhood of J.

By multiplying with a suitable cut-off function we can assume that $u\in\mathscr{E}'$. Choose $\psi(x, \xi)\in C^\infty(\mathbf{R}^{2n+2})$ homogeneous of degree 0 for large ξ, and with support so close to the rays through J that the symbol of $P-P'$ (P' defined by (2.1)) is equal to 0 in supp ψ, $u\in H_{(-1/15)}$ and $Pu\in H_{(\varepsilon)}$ in supp ψ, and $|\xi_0|\leqq C|\xi'|+1$ in supp ψ. We also want $|x_0|<1$ in supp ψ, $\psi=1$ on the part of the rays through J where we do not already know that $u\in H_{(0)}$, and finally $u\in H_{(\varepsilon)}$ where the support of grad ψ meets the rays through J.

It is then clear that it suffices to prove that $v=\psi^w u\in H_{(0)}$ near J. The assumptions on ψ imply that $v\in H_{(-1/15)}$ and $Pv=\psi^w Pu+[P, \psi^w]u\in H_{(-1/15)}$. Since $WF(Pv)\subseteq WF(v)\subseteq \text{supp}\,\psi$, Lemma 2.3 gives that v and $Pv\in H'_{(-1/15)}$.

Now we prove that $Pv\in H'_{(\varepsilon)}$ on $\pi_0(J)$. Since $u\in H_{(\varepsilon)}$ on supp (grad ψ)$\cap J$, we have $[P, \psi^w]u\in H_{(\varepsilon)}$ on J. It is also clear that $[P, \psi^w]u\in H_{(\varepsilon)}$ on $\pi_0^{-1}(\pi_0(J))$. In fact, since P is of the form (2.1) in supp ψ, P is non-characteristic in supp ψ where $\xi_0\neq 0$. Since $Pu\in H_{(\varepsilon)}$ in supp ψ, we obtain $u\in H_{(1+\varepsilon)}$ in supp ψ where $\xi_0\neq 0$. Thus, $[P, \psi^w]u\in H_{(\varepsilon)}$ on $\pi_0^{-1}(\pi_0(J))$, and since $\psi^w Pu\in H_{(\varepsilon)}$, we get $Pv\in H_{(\varepsilon)}$ on $\pi_0^{-1}(\pi_0(J))$. Lemma 2.3 then gives that $Pv\in H'_{(\varepsilon)}$ on $\pi_0(J)$.

Now, the symbol of $P-P'$ is equal to 0 in supp ψ, and the composition $(P-P')\psi^w$ can be computed by the standard calculus. Thus, $(P-P')\psi^w u\in H_{(t)}\subseteq H'_{(t)}$ for all positive t, so $P'v\in H'_{(-1/15)}$, and $P'v\in H'_{(\varepsilon)}$ on $\pi_0(J)$. By multiplying with a suitable cut-off function in x_0 we can obtain that $v=0$ when $|x_0|>1$. Proposition 2.5 gives that $v\in H'_{(0)}$ on $\pi_0(J)$, so Lemma 2.3 implies that $v\in H_{(0)}$ on J. This proves that Proposition 2.5 implies Proposition 2.1.

In the proof of Proposition 2.5 we shall make a change of scale in the x_0 variable and cut off near $\pi_0(J)$. To get uniform estimates we must vary the metric. In what follows, we shall denote by g any metric of the form $g=m\cdot g_0$, where $m(\xi')$ is independent of the x' variables and $1\leqq m\leqq h_0^{-1}$. This implies that $g_0\leqq g$ and $\sup g/g^\sigma=h^2=(m\cdot h_0)^2\leqq 1$. Now, since g is conformal to g_0, the following lemma shows that g is σ temperate if it is slowly varying, and that we can get a bound on the constants in the definition.

Lemma 2.6. *Assume that $G=m\cdot g_0$ is uniformly slowly varying, that $1\leqq m\leqq h_0^{-1}$, and that M is uniformly G continuous satisfying $1\leqq M\leqq H^{-1}=(m\cdot h_0)^{-1}$. Then G is uniformly σ temperate and M is uniformly σ, G temperate.*

Proof of Lemma 2.6. Since the triangle inequality gives

$$g_{0,w_1}(t) \leqq 2g_{0,w_2}(t)\big(1+g_{0,w_1}(w_1-w_2)\big), \quad w_j\in T^*\mathbf{R}^n,$$

it suffices to prove that m and M are uniformly σ, G temperate, that is,

$$m(w_1) \leqq Cm(w_2)\big(1+G^\sigma_{w_1}(w_2-w_1)\big)^N$$

and

$$M(w_1) \leqq CM(w_2)\big(1+G^\sigma_{w_1}(w_2-w_1)\big)^N.$$

Now, $g_0\leqq G$ implies that m is G continuous, so it suffices to consider the case $G_{w_j}(w_1-w_2)\geqq c_0>0$, $j=1, 2$. Then we find

$$M(w_1) \leqq H^{-1}(w_1) \leqq \big(G^\sigma_{w_1}(w_1-w_2)/c_0\big)^{1/2}$$

which proves that M is σ, G temperate.

To prove that m is σ, G temperate we note that if $g_{0,w_1}(w_1-w_2)\leqq c_1$ and c_1 is small enough, then $g_{0,w_1}\geqq g_{0,w_2}/C$. This implies that

$$m(w_2)\, g_{0,w_1}(w_1-w_2) \geqq G_{w_2}(w_1-w_2)/C \geqq c_0/C,$$

so we obtain

$$m(w_2)\, G^\sigma_{w_1}(w_1-w_2) = m(w_2)\, g_{0,w_1}(w_1-w_2)/m(w_1)\, h_0^2(w_1) \geqq c_0 m(w_1)/C$$

since $m\leqq h_0^{-1}$.

When $g_{0,w_1}(w_1-w_2)\geqq c_1$ we find

$$m(w_1) \leqq h_0^{-1}(w_1) \leqq G^\sigma_{w_1}(w_1-w_2)/c_1$$

since $h_0^{-1}g_0\leqq G^\sigma$. This proves that m is σ, G temperate and finishes the proof of Lemma 2.6.

Remark. Note that in Lemma 2.6 we did not have to assume that m and M were independent of the x variables.

We shall use the norm

$$\|u\|''_{(s)} = \Big((2\pi)^{-n-1}\int |\hat{u}(\xi)|^2 h(\xi')^{-2s}\, d\xi\Big)^{1/2},$$

which depends on the metric g. This norm is well suited to the calculus with symbols in $S(h^k, g)$. Since $m\geqq 1$ we obtain

$$\|u\|''_{(s)} \leqq \|u\|'_{(s)},$$

when s is positive. For convenience we put $\|u\|=\|u\|''_{(0)}$.

Now we state the *a priori* estimates we are going to use in the proof of Proposition 2.5.

Proposition 2.7. *Assume that* $P_1=D_0+iF_1^w(x, D')+R_1^w(x, D')$, *where* F_1 *and* $R_1\in C^\infty(\mathbf{R}^{2n+1})$ *are bounded functions of* x_0 *with values in* $S(h^{-1}, g)$ *and* $S(1, g)$

respectively, F_1 is real and does not change sign for fixed (x', ξ'). Assume that φ and $\psi \in S(1, g)$ and that $\psi=1$ on supp φ. *Then for every $\varepsilon>0$ there exists $\delta>0$ such that*

$$\|\varphi^w u\| \leqq C(\|\psi^w P_1 u\|''_{(\varepsilon)} + \|u\|''_{(-1/15)} + \|P_1 u\|''_{(-16/15)}) \tag{2.2}$$

if $u \in \mathcal{S}(\mathbf{R}^{n+1})$ and $u=0$ where $|x_0|>\delta$. Here δ and C do not depend on g and P_1 if g is uniformly σ temperate and if we have fixed bounds on the seminorms of the symbol of P_1.

Proof that Proposition 2.7 implies Proposition 2.5. Note that in Proposition 2.7 we do not need the assumption that F_1 and R_1 vanish on J. However, we need the assumption that u vanishes when $|x_0|>\delta$, where δ depends on ε and on the bounds on the symbol and the metric. We are going to reduce the proof of Proposition 2.5 to the estimate (2.2) by making a change of scale in the x_0 variable. In order to get a fixed bound on the symbol after the change of scale, we cut off the symbol in a small neighborhood of J and use the fact that F and R vanish on J. This motivates the following choice of metric. When $T \geqq 1$ we put

$$g_{(T)} = \min(T^2, h_0^{-1})\, g_0$$

and

$$h_{(T)} = (\sup g_{(T)}/g^\sigma_{(T)})^{1/2} = \min(T^2 h_0, 1) \leqq 1.$$

Since h_0^{-1} is g_0 continuous we find that $\min(T^2, h_0^{-1})$ is uniformly g_0 continuous. Now $g_0 \leqq g_{(T)}$ implies that $g_{(T)}$ is uniformly slowly varying, so Lemma 2.6 gives that $g_{(T)}$ is uniformly σ temperate.

We shall now microlocalize the symbol in a $g_{(T)}$ neighborhood of $\pi_0(J)$. Choose $\varphi_{(T)}$, $\psi_{(T)}$ and $\chi_{(T)} \in S(1, g_{(T)})$, with fixed bounds on each seminorm and homogeneous of degree 0 for large ξ', such that the symbols have support where $T^2 \leqq h_0^{-1}$ and where the $g_{(T)}$ distance to the ray through $(0, \xi')$ is less than a fixed constant. We also want $\varphi_{(T)}=1$ on the part of the ray through $(0, \xi')$ where $CT^2 \leqq h_0^{-1}$, $\psi_{(T)}=1$ on supp $\varphi_{(T)}$ and $\chi_{(T)}=1$ on supp $\psi_{(T)}$. Put

$$P_{(r)} = D_0 + iF^w_{(T)} + R^w_{(T)} \tag{2.3}$$

where

$$F_{(T)}(x, \xi') = T\chi_{(T)}(x', \xi')\, F(Tx_0, x', \xi')$$

and

$$R_{(T)}(x, \xi') = T\chi_{(T)}(x', \xi')\, R(Tx_0, x', \xi').$$

Then $R_{(T)}$ and $F_{(T)} \in C^\infty(\mathbf{R}^{2n+1})$.

Lemma 2.8. *$F_{(T)}$ and $R_{(T)}$ are uniformly bounded functions of x_0 with values in $S(h_{(T)}^{-1}, g_{(T)})$ and $S(1, g_{(T)})$ respectively, when $T \geqq 1$ and $|x_0| \leqq T^{-1}(1+T^{-1})$.*

Proof of Lemma 2.8. Since $F_{(T)}$ and $R_{(T)}$ have support where $g_{(T)}=T^2 g_0$, it suffices to prove that

$$|\chi_{(T)}F|_k^{g_0} \leqq C_k T^{k-3} h_0^{-1} \tag{2.4}$$

and

$$|\chi_{(T)}R|_k^{g_0} \leqq C_k T^{k-1} \quad \text{when} \quad |x_0| \leqq 1+T^{-1}. \tag{2.5}$$

Since $F\in S(h_0^{-1}, g_0)$ uniformly in x_0 and F vanishes of order 3 on J, we obtain from Taylor's formula that

$$|F|_k^{g_0} \leqq C_k T^{k-3} h_0^{-1} \quad \text{in} \quad \operatorname{supp}\chi_{(T)} \quad \text{when} \quad |x_0| \leqq 1+T^{-1},$$

because $\chi_{(T)}$ has support where the g_0 distance to the ray through $(0, \xi')$ is less than C/T. Since $R\in S(1, g_0)$ uniformly in x_0 and vanishes on J, the same argument gives

$$|R|_k^{g_0} \leqq C'_k T^{k-1} \quad \text{in} \quad \operatorname{supp}\chi_{(T)} \quad \text{when} \quad |x_0| \leqq 1+T^{-1}.$$

Now, $\chi_{(T)}$ has fixed bounds in $S(1, g_{(T)})$, thus

$$|\chi_{(T)}|_k^{g_0} \leqq C''_k T^k,$$

so Leibniz' rule gives (2.4) and (2.5), which proves the lemma.

End of proof that Proposition 2.7 implies Proposition 2.5. Assume that $u\in\mathcal{S}(\mathbf{R}^{n+1})$ and that $u=0$ when $|x_0|\geqq 1$. Let $u_{(T)}(x)=u(Tx_0, x')$, so that $u_{(T)}=0$ when $|x_0|\geqq 1/T$. Proposition 2.7 and Lemma 2.8 give that for each $\varepsilon>0$ there exists $T_\varepsilon\geqq 1$ such that

$$\|\varphi_{(T)}^w u_{(T)}\| \leqq C(\|\psi_{(T)}^w P_{(T)}u_{(T)}\|''_{(T,\varepsilon)}+\|u_{(T)}\|''_{(T,-1/15)}+\|P_{(T)}u_{(T)}\|''_{(T,-16/15)}), \tag{2.6}$$

if $T\geqq T_\varepsilon$, where

$$\|u\|''_{(T,s)} = \left((2\pi)^{-n-1}\int |\hat{u}(\xi)|^2 h_{(T)}(\xi')^{-2s}\, d\xi\right)^{1/2}.$$

Now we have

$$\min(T^{-2s}, 1)\|u\|'_{(s)} \leqq \|u\|''_{(T,s)} \leqq \max(T^{-2s},1)\,\|u\|'_{(s)} \tag{2.7}$$

so if we make a change of scale in the x_0 variable, writing Tx_0 instead of x_0, we find

$$\|\varphi_{(T)}^w u\| \leqq C_T(\|\psi_{(T)}^w \bar{P}_{(T)}u\|'_{(\varepsilon)}+\|u\|'_{(-1/15)}+\|\bar{P}_{(T)}u\|'_{(-16/15)}) \tag{2.8}$$

if $T\geqq T_\varepsilon$, where

$$\bar{P}_{(T)}(x, D) = D_0+i(\chi_{(T)}F)^w(x, D')+(\chi_{(T)}R)^w(x, D').$$

If we replace $\bar{P}_{(T)}$ with $P'(x, D)=D_0+iF^w(x, D')+R^w(x, D')$ we obtain, since $\chi_{(T)}=1$ on $\operatorname{supp}\psi_{(T)}$ and $\bar{P}_{(T)}-P'$ is of order 1,

$$\|\varphi^w_{(T)}u\| \leqq C'_T(\|\psi^w_{(T)}P'u\|'_{(\varepsilon)}+\|u\|'_{(-1/15)}+\|P'u\|'_{(-1/15)}), \tag{2.9}$$

when $T\geqq T_\varepsilon$ and $u\in\mathcal{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq 1$.

To prove Proposition 2.5 we must extend (2.9) to all $u\in H'_{(-1/15)}$ satisfying the hypotheses made on v in the proposition. It is clear that by continuity we may extend (2.9) to those $u\in H'_{(1+\varepsilon)}$ with support where $|x_0|\leqq 1$, such that $D_0u\in H'_{(\varepsilon)}$. Let

$$C_t(\xi') = (1+t|\xi'|^2)^{-1}$$

when $0<t\leqq 1$. It is then easy to see that C_t is a weight for g_0 which is uniformly σ, g_0 temperate and that the seminorms of C_t in $S(C_t, g_0)$ have fixed bounds.

Put $u_t=C_t^wu$ and

$$\|u\|'_{(t,s)} = \|C_t^wu\|'_{(s)}.$$

This norm is equivalent to $\|u\|'_{(s-2)}$ when $t>0$ and tends to $\|u\|'_{(s)}$ when $t\to 0$. If $u\in H'_{(-1/15)}$ and $P'u\in H'_{(-1/15)}$ then $D_0u\in H'_{(-16/15)}$. This implies that $u_t\in H'_{(1+\varepsilon)}$ and $D_0u=C_t^wD_0u_t\in H'_{(\varepsilon)}$ if $t>0$ and $\varepsilon\leqq 14/15$, which we have assumed.

Let $P_t=C_t^wP'(C_t^{-1})^w$. Then the symbol of P_t is equal to

$$\xi_0+iF(x,\xi')-(\{F, C_t\}/C_t)(x,\xi')+R(x,\xi')$$

apart from terms with fixed bounds in $S(h_0, g_0)$, when $0<t\leqq 1$. Ignoring these terms for a moment, we find that P_t fulfills the requirements in Proposition 2.5, since F vanishes of degree 3 on J. Hence

$$\|\varphi^w_{(T)}u_t\| \leqq C''_T(\|\psi^w_{(T)}P_tu_t\|'_{(\varepsilon)}+\|u_t\|'_{(-1/15)}+\|P_tu_t\|'_{(-1/15)}) \tag{2.10}$$

if T is large enough, since the terms with symbols bounded in $S(h_0, g_0)$ can be estimated with $\|u_t\|'_{(-1/15)}$.

Now, the symbols of $[\varphi^w_{(T)}, C_t^w]$ and $[\psi^w_{(T)}, C_t^w]$ are uniformly bounded in $S(h_{(T)}, g_{(T)})$ when $0<t\leqq 1$, so we find that

$$\|\varphi^w_{(T)}u\|'_{(t,0)} \leqq C'''_T(\|\psi^w_{(T)}P'u\|'_{(t,\varepsilon)}+\|u\|'_{(-1/15)}+\|P'u\|'_{(-1/15)}), \tag{2.11}$$

if T is large enough, u and $P'u\in H'_{(-1/15)}$ and u has support where $|x_0|\leqq 1$.

If we also assume that $\xi'\neq 0$ in $WF(u)$ and that $P'u\in H'_{(\varepsilon)}$ on $\pi_0(J)$, then we find that $\psi^w_{(T)}P'u\in H'_{(\varepsilon)}$ for large T. Thus, for large enough T, the right-hand side of (2.11) is bounded when $t\to 0$, which implies that $\varphi^w_{(T)}u\in H'_{(0)}$ then. This means that $u\in H'_{(0)}$ on $\pi_0(J)$, which proves that Proposition 2.7 implies Proposition 2.5.

Proposition 2.7 will be proved in Section 7.

3. The metric of Beals and Fefferman

In this section we shall define a metric which is a modification of the one used by Beals and Fefferman [1] to prove local solvability for operators satisfying the condition (P). The results in this section were essentially proved by Beals and Fefferman [2].

Assume that $g=m\cdot g_0$ is σ temperate, where $1\leqq m\leqq h_0^{-1}$, then $g_0\leqq g$ and $\sup g/g^\sigma=h^2\leqq 1$. Assume that $F(t,w)\in C^\infty(\mathbf{R}\times T^*\mathbf{R}^n)$ is a bounded function of $t\in\mathbf{R}$ with values in $S(h^{-1},g)$. By normalizing we may assume that $|F|\leqq h^{-1}$ and $|F|_1^g\leqq h^{-1}$ for all t and w. Now we want to know for which other metrics $G=Hg/h$, where $h\leqq H\leqq h^{6/7}$, we have that $F(t,\cdot)\in S(H^{-1},G)$ uniformly in t. The reason for taking $H\leqq h^{6/7}$ is that $\sup G/G^\sigma=H^2\leqq h^{12/7}$ then, so we obtain a good calculus in this metric. Now $F(t,\cdot)\in S(H^{-1},G)$ uniformly in t means that

$$|F|_j^G=(h/H)^{j/2}|F|_j^g\leqq C_jH^{-1},$$

that is,

$$|F|_j^g\leqq C_j(H/h)^{j/2-1}h^{-1}.$$

Since we want $H\geqq h$, this condition is automatically fulfilled if $j\geqq 2$, so it suffices that

$$|F|\leqq H^{-1}$$

and

$$|F|_1^g\leqq (hH)^{-1/2}.$$

If we choose

$$H^{-1}=\max\big(h^{-6/7},\sup_t|F|,(\sup_t|F|_1^g)^2h\big) \tag{3.1}$$

we find that F is a bounded function of t with values in $S(H^{-1},G)$ where $H^2=\sup G/G^\sigma$ satisfies $h\leqq H\leqq h^{6/7}$. This metric has the property that, if F does not change sign for fixed w, we can localize with respect to G so that each localization of F either can be factored, is semibounded or is of lower order.

Proposition 3.1. *Assume that* $g=m\cdot g_0$ *is uniformly* σ *temperate, and that* $1\leqq m\leqq h_0^{-1}$, *which implies* $\sup g/g^\sigma=h^2\leqq 1$. *Assume that* F *has fixed bounds in* $S(h^{-1},g)$ *for all* $t\in\mathbf{R}$, *and that* $|F|\leqq h^{-1}$ *and* $|F|_1^g\leqq h^{-1}$. *Let* $G=Hg/h$, *where* H *is defined by* (3.1). *Then* G *is uniformly* σ *temperate and* $\sup G/G^\sigma=H^2\leqq 1$.

Proof. It is clear that $H\leqq h^{6/7}\leqq 1$. Now $G=Hg/h=Hg_0/h_0$ is conformal to g_0 and $1\leqq h/h_0\leqq H/h_0\leqq h_0^{-1}$, so Lemma 2.6 gives that it suffices to prove that G is uniformly slowly varying. Since $g\leqq G$ we find that h is G continuous, so it remains to prove that H is G continuous.

Let $h_1=h(w_1)$ and $h_1/H(w_1)=r$. Choose orthonormal coordinates z with respect to g_{w_1} with the origin at w_1 and let $f(t,z)=h_1F(t,w)$. Since $F(t,\cdot)$ is uniformly bounded in $S(h^{-1}, g)$, we find that f is a bounded function of t with values in $C^\infty(U)$, where $U=\{z\in\mathbf{R}^{2n}: |z|<C\}$. The neighborhood $\{z: |z|<cr^{1/2}\}$ in $\mathbf{R}^{2n}$ corresponds to the G neighborhood $\{w\in T^*\mathbf{R}^n: G_{w_1}(w-w_1)<c^2\}$ of w_1, so it suffices to prove that

$$E(z) = \max\big(h_1^{1/7}, \sup_t |f(t, z)|, \sup_t |\mathrm{grad}_z f(t, z)|^2\big)$$

only varies with a fixed factor when $|z|<cr^{1/2}$. We need the following elementary

Lemma 3.2. *Assume that $f\in C^\infty(\mathbf{R}^{2n})$ has a fixed bound on the second derivatives when $|z|<C$. If*

$$\max\big(|f(0)|, |\mathrm{grad} f(0)|^2\big) = s \leqq r \leqq C',$$

then it follows that there exist c_1 and C_1 such that

$$\max\big(|f(z)|, |\mathrm{grad} f(z)|^2\big) \leqq C_1 r$$

when $|z|<c_1 r^{1/2}$, and

$$\max\big(|f(z)|, |\mathrm{grad} f(z)|^2\big) \geqq s/C_1$$

when $|z|<c_1 s^{1/2}$.

Proof of Lemma 3.2. The upper bound is an immediate consequence of Taylor's formula, since

$$|f(z)| \leqq |f(0)|+|(\mathrm{grad} f(0), z)|+C|z|^2 \leqq C_1 r$$

and

$$|\mathrm{grad} f(z)| \leqq |\mathrm{grad} f(0)|+2C|z| \leqq C_1 r^{1/2}$$

if $|z|\leqq c_1 r^{1/2}$. To get the lower bound we observe that in the case $s=|f(0)|\geqq |\mathrm{grad} f(0)|^2$, we obtain

$$|f(z)| \geqq |f(0)|-|(\mathrm{grad} f(0), z)|-C|z|^2 \geqq s/C_1$$

if $|z|\leqq c_1 s^{1/2}$. The corresponding argument works in the case $s=|\mathrm{grad} f(0)|^2$, which proves the lemma.

End of proof of Proposition 3.1. Since

$$\max\big(|f(t, 0)|, |\mathrm{grad}_z f(t, 0)|^2\big) \leqq E(0) = r \quad \text{for all } t,$$

Lemma 3.2 gives that

$$\max\big(|f(t, z)|, |\mathrm{grad}_z f(t, z)|^2\big) \leqq C_1 r$$

if $|z|\leqq c_1 r^{1/2}$. Now $h_1^{1/7}\leqq r$ so we obtain $E(z)\leqq C_1 E(0)$.

To get the lower bound, we note that in the case $r=h_1^{1/7}$ it follows that $E(z)\geqq h_1^{1/7}=E(0)$. In the case

$$\max\big(\sup_t |f(t, 0)|, \sup_t |\mathrm{grad}_z f(t, 0)|^2\big) = r$$

there exists $t_0 \in \mathbf{R}$ such that

$$\max\left(|f(t_0, 0)|,\ |\mathrm{grad}_z f(t_0, 0)|^2\right) \geqq r/2.$$

Lemma 3.2 then gives

$$r/C \leqq \max\left(|f(t_0, z)|,\ |\mathrm{grad}_z f(t_0, z)|^2\right) \leqq E(z)$$

when $|z| \leqq cr^{1/2}$, which gives the lower bound and finishes the proof of Proposition 3.1.

Proposition 3.3. *The assumptions in Proposition* 3.1 *imply that F is a bounded function of* $t \in \mathbf{R}$ *with values in* $S(H^{-1}, G)$. *If F is real and does not change sign for fixed w, and* $\delta > 0$ *is small enough, then in each G neighborhood* $\Omega_{w_0, \delta} = \{(t, w)\colon G_{w_0}(w - w_0) < \delta^2\}$ *we have one of the following cases:*
i) $h^{6/7}/C \leqq H \leqq h^{6/7}$,
ii) *F has constant sign,*
iii) $F(t, w) = a(t, w)b(w)$, *where* $0 \leqq a \in C^\infty$ *is a uniformly bounded function of t with values in* $S(1, G)$, *b has fixed bounds in* $S(H^{-1}, G)$ *and* $H|b|_1^g \geqq c > 0$ *in* $\Omega_{w_0, \delta}$.

Proof. If $H^{-1}(w_0) = h^{-6/7}(w_0)$ then we get the case i) for sufficiently small δ, because G varies slowly and h is G continuous. Thus we may assume that

$$H^{-1} = \max\left(\sup_t |F|, (\sup_t |F|_1^g)^2 h\right) \quad \text{when} \quad w = w_0. \tag{3.2}$$

Choose G_{w_0} orthonormal coordinates z with the origin at w_0. Let $f(t, z) = H_0 F(t, w)$, where $H_0 = H(w_0)$. Then f is a bounded function of t with values in $C^\infty(U)$, $U = \{z\colon |z| < C\}$. Now, (3.2) implies

$$1 = \max\left(\sup_t |f(t, 0)|, \sup_t |\mathrm{grad}_z f(t, 0)|^2\right). \tag{3.3}$$

If $1 = \sup_t |f(t, 0)|$, then we can find $t_0 \in \mathbf{R}$ such that $|f(t_0, 0)| \geqq 1/2$. Since f is real-valued and $f(t_0, \cdot)$ has fixed bounds in $C^\infty(U)$, we can find δ so small that either $f(t_0, z) > 0$ or $f(t_0, z) < 0$ when $|z| < \delta$. Now f does not change sign for fixed z, which gives us the case ii).

If $1 = \sup_t |\mathrm{grad}_z f(t, 0)|^2$, we can find t_0 such that $|\mathrm{grad}_z f(t_0, 0)| \geqq 1/2$. Since $f(t_0, \cdot)$ has fixed bounds in $C^\infty(U)$, the implicit function theorem gives that we can choose $f(t_0, z) = \zeta_1$ as a local coordinate when $|z| < 2\delta$. Then f must have the same sign as ζ_1, since f does not change sign for fixed z, thus $f = 0$ when $\zeta_1 = 0$. Taylor's formula gives

$$f(t, z) = e(t, z)\zeta_1 \quad \text{when} \quad |z| < 2\delta, \tag{3.4}$$

where $e \geqq 0$ is a bounded function of t with values in $C^\infty(U_1)$, $U_1 = \{z\colon |z| < 2\delta\}$.

If we differentiate the equation (3.4) we find that $D_t^k e \in {}_t^k C^\infty(U_1)$ for all k. Choose a cut-off function $\chi \in C_0^\infty(U_1)$ such that $\chi \geqq 0$ and $\chi(z)=1$ when $|z|<\delta$. Let $a(t, w)=\chi(z)e(t, z)$ in U_1 and equal to 0 otherwise, then we obtain the case iii) with $b(w)=F(t_0, w)$. This completes the proof of Proposition 3.3.

4. Estimates for the localized operators

We shall now state and prove the estimates which will be used in the proof of Proposition 2.7. The estimates are adapted to the localizations of the operator corresponding to the Beals—Fefferman metric defined in Section 3. Therefore we shall use symbols with values in ℓ^2 and $\mathscr{L}(\ell^2, \ell^2)$. The results are refinements of the estimates in Hörmander [3], including the estimate of Beals and Fefferman (Proposition 4.3).

Assume that G is a σ temperate metric in $T^*\mathbf{R}^n$, such that $\sup G/G^\sigma = H^2 \leqq 1$. To begin with we do not make any further restrictions on G. In what follows, the estimates will not depend on the metric G, as long as G fulfills the requirements stated and is uniformly σ temperate.

Proposition 4.1. *Assume that* $0 \leqq a(x', \xi') \in S(H^{-6/5}, G)$. *Then there exists a constant* C *such that*

$$(a^w u, u) \geqq -C\|u\|^2, \tag{4.1}$$

when $u \in \mathscr{S}(\mathbf{R}^n)$.

Proof. By regularizing we may assume that $H \in S(H, G)$. Put

$$b = aH^{1/5} \in S(H^{-1}, G).$$

Then $b \geqq 0$, so Theorem 6.2 in [4] gives $b_1(x', \xi') \in S(H^{1/5}, G)$ such that

$$(b^w u, u) \geqq (b_1^w u, u) \tag{4.2}$$

when $u \in \mathscr{S}(\mathbf{R}^n)$. Put $v=c^w u$, where $c=H^{-1/10}$. Since $(c^w)*b_1^w c^w$ has symbol in $S(1, G)$ we obtain

$$|(b_1^w v, v)| \leqq C_0\|u\|^2. \tag{4.3}$$

The calculus gives that the symbol for $(c^w)*b^w c^w$ is in $S(1, G)$ apart from the first terms given by

$$bc^2+(1/2i)c\{b, c\}+1/2i\{c, bc\} = a$$

so we obtain $(c^w)*b^w c^w = a^w + R^w$, where $R \in S(1, G)$. This implies

$$(a^w u, u) = (b^w v, v)-(R^w u, u) \geqq -C_1\|u\|^2 \tag{4.4}$$

when $u \in \mathscr{S}(\mathbf{R}^n)$, which proves the proposition.

In what follows we assume, as in Section 3, that $G=Hg/h$, where $h\leqq H\leqq h^{6/7}$, $g=m\cdot g_0$ is uniformly σ temperate and $1\leqq m(\xi')\leqq h_0^{-1}(\xi')$ is independent of the x' variables. Since $1\leqq h^{-6/7}\leqq H^{-1}$, Lemma 2.6 gives that $h^{-6/7}$, hence h, is a weight for G, for it is obviously G continuous.

For convenience, we change notations and put

$$\|u\|_{(s)}=\left((2\pi)^{-n-1}\int|\hat{u}(\xi)|^2h(\xi')^{-2s}\,d\xi\right)^{1/2}, \tag{4.5}$$

when $u\in\mathscr{S}(\mathbf{R}^{n+1})$. This norm corresponds to the norm $\|\cdot\|''_{(s)}$ in Section 2. Since $S(H^s,G)$ is uniformly bounded in $S(h^{s\cdot 6/7},G)$ when s is positive, we find that

$$\|R^wu\|\leqq C\|u\|_{(-h\cdot 6/7)} \tag{4.6}$$

if $R\in S(H^s,G)$ and $s\geqq 0$. As noted in Hörmander [4, p. 393] we also have the estimate (4.6) when R takes values in a Hilbert space. For example,

$$\sum\|R_j^wu\|^2\leqq C\|u\|_{(-s)}^2$$

when $\{R_j\}\in S(h^s,G)$ with values in ℓ^2. It is easy to see that

$$|(R^wu,u)|\leqq C\|u\|_{(s/2)}^2$$

when $R\in S(h^{-s},G)$. In fact, by choosing $c\in S(h^{s/2},g)$, independent of the x' variables such that $c(\xi')\geqq h^{s/2}(\xi')$ we obtain

$$|(R^wu,u)|=\left|((c^w)*R^wu,(c^{-1})^wu)\right|\leqq C\|u\|_{(s/2)}^2.$$

Let $G_1=H^{-1/6}G$ and $H_1^2=\sup G_1/G_1^\sigma=(H^{5/6})^2\leqq 1$. It is clear that G_1 is uniformly slowly varying, for $G_1\geqq G$. Since G_1 is conformal to the homogeneous metric g_0 and $g_0\leqq G_1\leqq G_1^\sigma$, Lemma 2.6 gives that G_1 is uniformly σ temperate.

The purpose of the metric G_1 is that, since $S(H^{-1},G_1)=S(H_1^{-6/5},G_1)$, we may localize a symbol in $S(H^{-1},G)$ with symbols in $S(1,G_1)$ and then apply Proposition 4.1 if the localized symbol is non-negative.

Proposition 4.2. *Assume that* $P(x,D)=D_0+iq^w(x,D')+r^w(x,D')$, *where* q *and* $r\in C^\infty(\mathbf{R}^{2n+1})$ *are bounded functions of* $x_0\in\mathbf{R}$ *with values in* $S(H^{-1},G)$ *and* $S(1,G)$ *respectively, and* q *is real. Assume that* $\varepsilon>0$ *and* $\{\varphi_j\}\in S(h^{-\varepsilon},G_1)$ *is real with values in* ℓ^2, *where* $G_1=H^{-1/6}G$. *If one can find* ψ_j *uniformly bounded in* $S(1,G_1)$, *such that* $\psi_j=1$ *on* supp φ_j and $\psi_jq\geqq 0$, *then for sufficiently small* ε *and* δ *we obtain that*

$$\sum\|\varphi_j^wu\|^2\leqq C\delta\sum\|\varphi_j^wPu\|^2+C\|u\|_{(-1/4)}^2, \tag{4.7}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ *has support where* $|x_0|\leqq\delta$.

Proof. We shall regard $\{q\}$, $\{r\}$ and $\{\psi_j\}$ as symbols with diagonal elements in $\mathscr{L}(\ell^2,\ell^2)$ as values, and $\{\varphi_j\}$ as having values in $\mathscr{L}(\ell^2,\mathbf{C})$ or $\mathscr{L}(\mathbf{C},\ell^2)$. In what follows, we shall often identify $\mathscr{L}(\ell^2, \mathbf{C})$ and $\mathscr{L}(\mathbf{C}, \ell^2)$ with ℓ^2. For example, when computing the symbol of the commutator $\{[\psi_j^w, \varphi_j^w]\}$, one term has values in $\mathscr{L}(\mathbf{C}, \ell^2)$ and the other in $\mathscr{L}(\ell^2, \mathbf{C})$ but we shall consider the symbol as having values in ℓ^2.

Now, the estimate (4.7) is stable for bounded perturbations so it suffices to prove the case $r=0$. In fact, if we have the estimate (4.7) for $P'=D_0+iq^w$ then we obtain

$$\sum \|\varphi_j^w u\|^2 \leqq \delta C \sum \|\varphi_j^w P' u\|^2 + C\|u\|^2_{(-1/4)}$$
$$\leqq \delta C'\big(\sum \|\varphi_j^w Pu\|^2 + \sum \|\varphi_j^w u\|^2 + \sum \|R_j^w u\|^2\big) + C\|u\|^2_{(-1/4)}.$$

Here $R_j^w = [\varphi_j^w, r^w]$ so $\{R_j\} \in S(H_1 h^{-\varepsilon}, G_1) \subseteqq S(h^{5/7-\varepsilon}, G_1)$, which implies that

$$\sum \|R_j^w u\|^2 \leqq C\|u\|^2_{(-1/4)}$$

if ε is small enough. For small δ we obtain the estimate for the perturbated operator.

Thus we assume that $P(x, D) = D_0 + iq^w(x, D')$ in what follows. Put $\Phi_j(x, \xi') = \exp(k\cdot x_0)\varphi_j(x', \xi')$. Then $\{\Phi_j\} \in S(h^{-\varepsilon}, G_1)$ uniformly when $|x_0| \leqq 1/k$. Since we use the calculus in $\mathbf{R}^n$, there is no difficulty in defining $\Phi_j^w = \exp(k\cdot x_0)\varphi_j^w$.

In what follows, we assume that $u \in \mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0| \leqq 1/k$. Now we have

$$(\Phi_j^w Pu, \Phi_j^w u) = ([\Phi_j^w, P]u, \Phi_j^w u) + (P\Phi_j^w u, \Phi_j^w u),$$

which implies that

$$\operatorname{Im}(\Phi_j^w Pu, \Phi_j^w u) = \operatorname{Re}([\Phi_j^w, q^w]u, \Phi_j^w u) + \operatorname{Im}([\Phi_j^w, D_0]u, \Phi_j^w u) + (q^w \Phi_j^w u, \Phi_j^w u), \tag{4.8}$$

because $q(x, \xi')$ is real, which makes $(q^w v, v)$ real.

Now $0 \leqq \psi_j q \in S(H^{-1}, G_1) = S(H_1^{-6/5}, G_1)$ so Proposition 4.1 gives

$$((\psi_j q)^w v, v) \geqq -C\|v\|^2 \quad \text{when} \quad v \in \mathscr{S}(\mathbf{R}^{n+1}).$$

The calculus gives that if $\bar{R}_j^w = (\psi_j q)^w \Phi_j^w - q^w \Phi_j^w$, then $\{\bar{R}_j\} \in S(H_1^2 H^{-1} h^{-\varepsilon}, G_1) \subseteqq S(h^{4/7-\varepsilon}, G_1)$, since $H_1^2 H^{-1} = H^{4/6} \leqq h^{4/7}$. Thus we get

$$\sum (q^w \Phi_j^w u, \Phi_j^w u) \geqq -C\sum \|\Phi_j^w u\|^2 - \sum (\bar{R}_j^w u, \Phi_j^w u) \geqq -C'\big(\sum \|\Phi_j^w u\|^2 + \|u\|^2_{(-1/4)}\big) \tag{4.9}$$

if ε is small enough.

The calculus with symbols with values in ℓ^2 and $\mathscr{L}(\ell^2, \ell^2)$ gives that the symbol of $\sum (\Phi_j^w)*[\Phi_j^w, q^w]$ is equal to $-i\sum \Phi_j\{\Phi_j, q\}$ apart from an error term in $S(h^{4/7-2\varepsilon}, G_1)$. (Here $\{\Phi_j, q\}$ is the Poisson bracket of Φ_j and q.) Since this symbol is imaginary, we find that

$$\operatorname{Re}\sum ([\Phi_j^w, q^w]u, \Phi_j^w u) = (R^w u, u) \tag{4.10}$$

where $R\in S(h^{4/7-2\varepsilon}, G_1)$. As mentioned before, this implies

$$|(R^w u, u)| \leqq C\|u\|^2_{(-1/4)} \tag{4.11}$$

if ε is small enough. Since $[\Phi_j^w, D_0]=ik\Phi_j^w$, we find that

$$\operatorname{Im}([\Phi_j^w, D_0]u, \Phi_j^w u) = k\|\Phi_j^w u\|^2. \tag{4.12}$$

Now, (4.8)—(4.12) imply

$$(k-C'-1)\sum\|\Phi_j^w u\|^2 \leqq \sum\|\Phi_j^w Pu\|^2+C''\|u\|^2_{(-1/4)} \tag{4.13}$$

if ε is small enough and $u=0$ when $|x_0|>1/k$.

We have $\Phi_j^w(x, D')=\exp(k\cdot x_0)\varphi_j^w(x', D')$ so if k is large enough and $u\in\mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq 1/k=\delta$, then we get the estimate (4.7). This completes the proof of the proposition.

Note that, by writing $-x_0$ instead of x_0, we get the same estimate if $\psi_j q$ is non-positive instead of non-negative.

Proposition 4.3. *Assume that*

$$P(x, D) = D_0+iq^w(x, D')+r^w(x, D')$$

where $q(x, \xi')$ *and* $r(x, \xi')\in C^\infty(\mathbf{R}^{2n+1})$, $q(x, \xi')$ *is real and does not change sign for fixed* (x', ξ'), *and for some constants a and b we have*

$$|D^\alpha_{x'} D^\beta_{\xi'} q| \leqq C_{\alpha,\beta}a^{1-|\alpha|}b^{1-|\beta|},$$

$$|D^\alpha_{x'} D^\beta_{\xi'} r| \leqq C'_{\alpha,\beta}a^{-|\alpha|}b^{-|\beta|},$$

where $a\cdot b\geqq 1$.

Then it follows that for sufficiently small δ *we have the estimate*

$$\|u\| \leqq \delta C_0\|Pu\| \tag{4.14}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ *and* $u=0$ *where* $|x_0|>\delta$.

Proof. As in the proof of Proposition 4.2, the estimate (4.14) is stable for bounded perturbations. Since the theorem of Calderón and Vaillancourt gives that r^w is bounded in L^2, it suffices to prove the proposition when $r=0$. By making a linear symplectic transformation, we can assume that $a=1$, and then the proof is given by Beals and Fefferman [1].

5. Uniform local solvability of the Hamilton operator

When it is possible to factor the imaginary part of the principal symbol, that is, case iii) in Proposition 3.3, then in order to localize the estimate of Beals—Fefferman (Proposition 4.3) we have to construct operators which approximately commute with P. This means that the corresponding symbol m must satisfy the equation

$$H_p m = 0 \quad \text{when} \quad p = 0, \tag{5.1}$$

where H_p is the Hamilton field of the principal symbol p. In the case

$$p(x, \xi) = \xi_0 + ia(x, \xi')b(x', \xi'), \quad (x_0, x') \in \mathbf{R}^{n+1},$$

where $a(x, \xi') \geqq 0$, we obtain $H_p = \partial_0 + iaH_b$ when $b=0$. We shall construct symbols m satisfying (5.1) when $b=0$. It will then be possible to estimate the corresponding commutator by cutting off near $b^{-1}(0)$ and using Proposition 4.2.

Since the purpose is to construct symbols, it is important to obtain non-trivial solutions to (5.1) which are defined in a fixed neighborhood of the origin and have fixed bounds on every x', ξ' derivative (in suitable coordinates). Hörmander [3, Section 4] has proved the existence of m satisfying (5.1) when $b=0$. In this section, we are going to make his results uniform.

Let $B^\infty(\mathbf{R}^k)$ be the Fréchet space of functions in $C^\infty(\mathbf{R}^k)$ with uniform bound in $\mathbf{R}^k$ for each derivative.

Definition 5.1. We say that $Q \in W$ if

$$Qu = \partial_0 u + ia(x)\partial_1 u$$

where $0 \leqq a(x) \in C^\infty(\mathbf{R}^{k+1})$ and $\{a(x_0, \cdot), x_0 \in \mathbf{R}\}$ is bounded in $B^\infty(\mathbf{R}^k)$. A set M in W is called bounded if we have uniform bounds in $B^\infty(\mathbf{R}^k)$ on $\{a(x_0, \cdot), x_0 \in \mathbf{R}\}$ when $\partial_0 + ia\partial_1 = Q \in M$.

In the case iii) in Proposition 3.3, it is clear that, in suitable G orthonormal coordinates, the Hamilton field on the bicharacteristics is bounded in W.

Since we are going to construct solutions which generate the analytic structure in the (reduced) bicharacteristics (see Hörmander [3, Section 4]) it is necessary to solve the inhomogeneous equation.

Proposition 5.2. *For each bounded set M in W, we can find a neighborhood Ω of the origin in $\mathbf{R}^{k+1}$, such that for each bounded set F_1 in $B^\infty(\mathbf{R}^k)$ there exists a bounded set F_2 in $B^\infty(\mathbf{R}^k)$, such that if $Q \in M$, $f \in C^\infty(\mathbf{R}^{k+1})$ and $f(x_0, \cdot) \in F_1$ when $x_0 \in \mathbf{R}$, then the equation*

$$Qu = f \quad \text{in } \Omega,$$

has a solution $u \in C^\infty(\mathbf{R}^{k+1})$ with the property that $u(x_0, \cdot) \in F_2$ when $x_0 \in \mathbf{R}$.

The proposition will be proved by using suitable *a priori* estimates and the Hahn—Banach theorem. To begin with we need the following L^2 estimate.

Lemma 5.3. *For each bounded set M in W there exists $\delta>0$ and a neighborhood U of the origin in $\mathbf{R}^k$ such that*

$$\|u\| \leqq C_0\|Q^* u\| \tag{5.2}$$

when $Q\in M$ and $u\in C_0^\infty((-\delta, \delta)\times U)$.

Proof of Lemma 5.3. First we observe that if $x''=(x_2, \ldots, x_k)$ then

$$\{a(x_0, \cdot, x''),\ (x_0, x'')\in\mathbf{R}^k\}$$

is uniformly bounded in $B^\infty(\mathbf{R})$ when $Q=\partial_0+ia\partial_1\in M$. Thus it suffices to prove the estimate (5.2) when $k=1$. In that case we put

$$\|u\|'_{(s)} = (2\pi)^{-1}\Big(\int |\hat{u}(\xi)|^2(1+\xi_1^2)^s\, d\xi\Big)^{1/2}$$

if $u\in\mathcal{S}(\mathbf{R}^2)$.

Choose $\varphi_j(t)\in C^\infty(\mathbf{R})$, $j=0, 1, 2$, such that $\sum\varphi_j^2=1$, φ_j has support when $(-1)^j t\geqq 1/2$, $j\neq 0$, and $|t|\leqq 1$ in $\operatorname{supp}\varphi_0$. It is obvious that $\varphi_j(\xi_1)\in S(1, g_0)$, where g_0 is the following metric in $T^*\mathbf{R}$,

$$g_{0, x_1, \xi_1}(t, \tau) = |t|^2+|\tau|^2/(1+\xi_1^2).$$

It follows that

$$\sum\|\varphi_j^w u\| \leqq C\|u\| \leqq C'\big(\sum\|\varphi_j^w u\|+\|u\|'_{(-1)}\big) \tag{5.3}$$

and

$$\|\varphi_0^w u\| \leqq C\|u\|'_{(-1)} \tag{5.4}$$

if $u\in C_0^\infty(\mathbf{R}^2)$.

Now we note that iQ^* fulfills the requirements on P in the case $n=1$ in Proposition 4.2. In addition, we have uniform bounds on the symbols when $Q\in M$, because $a(x_0, \cdot)$ is uniformly bounded in $B^\infty(\mathbf{R})$ then. Since $\varphi_j(\xi_1)$ has support when $(-1)^j\xi_1\geqq 1/2, j\neq 0,$ Proposition 4.2 gives, if $\delta>0$ is small enough, that

$$\|\varphi_j^w u\| \leqq C_0'(\|\varphi_j^w Q^* u\|+\|u\|'_{(-1/4)}) \tag{5.5}$$

if $u\in\mathcal{S}(\mathbf{R}^2)$ and $u=0$ where $|x_0|>\delta$.

If we combine (5.3)—(5.5), we obtain

$$\|u\| \leqq C_1'(\|Q^* u\|+\|u\|'_{(-1/4)}) \tag{5.6}$$

if $u\in\mathcal{S}(\mathbf{R}^2)$ and $u=0$ where $|x_0|>\delta$. Now, if $c>0$ is small enough, we have

$$\|u\|'_{(-1/4)} \leqq (2C_1')^{-1}\|u\|, \tag{5.7}$$

if $u\in C_0^\infty(\mathbf{R}^2)$ has support where $|x_1|<c$. In fact,

$$(\|u\|'_{(-1/4)})^2 = \int F(x_1-y_1)u(x_0, x_1)\bar{u}(x_0, y_1)\, dx_0\, dx_1\, dy_1,$$

where $F(x_1)\in L^1_{loc}(\mathbf{R})$ is the inverse Fourier transform of $(1+\xi_1^2)^{-1/4}$. If we choose c small enough, we get

$$\int_{-2c}^{2c} |F(t)|\,dt \leqq (2C_1')^{-2},$$

which proves (5.7). Now, by combining (5.6) and (5.7) we obtain (5.2), which proves Lemma 5.3.

Proof of Proposition 5.2. We shall use the $H'_{(s)}$-norms $\|\cdot\|'_{(s)}$, which were defined in Section 2. Now it is clear that, after multiplication with a suitable cut-off function, we may assume that f has compact support. Since $\{f(x_0, \cdot),\ x_0\in\mathbf{R}\}$ is contained in F_1, which is bounded in $B^\infty(\mathbf{R}^k)$, we find that f belongs to a bounded set $E\subseteqq H'_{(\infty)}=\cap H'_{(s)}$, that is,

$$(5.8) \qquad \|f\|'_{(s)} \leqq C'_s, \quad s\in\mathbf{R}.$$

Now, Lemma 5.3 gives $\delta>0$ and a neighborhood U of the origin in $\mathbf{R}^k$, such that

$$(5.9) \qquad \|u\| \leqq C_0\|Q^*u\|, \quad \text{when} \quad u\in C_0^\infty((-\delta,\delta)\times U).$$

Choose an open neighborhood V of the origin in $\mathbf{R}^k$, such that $\bar V\subset U$, and choose a fundamental decreasing system of neighborhoods of $\bar V$ in U,

$$\bar V\subset \ldots \subset\subset U_2 \subset\subset U_1 \subset\subset U_0 = U.$$

Let

$$m_N(\xi') = \prod_1^N (1+\varepsilon_j^2|\xi'|^2)^{1/2} = (E_N(\xi'))^{-1}$$

and $m_0=E_0=1$. By induction we are going to prove that we can choose $\varepsilon_j\in(0,1)$, such that for every N

$$(5.10) \qquad \|f\|_{m_N} = \|m_N^w f\| \leqq C_0' \prod_1^N (1+2^{-j}), \quad \text{when} \quad f\in E,$$

and

$$(5.11) \qquad \|u\|_{E_N} = \|E_N^w u\| \leqq C_0 \prod_1^N (1+2^{-j})\|Q^*u\|_{E_N},$$

when $u\in C_0^\infty((-\delta,\delta)\times U_N)$.

When $N=0$ these estimates follow from (5.8) and (5.9). Now, if (5.10) is fulfilled for some N, then

$$\|f\|^2_{m_{N+1}} \leqq \|f\|^2_{m_N} + \varepsilon^2_{N+1}(\|f\|'_{(N+1)})^2$$
$$\leqq C_0' \prod_1^N (1+2^{-j}) + (\varepsilon_{N+1}C'_{N+1})^2,$$

so by choosing ε_{N+1} small enough we obtain (5.10) with N replaced by $N+1$. To prove (5.11) we need the following lemma which will be proved later in this section.

Lemma 5.4. *Assume that for some* N

$$\|u\|_{E_N} \leqq K_N\|Q^*u\|_{E_N}, \quad \textit{when} \quad u\in C_0^\infty((-\delta,\delta)\times U_N).$$

Then for every $\varrho>0$ *there exists* $\varepsilon>0$, *such that*

$$\|u\|_{E_{N+1}} \leqq K_N(1+\varrho)\|Q^* u\|_{E_{N+1}},$$

when $u\in C_0^\infty((-\delta,\delta)\times U_{N+1})$, *if* $\varepsilon_{N+1}\leqq\varepsilon$.

End of proof of Proposition 5.2. By induction we obtain $\varepsilon_j\in(0,1)$, such that (5.10) and (5.11) are fulfilled for every N. With this choice of ε_j, let $F\subseteqq H'_{(\infty)}$ be the Banach space with the following norm:

$$\|u\|_m^2 = (2\pi)^{-k-1}\int |\hat{u}(\xi)|^2 m(\xi')^2\,d\xi,$$

where

$$m^2(\xi') = \Pi_1^\infty (1+\varepsilon_j^2|\xi'|^2),$$

which is then convergent for all ξ'. Then F is the dual space of $C_0^\infty(\mathbf{R}^{k+1})$ with the norm

$$\|v\|_{1/m}^2 = (2\pi)^{-k-1}\int |\hat{v}(\xi)|^2 m(\xi')^{-2}\,d\xi.$$

Letting $N\to\infty$ in (5.10) and (5.11) we find

(5.12) $$\|f\|_m \leqq eC_0', \quad \text{when } f\in E,$$

and

(5.13) $$\|v\|_{1/m} \leqq eC_0\|Q^* v\|_{1/m}, \quad \text{when } v\in C_0^\infty(\Omega),$$

where $\Omega=(-\delta,\delta)\times V$.

If we apply the Hahn—Banach theorem to the mapping

$$L\ni Q^* v \to (v,f)$$

where $L=\{Q^* v,\ v\in C_0^\infty(\Omega)\}$, we get $u\in F$ such that

(5.14) $$\|u\|_m \leqq eC_0\|f\|_m \leqq e^2 C_0 C_0',$$

and

(5.15) $$Qu = f \quad \text{in } \Omega.$$

This implies that u belongs to a bounded set in $H'_{(\infty)}$, since

$$\|u\|'_{(N)} \leqq (\varepsilon_1\ldots\varepsilon_N)^{-1}\|u\|_m, \quad N>0.$$

The equation (5.15) gives that $\partial_0 u$ has fixed bounds in $H'_{(\infty)}(\Omega)$, that is, locally in Ω. It is then clear that we get fixed bounds for each x' derivative of u locally in Ω. If we differentiate the equation (5.15), we successively find that each derivative of u is in $L^2(\Omega)$, which implies that $u\in C^\infty(\Omega)$. If we multiply u with a suitable cut-off function and choose a smaller Ω, we get $u\in C^\infty(\mathbf{R}^{k+1})$ satisfying (5.15), such that $u(x_0,\cdot)\in F_2$, a fixed bounded set in $B^\infty(\mathbf{R}^k)$, when $x_0\in\mathbf{R}$. This completes the proof of Proposition 5.2.

Proof of Lemma 5.4. We are going to use a metric g_ε in $T^*\mathbf{R}^k$, which depends on the parameter ε,

$$g_\varepsilon(t, \tau) = |t|^2+\varepsilon^2|\tau|^2/(1+\varepsilon^2|\xi'|^2) \quad \text{at} \quad (x', \xi').$$

We assume that $0<\varepsilon\leqq 1$. It is then easy to see that g_ε is uniformly σ temperate. We find

$$\sup g_\varepsilon/g_\varepsilon^\sigma = h_\varepsilon^2 = \varepsilon^2/(1+\varepsilon^2|\xi'|^2) \leqq \varepsilon^2 \quad \text{at} \quad (x', \xi').$$

It is easy to see that h_ε is uniformly σ, g_γ temperate when $\varepsilon\leqq\gamma$, which implies that $E_N=\prod_1^N (h_{\varepsilon_j}/\varepsilon_j)$ is a weight for g_1.

Choose $\psi(x')\in C_0^\infty(U_N)$ such that $0\leqq\psi\leqq 1$ and $\psi(x')=1$ when $x'\in U_{N+1}$. Let

$$F_\varepsilon(\xi') = (1+\varepsilon^2|\xi'|^2)^{-1/2} = h_\varepsilon(\xi')/\varepsilon,$$

and $v=\psi F_\varepsilon^w u$, when $u\in C_0^\infty((-\delta, \delta)\times U_{N+1})$. Then $v\in C_0^\infty((-\delta, \delta)\times U_N)$, and, if $E_{N+1}=E_N\cdot F_\varepsilon$,

$$E_N^w v = E_N^w[\psi, F_\varepsilon^w]u+E_{N+1}^w u,$$

since $\psi=1$ on $\operatorname{supp} u$. The symbol of $[\psi, F_\varepsilon^w]$ is bounded in $S(h_\varepsilon F_\varepsilon, g_\varepsilon)$, uniformly in ε, which implies that the symbol of $E_N^w[\psi, F_N^w](E_{N+1}^{-1})^w$ is bounded in $S(h_\varepsilon, g_1)$ uniformly with respect to ε. Since $h_\varepsilon\leqq\varepsilon$ it follows that

(5.16) $$\|u\|_{E_{N+1}} \leqq \|v\|_{E_N}+\varepsilon C\|u\|_{E_{N+1}} \quad \text{when} \quad \varepsilon\leqq 1.$$

Since $v\in C_0^\infty((-\delta, \delta)\times U_N)$ the hypothesis of the lemma gives

(5.17) $$\|v\|_{E_N} \leqq K_N\|Q^*v\|_{E_N}.$$

We are now going to estimate the right-hand side of (5.17). Since $\psi=1$ on $\operatorname{supp} u$ we find

(5.18) $$E_N^w Q^*v = E_N^w Q^*[\psi, F_\varepsilon^w]u+E_N^w[Q^*, F_\varepsilon^w]u+E_{N+1}^w Q^*u.$$

The first term on the right-hand side of (5.18) is

(5.19) $$E_N^w Q^*[\psi, F_\varepsilon^w]u = E_N^w\big[Q^*, [\psi, F_\varepsilon^w]\big]u+E_N^w[\psi, F_\varepsilon^w]Q^*u.$$

Since the symbol of $[\psi, F_\varepsilon^w]$ is uniformly bounded in $S(h_\varepsilon F_\varepsilon, g_\varepsilon)$, and $\xi_1\in S(h_\varepsilon^{-1}, g_\varepsilon)$ uniformly in ε, we obtain that the symbol of $E_N^w\big[Q^*, [\psi, F_\varepsilon^w]\big]$ is uniformly bounded in $S(h_\varepsilon E_{N+1}, g_1)$. This implies

(5.20) $$\big\|E_N^w\big[Q^*, [\psi, F_\varepsilon^w]\big]u\big\| \leqq \varepsilon C\|u\|_{E_{N+1}}.$$

As before, we find

(5.21) $$\|E_N^w[\psi, F_\varepsilon^w]Q^*u\| \leqq \varepsilon C\|Q^*u\|_{E_{N+1}}.$$

The second term on the right-hand side of (5.18) is

(5.22) $$E_N^w[Q^*, F_\varepsilon^w]u = \big[E_N^w, [Q^*, F_\varepsilon^w]\big]u+[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w E_{N+1}^w u.$$

The symbol of $[Q^*, F_\varepsilon^w]$ is in $S(h_\varepsilon F_\varepsilon, g_\varepsilon)$ apart from the first term, which is $i\{a(x), F_\varepsilon(\xi')\}\xi_1 = b_\varepsilon(x, \xi')\xi_1$, where $b_\varepsilon \in S(h_\varepsilon F_\varepsilon, g_\varepsilon)$ uniformly in ε and x_0. Then the symbol of

$$[E_N^w, [Q^*, F_\varepsilon^w]] - [E_N^w, b_\varepsilon^w] D_1$$

is uniformly bounded in $S(h_\varepsilon E_{N+1}, g_1)$ when $\varepsilon \leqq 1$. Since the symbol of $[E_N^w, b_\varepsilon^w]$ is uniformly bounded in $S(h_1 E_N h_\varepsilon F_\varepsilon, g_1)$, the symbol of $[E_N^w, [Q^*, F_\varepsilon^w]]$ is uniformly bounded in $S(h_\varepsilon E_{N+1}, g_1)$, which implies

$$\| [E_N^w, [Q^*, F_\varepsilon^w]] u \| \leqq \varepsilon C \|u\|_{E_{N+1}}. \tag{5.23}$$

To estimate the second term on the right-hand side of (5.22) we need the following

Lemma 5.5. *Assume that M is a bounded set in W. Then*

$$\|[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w v\| \leqq C\varepsilon^{1/4}(\|Q^* v\| + \|v\|) \tag{5.24}$$

if $v \in \mathscr{S}(\mathbf{R}^{k+1})$, $Q \in M$ *and* $F_\varepsilon(\xi') = (1+\varepsilon^2|\xi'|^2)^{-1/2}$.

End of proof of Lemma 5.4. The estimate (5.24) implies

$$\|[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w E_{N+1}^w u\| \leqq C'\varepsilon^{1/4}(\|Q^* u\|_{E_{N+1}} + \|u\|_{E_{N+1}}) \tag{5.25}$$

since the symbol of the commutator $[Q^*, E_{N+1}^w]$ is uniformly bounded in $S(E_{N+1}, g_1)$.

If we combine the estimates above, we obtain for some constant C

$$\|Q^* v\|_{E_N} \leqq (1+C\varepsilon^{1/4})\|Q^* u\|_{E_{N+1}} + C\varepsilon^{1/4}\|u\|_{E_{N+1}}. \tag{5.26}$$

Together with (5.16) and (5.17), this implies

$$\|u\|_{E_{N+1}} \leqq K_N(1+C\varepsilon^{1/4})\|Q^* u\|_{E_{N+1}} + C'\varepsilon^{1/4}\|u\|_{E_{N+1}}. \tag{5.27}$$

If $\varepsilon \leqq 1$ is sufficiently small, then

$$\|u\|_{E_{N+1}} \leqq K_N(1+C''\varepsilon^{1/4})\|Q^* u\|_{E_{N+1}},$$

which proves Lemma 5.4.

Proof of Lemma 5.5. We shall use the norms

$$\|u\|_{(s,\varepsilon)} = \|(h_\varepsilon^{-s})^w u\|, \quad s \in \mathbf{R}.$$

Then we have

$$\|u\|_{(-s,\varepsilon)} \leqq \varepsilon^s \|u\|, \quad \text{if} \quad s \geqq 0.$$

We shall prove the estimate

$$\|[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w v\| \leqq C(\|Q^* v\|_{(-1/4,\varepsilon)} + \|v\|_{(-1/4,\varepsilon)}) \tag{5.28}$$

when $v \in \mathscr{S}(\mathbf{R}^{k+1})$. This will give us (5.24).

The symbol of $[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w$ is in $S(h_\varepsilon, g_\varepsilon)$ apart from the first term, which is

$$A_\varepsilon(x, \xi') = i\{a(x), F_\varepsilon(\xi')\}\xi_1/F_\varepsilon(\xi')$$
$$= i \sum_k \partial_{x_k} a(x)\xi_k \xi_1 h_\varepsilon^2(\xi').$$

Thus, we find

(5.29) $$\|[Q^*, F_\varepsilon^w](F_\varepsilon^{-1})^w v\| \leqq \|A_\varepsilon^w v\| + C\|v\|_{(-1,\varepsilon)}.$$

Since $0 \leqq a \leqq C$ and $|\partial_x^2, a| \leqq C$, it follows that

$$|\text{grad } a| \leqq 2C^{1/2}a^{1/2}.$$

We are going to prove (5.28) by estimating an approximate square root of $a(x)\xi_1$. The motivation for this is that, since

$$\text{Re}(Q^* u, u) = -(aD_1 u, u) - (1/2)((D_1 a)u, u)$$

we obtain

(5.30) $$|(aD_1 u, u)| \leqq C(\|Q^* u\|^2 + \|u\|^2).$$

We can choose $\psi_j(\xi') \in S(1, h_\varepsilon^{-1/2} g_\varepsilon)$, $j=0, 1, 2$, such that $\sum \psi_j^2(\xi')=1$, ψ_j has support where $(-1)^j \xi_1 \geqq c \cdot h_\varepsilon^{-3/4}$ and ψ_0 has support where $|\xi_1| \leqq C \cdot h_\varepsilon^{-3/4}$. In fact, with $\varphi_j \in C^\infty(\mathbf{R})$ as in the proof of Lemma 5.3, one may use $\psi_j(\xi') = \varphi_j(\xi_1 h_\varepsilon^{3/4}(\xi'))$. It is easy to see that $h_\varepsilon^{-1/2} g_\varepsilon$ is uniformly σ temperate when $0 < \varepsilon \leqq 1$. Note that $h_\varepsilon^{-1/2} g_\varepsilon = h_\varepsilon^{-1/2}|dx'|^2 + h_\varepsilon^{3/2}|d\xi'|^2$ so the quotient with the dual metric is $(h_\varepsilon^{1/2})^2$.

We now partition the symbol

$$A_\varepsilon^w = \sum (A_\varepsilon \psi_j^2)^w.$$

It is clear that $\xi_1 \psi_0^2(\xi') \in S(h_\varepsilon^{-3/4}, h_\varepsilon^{-1/2} g_\varepsilon)$, since $|\xi_1| \leqq C \cdot h_\varepsilon^{-3/4}$ in $\text{supp}\, \psi_0$ and differentiation cannot lead to loss of more than one such favorable factor. Thus, $A_\varepsilon \psi_0^2 \in S(h_\varepsilon^{1/4}, h_\varepsilon^{-1/2} g_\varepsilon)$, which implies

(5.31) $$\|(A_\varepsilon \psi_0^2)^w v\| \leqq C\|v\|_{(-1/4,\varepsilon)}.$$

Now we factor $A_\varepsilon \psi_j^2$ when $j \neq 0$. We have

$$B_j(\xi') = ((-1)^j \xi_1)^{1/2} \psi_j(\xi') \in S(h_\varepsilon^{-1/2}, h_\varepsilon^{-1/2} g_\varepsilon)$$

since $(-1)^j \xi_1 \geqq c h_\varepsilon^{-3/4} \geqq c$ in $\text{supp}\, \psi_j$. Let

$$C_{j,k}(\xi') = B_j(\xi')\xi_k h_\varepsilon^2(\xi') \in S(h_\varepsilon^{1/2}, h_\varepsilon^{-1/2} g_\varepsilon), \quad j \neq 0.$$

Then

$$i \sum_k (\partial_{x_k} a) B_j C_{j,k} = (-1)^j A_\varepsilon \psi_j^2,$$

and the symbol of $i \sum_k (\partial_{x_k} a) B_j^w C_{j,k}^w$ is equal to $(-1)^j A_\varepsilon \psi_j^2$, $j \neq 0$, apart from terms in $S(h_\varepsilon^{1/2}, h_\varepsilon^{-1/2} g_\varepsilon)$. Thus

(5.32) $$\|A_\varepsilon^w v\| \leqq \sum_{j \neq 0} \|(\partial_{x_k} a) B_j^w C_{j,k}^w v\| + C\|v\|_{(-1/4,\varepsilon)}.$$

Now, we have $|\partial_{x_k} a| \leqq 2C^{1/2}a^{1/2}$, and

$$\|a^{1/2}B_j^w u\|^2 = (aB_j^w u, B_j^w u) = ([B_j^w, a]B_j^w u, u)$$
$$+(-1)^j\big(([a, \psi_j^w]D_1\psi_j^w u, u)+(aD_1\psi_j^w u, \psi_j^w u)\big),$$

where the symbols of $[B_j^w, a]$ and $[a, \psi_j^w]$ are in $S(1, h_\varepsilon^{-1/2}g_\varepsilon)$ and $S(h_\varepsilon^{1/2}, h_\varepsilon^{-1/2}g_\varepsilon)$ respectively. If we use the estimate (5.30) we find

$$\|a^{1/2}B_j^w u\| \leqq C(\|Q^*\psi_j^w u\|+\|u\|_{(1/4,\varepsilon)})$$
$$\leqq C'(\|Q^* u\|+\|u\|_{(1/4,\varepsilon)}),$$

since the commutator $[Q^*, \psi_j^w]$ is in $S(1, h_\varepsilon^{-1/2}g_\varepsilon)$ apart from the first term, which is $i\{a, \psi_j\}\xi_1 \in S(h_\varepsilon^{-1/4}, h_\varepsilon^{-1/2}g_\varepsilon)$. This implies

$$\|(\partial_{x_k}a)B_j^w C_{j,k}^w v\| \leqq C(\|Q^* C_{j,k}^w v\|+\|C_{j,k}^w v\|_{(1/4,\varepsilon)})$$
$$\leqq C'(\|Q^* v\|_{(-1/4,\varepsilon)}+\|v\|_{(-1/4,\varepsilon)}). \tag{5.33}$$

In fact, $C_{j,k} \in S(h_\varepsilon^{1/2}, h_\varepsilon^{-1/2}g_\varepsilon)$ so the symbol of the commutator $[Q^*, C_{j,k}^w]$ is in $S(h_\varepsilon^{1/2}, h_\varepsilon^{-1/2}g_\varepsilon)$ apart from the first term, which is $i\{a, C_{j,k}\}\xi_1 \in S(h_\varepsilon^{1/4}, h_\varepsilon^{-1/2}g_\varepsilon)$.

Combining (5.29), (5.32) and (5.33), we get (5.28), which proves Lemma 5.5.

We shall now state the result, which we are going to use in later sections.

Corollary 5.6. *For each bounded set M in W, we can find a neighborhood Ω of the origin in $\mathbf{R}^{k+1}$ and a bounded set F in $B^\infty(\mathbf{R}^k)$, so that if $Q \in M$ then there exists a solution $u \in C^\infty(\mathbf{R}^{k+1})$ to the equation*

$$Qu = \partial_0 u + ia\partial_1 u = 0 \quad \text{in } \Omega, \tag{5.34}$$

such that $\partial_1 u = \exp(w)$ in Ω, where $|w| < \pi/6$, and $u(x_0, \cdot) \in F$ for all x_0. We also get a fixed bound on grad u *and* grad $(\partial_1 u)$ *in Ω.*

Proof. First we note that if u solves (5.34) and $\partial_1 u = \exp(w)$, where $|w| < \pi/6$, then we get a fixed bound on $\partial_0 u$ in Ω. If we differentiate the equation (5.34) with respect to x_1, letting $v = \partial_1 u$, then we get the equation

$$Qv = \partial_0 v + ia\partial_1 v = -i(\partial_1 a)v \quad \text{in } \Omega, \tag{5.35}$$

which we want to solve with $v = \exp(w) \in C^\infty(\mathbf{R}^{k+1})$, where $|w| < \pi/6$ in Ω and $w(x_0, \cdot)$ is contained in a bounded set in $B^\infty(\mathbf{R}^k)$. As before, we also get a bound on $\partial_0 v$ then.

If this is possible, then by integrating (5.35) with respect to x_1 in a smaller neighborhood Ω putting

$$u(x) = \int_0^{x_1} v(x_0, t, x'')\,dt,$$

we get

$$Qu = \partial_0 u + ia\partial_1 u = f \quad \text{in } \Omega,$$

where $f(x)=i(av)(x_0, 0, x'')\in C^\infty(\mathbf{R}^{k+1})$ is independent of x_1, and $f(x_0, \cdot)$ is bounded in $B^\infty(\mathbf{R}^k)$ when $x_0\in\mathbf{R}$. Then

$$u(x)-\int_0^{x_0} f(t, x'')\,dt,$$

multiplied by a suitable cut-off function, is a solution to (5.34) with the desired properties if we take a smaller Ω.

Thus, it suffices to solve (5.35) with $v=\exp(w)$. Then we obtain the following equation for w

$$Qw = \partial_0 w+ia\partial_1 w = -i\partial_1 a \quad \text{in } \Omega, \tag{5.36}$$

and we want to find a solution $w\in C^\infty(\mathbf{R}^{k+1})$, such that $|w|<\pi/6$ in Ω and $w(x_0, \cdot)$ is contained in a bounded set in $B^\infty(\mathbf{R}^k)$.

Now, since $Q\in M$, $\partial_1 a(x_0, \cdot)$ is contained in a bounded set in $B^\infty(\mathbf{R}^k)$, so Proposition 5.2 gives a neighborhood Ω of the origin in $\mathbf{R}^{k+1}$ and a solution $w(x)\in C^\infty(\mathbf{R}^{k+1})$ to (5.36) such that $w(x_0, \cdot)$ is contained in a bounded set in $B^\infty(\mathbf{R}^k)$. As before, we get a bound on $\operatorname{grad} w$ in Ω. By subtracting $w(0)$ and taking a smaller Ω, we obtain $|w|<\pi/6$ in Ω. Then $v=\exp(w)$ is a solution to (5.35) with the desired properties. This completes the proof of Corollary 5.6.

6. The construction of a local weight function

In this section, we continue the work in Section 5 to construct special solutions to the equation

$$Qv = \partial_0 v+ia\partial_1 v = 0 \tag{6.1}$$

when Q is in a bounded set in W (see Definition 5.1). According to Hörmander [3, Theorem 4.6], such a solution must be an analytic function of the solution u to (6.1) given by Corollary 5.6. Thus, we have to construct analytic functions with certain properties in varying domains.

Proposition 6.1. *For each bounded set M in W and constants $\varepsilon, c>0$ there exist positive constants $\delta, \varrho, C_\alpha, c_0<c_1<c_2<c$, a neighborhood*

$$\Omega = \{x\in\mathbf{R}^{k+1}\colon\ |x_0| < \delta,\ |x_1| < c_2,\ |x''| < \varrho\}$$

of the origin, and for each $Q\in M$ a solution $v\in C^\infty(\Omega)$ to the equation

$$Qv = \partial_0 v+ia\partial_1 v = 0 \quad \textit{in } \Omega,$$

such that

(6.2) $$\operatorname{Re} v \geqq 0 \quad in\ \Omega,$$

(6.3) $$\operatorname{Re} v \leqq \varepsilon \quad in\ \{x\in\Omega\colon\ |x_1| \leqq c_0\},$$

(6.4) $$\operatorname{Re} v \geqq 1 \quad in\ \{x\in\Omega\colon\ |x_1| \geqq c_1\},$$

and

(6.5) $$|D^{\alpha}_{x'} v| \leqq C_\alpha \quad in\ \Omega\ for\ all\ \alpha,$$

which implies that $|D_0 v| \leqq C$ *in* Ω.

When we prove Proposition 6.1 it is no restriction to assume that Ω is contained in the neighborhood given by Corollary 5.6. Thus we may assume that for each $Q\in M$ we can find $u\in C^\infty(\mathbf{R}^{k+1})$ satisfying (6.1) in Ω, such that $\partial_1 u = \exp(w)$ in Ω, where $|w|<\pi/6$, and we have a fixed bound for each x' derivative of u. We choose $\varkappa \leqq c$ so small that u is defined in the set

$$\{x\in\mathbf{R}^{k+1}\colon\ \max(|x_0|, |x_1|, |x''|) \leqq \varkappa\}.$$

When $Q\in M$ we take this solution $u=u_Q$ to (6.1) and let

$$\omega_d = \{u(x)\in\mathbf{C}\colon\ |x_0| \leqq \delta,\ |x_1| \leqq d \text{ and } x'' = 0\},$$

which is defined when δ and d are less than $\varkappa$. When $K\subseteq\mathbf{C}$ we denote by $N_\lambda(K)$ the set of points in $\mathbf{C}$ having euclidean distance to K, $\operatorname{dist}(z, K)$, less than λ.

We shall prove Proposition 6.1 by constructing the solution v when $x''=0$ and then perturb with the parameters x''. As indicated before we shall do this by constructing analytic functions in $N_\lambda(\omega_{c_2})$.

Proposition 6.2. *For each bounded set M in W and constant $\varepsilon>0$ there exist positive constants $\delta<\varkappa$, λ, C_k and $c_0<c_1<c_2<\varkappa$ such that for each $Q\in M$ there exists an analytic function $f(z)$ in $N_\lambda(\omega_{c_2})$ such that*

(6.6) $$\operatorname{Re} f \geqq 0 \quad in\ N_\lambda(\omega_{c_2}),$$

(6.7) $$\operatorname{Re} f \leqq \varepsilon \quad in\ N_\lambda(\omega_{c_0}),$$

(6.8) $$\operatorname{Re} f \geqq 1 \quad in\ N_\lambda(\omega_{c_2}\setminus\omega_{c_1}),$$

and

(6.9) $$|D^k_z f| \leqq C_k \quad in\ N_\lambda(\omega_{c_2})\ for\ all\ k \geqq 0.$$

Proof that Proposition 6.2 implies Proposition 6.1. Let $v(x)=f(u(x))$. Then v is defined in Ω if ϱ is small enough. Since we have a fixed bound on $\operatorname{grad} u$ we can choose ϱ independent on $Q\in M$ so that $u(x)\in N_\lambda(\omega_{c_2})$ when $x\in\Omega$, $u(x)\in N_\lambda(\omega_{c_0})$ if $x\in\Omega$ and $|x_1|\leqq c_0$ and $u(x)\in N_\lambda(\omega_{c_2}\setminus\omega_{c_1})$ if $x\in\Omega$ and $|x_1|\geqq c_1$. Then (6.6)—(6.9) imply (6.2)—(6.5), which proves that Proposition 6.2 implies Proposition 6.1.

Proof of Proposition 6.2. Let

$$\gamma_j = \{u(x)\in\mathbf{C}\colon |x_0| \leqq \delta,\ x_1 = (-1)^j \varkappa,\ x'' = 0\},\quad j = 1, 2,$$

and $\gamma=\gamma_1\cup\gamma_2$. At first sight it seems natural to take $\operatorname{Re} f$ equal to the harmonic measure of γ. However, since we have no lower bound on the arc length of γ, we must modify the construction.

Let $\Gamma=\Gamma_1\cup\Gamma_2$, where Γ_j is the union of γ_j and the line segment $\{u(\delta, (-1)^j\varkappa, 0)+i\delta t\colon t\in(0, 1)\}$ of length δ. We need the following geometrical result.

Lemma 6.3. *With the definitions above, we have*

i) $\delta\leqq|\Gamma_j|\leqq C\delta$, *where* $|\Gamma_j|$ *is the arc length of* Γ_j,

ii) $(\varkappa-d)/2e\leqq\operatorname{dist}(z, \Gamma)\leqq\operatorname{dist}(z, \gamma)\leqq e(\varkappa-d)$ *when* $z=u(\bar{x}_0, \bar{x}_1, 0)$ *where* $|\bar{x}_0|\leqq\delta$ *and* $|\bar{x}_1|=d\leqq\varkappa$.

Proof of the lemma. If we parametrize γ_j with $[-\delta, \delta]\ni t\to u(t, (-1)^j\varkappa, 0)\in\gamma_j$, we obtain that $|\gamma_j|\leqq C\delta$, since $\partial_0 u$ has a fixed bound then. Now we have $|\Gamma_j|=|\gamma_j|+\delta$, which gives i).

Let $w_j=u(\bar{x}_0, (-1)^j\varkappa, 0)\in\gamma_j, j=1, 2$. Then the arc length of the curve between z and w_j,

$$[0, 1]\ni s \to u(\bar{x}_0, x_1+s((-1)^j\varkappa-\bar{x}_1), 0),\quad j = 1, 2, \tag{6.10}$$

is bounded by $e|\bar{x}_1-(-1)^j\varkappa|$ since $|\partial_1 u|<e$ then. Now, the distance between z and γ must be shorter, which proves the upper bound in ii).

To get the lower bound we observe that since the argument $\operatorname{Arg}(\partial_1 u)$ of $\partial_1 u$ has absolute value less than $\pi/6$ and $|\partial_1 u|>e^{-1}$, we obtain $\operatorname{Re}(\partial_1 u)>3^{1/2}/2e$. By projecting the curves (6.10) on the real axis, we find

$$\operatorname{Re}\big((-1)^j(w_j-z)\big) > 3^{1/2}|\bar{x}_1-(-1)^j\varkappa|/2e,\quad j = 1, 2. \tag{6.11}$$

Since $|\operatorname{Arg}(\partial_1 u)|<\pi/6$ we have

$$\big|\operatorname{Arg}\big((-1)^j(w_j-z)\big)\big| < \pi/6,\quad j = 1, 2. \tag{6.12}$$

We also obtain

$$\pi/3 \leqq |\operatorname{Arg}(w-w_j)| \leqq 2\pi/3 \tag{6.13}$$

when $w\in\Gamma_j\setminus w_j$, since $\partial_0 u=-ia\partial_1 u$, where $a\geqq 0$.

If we combine (6.11)—(6.13) we find that

$$\operatorname{dist}(z, w) \geqq \big(3^{1/2}(\varkappa-|\bar{x}_1|)/2e\big)\tan(\pi/6) = (\varkappa-d)/2e$$

when $w\in\Gamma$, which proves Lemma 6.3.

End of proof of Proposition 6.2. Put

$$f(z) = 2^{3/2}\delta^{-1/2}\Big(\int_{\Gamma_1}(z-w(s))^{-1/2}\,ds+\int_{\Gamma_2}(w(s)-z)^{-1/2}\,ds\Big)$$

where we have chosen the branch of the square root in $\mathbf{C}\setminus\mathbf{R}^-$ which is real on $\mathbf{R}^+$ and we have integrated with respect to the arc length. We find that f is analytic in a neighborhood of ω_d, if $d<\varkappa$. If $z\in\omega_{c_0}$, $c_0<\varkappa$, then dist $(z, \Gamma)\geqq(\varkappa-c_0)/2e>0$. Thus we obtain that dist $(z, \Gamma)\geqq(\varkappa-c_0)/3e$ if $z\in N_\lambda(\omega_{c_0})$ and λ is small enough. This implies

$$\operatorname{Re} f(z) \leqq C\delta^{-1/2}(\varkappa-c_0)^{-1/2}\delta \leqq C\delta^{1/2}(\varkappa-c_0)^{-1/2},$$

when $z\in N_\lambda(\omega_{c_0})$. For fixed $c_0<\varkappa$ this can be made smaller than ε by choosing δ small enough. We fix $c_0<\varkappa$ and $\delta<\varkappa$ so that this is the case.

Choose c_1 and c_2 so that $\max(c_0, \varkappa-\delta/e)<c_1<c_2<\varkappa$. Since

$$\left|\operatorname{Arg}\left((-1)^j(w-z)\right)\right| < 2\pi/3$$

and $|w-z|\geqq(\varkappa-c_2)/2e$ when $z\in\omega_{c_2}$ and $w\in\Gamma_j$, we find that $\operatorname{Re} f\geqq 0$ in $N_\lambda(\omega_{c_2})$ if λ is small enough. As above we can estimate

$$|D_z^k f(z)| \leqq C_k\delta^{1/2}r^{-1/2-k} \quad \text{when} \quad \operatorname{dist}(z, \Gamma) \geqq r,$$

so it is clear that we get (6.9) for sufficiently small λ. It remains to prove (6.8) for small λ. If $z\in\omega_{c_2}\setminus\omega_{c_1}$ then Lemma 6.3 gives that dist $(z, \gamma)<e(\varkappa-c_1)<\delta$. Thus, for sufficiently small λ we find that dist $(z, \gamma)<\delta$ if $z\in N_\lambda(\omega_{c_2}\setminus\omega_{c_1})$. So to prove (6.8) it suffices to show that $\operatorname{Re} f(z)\geqq 1$ if dist $(z, \gamma)<\delta$ and $\left|\operatorname{Arg}\left((-1)^j(w-z)\right)\right|< 2\pi/3$ for all $w\in\Gamma_j$.

In this case we can find $w_0\in\gamma$ such that $|z-w_0|\leqq\delta$. Assume for example that $w_0\in\gamma_1$; the same argument works if $w_0\in\gamma_2$. Let γ_0 be the component of $\Gamma_1\setminus\{w_0\}$ which has arc length at least δ. Since $|\operatorname{Arg}(\partial_1 u)|<\pi/6$ and $\partial_0 u=-ia\partial_1 u$, we can parametrize γ_0 with the distance to w_0.

Since $\left|\operatorname{Arg}\left((z-w)^{-1/2}\right)\right|<\pi/3$ when $w\in\gamma_0$, we find

$$2\operatorname{Re}\left((z-w)^{-1/2}\right) \geqq |z-w|^{-1/2} \geqq (|z-w_0|+|w-w_0|)^{-1/2}$$

when $w\in\gamma_0$. This implies

$$\operatorname{Re} f(z) \geqq 2^{3/2}\delta^{-1/2}\int_{\gamma_0}\operatorname{Re}\left((z-w(t))^{-1/2}\right)dt$$

$$\geqq (2/\delta)^{1/2}\int_0^\delta(\delta+t)^{-1/2}\,dt \geqq 1$$

since $|z-w_0|\leqq\delta$. This completes the proof of Proposition 6.2.

7. Proof of Proposition 2.7

Let $P(x, D)=D_0+iF^w(x, D')+R^w(x, D')$, where F and $R\in C^\infty(\mathbf{R}^{2n+1})$ are bounded functions of $x_0\in\mathbf{R}$ with values in $S(h^{-1}, g)$ and $S(1, g)$ respectively, F is real and does not change sign for fixed (x', ξ'). Here the metric g is σ temperate, conformal to the metric $|dx'|^2+|d\xi'|^2/(1+|\xi'|^2)$ in $T^*\mathbf{R}^n$, which we denote by g_0, so that $g=m\cdot g_0$, where $1\leqq m^2(\xi')\leqq(1+|\xi'|^2)$ is independent of the x' variables. Then we find that $h^2=\sup g/g^\sigma\leqq 1$. As in Sections 2 and 4, we shall use the norms

$$\|u\|_{(s)}=\left((2\pi)^{-n-1}\int|\hat{u}(\xi')|^2h(\xi')^{-2s}\,d\xi\right)^{1/2}$$

when $u\in\mathcal{S}(\mathbf{R}^{n+1})$. Assume that φ and $\psi\in S(1, g)$ and that $\psi=1$ on $\operatorname{supp}\varphi$. Then we shall prove that for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\|\varphi^w u\|\leqq C(\|\psi^w Pu\|_{(\varepsilon)}+\|u\|_{(-1/15)}+\|Pu\|_{(-16/15)}) \tag{7.1}$$

if $u\in\mathcal{S}(\mathbf{R}^n)$ has support where $|x_0|\leqq\delta$. We shall do this by localizing in the Beals—Fefferman metric defined in Section 3 and using the estimates in Section 4. It will follow from the proof that δ and C do not depend on P and g as long as the symbol of P has a fixed bound on every seminorm and g is uniformly σ temperate.

Since $F(x, \xi')\in S(h^{-1}, g)$ uniformly in x_0, we may normalize F so that $|F|\leqq h^{-1}$ and $|F|_1^g\leqq h^{-1}$. We define the Beals—Fefferman metric $G=Hg/h$, where H is defined by

$$H^{-1}=\max\left(h^{-6/7}, \sup_{x_0}|F|, (\sup_{x_0}|F|_1^g)^2h\right).$$

Then $h\leqq H\leqq h^{6/7}\leqq 1$ so $g\leqq G\leqq h^{-1/7}g$ and $\sup G/G^\sigma=H^2\leqq 1$. Proposition 3.1 gives that G is uniformly σ temperate if this is the case for g, and F has a fixed bound on every seminorm in $S(h^{-1}, g)$. Then we also obtain that F has a fixed bound on every seminorm in $S(H^{-1}, G)$, and since $g\leqq G$ we find that a bounded set in $S(1, g)$ is uniformly bounded in $S(1, G)$.

Choose a symbol $\{\varphi_k\}\in S(1, G)$ with values in ℓ^2 so that $\sum\varphi_k^2=1$ on $\operatorname{supp}\varphi$. We can choose the support of φ_k so close to (x_k', ξ_k') that G only varies with a fixed factor in $\operatorname{supp}\varphi_k$. If we put $G_k=G_{x_k', \xi_k'}$ and compose suitable cut-off functions with the G_k distance to (x_k', ξ_k'), it follows from [4, Lemma 2.5] that we can construct symbols $\{\psi_k\}$ and $\{\chi_k\}\in S(1, G)$ with values in ℓ^2, such that $\psi_k=1$ on $\operatorname{supp}\varphi_k$ and $\chi_k=1$ on $\operatorname{supp}\psi_k$.

It is clear that we can choose these symbols non-negative and with support so small that G and g only vary with a fixed factor in $\operatorname{supp}\chi_k$ and that we have one of the cases i)—iii) in Proposition 3.3 there. By shrinking the supports we may assume that $\psi\geqq 1/2$ on $\operatorname{supp}\chi_k$, for all k. Later in the proof (see case III) below) we shall pose additional restrictions on $\operatorname{supp}\varphi_k$, but this will only change the seminorms of the symbols (see [4, Lemma 2.5]).

Let $P_k(x, D)=D_0+iF_k^w(x, D')+R_k^w(x, D')$ where $F_k=\chi_k F$ and $R_k=\chi_k R$. Then it is clear that $F_k \in S(H^{-1}, G)$ and $R_k \in S(1, G)$ uniformly in x_0 and k. In what follows, we shall often consider $\{P_k\}$ as an operator with symbol having diagonal elements in $\mathscr{L}(\ell^2, \ell^2)$ as values.

Since the metric G is conformal to the metric g_0, it follows that the operators P_k satisfy the conditions in Proposition 4.3. In fact, we can take $a=(G_k(e, 0))^{-1/2}$ and $b=(G_k(0, e))^{-1/2}$, where e is an arbitrary unit vector in $\mathbf{R}^n$, since then $a\cdot b= H_k^{-1}=H^{-1}(x_k', \xi_k')\geqq 1$.

Now, the choice of χ_k and Proposition 3.3 imply that we have one of the following cases

I) $h^{6/7}/C\leqq H\leqq h^{6/7}$ in $\operatorname{supp} F_k$,

II) F_k has constant sign,

III) $F_k(x, \xi')=a_k(x, \xi')b_k(x', \xi')$, where $0\leqq a_k\in C^\infty(\mathbf{R}^{2n+1})$ is uniformly bounded in $S(1, G)$ when $x_0\in\mathbf{R}$, b_k is uniformly bounded in $S(H^{-1}, G)$ and $H_k|b_k|_1^G\geqq c>0$ in $\operatorname{supp} a_k$.

Since $\sum \varphi_k^2=1$ on $\operatorname{supp}\varphi$, the calculus gives as in the proof of Proposition 4.1 that

$$\|\varphi^w u\|^2 = \sum \|\varphi_k^w \varphi^w u\|^2+(r^w u, u) \tag{7.2}$$

where $r\in S(H^2, G)\subseteq S(h^{12/7}, G)$, so we obtain

$$|(r^w u, u)| \leqq C\|u\|^2_{(-6/7)}. \tag{7.3}$$

Thus it suffices to prove that for every $\varepsilon>0$ there exists $\delta>0$ such that

$$\sum \|\varphi_k^w \varphi^w u\|^2 \leqq C(\|\psi^w Pu\|^2_{(\varepsilon)}+\|u\|^2_{(-1/15)}+\|Pu\|^2_{(-16/15)}+\delta^2\|\varphi^w u\|^2) \tag{7.4}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq\delta$. In fact, for sufficiently small δ the estimates (7.2)—(7.4) imply (7.1). We shall prove (7.4) by estimating the terms in the cases I)—III).

I) Let K_1 be the set of all k for which $h^{6/7}/C\leqq H\leqq h^{6/7}$ in $\operatorname{supp} F_k$. Then $\{F_k\}_{K_1}\in S(h^{-6/7}, G)$ uniformly in x_0. Proposition 4.3 gives for sufficiently small δ, that

$$\|\varphi_k^w \varphi^w u\| \leqq \delta C\|P_k\varphi_k^w \varphi^w u\| \tag{7.5}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ and $u=0$ where $|x_0|>\delta$, for $\varphi_k^w\varphi^w u\in\mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq\delta$ then. Now we have

$$\|P_k\varphi_k^w \varphi^w u\| \leqq \|[P_k, \varphi_k^w]\varphi^w u\|+\|\varphi_k^w[P_k, \varphi^w]u\| + \|\varphi_k^w\varphi^w(P_k-P)u\|+\|\varphi_k^w\varphi^w Pu\|. \tag{7.6}$$

The calculus with symbols with values in ℓ^2 and $\mathscr{L}(\ell^2, \ell^2)$ (see Section 4) gives that $[P_k, \varphi_k^w]=E_k^w$, where $\{E_k\}\in S(1, G)$ uniformly in x_0 with values in ℓ^2. Thus we find

$$\sum \|[P_k, \varphi_k^w]\varphi^w u\|^2 \leqq C\|\varphi^w u\|^2. \tag{7.7}$$

We also obtain $[P_k, \varphi^w]=A_k^w+B_k^w$, where $A_k=\{F_k, \varphi\}$ and $\{B_k\}\in S(H, G)$ uniformly in x_0. Since $\{F_k\}_{K_1}\in S(h^{-6/7}, G)$ uniformly in x_0 and $\varphi\in S(1, g)$ we find that $\{A_k\}_{K_1}\in S(h^{1/14}, G)$ uniformly in x_0. This implies

$$\sum_{K_1}\|\varphi_k^w[P_k, \varphi^w]u\|^2 \leqq C\sum_{K_1}\|[P_k, \varphi^w]u\|^2 \leqq C'\|u\|_{(-1/14)}^2. \tag{7.8}$$

If we consider $\{(P_k-P)\}$ as an operator with diagonal elements in $\mathscr{L}(\ell^2, \ell^2)$ as symbol values we find, since $\chi_k=1$ on $\operatorname{supp}\varphi_k$, that $\varphi_k^w\varphi^w(P_k-P)=r_k^w$, where $\{r_k\}\in S(H, G)$ uniformly in x_0, with values in ℓ^2. Thus we obtain

$$\sum\|\varphi_k^w\varphi^w(P_k-P)u\|^2 \leqq C\|u\|_{(-6/7)}^2. \tag{7.9}$$

Since $\psi=1$ on $\operatorname{supp}\varphi$ we find that $\varphi^w=\varphi^w\psi^w+r^w$, where $r\in S(H^2, G)$. This implies that

$$\sum\|\varphi_k^w\varphi^w Pu\|^2 \leqq C\|\varphi^w Pu\|^2 \leqq C'(\|\psi^w Pu\|^2+\|Pu\|_{(-16/15)}^2). \tag{7.10}$$

If we combine the estimates (7.5)—(7.10) we obtain

$$\sum_{K_1}\|\varphi_k^w\varphi^w u\|^2 \leqq \delta^2 C(\|\psi^w Pu\|^2+\|Pu\|_{(-16/15)}^2+\|u\|_{(-1/15)}^2+\|\varphi^w u\|^2) \tag{7.11}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq\delta$, and δ is small enough. This gives the desired estimate in case I).

Before continuing with the other cases we observe that since

$$\varphi_k^w\varphi^w u = \varphi^w\varphi_k^w u+[\varphi_k^w, \varphi^w]u,$$

where $\{[\varphi_k^w, \varphi^w]\}$ has symbol in $S(H, G)$ with values in ℓ^2, we obtain

$$\sum\|\varphi_k^w\varphi^w u\|^2 \leqq C\big(\sum\|\varphi_k^w u\|^2+\|u\|_{(-6/7)}^2\big).$$

Thus it suffices to estimate $\sum\|\varphi_k^w u\|^2$ in what follows.

II) Let K_2 be the set of all k for which F_k has constant sign. Since $F_k=\chi_k F$ and $\chi_k=1$ on $\operatorname{supp}\varphi_k$ Proposition 4.2 gives, if δ is small enough, that

$$\sum_{K_2}\|\varphi_k^w u\|^2 \leqq \delta C\sum_{K_2}\|\varphi_k^w Pu\|^2+C\|u\|_{(-1/4)}^2 \tag{7.12}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ and $u=0$ where $|x_0|>\delta$. Now $\psi\geqq 1/2$ on $\operatorname{supp}\varphi_k$, so

$$\varphi_k^w = (\varphi_k/\psi-(2i)^{-1}\{\varphi_k, \psi\}/\psi^2)^w\psi^w+\bar{r}_k^w,$$

where $\{\bar{r}_k\}\in S(H^2, G)$ with values in ℓ^2. This implies

$$\sum\|\varphi_k^w Pu\|^2 \leqq C(\|\psi^w Pu\|^2+\|Pu\|_{(-16/15)}^2) \tag{7.13}$$

which combined with (7.12) give the desired estimate in case II).

III) Let K_3 be the set of all k for which $F_k=a_k b_k$, where $0\leqq a_k\in C^\infty(\mathbf{R}^{2n+1})$ is uniformly bounded in $S(1, G)$ when $x_0\in\mathbf{R}$, b_k is uniformly bounded in $S(H^{-1}, G)$ and $H_k|b_k|_1^G\geqq c>0$ in $\operatorname{supp}a_k$. In this case we shall localize the estimate of Beals and Fefferman (Proposition 4.3) by using symbols which are elliptic in $\operatorname{supp}\varphi_k$ and which approximately commute with P_k.

Lemma 7.1. *For every* $\varepsilon>0$ *there exist positive constants* δ_0, ϱ_0 *so that if* $k\in K_3$ *there exists* $m_k(x, \xi')\in S(h^{-\varepsilon}, G)$ *uniformly when* $|x_0|<\delta_0$, *with support where* $\psi_k=1$, *such that*

a) $\partial_0 m_k+ia_k H_{b_k} m_k=A_k\in S(h^{1-\varepsilon}, G)$ *uniformly when* $|x_0|<\delta_0$,

b) $|1/m_k|_j^G\leqq C_j$ *where the* G_k *distance to* (x_k', ξ_k') *is less than* ϱ_0 *and* $|x_0|<\delta_0$.

Proof of Lemma 7.1. Choose G_k orthonormal coordinates $z'=(z_1, z'')$ with the origin at (x_k', ξ_k') and let $z=(z_0, z')=(x_0, z')$. Now $cH_k^{-1}\leqq |b_k|_1^G\leqq CH_k^{-1}$ in $\operatorname{supp} a_k$ and

$$G_k^\sigma(H_{b_k}) = \sup_w |\sigma(H_{b_k}, w)|^2/G_k(w) = \sup_w |(db_k, w)|^2/G_k(w) = (|b_k|_1^{G_k})^2.$$

Since $G_k=H_k^2 G_k^\sigma$ we find that the Hamilton field H_{b_k} transforms to a vector field with fixed upper and lower bounds in a fixed neighborhood of the origin. Since $\partial^\alpha H_{b_k}=H_{\partial^\alpha b_k}$ we also get fixed bounds for all derivatives of the vector field.

Let $\bar a_k(z)=a_k(x, \xi')$. Then it is clear that $0\leqq \bar a_k\in C^\infty(\mathbf{R}^{2n+1})$ with fixed bounds for each z' derivative. By a change of z' variables we may transform H_{b_k} to ∂_{z_1} in $\operatorname{supp} \bar a_k$. Observe that G_k remains uniformly equivalent to the euclidean metric in these coordinates. It is also clear that $\psi_k=1$ where $|z'|\leqq\sqrt{2}\lambda$, for some fixed positive constant λ.

Let $Q_k=\partial_0+i\bar a_k\partial_1$. Then it is obvious that Q_k is bounded in W when $k\in K_3$ (see Definition 5.1). Thus Proposition 6.1 gives positive constants δ_0, ϱ and $c_0< c_1<c_2<\lambda$, a neighborhood $\Omega=\{z\colon |z_0|<\delta_0,\ |z_1|<c_2,\ |z''|<\varrho\}$ of the origin and a solution $v_k\in C^\infty(\Omega)$ to the equation

$$Q_k v_k = 0 \quad \text{in } \Omega, \tag{7.14}$$

such that v_k has a fixed bound for each z' derivative,

$$\operatorname{Re} v_k \geqq 0 \quad \text{in } \Omega, \tag{7.15}$$

$$\operatorname{Re} v_k \leqq \varepsilon/3 \quad \text{in} \quad \{z\in\Omega\colon |z_1| \leqq c_0\}, \tag{7.16}$$

and

$$\operatorname{Re} v_k \geqq 1 \quad \text{in} \quad \{z\in\Omega\colon |z_1| \geqq c_1\}. \tag{7.17}$$

By choosing a smaller ϱ we obtain $\varrho<\lambda$, thus $\psi_k=1$ in Ω. Choose $\Phi(t)\in C_0^\infty(\mathbf{R})$ with support where $|t|<c_2$ such that $\Phi(t)=1$ where $|t|\leqq c_1$, and let

$$m_k(x, \xi') = \Phi(z_1)\Phi(|z''|\,c_2/\varrho)\, h_k^{v_k(z)-\varepsilon/2}$$

where $h_k=h(x_k', \xi_k')$. Then m_k has support where $\psi_k=1$ and is uniformly bounded in $S(h^{-\varepsilon}, G)$ when $|x_0|<\delta_0$. In fact, $\operatorname{Re} v_k\geqq 0$ in $\operatorname{supp} m_k$ when $|x_0|<\delta_0$, which implies that $|m_k|\leqq Ch_k^{-\varepsilon/2}$, and differentiation with respect to z_j, $j\geqq 1$, can only produce factors bounded by $|\log h_k|$. Since $\Phi'(t)$ has support where $c_1 \leqq |t| < c_2$ and v_k satisfies (7.14) in $\operatorname{supp} m_k$ when $|x_0| < \delta_0$, we obtain a) with

$$A_k(x, \xi') = i\bar{a}_k(z)\Phi'(z_1)\Phi(|z''|c_2/\varrho)h_k^{v_k(z)-\varepsilon/2}.$$

Now $\operatorname{Re} v_k \geqq 1$ in $\operatorname{supp} A_k$ which, as before, gives that $A_k \in S(h^{1-\varepsilon}, G)$ uniformly when $|x_0| < \delta_0$. From (7.16) it follows that $|1/m_k| \leqq h_k^{\varepsilon/6}$ when the G_k distance to (x_k', ξ_k') is less than a fixed constant ϱ_0. Since differentiation of m_k^{-1} can only give factors bounded by $|\log h_k|$, we obtain b), which finishes the proof of Lemma 7.1.

When $\varepsilon > 0$ and $k \in K_3$ we choose m_k as in Lemma 7.1. Since $\psi_k = 1$ on $\operatorname{supp} m_k$ we have a bound on the number of overlapping supports. Thus we obtain that $\{m_k\}_{K_3} \in S(h^{-\varepsilon}, G)$ and $\{A_k\}_{K_3} \in S(h^{1-\varepsilon}, G)$ with values in ℓ^2, when $|x_0| < \delta_0$.

Now we pose the additional condition that φ_k shall have support where the G_k distance to (x_k', ξ_k') is less than ϱ_0, so that $|1/m_k|_j^G \leqq C_j$ in $\operatorname{supp} \varphi_k$ if $k \in K_3$ and $|x_0| < \delta_0$. This we could have required from the beginning, but it would have been difficult to motivate then. With this condition we obtain that $\varphi_k/m_k \in S(1, G)$ uniformly when $|x_0| < \delta_0$. Thus we find

$$\sum_{K_3} \|\varphi_k^w u\|^2 \leqq C\left(\sum_{K_3} \|m_k^w u\|^2 + \|u\|^2_{(\varepsilon-6/7)}\right). \tag{7.18}$$

if $u \in \mathscr{S}(\mathbf{R}^{n+1})$ and $|x_0| < \delta_0$ in $\operatorname{supp} u$. It remains to estimate $\sum_{K_3} \|m_k^w u\|^2$ if u has support where $|x_0| \leqq \delta < \delta_0$, which we assume in what follows.

Proposition 4.3 gives that if δ is small enough then

$$\|m_k^w u\| \leqq \delta C \|P_k m_k^w u\| \tag{7.19}$$

if $u \in \mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0| \leqq \delta$. Now we have

$$\|P_k m_k^w u\| \leqq \|[P_k, m_k^w]u\| + \|m_k^w(P_k - P)u\| + \|m_k^w Pu\|. \tag{7.20}$$

Since $\psi \geqq 1/2$ in $\operatorname{supp} m_k$ we obtain as in the proof of (7.13) that $m_k^w = T_k^w \psi^w + r_k^w$, where $\{T_k\}_{K_3} \in S(h^{-\varepsilon}, G)$ and $\{r_k\}_{K_3} \in S(H^2 h^{-\varepsilon}, G)$ with values in ℓ^2 when $|x_0| < \delta_0$. This implies

$$\sum \|m_k^w Pu\|^2 \leqq C(\|\psi^w Pu\|^2_{(\varepsilon)} + \|Pu\|^2_{(-16/15)}) \tag{7.21}$$

if ε is small enough. Since $\chi_k = 1$ on $\operatorname{supp} m_k$ we obtain as in the proof of (7.9)

$$\sum \|m_k^w(P_k - P)u\|^2 \leqq C\|u\|^2_{(-1/15)}. \tag{7.22}$$

The calculus with symbols with values in ℓ^2 and $\mathscr{L}(\ell^2, \ell^2)$ gives that the symbol of the commutator $[P_k, m_k^w]$ is equal to $-iH_p m_k + C_k$, where $\{C_k\} \in S(h^{-\varepsilon}H, G)$, when $|x_0| < \delta_0$, with values in ℓ^2. Now $H_p m_k = \partial_0 m_k + i\{F_k, m_k\} = \partial_0 m_k + ia_k H_{b_k} m_k + ib_k\{a_k, m_k\} = A_k + B_k$. Here we know that $\{A_k\}_{K_3} \in S(h^{1-\varepsilon}, G)$ with values in ℓ^2, and since $B_k = ib_k\{a_k, m_k\}$ we find that $\{B_k\}_{K_3} \in S(h^{-\varepsilon}, G)$ uniformly, when

$|x_0|<\delta_0$, with values in ℓ^2. Thus we obtain

$$\|[P_k, m_k^w]u\|^2 \leqq 3\sum(\|A_k^w u\|^2+\|B_k^w u\|^2+\|C_k^w u\|^2) \\ \leqq 3\sum\|B_k^w u\|^2+C\|u\|^2_{(\varepsilon-6/7)}. \tag{7.23}$$

Thus it suffices to estimate $\sum\|B_k^w u\|^2$. This will be possible since the symbol B_k vanishes on the zero set of b_k. First we make a factorization so that we get a symbol which does not depend on x_0.

Since $\psi_k=1$ on $\operatorname{supp} m_k$ we can write $B_k=E_k\cdot M_k$, where $E_k=ih_k^\varepsilon\{a_k, m_k\}/H_k$ and $M_k=\psi_k b_k h_k^{-\varepsilon}H_k$, so that $\{E_k\}\in S(1, G)$ uniformly when $|x_0|<\delta_0$ and $\{M_k\}\in S(h^{-\varepsilon}, G)$. Then we obtain

$$\sum\|B_k^w u\|^2 \leqq C\big(\sum\|M_k^w u\|^2+\|u\|^2_{(\varepsilon-6/7)}\big). \tag{7.24}$$

To estimate $\sum\|M_k^w u\|^2$ we need the following lemma which will be proved later.

Lemma 7.2. *Under the assumptions above there exists $\delta>0$ so that for ε small enough we have the estimate*

$$\sum_{K_3}\|M_k^w u\|^2 \leqq C(\|\psi^w Pu\|^2_{(\varepsilon)}+\|u\|^2_{(-1/15)}+\|Pu\|^2_{(-16/15)}) \tag{7.25}$$

if $u\in\mathcal{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq\delta$.

If we combine the estimates (7.18)—(7.25), using Lemma 7.2, we obtain for sufficiently small ε and δ

$$\sum_{K_3}\|\varphi_k^w u\|^2 \leqq C(\delta^2\|\psi^w Pu\|^2_{(\varepsilon)}+\|u\|^2_{(-1/15)}+\|Pu\|^2_{(-16/15)}) \tag{7.26}$$

if $u\in\mathcal{S}(\mathbf{R}^{n+1})$ and $|x_0|\leqq\delta$ in $\operatorname{supp} u$. This gives the desired estimate in case III) and finishes the proof of Proposition 2.7.

Proof of Lemma 7.2. We shall prove the estimate (7.25) by cutting off $M_k=\psi_k H_k b_k h_k^{-\varepsilon}$ near the zero set and use the estimate of Proposition 4.2. Choose $\Phi(t)\in C_0^\infty(\mathbf{R})$ with support where $|t|\leqq 1$ such that $\Phi(t)\leqq 1$ with equality where $|t|<1/2$, and let

$$\theta_k = \Phi(H_k^{-1/12}h_k^\varepsilon M_k) = \Phi(H_k^{11/12}\psi_k b_k).$$

Then we obtain that $\theta_k\in S(1, G_1)$ uniformly, where $G_1=H^{-1/6}G$, since differentiation with respect to unit vectors in the G metric can only produce factors bounded by $H_k^{-1/12}$.

Let $M_{k,0}=\theta_k M_k$. Then $M_{k,0}\in S(H^{1/12}h^{-\varepsilon}, G_1)$ uniformly, since $|M_k|\leqq H_k^{1/12}h_k^{-\varepsilon}$ in $\operatorname{supp}\theta_k$ and

$$|M_k|_j^{G_1} = H^{j/12}|M_k|_j^G \leqq C_j H^{1/12}h^{-\varepsilon}$$

if $j>0$. Now $(1-\theta_k)M_k=M_{k,1}+M_{k,2}$ where $M_{k,i}\in S(h^{-\varepsilon}, G_1)$ uniformly and has support where $(-1)^i F\geqq 0$, because $\theta_k=1$ in a neighborhood of the zero set of

b_k and $F=a_k\cdot b_k$ in $\operatorname{supp} M_k$. Since $\chi_k=1$ on $\operatorname{supp} M_k$ and $\{\chi_k\}\in S(1, G)$, we have a bound on the number of overlapping $\operatorname{supp} M_k$, which gives that $\{M_{k,0}\}\in S(H^{1/12}h^{-\varepsilon}, G_1)$ and $\{M_{k,i}\}\in S(h^{-\varepsilon}, G_1)$ with values in ℓ^2.

Now we have

$$\sum_k \|M_k^w u\|^2 \leqq 3 \sum_{k,i} \|M_{k,i}^w u\|^2. \tag{7.27}$$

Since $\{M_{k,0}\}\in S(H^{1/12}h^{-\varepsilon}, G_1)$ and $H^{1/12}\leqq h^{1/14}$ we obtain

$$\sum \|M_{k,0}^w u\|^2 \leqq C\|u\|^2_{(-1/15)} \tag{7.28}$$

if ε is small enough. We shall now estimate the remaining terms by using Proposition 4.2. To do this we need to know that F has constant sign in a G_1 neighborhood of $\operatorname{supp} M_{k,i}$. Let

$$\tilde{\theta}_k = \Phi(2H_k^{-1/12}h_k^\varepsilon M_k).$$

Then, as before, we obtain that $\tilde{\theta}_k\in S(1, G_1)$ uniformly. Now $\tilde{\theta}_k\leqq 1$ with equality in a neighborhood of the zeros of b_k and $\theta_k=1$ on $\operatorname{supp}\tilde{\theta}_k$. Thus we obtain that $(1-\tilde{\theta}_k)=1$ on $\operatorname{supp}(1-\theta_k)$ and $(1-\tilde{\theta}_k)\geqq 0$ with equality in a neighborhood of the zeros of b_k. Since $\chi_k=1$ on $\operatorname{supp} M_k$ we find

$$(1-\tilde{\theta}_k)\chi_k = \chi_{k,1}+\chi_{k,2}$$

where $\chi_{k,i}$ has support where $(-1)^i b_k\geqq 0$ and $\chi_{k,i}=1$ on $\operatorname{supp} M_{k,i}$. Now $\chi_k\geqq 0$ and $F=a\cdot b_k$ in $\operatorname{supp}\chi_k$, where $a\geqq 0$, so we find that $(-1)^i\chi_{k,i}F\geqq 0$. Since χ_k and $(1-\tilde{\theta}_k)$ are uniformly bounded in $S(1, G_1)$ we obtain that $\chi_{k,1}$ and $\chi_{k,2}$ are uniformly bounded in $S(1, G_1)$ because they have disjoint support.

Proposition 4.2 then gives for sufficiently small ε and δ that

$$\sum_{i\neq 0}\sum_k \|M_{k,i}^w u\|^2 \leqq \delta C\sum \|M_{k,i}^w Pu\|^2 + C\|u\|^2_{(-1/4)} \tag{7.29}$$

if $u\in\mathscr{S}(\mathbf{R}^{n+1})$ has support where $|x_0|\leqq\delta$. Since $\psi\geqq 1/2$ on $\operatorname{supp} M_{k,i}$, the calculus gives as in the proof of (7.13) that

$$M_{k,i}^w = S_{k,i}^w\psi^w + R_{k,i}^w$$

where $\{S_{k,i}\}\in S(h^{-\varepsilon}, G_1)$ and $\{R_{k,i}\}\in S(H_1^3 h^{-\varepsilon}, G_1)\subseteq S(h^2, G_1)$ for small ε because $H_1=H^{5/6}\leqq h^{5/7}$. Thus we obtain

$$\sum_k \|M_{k,i}^w Pu\|^2 \leqq C(\|\psi^w Pu\|^2_{(\varepsilon)} + \|Pu\|^2_{(-16/15)}),\quad i\neq 0, \tag{7.30}$$

for small ε. If we combine the estimates (7.27)—(7.30) we obtain (7.25), which finishes the proof of Lemma 7.2.

References

1. BEALS, R. and FEFFERMAN, C., On local solvability of linear partial differential equations, *Ann. of Math.* **97** (1973), 482—498.
2. BEALS, R. and FEFFERMAN, C., Spatially inhomogeneous pseudodifferential operators I, *Comm. Pure Appl. Math.* **27** (1974), 1—24.
3. HÖRMANDER, L., Propagation of singularities and semiglobal existence theorems for (pseudo)-differential operators of principal type, *Ann. of Math.* **108** (1978), 569—609.
4. HÖRMANDER, L., The Weyl calculus of pseudo-differential operators, *Comm. Pure Appl. Math.* **32** (1979), 359—443.
5. NIRENBERG, L. and TREVES, F., On local solvability of linear partial differential equations; Part I: Necessary conditions, Part II: Sufficient conditions, Correction, *Comm. Pure Appl. Math.* **23** (1970), 1—38 and 459—509, and **24** (1971), 279—288.
6. SJÖSTRAND, J., Operators of principal type with interior boundary conditions, *Acta Math.* **130** (1973), 1—51.

Received March 13, 1981

Nils Dencker
Department of Mathematics
University of Lund
Box 725
S-220 07 LUND
Sweden