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# Sharp estimates for $\bar{\partial}$ on convex domains of finite type

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**Abstract.** Let $\Omega$ be a bounded convex domain in $\mathbf{C}^n$, with smooth boundary of finite type $m$.

The equation $\bar{\partial}u=f$ is solved in $\Omega$ with sharp estimates: if $f$ has bounded coefficients, the coefficients of our solution $u$ are in the Lipschitz space $\Lambda^{1/m}(\Omega)$. Optimal estimates are also given when data have coefficients belonging to $L^p(\Omega)$, $p\geq 1$.

We solve the $\bar{\partial}$-equation by means of integral operators whose kernels are not based on the choice of a "good" support function. Weighted kernels are used; in order to reflect the geometry of $b\Omega$, we introduce a weight expressed in terms of the Bergman kernel of $\Omega$.

## 1. Introduction and statement of results

This paper aims at illustrating the following: if one wants to solve the $\bar{\partial}$-equation with estimates via integral operators, one may choose integral kernels whose construction is not based on the use of a suitable support function.

The main applications we give are optimal Hölder and $L^p$ estimates for the Cauchy–Riemann equation in smoothly bounded convex domains of finite type in $\mathbf{C}^n$. It was previously announced in [C1] that the sharp Hölder estimate $(\mathcal{H}_{1/m})$ defined below holds in such domains.

We say that $(\mathcal{H}_{1/m})$ holds in a pseudoconvex domain $\Omega$ of finite type $m$ if the $\bar{\partial}$-equation has a solution with coefficients in the Lipschitz space $\Lambda^{1/m}(\Omega)$ for data with bounded coefficients. Such an estimate is sharp as was proved in [K1].

We restrict ourselves to the case of weakly pseudoconvex domains $\Omega$ of finite type in order to place our result in a historical context.

The case of smoothly bounded pseudoconvex domains of finite type in $\mathbf{C}^2$ is well understood and we refer the reader to [FK] for a survey of related results.

New difficulties arise when one tries to find optimal estimates for the $\bar{\partial}$-equation or the $\bar{\partial}$-Neumann problem in pseudoconvex domains of finite type in the higher dimensional case $n>2$. Results obtained until now always need additional hypotheses on the domain.

Using methods similar to those in [FK], Fefferman, Kohn and Machedon [FKM] proved Hölder estimates for several operators linked to the $\bar{\partial}_b$-Neumann problem for pseudoconvex domains of finite type $m$ with a diagonalizable Levi form: for the $\bar{\partial}$-equation in such domains, solutions in the Lipschitz spaces $\Lambda^{s-\varepsilon+1/m}$ (where $s+1/m$ is not an integer and $\varepsilon>0$ is arbitrarily small) have been obtained for data with coefficients in $\Lambda^s$.

Regarding the other known results, an additional assumption of convexity has been made. All these results are based on $\bar{\partial}$-solving integral operators.

The first results in this direction were due to Range [R]; in the case of the complex ellipsoids in $\mathbf{C}^n$, Range has obtained an almost optimal Hölder estimate (and for $n=2$ the sharp estimate $(\mathcal{H}_{1/m})$). Diederich, Fornæss and Wiegerinck [DFW] treated the real ellipsoids case; they constructed a new holomorphic support function well adapted to the geometry of the boundary of such domains enabling them to get $(\mathcal{H}_{1/m})$.

More recently Bruna, Charpentier and Dupain [BCD] dealt with the equation $i\partial\bar{\partial}u=\vartheta$ on a bounded convex domain $\Omega$ of finite type; so they had to solve the $\bar{\partial}$-equation in $\Omega$ with precise estimates. They obtained $(\mathcal{H}_{1/m})$ under an additional condition $(\star)$ of strict-type on $\Omega$, i.e. the condition $(\star)$ holds if there exists a constant $c$ such that for all boundary points $z$, all unit vectors $v$ in the complex-tangent space $T_z^c(b\Omega)$ and all small real $t$, one has

$$\frac{1}{c}\varrho(z+tv)\le\varrho(z+itv)\le c\varrho(z+tv).$$

Estimates in other norms or regarding more specific domains are treated in [BC], [S], [CKM] and [Mz].

The above results refer to specific convex domains of finite type, while being all based on an explicit (or fairly explicit) support function choice.(1)

For estimates for the $\bar{\partial}$-equation, integral formulas are most convenient when the kernels involved are expressed in terms of tools reflecting the geometry of the domains. These tools are for instance Leray maps, support functions and weight factors.

The Bergman and Szegő kernels reflect the geometry of a domain. Thus, it seems natural to construct the kernel $K$ of a $\bar{\partial}$ resolving operator with a chosen support function and/or a weight factor enabling to construct $K$ in terms of the

(1) Using the support function recently constructed by K. Diederich and J. E. Fornæss [DF] for convex domains of finite type, K. Diederich, B. Fischer and J. E. Fornæss [DFF] have recently given a different proof of our Theorem 1.1.

Szegő and/or the Bergman kernels of the domain under question. As soon as we focus on a class of domains for which we can precisely estimate both these geometric kernels and their derivatives, the integral formulas thus obtained can be very flexibly used.

In this paper we handle the $\bar{\partial}$-problem in a smoothly bounded convex domain of finite type in $\mathbf{C}^n$ by means of integral formulas. We use a Koppelman–Berndtsson–Andersson type weighted kernel choosing a weight in terms of the Bergman kernel of the domain. By using all the precise estimates on the Bergman kernel known in this setting—thanks to the work of McNeal [M1]—we prove the following results.

**Theorem 1.1.** *Let $\Omega\Subset\mathbf{C}^n$ be a convex domain of finite type $m$ in the D'Angelo sense with a $C^\infty$-smooth boundary. For $1\le q\le n-1$, there exists a constant $C$ such that for every $\bar{\partial}$-closed form $f$ on $\Omega$ of bidegree $(n,q)$ with bounded coefficients, the equation $\bar{\partial}u=f$ has a solution $u$ which satisfies*

$$\|u\|_{\Lambda^{1/m}(\Omega)}\le C\|f\|_\infty.$$

Here, $\Lambda^{1/m}(\Omega)$ denotes the usual norm in $\Lambda^{1/m}_{n,q-1}(\Omega)$ while $\|\cdot\|_\infty$ denotes the sup norm in $L^\infty_{n,q}(\Omega)$.

We also get results for data in other Lebesgue spaces.

The BMO space involved in the following theorem is the isotropic one defined by means of euclidian balls.

**Theorem 1.2.** *Under the assumptions of Theorem* 1.1 *for the domain $\Omega$, the equation $\bar{\partial}u=f$, for $f$ a $\bar{\partial}$-closed $(n,q)$-form with coefficients in $L^p(\Omega)$, has a solution $u$ with coefficients belonging to*

(a) $L^s(\Omega)$, *where* $1/s=1/p-1/(mn+2)$, *if* $1\le p<mn+2$;
(b) $\Lambda^\alpha(\Omega)$, *where* $\alpha=1/m-(n+2/m)/p$, *if* $p>mn+2$;
(c) $\mathrm{BMO}(\Omega)$, *if* $p=mn+2$.

*Remark.* In Theorems 1.1 and 1.2, we consider forms of bidegree $(n,\cdot)$; this makes it easier to get estimates for our solution.

**Corollary 1.3.** *Under the assumptions of Theorem* 1.1, *the canonical solution of the equation $\bar{\partial}u=f$, where $f$ is a $\bar{\partial}$-closed $(0,1)$-form on $\Omega$ with coefficients in $L^p(\Omega)$, belongs to*

(a) $L^s(\Omega)$, *where* $1/s=1/p-1/(mn+2)$, *if* $1\le p<mn+2$;
(b) $\Lambda^\alpha(\Omega)$, *where* $\alpha=1/m-(n+2/m)/p$, *if* $mn+2<p\le+\infty$.

*Proof.* If $u$ is the solution of the equation $\bar{\partial}u=f$ given by Theorem 1.1 the canonical solution of this equation is $u-\mathcal{P}u$, where $\mathcal{P}$ is the Bergman projection operator of $\Omega$. The corollary follows thus immediately from Theorem 1.2 and some continuity results on $\mathcal{P}$ proved by McNeal and Stein in [M2] and [MS], more precisely the continuity of the operator $\mathcal{P}$ from $L^s(\Omega)$ to $L^s(\Omega)$ when $1<s<+\infty$, and from $\Lambda^\alpha(\Omega)$ to $\Lambda^\alpha(\Omega)$ when $\alpha>0$. $\square$

The plan of the paper is as follows.

In Section 2 we define a weighted kernel with a suitable weight reflecting the boundary geometry and prove an integral representation formula for forms with coefficients of class $C^1$ up to the boundary. We obtain an integral operator solving the $\bar\partial$-equation. The differentiability assumption on the forms is, of course, superfluous as explained at the beginning of Section 4. In Section 3 we define the notation we shall use for the estimates and recall the needed results of McNeal on the Bergman kernel. Section 4 is devoted to the proof of Theorem 1.1; the main estimates for our kernel are given there. In the last section, we sketch the proof of Theorem 1.2; most of the computations are similar to those in Section 4 and we just present what has to be changed.

I take the opportunity to mention here that I gave a talk about the main ingredients of the proof of Theorem 1.1 in Warsaw in July 97 (International Conference: Complex Analysis and Applications).

The contents of the present paper (with minor changes) were distributed in the preprint [C2].

In another paper [C3] we prove weighted $L^p$ estimates and boundary $L^1$ estimates for the solution of the $\bar\partial$-equation in bounded convex domains of finite type in $\mathbf{C}^n$ and give applications to the zero sets of functions in some classes of Nevanlinna-type.

## 2. A representation formula for forms

Let $\Omega$ be a bounded convex domain in $\mathbf{C}^n$ with a $C^\infty$-smooth boundary. Suppose every $p\in b\Omega$ is a point of finite type $\le m$, in the sense of D'Angelo.

Following [BCD] we may assume that $0\in\Omega$ and will choose as defining function for $\Omega$ the function $\varrho=g-1$, where $g$ is the gauge function of $\Omega$; $\varrho$ is of class $C^\infty$ on $\Omega\setminus\{0\}$.

By $\mathcal{B}(\zeta,z)$ we will denote the Bergman kernel for the domain $\Omega$; $\mathcal{B}(\zeta,z)$ is holomorphic in $z$, antiholomorphic in $\zeta$; under the assumptions made on $\Omega$, $\mathcal{B}(\cdot\,,\cdot)$ is of class $C^\infty$ on $\overline\Omega\times\overline\Omega\setminus\Delta_{b\Omega}$, where $\Delta_{b\Omega}$ denotes the diagonal of $b\Omega\times b\Omega$ ([M1]).

Let

$$\widetilde{Q}=\widetilde{Q}(\zeta,z)=\frac{1}{\mathcal{B}(\zeta,\zeta)}\int_0^1(\partial_Z\mathcal{B})(\zeta,\zeta+t(z-\zeta))\,dt, \tag{2.1}$$

where

$$(\partial_Z\mathcal{B})(\zeta,\zeta+t(z-\zeta))=\sum_{j=1}^{n}\frac{\partial\mathcal{B}}{\partial Z_j}(\zeta,\zeta+t(z-\zeta))\,dz_j$$

and $\partial/\partial Z_j$ denotes a derivative with respect to the second variable.

Let $N_0\geq 2n$ be a positive integer. We define for $(\zeta,z)\in\Omega\times\bar\Omega\setminus\Delta$, where $\Delta$ denotes the diagonal of $\mathbf{C}^n\times\mathbf{C}^n$,

$$\begin{aligned}K(\zeta,z)&=\sum_{k=0}^{n-1}c(k,n)\left(\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)}\right)^{N_0-k}\frac{(\partial_z|\zeta-z|^2)\wedge(\bar\partial_\zeta\widetilde{Q})^k\wedge(d\partial_z|\zeta-z|^2)^{n-k-1}}{|\zeta-z|^{2n-2k}}\\&=\sum_{k=0}^{n-1}c(k,n)K^{(k)}(\zeta,z),\end{aligned}\tag{2.2}$$

where $c(k,n)=-(-1)^{n(n-1)/2}\binom{N_0}{k}$.

**Proposition 2.1.** *If $f$ is an $(n,q)$-form with coefficients in $C^1(\bar\Omega)$, $q\geq1$, then for $z\in\Omega$,*

$$f(z)=C(n,q)\left(\bar\partial_z\int_\Omega f(\zeta)\wedge K(\zeta,z)+(-1)^{n+q-1}\int_\Omega\bar\partial f(\zeta)\wedge K(\zeta,z)\right).$$

*Remark.* The above proposition is nothing else than an integral formula with weight factors of the Berndtsson–Andersson type and we refer to [BA] for details about such homotopy formulas with weighted kernels.

*Proof of the proposition.* We are going to introduce a Koppelman–Berndtsson–Andersson kernel $\breve{K}$.

Let

$$Q_j(\zeta,z)=\frac{1}{\mathcal{B}(\zeta,\zeta)}\int_0^1\frac{\partial\mathcal{B}}{\partial Z_j}(\zeta,\zeta+t(z-\zeta))\,dt.\tag{2.3}$$

Convexity of $\Omega$ implies that we can write, for all $(\zeta,z)\in\Omega\times\Omega$,

$$\langle Q,z-\zeta\rangle:=\sum_{j=1}^{n}Q_j(\zeta,z)(z_j-\zeta_j)=\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)}-1.\tag{2.4}$$

Let

$$\begin{aligned}\breve{Q}&=\sum_{j=1}^{n}Q_j(\zeta,z)\,d(\zeta_j-z_j),\\ s&=\sum_{j=1}^{n}(\bar\zeta_j-\bar z_j)\,d(\zeta_j-z_j)=\partial(|\zeta-z|^2).\end{aligned}$$

For the kernel $\breve{K}$, the ingredients are as follows:

(1) The maps $Q_j$, $j=1,\dots,n$, defined above, which are of class $C^\infty$ on $\Omega\times\bar{\Omega}$.

(2) A holomorphic function of one variable in a simply connected domain that contains the image of $\Omega\times\bar{\Omega}$ under the map $(\zeta,z)\mapsto 1+\langle Q,z-\zeta\rangle$ and such that $H(1)=1$; we choose $H(\alpha)=\alpha^{N_0}$.

(3) A suitable section of the Cauchy–Leray bundle over $\bar{\Omega}\times\bar{\Omega}\setminus\Delta$; the Bochner–Martinelli one is convenient here.

Let for $(\zeta,z)\in\Omega\times\bar{\Omega}\setminus\Delta$

$$\breve{K}(\zeta,z)=\sum_{k=0}^{n-1}c'(k,n)H^{(k)}(1+\langle Q,z-\zeta\rangle)\frac{s\wedge(d\breve{Q})^k\wedge(ds)^{n-1-k}}{|\zeta-z|^{2n-2k}},$$

where

$$c'(k,n)=-\frac{(-1)^{n(n-1)/2}}{k!},$$
$$\breve{P}(\zeta,z)=\frac{(-1)^{n(n-1)/2}}{n!}H^{(n)}(1+\langle Q,z-\zeta\rangle)(d\breve{Q})^n.$$

Define for $0<\varepsilon<\varepsilon_0\ll 1$, $\Omega_\varepsilon=\{z\in\Omega\,|\,\varrho(z)<-\varepsilon\}$. Recall (cf. [BA]) that

$$d_{\zeta,z}\breve{K}=[\Delta]+\breve{P},$$

where $[\Delta]$ denotes the current of integration over $\Delta$. Applying the main result of [BA], we thus get a formula of Koppelman–Berndtsson–Andersson in $\Omega_\varepsilon$: if $f\in C^1_{n,q}(\bar{\Omega})$, $q\geq 1$ and $0<\varepsilon<\varepsilon_0$, then for $z\in\Omega_\varepsilon$,

(2.5)
$$f(z)=C(n,q)\bigg(\int_{b\Omega_\varepsilon}f(\zeta)\wedge\breve{K}(\zeta,z)+(-1)^{n+q-1}\int_{\Omega_\varepsilon}\bar{\partial}f\wedge\breve{K}-\bar{\partial}_z\int_{\Omega_\varepsilon}f\wedge\breve{K}-\int_{\Omega_\varepsilon}f\wedge\breve{P}\bigg).$$

Notation: for a kernel $L(\zeta,z)$, we will as usual denote by $L_{p,q}(\zeta,z)$ the component of $L$ which is of bidegree $(p,q)$ in $z$.

The maps $Q_j$ are holomorphic in $z$, thus the components $\breve{P}_{p,q}=0$ for $q\geq 1$; so the last integral in the right-hand side of (2.5) is zero.

Recall that the Bergman kernel is $C^\infty$-smooth on $\bar{\Omega}\times\bar{\Omega}\setminus\Delta_{b\Omega}$, so for any domain $\Gamma\Subset\Omega$ and any differential operator $D^\alpha_{\zeta,z}$, there exist constants $c_0=C(\Gamma)$, $c_\alpha=C(\Gamma,D^\alpha)$ such that

$$|\mathcal{B}(\zeta,z)|\leq c_0,\ |D^\alpha_{\zeta,z}\mathcal{B}(\zeta,z)|\leq c_\alpha\quad\text{for } (\zeta,z)\in\bar{\Omega}\times\Gamma.$$

We also have, $\mathcal{B}(\zeta,\zeta)>0$ for $\zeta\in\Omega$, so $\inf_{-\varrho(\zeta)\geq\varepsilon_1}\mathcal{B}(\zeta,\zeta)\geq c_1>0$.

Besides, (cf. [M1]) if $0<-\varrho(\zeta)<\varepsilon_1$ with $\varepsilon_1$ chosen small enough,

$$\frac{1}{\mathcal{B}(\zeta,\zeta)}\leq C(\varepsilon_1)(-\varrho(\zeta))^{2+(2n-2)/m}\leq C(\varepsilon_1)(-\varrho(\zeta))^2.$$

We also use the estimate $|\bar\partial\breve{Q}|\lesssim(-\varrho(\zeta))^{-2}$ for $-\varrho(\zeta)\ll\varepsilon_1$, which is proved later in Section 4—cf. (4.11). We thus obtain for $0<\varepsilon<\min(\varepsilon_1,\varepsilon_0)$,

$$\sup_{\substack{z\in\Gamma\\ \zeta\in b\Omega_\varepsilon}}|\breve{K}(\zeta,z)|\leq C\varepsilon$$

and therefore the integral over the boundary $b\Omega_\varepsilon$ in (2.5) tends to 0, as $\varepsilon\to0$.

Obviously, we can use a standard limiting argument regarding the second integral and the third one in the right-hand side of (2.5).

This completes the proof of the proposition if we remark that $\breve{K}_{(n,\cdot)}(\zeta,z)\wedge f(\zeta)=K_{(n,\cdot)}(\zeta,z)\wedge f(\zeta)$, for every form $f$ of bidegree $(n,\cdot)$. $\square$

## 3. Notation. Review of some estimates for the Bergman kernel

For the reader's convenience, in this paragraph we recall some estimates on the Bergman kernel obtained by McNeal in [M1] for a domain $\Omega$, when $\Omega$ is a smoothly bounded convex domain of finite type in $\mathbf{C}^n$. Incidentally, some notation will also be made precise.

In the sequel, we will use the standard notation $A\lesssim B$, for $A$ and $B$ functions of several variables, to denote that $A\leq CB$ for a constant $C$ independent of certain parameters which will be clear in the context. Of course $A\approx B$ will mean $A\lesssim B$ and $B\lesssim A$.

To begin with, there are some related geometric objects and quantities.

For $\eta>0$ and $v\in\mathbf{C}^n$, $|v|=1$, McNeal has introduced the quantity $\sigma(z,v,\eta)$ (where $z\in\bar\Omega$, $z$ close to $b\Omega$), which measures the radius of the largest complex disc, centered at $z$, in the direction $v$, which lies entirely in the domain $\{\varrho<\varrho(z)+\eta\}$. More precisely

$$\sigma(z,v,\eta)=\sup\{r>0\mid\varrho(z+\lambda v)-\varrho(z)\leq\eta,\ |\lambda|\leq r\}.$$

We will need some properties of $\sigma(z,v,\eta)$, $\eta>0$, $v\in\mathbf{C}^n$, $|v|=1$, $z\in\bar\Omega$, where $b\Omega$ is supposed to be of finite type $\leq m$,

(3.1) $\qquad \sigma(z,v,\eta)=O(\eta^{1/m})\quad$ and $\quad\sigma(z,v,\eta)\gtrsim\eta$ uniformly in $z$ and $v$.

For $\eta_1 \le \eta_2$, we have uniformly in $z$ and $v$

$$\left(\frac{\eta_1}{\eta_2}\right)^{1/2} \sigma(z,v,\eta_2) \lesssim \sigma(z,v,\eta_1) \lesssim \left(\frac{\eta_1}{\eta_2}\right)^{1/m} \sigma(z,v,\eta_2). \tag{3.2}$$

We recall now the notion of *$\eta$-extremal basis of McNeal* as done in [BCD]; we will follow the presentation given in [BCD].

Let $z \in \overline{\Omega}$ close to $b\Omega$ and $\eta > 0$ be fixed. We proceed to choose a certain orthonormal basis $(v_j)_{j=1}^n$ of the tangent space $T_z(\mathbf{C}^n)$. The first vector $v_1$ is the unit vector of the direction of the gradient vector at $z$; chosen $v_1, \dots, v_{i-1}$, we choose $v_i$ to be a unit vector realizing the maximum of $\sigma(z,v,\eta)$ among the unit vectors orthogonal in $\mathbf{C}^n$ to $v_1, \dots, v_{i-1}$. Of course the obtained basis $(v_j)_{j=1}^n$ of $T_z$ depends on both $z$ and $\eta$.

The polydisc $P(z,\eta)$ of McNeal centered at $z$, with radius $\eta$ is defined as

$$P(z,\eta) = \left\{ w = z + \sum_{j=1}^n w_j v_j \;\middle|\; |w_j| \le c\sigma(z,v_j,\eta) \right\},$$

where the constant $c = c(n)$ is chosen such that $w \in P(z,\eta) \Rightarrow |\varrho(w) - \varrho(z)| \le \eta$.

The construction of McNeal's polydiscs makes $\Omega$ a space of homogeneous type. Recall some properties of these polydiscs (cf. [M1], [BCD] for details).

We have for suitable uniform constants $\gamma$ and $\gamma'$, $z + \lambda v \in P(z,\eta)$ whenever $|\lambda| \le \gamma\sigma(z,v,\eta)$, and $|\lambda| \le \gamma'\sigma(z,v,\eta)$ as soon as $z + \lambda v \in P(z,\eta)$.

For each constant $C > 0$, there exists $b = b(C)$ such that

$$\begin{gathered} P(z, C\eta) \subset bP(z,\eta), \\ CP(z,\eta) \subset P(z, b\eta), \\ \operatorname{Vol} P(z, C\eta) \approx \operatorname{Vol} P(z,\eta). \end{gathered} \tag{3.3}$$

There exists a constant $C_1$ independent of $\zeta, z \in \mathcal{U} \cap \Omega$ and $\eta > 0$ (where $\mathcal{U}$ is defined below—cf. (3.6)) such that if $P(z,\eta) \cap P(\zeta,\eta) \ne \emptyset$, then $P(z,\eta) \subset C_1 P(\zeta,\eta)$.

We have, with uniform constants,

$$\begin{aligned} \sigma(\zeta,v,\eta) &\approx \sigma(z,v,\eta) && \text{for } \zeta \in P(z,\eta), \\ \operatorname{Vol} P(\zeta,\eta) &\approx \operatorname{Vol} P(z,\eta), && \text{if } P(z,\eta) \cap P(\zeta,\eta) \ne \emptyset. \end{aligned} \tag{3.4}$$

If $(v_j)_{j=1}^n$ is an $\eta$-extremal basis of McNeal at $z$ then

$$\operatorname{Vol} P(z,\eta) \approx \prod_{j=1}^n \sigma(z,v_j,\eta)^2. \tag{3.5}$$

We are going to recall the estimates of the Bergman kernel proved in [M1] using the reformulation given by McNeal and Stein in [MS].

For $z\in b\Omega$, $\eta>0$, $T(z,\eta):=P(z,\eta)\cap\Omega$ is called the *tent at* $z$ *of radius* $\eta$.

Let $S^n$ be the unit sphere in $\mathbf{C}^n$. For $v\in S^n$ and $\varphi\in C^\infty(\Omega)$, let $D_v\varphi$ denote the directional derivative of $\varphi$ in the direction $v$.

For $k=(k_1,\dots,k_q)\in\mathbf{N}^q$ and $\Lambda=(\lambda_1,\dots,\lambda_q)$, where $\lambda_j\in S^n$, $j=1,\dots,q$, we will be using standard multi-index notation, i.e. $D_\Lambda^k$ denotes the differential operator $D_{\lambda_1}^{k_1}\dots D_{\lambda_q}^{k_q}$ and $\sigma(z,\Lambda,\eta)^k=\sigma(z,\lambda_1,\eta)^{k_1}\dots\sigma(z,\lambda_q,\eta)^{k_q}$ for $z\in\Omega$.

The following important result of McNeal ([M1], [MS]) is crucial for all that follows regarding the estimates we want to get.

For every $p\in b\Omega$, there exists a neighborhood $\mathcal{U}(p)$ such that for $\zeta,z\in\mathcal{U}\cap\Omega$, $k$ and $s$ multi-indices, $\Lambda$ and $\Lambda'$ multi-unit vectors,

$$|\bar{D}_\Lambda^k D_{\Lambda'}^s \mathcal{B}(\zeta,z)|\le C(k,s)\frac{\sigma(\zeta,\Lambda,\varepsilon)^{-k}\sigma(\zeta,\Lambda',\varepsilon)^{-s}}{\operatorname{Vol}T_{\zeta,z}}, \tag{3.6}$$

where $\operatorname{Vol}T_{\zeta,z}$ is the volume of the smallest tent containing both $\zeta$ and $z$, $\varepsilon=\varepsilon(\zeta,z)$ the radius of this tent (*smallest* means of smallest volume).

One has

$$\text{if } \varepsilon=\varepsilon(\zeta,z), \text{ then } \sigma(\zeta,\lambda,\varepsilon)\approx\sigma(z,\lambda,\varepsilon) \text{ for } \lambda\in S^n, \tag{3.7}$$

$$\varepsilon=\varepsilon(\zeta,z)\approx|\varrho(\zeta)|+|\varrho(z)|+\mathcal{M}(\zeta,z), \tag{3.8}$$

where $\mathcal{M}(\zeta,z)$ is the quasi-distance of McNeal; up to uniform constant multiples

$$\mathcal{M}(z,\zeta)\approx\mathcal{M}(\zeta,z)=\inf\{\eta\mid\zeta\in P(z,\eta)\}\quad\text{for } |\zeta-z|\ll1,\ \zeta \text{ close to } b\Omega. \tag{3.9}$$

McNeal has also estimated the Bergman kernel function from below on the diagonal.

For every $p\in b\Omega$, there exists a neighborhood $\mathcal{U}'(p)$ of $p$ such that

$$\mathcal{B}(\zeta,\zeta)\gtrsim\frac{1}{\operatorname{Vol}P(\zeta,\delta)},\quad \zeta\in\mathcal{U}'(p)\cap\Omega, \tag{3.10}$$

where $\delta=\delta(\zeta)=\frac12|\varrho(\zeta)|$. Without loss of generality, we may assume that $\mathcal{U}'(p)=\mathcal{U}(p)$ for $p\in b\Omega$. We suppose this is the case in all that follows.

At last let us give an upper bound on the function $|\mathcal{B}(\zeta,z)|/\mathcal{B}(\zeta,\zeta)$. The estimate given below does not appear in [M1] but is implicit in McNeal's paper; it is very easy to get.

The size of $P(\zeta,\varepsilon(\zeta,z))$ (resp. $P(\zeta,\delta(\zeta))$ in the normal direction is, up to uniform constant multiples, $\varepsilon(\zeta,z)$ (resp. $\delta(\zeta)$). We have $P(\zeta,\delta(\zeta))\subset CP(\zeta,\varepsilon(\zeta,z))$ uniformly in $\zeta$ and $z$, and

$$\operatorname{Area}[P(\zeta,\delta(\zeta))\cap\{\varrho=\varrho(\zeta)\}]\lesssim\operatorname{Area}[P(\zeta,\varepsilon(\zeta,z))\cap\{\varrho=\varrho(\zeta)\}].$$

Let $p\in b\Omega$, $\mathcal{U}=\mathcal{U}(p)$. We obtain by using (3.6) and (3.10),

$$\frac{|\mathcal{B}(\zeta,z)|}{\mathcal{B}(\zeta,\zeta)} \lesssim \frac{\operatorname{Vol} P(\zeta,\delta)}{\operatorname{Vol} T_{\zeta,z}} \approx \frac{\operatorname{Vol} P(\zeta,\delta)}{\operatorname{Vol} P(\zeta,\varepsilon(\zeta,z))} \lesssim \frac{\delta(\zeta)}{\varepsilon(\zeta,z)}, \quad \zeta,z\in\mathcal{U}\cap\Omega. \tag{3.11}$$

*Remark.* We could have obtained the better bound $(\delta(\zeta)/\varepsilon(\zeta,z))^2$. This latter bound is not useful in order to estimate our kernel $K$ defined in (2.2) because we can choose $N_0$ as big as we want.

## 4. Proof of Theorem 1.1

A classical approximation argument reduces the proof of Theorems 1.1 and 1.2 to the case of forms which have coefficients in $C^1(\bar{\Omega})$. There is no difficulty here in getting constants independent of $\eta$ in all the estimates, if one approximates for instance a form in $L^1_{n,q}(\Omega)$ by forms smooth in exhausting subdomains $\Omega_\eta$ ($\eta$ near 1) which are homothetic to $\Omega$. Define, for $f$ an $(n,\cdot\,)$-form

$$\Theta f(z) = \int_\Omega f(\zeta)\wedge K(\zeta,z). \tag{4.1}$$

The form $\Theta f$ is a solution of the equation $\bar\partial u=f$ for $f$ a $\bar\partial$-closed form in $C^1_{n,q}(\bar\Omega)$ (cf. Proposition 2.1). The aim of this paragraph is to prove the continuity of $\Theta$ from $L^\infty_{n,q}(\Omega)$ to $\Lambda^{1/m}_{n,q-1}(\Omega)$.

### 4.1. Estimate of $\int_\Omega f(\zeta)\wedge K^{(0)}(\zeta,z)$

**Proposition 4.1.** *For $0<\alpha<1$, there exists a constant $C=C(\alpha,\Omega)$ such that for every $f\in C^1_{n,q}(\bar\Omega)$,*

$$\left\|\int_\Omega f(\zeta)\wedge K^{(0)}(\zeta,z)\right\|_{\Lambda^\alpha(\Omega)} \le C\|f\|_\infty.$$

Let $[BM](\zeta,z)$ denote the kernel obtained by adding all the components of bidegree $(n,\cdot\,)$ in $z$ of the Bochner–Martinelli kernel. From (2.2) we can write

$$K^{(0)} = [BM] + \sum_{j=0}^{N_0-1} E_j, \quad \text{where } E_j(\zeta,z) = \left(\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)} - 1\right)\left(\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)}\right)^j [BM](\zeta,z).$$

It is well known that the result given by Proposition 4.1 is true if we use the Bochner–Martinelli kernel instead of $K^{(0)}(\zeta,z)$.

Let

$$E_j^*(f)(z)=\int_\Omega f(\zeta)\wedge E_j(\zeta,z),\quad z\in\Omega,\ j=0,\dots,N_0-1.$$

In order to prove that $E_j^*$, $1\le j\le N_0-1$, is a bounded linear operator from $L^\infty_{n,q}(\Omega)$ to $\Lambda^\alpha_{n,q-1}(\Omega)$, $0<\alpha<1$, we will use the following very classical lemma of Hardy–Littlewood.

**Lemma 4.2.** *If $g\in C^1(\Omega)\cap L^\infty(\Omega)$ and if for $0<\alpha<1$ and for some real constant $C$, $|\nabla g(z)|\le C\operatorname{dist}(z,b\Omega)^{-1+\alpha}$, $z\in\Omega$, then $g\in\Lambda^\alpha(\Omega)$ and $\|g\|_{\Lambda^\alpha(\Omega)}\lesssim C$.*

In all that follows we will assume that $z\in\mathcal U$, $\zeta\in\mathcal W=\frac12\mathcal U$, where $\mathcal U$ is one of the neighborhoods $\mathcal U(p)$ defined in Section 3. The smoothness of the Bergman kernel off the boundary diagonal and the known properties of the Bochner–Martinelli kernel both insure that the right estimates hold in all the other cases.

Let $D_z$ be any derivative with respect to the variable $z$. We have from (3.1), (3.6), (3.10) and (3.11),

$$\frac{|D_z\mathcal B(\zeta,z)|}{\mathcal B(\zeta,\zeta)}\lesssim\frac{\delta(\zeta)}{\varepsilon(\zeta,z)^2},\quad z\in\mathcal U\cap\Omega,\ \zeta\in\mathcal W\cap\Omega. \tag{4.2}$$

We also have from (2.3), (2.4), (3.1), (3.6) and (3.8) the following estimates

$$\left|\frac{\mathcal B(\zeta,z)-\mathcal B(\zeta,\zeta)}{\mathcal B(\zeta,\zeta)}\right|\lesssim\frac{|\zeta-z|}{\mathcal B(\zeta,\zeta)}\int_0^1\frac{dt}{\varepsilon(\zeta,z_t)\operatorname{Vol}T_{\zeta,z_t}}, \tag{4.3}$$

$$\begin{gathered}\varepsilon(\zeta,z_t)\approx|\varrho(\zeta)|+|\varrho(z_t)|+\mathcal M(\zeta,z_t)\gtrsim\delta(\zeta),\\ \inf_{0\le t\le1}\operatorname{Vol}T_{\zeta,z_t}\gtrsim\operatorname{Vol}T_\zeta\approx\operatorname{Vol}P(\zeta,\delta(\zeta)),\end{gathered} \tag{4.4}$$

where $z_t=\zeta+t(z-\zeta)$ and $T_\zeta$ is a smallest tent (i.e. of smallest volume) containing $\zeta$.

Collecting the estimates above, using (3.11) and observing that

$$\varepsilon(\zeta,z)\gtrsim|\varrho(\zeta)|+|\varrho(z)|\gtrsim|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|,\quad \zeta,z\in\Omega\cap\mathcal U, \tag{4.5}$$

we can write for $j=1,\dots,N_0-1$, $z\in\mathcal U\cap\Omega$ and $\zeta\in\mathcal W\cap\Omega$,

$$\begin{aligned}|\nabla E_j(\zeta,z)|&\lesssim\frac{\delta(\zeta)}{\varepsilon(\zeta,z)^2|\zeta-z|^{2n-1}}+\left|\frac{\mathcal B(\zeta,z)}{\mathcal B(\zeta,\zeta)}-1\right|\left|\frac{\mathcal B(\zeta,z)}{\mathcal B(\zeta,\zeta)}\right|^j\frac{1}{|\zeta-z|^{2n}},\\ |\nabla E_j(\zeta,z)|&\lesssim\frac{1}{(|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|)|\zeta-z|^{2n-1}}.\end{aligned}$$

Let $0<\alpha<1$. If $z$ is moving in $\mathcal{U}$, we have, by integrating first with respect to $t_1=\varrho(\zeta)-\varrho(z)$, and then using polar coordinates to integrate in the remaining variables,

$$\begin{aligned}\int_{\mathcal{W}\cap\Omega}|\nabla_z E_j(\zeta,z)|\,d\lambda(\zeta)&\lesssim\frac{1}{|\varrho(z)|^{1-\alpha}}\int_{\mathcal{W}\cap\Omega}\frac{d\lambda(\zeta)}{|\varrho(\zeta)-\varrho(z)|^{\alpha+b}|\zeta-z|^{2n-1-b}}\\&\lesssim\frac{1}{|\varrho(z)|^{1-\alpha}},\end{aligned}$$

where $0<b<1-\alpha$. Clearly $\|E_j^*f\|_\infty\lesssim\|f\|_\infty$, $1\le j\le N_0-1$.

The operator $E_0^*$ requires a separate analysis.

**Claim.** *Let* $\alpha\in]0,1[$, *then*

$$|E_0^*f(z)-E_0^*f(w)|\lesssim\|f\|_\infty|z-w|^\alpha\quad\textit{uniformly in } z,w\in\Omega,\ f\in L^\infty_{n,q}(\Omega).$$

For $z$ close to $b\Omega$, let $\pi(z)$ denote the point on $b\Omega$ where the integral curve of $\operatorname{grad}\varrho$ through $z$ meets $b\Omega$; we define for $\zeta$ and $z$ close together

$$z'=z-|z-w|\nu_{\pi(z)}\quad\text{and}\quad w'=w-|z-w|\nu_{\pi(w)},$$

where $\nu_{\pi(z)}$ (resp. $\nu_{\pi(w)}$) is the unit outward normal vector at $\pi(z)$ (resp. $\pi(w)$) to $b\Omega$. We estimate the expressions $|E_0^*f(z)-E_0^*f(z')|$ and $|E_0^*f(z')-E_0^*f(w')|$ separately. The process involved is classical (cf. [K2] for instance), it suffices to adapt it to our context and we omit the details.

### 4.2. Estimates involving the type m

**Proposition 4.3.** *There exists a constant $C$ such that, for every $f\in L^\infty_{n,q}(\Omega)$,*

$$\left\|\int_\Omega f(\zeta)\wedge K^{(k)}(\zeta,z)\right\|_{\Lambda^{1/m}(\Omega)}\le C\|f\|_\infty,\quad k=1,\dots,n-1.$$

It will be clear from the computations done in the proof of the proposition that we have $\|\int_\Omega f\wedge K^{(k)}\|_{L^\infty(\Omega)}\le C'\|f\|_\infty$. Applying the lemma of Hardy–Littlewood we deduce thus the above proposition from the following estimates for $k=1,\dots,n-1$,

$$\int_\Omega\|\nabla_z K^{(k)}(\zeta,z)\|\,d\lambda(\zeta)\lesssim|\varrho(z)|^{-1+1/m}.\tag{4.6}$$

Of course, we only need to prove (4.6) for $k=1$ and $k=n-1$.

**4.2.1.** Here, we consider $K^{(1)}(\zeta,z)$. We have from (2.2),

$$K^{(1)}(\zeta,z)=\left(\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)}\right)^{N_0-1}\frac{(\partial_z|\zeta-z|^2)(\bar\partial\widetilde{Q})(d\partial_z|\zeta-z|^2)^{n-2}}{|\zeta-z|^{2n-2}}.$$

Let $D_z$ be any derivative with respect to the variable $z$. Let us first estimate $D_z(\bar\partial\widetilde{Q})$,

$$\begin{aligned}\bar\partial\widetilde{Q}&=\bar\partial_\zeta\widetilde{Q}=-\frac{\bar\partial\mathcal{B}(\zeta,\zeta)}{\mathcal{B}(\zeta,\zeta)^2}\wedge R+\frac{1}{\mathcal{B}(\zeta,\zeta)}\bar\partial_\zeta R,\\ D_zR&=\sum_{j=1}^n\left[\int_0^1 t\left(D_Z\frac{\partial}{\partial Z_j}\mathcal{B}\right)(\zeta,z_t)\,dt\right]dz_j,\end{aligned}\tag{4.7}$$

where $R=\int_0^1(\partial_Z\mathcal{B})(\zeta,z_t)\,dt$, $z_t=\zeta+t(z-\zeta)$ and $D_Z$ means the derivative $D$ acting on the second variable.

From (3.6), (3.1) and (4.3), we get for $z\in\mathcal{U}\cap\Omega$ and $\zeta\in\mathcal{W}\cap\Omega$,

$$\begin{aligned}|D_zR|&\lesssim\frac{1}{\delta(\zeta)\operatorname{Vol}P(\zeta,\delta)}\int_0^1\frac{t\,dt}{\varepsilon(\zeta,z_t)},\\ |D_z\bar\partial_\zeta R|&\lesssim\frac{1}{\delta(\zeta)^2\operatorname{Vol}P(\zeta,\delta)}\int_0^1\frac{t\,dt}{\varepsilon(\zeta,z_t)}.\end{aligned}\tag{4.8}$$

Because of the convexity of $\varrho$ we have for $\zeta,z\in\mathcal{U}\cap\Omega$,

$$\varepsilon(\zeta,z_t)\gtrsim-\varrho(\zeta)-\varrho(z_t)\ge t(|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|)+|\varrho(\zeta)|,$$

and thus

$$\int_0^1\frac{t\,dt}{\varepsilon(\zeta,z_t)}=O\left(\frac{1}{|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|}\right),\quad \zeta,z\in\mathcal{U}\cap\Omega.\tag{4.9}$$

Using moreover (3.10) and (3.6) we thus obtain

$$|D_z\bar\partial\widetilde{Q}|\lesssim\frac{1}{\delta(\zeta)^2(|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|)}.$$

We get in an obvious way

$$|\bar\partial\widetilde{Q}|=O\left(\frac{1}{\delta(\zeta)^2}\right).\tag{4.10}$$

Collecting all the estimates above, using (3.11), (4.2) and (4.5), we can write for $z\in\mathcal{U}\cap\Omega$,

$$\int_{\mathcal{W}\cap\Omega} |\nabla_z K^{(1)}(\zeta,z)|\,d\zeta \lesssim I_1(z)+I_2(z),$$

where

$$I_1(z)=\int_{\mathcal{W}\cap\Omega}\frac{d\zeta}{\varepsilon(\zeta,z)^2|\zeta-z|^{2n-2}},$$
$$I_2(z)=\int_{\mathcal{W}\cap\Omega}\frac{d\zeta}{(|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|)\varepsilon(\zeta,z)^2|\zeta-z|^{2n-3}}.$$

We will estimate $I_2(z)$. The finite type hypothesis implies

$$\mathcal{M}(\zeta,z)\gtrsim|\zeta-z|^m.$$

In order to integrate over $\mathcal{W}\cap\Omega$ we choose an orthonormal basis $(e_j)_{j=1}^n$ of $T_z(\mathbf{C}^n)$ such that $e_1=\nabla\varrho(z)/\|\nabla\varrho(z)\|$. We will abuse the notation by continuing to call $(\zeta_1,\dots,\zeta_n)$ and $(z_1,\dots,z_n)$ the coordinates of $\zeta$ and $z$ with respect to the system of coordinates corresponding to the basis $(e_j)_{j=1}^n$. Writing $\zeta-z=(\zeta_1-z_1,\zeta'-z')$, we have for $\zeta,z\in\mathcal{U}\cap\Omega$,

$$\begin{aligned}\varepsilon(\zeta,z)&\gtrsim|\varrho(\zeta)|+|\varrho(z)|+\mathcal{M}(\zeta,z)\gtrsim|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|+|\zeta_1-z_1|+|\zeta-z|^m\\&\gtrsim|\varrho(z)|+|\varrho(\zeta)-\varrho(z)|+|\operatorname{Im}(\zeta_1-z_1)|+|\zeta'-z'|^m.\end{aligned}$$

The change of variables $t_1=\varrho(\zeta)-\varrho(z)$, $t_2=\operatorname{Im}(\zeta_1-z_1)$, $t'=\zeta'-z'$ gives

$$I_2(z)\lesssim\int_{\substack{|t_1|\le1\\|t_2|\le1\\|t'|\le1}}\frac{dt_1\,dt_2\,dt'}{(|\varrho(z)|+|t_2|+|t'|^m)^{2-\gamma}(|\varrho(z)|+|t_1|)^{1+\gamma}|t'|^{2n-3}},$$

where $0<\gamma<1-1/m$ and

$$I_2(z)=O(|\varrho(z)|^{-1+1/m}).$$

Similar and easy computations lead to the same estimate for $I_1(z)$.

**4.2.2.** In this paragraph we estimate the main term. Among the terms $K^{(k)}$ (cf. (2.2)), it is the term $K^{(n-1)}$ which has the most interplay with the geometry of the domain. In order to estimate $K^{(n-1)}$ one has to use all of the geometric information contained in the Bergman kernel (cf. Section 3).

To simplify notation, let $G=K^{(n-1)}$. It has already been observed at the beginning of Section 4.1 that we just have to prove the analogue of (4.6) for

$$\int_{\mathcal{W}\cap\Omega} |\nabla_z G(\zeta,z)|\, d\zeta, \quad z\in\mathcal{U}\cap\Omega.$$

There exists a constant $\beta>1$ such that

$$\mathcal{M}(\zeta,z)<\eta \implies \zeta\in P(z,\beta\eta), \quad z\in\Omega\cap\mathcal{U},\ 0<\eta\ll 1. \tag{4.11}$$

Define

$$\begin{aligned} \mathcal{C}_0=\mathcal{C}_0(z) &:= P(z,\beta\, d(z))\cap\mathcal{W}\cap\Omega, \\ \mathcal{C}_l(z) &:= \{\zeta\in\Omega\cap\mathcal{W}\,|\, 2^{l-1}d(z)\le\mathcal{M}(\zeta,z)<2^l\, d(z)\} \quad \text{for } l\ge 1, \end{aligned}$$

where $d(z)=\operatorname{dist}(z,b\Omega)$.

*Notation.* From now on we will often use the shorthand notation $d=d(z)$ and $\delta=\delta(\zeta)$.

Let $D_z$ be any derivative with respect to $z$.

In order to clarify our computations we proceed to first give an expression for $D_zG(\zeta,z)$ when $\zeta\in\mathcal{C}_0(z)$ and to prove $\int_{\mathcal{C}_0(z)}|\nabla_zG|\,d\zeta\lesssim|\varrho(z)|^{-1+1/m}$.

By $D_Z\mathcal{B}$ we will mean that the derivative is with respect to the second variable of $\mathcal{B}(\,\cdot\,,\cdot\,)$. For instance (cf. (2.1))

$$D_z\int_0^1(\partial_Z\mathcal{B})(\zeta,z_t)\,dt=\int_0^1 t(\partial_ZD_Z\mathcal{B})(\zeta,z_t)\,dt,$$

where $z_t=\zeta+t(z-\zeta)$. Let $(e_j)_{j=1}^n$ be a $\beta d$-extremal basis at $z$, we will also write $L_j=e_j$. Let us denote by $(L_1^*,\dots,L_n^*)$ the basis of $T_z^*(\mathbf{C}^n)$ which is the dual basis of $(L_1,\dots,L_n)$. By $L^{(w)}$ we will mean that the derivation $L$ acts on the variable $w$. If $L_j^{*(z)}=L_j^*=\sum_{k=1}^n\alpha_j^k(z)\,dz_k$, $L_j^{*(\zeta)}$ of course means $\sum_{k=1}^n\alpha_j^k(z)\,d\zeta_k$.

In writing $D_zG(\zeta,z)$ we express all the differential forms with respect to another basis. The derivatives $\partial_z$ and $\partial_Z$ will be expressed in terms of $L_1^{*(z)},\dots,L_n^{*(z)}$, and $\bar{\partial}_\zeta$ in terms of $\bar{L}_1^{*(\zeta)},\dots,\bar{L}_n^{*(\zeta)}$.

*Convention* 4.4. In any ambiguous case, $L^{(z)}\mathcal{Y}(\,\cdot\,,\cdot\,)$ means a derivative with respect to the second variable of $\mathcal{Y}(\,\cdot\,,\cdot\,)$; $\mathcal{Y}$ will be essentially $\mathcal{B}$ or some derivative of $\mathcal{B}$.

We have

$$G(\zeta,z)=\left(\frac{\mathcal{B}(\zeta,z)}{\mathcal{B}(\zeta,\zeta)}\right)^N \frac{(\bar\partial_\zeta\widetilde{Q})^{n-1}\partial_z|\zeta-z|^2}{|\zeta-z|^2}, \tag{4.12}$$

where $N=N_0-n+1$. We can write, using (4.7) and (4.8),

$$D_zG=\frac{\mathcal{B}(\zeta,z)^{N-1}}{\mathcal{B}(\zeta,\zeta)^{N+n-1}}(\Gamma_1(\zeta,z)+\Gamma_2(\zeta,z)), \tag{4.13}$$

with

$$\begin{aligned}\Gamma_1(\zeta,z)=&\frac{1}{\mathcal{B}(\zeta,\zeta)}\sum_{\substack{|I|=n\\|J|=n-1}} CD_z^{\theta(k_0)}\left(\frac{\mathcal{B}(\zeta,z)}{|\zeta-z|^2}\right)L_{i_0}^{(z)}D_z^{\theta(i_0)}(|\zeta-z|^2)(\bar L_{j_1}^{(\zeta)}\mathcal{B})(\zeta,\zeta)\\
&\times\int_0^1 t^{\theta(i_1)}(L_{i_1}^{(z)}D_Z^{\theta(i_1)}\mathcal{B})(\zeta,z_t)\,dt\\
&\times\prod_{k=2}^{n-1}\left(\int_0^1 t^{\theta(i_k)}(\bar L_{j_k}^{(\zeta)}L_{i_k}^{(z)}D_Z^{\theta(i_k)}\mathcal{B})(\zeta,z_t)\,dt\right)L_I^{*(z)}\wedge\bar L_J^{*(\zeta)},\\
\Gamma_2(\zeta,z)=&\sum_{\substack{|I|=n\\|J|=n-1}} CD_z^{\theta(k_0)}\left(\frac{\mathcal{B}(\zeta,z)}{|\zeta-z|^2}\right)L_{i_0}^{(z)}D_z^{\theta(i_0)}|\zeta-z|^2\\
&\times\prod_{k=1}^{n-1}\left(\int_0^1 t^{\theta(i_k)}(\bar L_{j_k}^{(\zeta)}L_{i_k}^{(z)}D_Z^{\theta(i_k)}\mathcal{B})(\zeta,z_t)\,dt\right)L_I^{*(z)}\wedge\bar L_J^{*(\zeta)},\end{aligned}$$

where $z_t=\zeta+t(z-\zeta)$ and for every (n+1)-tuple $(\theta(k_0),\theta(i_0),\dots,\theta(i_{n-1}))$ all terms but one (which is equal to 1) are equal to 0,

$$\begin{aligned}L_I^*&=L_{i_0}^*\wedge\dots\wedge L_{i_{n-1}}^*, &&\text{if } I=(i_0,i_1,\dots,i_{n-1})\in(\mathbf{N}^*)^n,\\
\bar L_J^*&=\bar L_{j_1}^*\wedge\dots\wedge\bar L_{j_{n-1}}^*, &&\text{if } J=(j_1,\dots,j_{n-1})\in(\mathbf{N}^*)^{n-1}.\end{aligned}$$

Using (3.6), (4.3) and (3.1), we have

$$\left|\int_0^1 t(L_{i_1}^{(z)}D_Z\mathcal{B})(\zeta,z_t)\,dt\right|\lesssim\frac{1}{\operatorname{Vol}P(\zeta,\delta)}\int_0^1\frac{t\,dt}{\sigma(\zeta,e_{i_1},\varepsilon(\zeta,z_t))\varepsilon(\zeta,z_t)}.$$

Furthermore, (3.2) and (4.3) imply $\sigma(\zeta,e_{i_1},\varepsilon(\zeta,z_t))\gtrsim\sigma(\zeta,e_{i_1},\delta)$. It follows, by using moreover (4.9),

$$\left|\int_0^1 t(L_{i_1}^{(z)}D_Z\mathcal{B})(\zeta,z_t)\,dt\right|\lesssim\frac{1}{\operatorname{Vol}P(\zeta,\delta)\sigma(\zeta,e_{i_1},\delta)(d+|\varrho(\zeta)-\varrho(z)|)}.$$

Using similar arguments we get

$$\left|\int_0^1 (L_{i_1}^{(z)}\mathcal{B})(\zeta,z_t)\,dt\right| \lesssim \frac{1}{\operatorname{Vol} P(\zeta,\delta)\sigma(\zeta,e_{i_1},\delta)},$$

$$\left|\int_0^1 t(\bar L_{j_k}^{(\zeta)} L_{i_k}^{(z)} D_Z\mathcal{B})(\zeta,z_t)\,dt\right| \lesssim \frac{1}{\operatorname{Vol} P(\zeta,\delta)\sigma(\zeta,e_{j_k},\delta)\sigma(\zeta,e_{i_k},\delta)(d+|\varrho(\zeta)-\varrho(z)|)}.$$

If one also takes into account (3.6), (3.10), (3.11), (4.2) and (4.5) the estimates above imply that $|\nabla_z G(\zeta,z)|$ is dominated by a sum of terms which are up to multiple constants

$$|\zeta-z|^{-1}\mathcal{F}_{I,J}(\zeta,z) \quad \text{or} \quad (d+|\varrho(\zeta)-\varrho(z)|)^{-1}\mathcal{F}_{I,J}(\zeta,z), \tag{4.14}$$

where

$$\mathcal{F}_{I,J} = \left(\frac{\delta(\zeta)}{\varepsilon(\zeta,z)}\right)^N \frac{1}{|\zeta-z|} \prod_{k=1}^{n-1} \frac{1}{\sigma(\zeta,e_{i_k},\delta)\sigma(\zeta,e_{j_k},\delta)},$$

$I=(i_1,\dots,i_{n-1})$ and $J=(j_1,\dots,j_{n-1})$ are multi-indices that satisfy

$$1\le i_1<\dots<i_{n-1}\le n \quad \text{and} \quad 1\le j_1<\dots<j_{n-1}\le n.$$

*Remark.* We have solved $\bar\partial u=f$ for $(n,\cdot)$-forms $f$, so for bidegree reasons forms like $L_I^{*(z)}$ appear in $\Gamma_1$ and $\Gamma_2$ (cf. (4.13)). This allows us to get a nice condition on $I$ in (4.14).

We need additional estimates on $\mathcal{F}_{I,J}$. Suppose $\zeta\in\mathcal{C}_0(z)$. We have thus $|\varrho(\zeta)-\varrho(z)|\le\beta d$ and $\delta=\delta(\zeta)\le c_1 d(z)=c_1 d$ (where $c_1$ is an absolute constant). So we obtain from (3.2) and (3.4)

$$\sigma(\zeta,e_{j_k},\delta)\gtrsim\left(\frac{\delta}{d}\right)^{1/2}\sigma(\zeta,e_{j_k},\beta d)\approx\left(\frac{\delta}{d}\right)^{1/2}\sigma(z,e_{j_k},\beta d).$$

From (3.5), we can thus write

$$\mathcal{F}_{I,J}(\zeta,z)\lesssim\left(\frac{\delta(\zeta)}{\varepsilon(\zeta,z)}\right)^N\left(\frac{d(z)}{\delta(\zeta)}\right)^{n-1}\frac{\sigma(z,e_{i_n},\beta d)\sigma(z,e_{j_n},\beta d)}{|\zeta-z|\operatorname{Vol} P(z,\beta d)}$$

with $i_n$, $j_n$ such that $\{i_1,\dots,i_n\}=\{j_1,\dots,j_n\}=\{1,\dots,n\}$.

The definition of the $\beta d$-extremal basis $(e_j)_{j=1}^n$ ensures that for $0<\beta d<1$,

$$\sigma(z,e_{i_n},\beta d)\le\sigma(z,e_2,\beta d), \quad i_n=1,\dots,n. \tag{4.15}$$

Let us write $\tau_j^{(0)}(z)=\tau_j(z,d):=\sigma(z,e_j,\beta d)$, $j=1,\dots,n$.

Let us choose $N\geq n$. We get thus from (3.8), for $\zeta\in\mathcal{C}_0(z)$,

$$\mathcal{F}_{I,J}(\zeta,z)\lesssim\frac{\delta(\zeta)}{\varepsilon(\zeta,z)}\frac{(\tau_2^{(0)}(z))^2}{|\zeta-z|\operatorname{Vol}P(z,\beta d)}\lesssim\frac{(\tau_2^{(0)}(z))^2}{|\zeta-z|\operatorname{Vol}P(z,\beta d)}. \tag{4.16}$$

Let us now estimate

$$\mathcal{I}_0(I,J)=\mathcal{I}_0:=\int_{\mathcal{C}_0(z)}\frac{\mathcal{F}_{I,J}(\zeta,z)\,d\zeta}{d+|\varrho(\zeta)-\varrho(z)|}.$$

Recall that $P(z,\beta d)$ is defined as

$$P(z,\beta d)=\left\{w=z+\sum_{j=1}^{n}w_je_j\;\middle|\;|w_j|\leq c\tau_j(z,d)\right\}.$$

In order to integrate over $P(z,\beta d)$ we can consider the system of coordinates $(t_1+it_2,w_2,\dots,w_n)$, where $t_1=\varrho(\zeta)-\varrho(z)$ and $t_2=\operatorname{Im}w_1$ (with $w_j=\langle\zeta-z,e_j\rangle$, $j=1,\dots,n$). We use the shorthand notation $\tau_j=\tau_j(z,d)$,

$$\mathcal{I}_0\lesssim\frac{\tau_2^2}{\operatorname{Vol}P(z,\beta d)}\int_{\substack{|t_1|+|t_2|\lesssim d\\|w_j|\lesssim\tau_j,j=2,\dots,n}}\frac{dt_1\,dt_2\,d\lambda(w_2)\dots d\lambda(w_n)}{(d+|t_1|)|w_2|},$$

$$\mathcal{I}_0\lesssim d\tau_2\left(\prod_{j=2}^{n}\tau_j^2\right)\frac{1}{\operatorname{Vol}P(z,\beta d)}\lesssim d(z)^{-1+1/m}. \tag{4.17}$$

(We have used (3.5), (3.1) and the estimate $\tau_1(z,d)\approx d$).

For the integral $\int_{P(z,\beta d)}|\zeta-z|^{-1}\mathcal{F}_{I,J}\,d\zeta$ we obtain the same estimate.

In order to prove the following

$$\int_{\mathcal{C}_l}|\nabla_zG(\zeta,z)|\,d\zeta\lesssim a_ld(z)^{-1+1/m},\quad l\geq1, \tag{4.18}$$

with $\sum_{l=0}^{\infty}a_l<+\infty$, we will use a method quite similar to the one used in the above given proof of (4.17) and we just explain what we have to change.

Let us consider, for $l$ fixed in $\mathbf{N}$, a $\beta2^ld$-extremal basis $(v_j^{(l)})_{j=1}^n$ at $z$, which we shortly denote by $(e_j)_{j=1}^n$ or $(L_j)_{j=1}^n$ instead of $(v_j^{(l)})_{j=1}^n$, and as above, let $(L_j^{(l)*})_{j=1}^n$, for brevity $(L_j^*)_{j=1}^n$, be the dual basis of $(L_j)_{j=1}^n$.

From (4.12) we deduce again an expression for $D_zG$ analogous to (4.13), and it is still true that in order to prove (4.18) it suffices to consider

$$\int_{\mathcal{C}_l(z)}\frac{\mathcal{F}_{I,J}\,d\lambda(\zeta)}{d+|\varrho(\zeta)-\varrho(z)|}\quad\text{and}\quad\int_{\mathcal{C}_l(z)}\frac{\mathcal{F}_{I,J}\,d\lambda(\zeta)}{|\zeta-z|},$$

where $\mathcal{F}_{I,J}$ is defined as in (4.14). Of course we keep in mind that our present choice of $(e_j)_{j=1}^n$ is now $(e_j)_{j=1}^n=(v_j^{(l)})_{j=1}^n$.

Suppose $\zeta\in\mathcal{C}_l(z)$. We have $|\varrho(\zeta)-\varrho(z)|\le\beta 2^l d$, $\delta=\delta(\zeta)\lesssim 2^l d$ and $\varepsilon(\zeta,z)\approx 2^l d$, (cf. (3.8)). From (3.2) we get

$$\sigma(\zeta,e_{j_k},\delta)\gtrsim\left(\frac{\delta}{2^l d}\right)^{1/2}\sigma(\zeta,e_{j_k},\beta 2^l d).$$

Since $\zeta\in\mathcal{C}_l(z)\subset P(z,\beta 2^{ld})$, we get using (3.5) and (3.4),

$$\operatorname{Vol}P(\zeta,2^l d)\approx\operatorname{Vol}P(z,\beta 2^l d)\approx\left(\prod_{j=1}^n\sigma(z,e_j,\beta 2^l d)\right)^2.$$

Collecting all the information above and denoting $\sigma(z,e_j,\beta 2^l d)$ by $\tau_j^{(l)}(z)$, we get as soon as $N\ge n+1$,

(4.19)

$$\mathcal{F}_{I,J}(\zeta,z)\lesssim\left(\frac{\delta(\zeta)}{\varepsilon(\zeta,z)}\right)\frac{\tau_2^{(l)}(z)^2}{|\zeta-z|\operatorname{Vol}P(z,\beta 2^l d)},\quad \zeta\in\mathcal{C}_l(z),\ z\in\mathcal{U}\cap\Omega,$$

and

$$\begin{aligned}\mathcal{I}_l(I,J):=\int_{\mathcal{C}_l}\frac{\mathcal{F}_{I,J}\,d\zeta}{d+|\varrho(\zeta)-\varrho(z)|}&\lesssim\frac{\tau_2^{(l)}(z)^2}{\operatorname{Vol}P(z,2^l d)}\int_{P(z,\beta 2^l d)}\frac{d\lambda(\zeta)}{(d+|\varrho(\zeta)-\varrho(z)|)|\zeta-z|}\\ &\lesssim l2^{(-1+1/m)l}d(z)^{-1+1/m}.\end{aligned}$$

This concludes the proof of (4.18).

## 5. The estimates of Theorem 1.2

**5.1.** The continuity of the operator $\Theta$ given by (4.1) from $L^1_{n,q}(\Omega)$ to $L^r_{n,q-1}(\Omega)$ (for $q\ge1$), where $r=(mn+2)/(mn+1)$, is easily deduced from the following proposition.

**Proposition 5.1.** *Let $r=(mn+2)/(mn+1)$. Then*

$$\sup_{\zeta\in\Omega}\int_\Omega|K^{(k)}(\zeta,z)|^r\,d\lambda(z)<+\infty,\quad k=0,\dots,n-1.$$

It suffices to prove the estimate for $k=0$, $k=1$ and $k=n-1$. For $k=0$, the result is standard, recalling that the weight $\mathcal{B}(\zeta,z)/\mathcal{B}(\zeta,\zeta)$ is uniformly bounded on $\Omega\times\Omega$. It is easy to deal with the case $k=1$, so we will just study the integral involving $G=K^{(n-1)}$.

Let $\mathcal{C}_0(\zeta)=P(\zeta,\beta\delta(\zeta))$, where $\beta$ is defined in (4.12), and for $l\geq 1$ let

$$\mathcal{C}_l(\zeta):=\{z\in\mathcal{U}\cap\Omega\,|\,2^{l-1}\delta(\zeta)\leq\mathcal{M}(z,\zeta)<2^l\delta(\zeta)\}.$$

We have $\mathcal{C}_l(\zeta)\subset P(\zeta,\beta 2^l\delta(\zeta))$, $l\geq 0$.

Let $(e_j^{(l)})_{j=1}^n$ be a $2^l\beta\delta(\zeta)$-extremal basis at $\zeta$. If $(w_1,\dots,w_n)$ is the new system of coordinates with respect to this basis, we write $L_j^{(l)}=\partial/\partial w_j$, and $(L_j^{(l)*})_{j=1}^n$ is the basis of $T_\zeta^*\mathbf{C}^n$ which is the dual basis of $(L_j^{(l)})_{j=1}^n$. In order to estimate $G(\zeta,z)$ for $z\in\mathcal{C}_l(\zeta)$ we will first proceed in the same way as in Subsection 4.2.2 (with analogous notation and Convention 4.4 except that the $(L_j^{(l)})_{j=1}^n$ basis here is $(e_j^{(l)})_{j=1}^n$). In formula (4.12) given for $G(\zeta,z)$ we will use the basis $(L_j^{(l)*(z)})_{j=1}^n$ (resp. $(\bar{L}_j^{(l)*(\zeta)})_{j=1}^n$) in order to express the forms $dz_k$ (resp. $d\bar{\zeta}_k$). Thus we get

$$|G(\zeta,z)|\lesssim\sum_{I,J}\mathcal{F}_{I,J}\quad\text{for } z\in\mathcal{C}_l(\zeta),$$

where $I=(i_1,\dots,i_{n-1})$, $J=(j_1,\dots,j_{n-1})$ are multi-indices with $1\leq i_1<\dots<i_{n-1}\leq n$ and $1\leq j_1<\dots<j_{n-1}\leq n$, and

$$\mathcal{F}_{I,J}=\left(\frac{\delta(\zeta)}{\varepsilon(\zeta,z)}\right)^N\left(|\zeta-z|\prod_{k=1}^{n-1}\sigma(\zeta,e_{i_k}^{(l)}(\zeta),\delta)\sigma(\zeta,e_{j_k}^{(l)}(\zeta),\delta)\right)^{-1}.\tag{5.1}$$

Using the estimates

$$\varepsilon(\zeta,z)\approx 2^l\delta(\zeta),\qquad\text{if } z\in\mathcal{C}_l(\zeta),\tag{5.2}$$

$$\sigma(\zeta,e_j^{(l)}(\zeta),\delta)\gtrsim 2^{-l/2}\sigma(\zeta,e_j^{(l)}(\zeta),\beta 2^l\delta)\quad\text{uniformly in }\zeta,\ j\text{ and }l,\tag{5.3}$$

$$\sigma(\zeta,e_j^{(l)}(\zeta),\delta)\leq\sigma(\zeta,e_2^{(l)}(\zeta),\delta)\qquad\text{for all } j,\tag{5.4}$$

we obtain as soon as $N\geq n$,

$$G(\zeta,z)\lesssim\frac{1}{2^l|\zeta-z|}\prod_{\substack{j=1\\ j\neq 2}}^{n}\frac{1}{\tau_j^{(l)}(\zeta)^2},\quad z\in\mathcal{C}_l(\zeta),\tag{5.5}$$

where

$$\tau_j^{(l)}(\zeta)=\sigma(\zeta,e_k^{(l)}(\zeta),\beta 2^l\delta(\zeta)).$$

From (5.5) and Lemma 5.2 below, we deduce immediately the estimate

$$\int_{\mathcal{C}_l(\zeta)}|G(\zeta,z)|^r\,d\lambda(z)=O(2^{-yl}),$$

where $y=[n(m-1)+2]/(mn+1)$. The required result regarding $G$ is thus proved.

**Lemma 5.2.** *We have uniformly in* $\zeta$

$$\mathcal{J}_l(\zeta):=\int_{\mathcal{C}_l(\zeta)}\frac{1}{|\zeta-z|^r}\prod_{j\neq 2}\frac{1}{\tau_j^{(l)}(\zeta)^{2r}}\,d\lambda(z)=O(2^{nl/(mn+1)}).$$

Here, for the product in the integrand of $\mathcal{J}_l$, we will be using a more convenient expression.

It is proved in [M1] that for $\zeta$ in $\Omega\cap\mathcal{U}$, $v$ a unit vector and $\eta>0$, one has

$$\sigma(\zeta,v,\eta)\approx\min_{1\leq p+q\leq m}\left(\frac{\eta}{|a_{p,q}(\zeta,v)|}\right)^{1/(p+q)},$$

where

$$a_{p,q}(\zeta,v)=\frac{\partial^{p+q}\varrho(\zeta+\lambda v)}{\partial\lambda^p\partial\bar{\lambda}^q}\bigg|_{\lambda=0}.$$

We can then choose, for every $(k,l)$ with $k\in\{2,\dots,n\}$ and $l\in\mathbf{N}$, integers $i_{k,l}$ and $j_{k,l}$ with $s_k:=i_{k,l}+j_{k,l}\geq 2$, such that

$$\frac{1}{\tau_k^{(l)}(\zeta)^2}\approx\left(\frac{A_k^{(l)}(\zeta)}{2^l\delta}\right)^{2/s_k}, \tag{5.6}$$

where $A_k^{(l)}(\zeta):=|a_{i_{k,l},j_{k,l}}(\zeta,e_k^{(l)})|$. Since $\tau_1^{(l)}(\zeta)\approx 2^l\delta$, we get

$$\prod_{j\neq 2}\frac{1}{\tau_k^{(l)}(\zeta)^2}\lesssim\frac{1}{(2^l\delta)^{2+\sum_{j=3}^n 2/s_j}}\prod_{j=3}^n A_j^{(l)}(\zeta)^{2/s_j}.$$

We can write, for $z\in\mathcal{C}_l(\zeta)\subset P(\zeta,\beta 2^l\delta)$,

$$z=\zeta+\sum_{j=1}^n\lambda_j e_j^{(l)}(\zeta),\quad |\lambda_j|\leq c_0\tau_j^{(l)}(\zeta). \tag{5.7}$$

We will abuse notation writing $\zeta-z=(\zeta_1-z_1,\dots,\zeta_n-z_n)$, where the coordinates $\zeta_k-z_k=\lambda_k$ are now the ones associated with the basis $(e_k^{(l)}(\zeta))_{k=1}^n$.

Using (5.6), (5.7) and the estimate $2^l\delta(\zeta)\gtrsim\mathcal{M}(\zeta,z)\gtrsim|\zeta_1-z_1|$ for $z\in\mathcal{C}_l$, we get

$$\varepsilon(\zeta,z)\approx 2^l\delta(\zeta)\gtrsim d(z)+\delta(\zeta)+|\zeta_1-z_1|+\sum_{j=3}^n A_j^{(l)}(\zeta)|\zeta_j-z_j|^{s_j},$$

$$\mathcal{J}_l(\zeta)\lesssim\int_{\substack{|\zeta-z|\ll 1\\ z\in P(\zeta,\beta 2^l\delta)}}\frac{\prod_{j=3}^n A_j^{(l)}(\zeta)^{2/s_j}\,d\lambda(z)}{|\zeta''-z''|^r\big(d(z)+\delta(\zeta)+|\zeta_1-z_1|+\sum_{j=3}^n A_j^{(l)}(\zeta)|\zeta_j-z_j|^{s_j}\big)^x},$$

where $\zeta''-z''=(\zeta_1-z_1,\zeta_2-z_2)$ and $x=2r+r\sum_{j=3}^n 2/s_j$.

Standard computations yield

$$\mathcal{J}_l\lesssim\int_{\substack{|\zeta_1-z_1|\le 2^l\delta\\ |\zeta_2-z_2|\le\tau_2^{(l)}}}\frac{d\lambda(z)}{|\zeta''-z''|^r(\delta(\zeta)+|\zeta_1-z_1|)^{2r+(r-1)\sum_{j\ge 3}2/s_j}}.$$

Since $s_j\ge 2$ for all $j\ge 3$, we get by using polar coordinates

$$\mathcal{J}_l\lesssim\int_{\substack{\varrho_1\le 2^l\delta\\ \varrho_2\le 2^{l/m}\delta^{1/m}}}\frac{\varrho_2^{1-r}\,d\varrho_1\,d\varrho_2}{(\delta(\zeta)+\varrho_1)^{1+(r-1)n}}.$$

The lemma is proved. □

**5.2.** In the proof of part (c) of Theorem 1.2, we will use the following classical lemma (cf. for instance [MS, Lemma 7]).

**Lemma 5.3.** *Let $g\in C^1(\Omega)$ be such that there exists $C>0$ for which $|\nabla g(z)|\le C\,\mathrm{dist}(z,b\Omega)^{-1}$, $z\in\Omega$, then $g\in\mathrm{BMO}(\Omega)$.*

Lemma 5.3, the Hölder inequality and Proposition 5.4 below yield the desired result.

**Proposition 5.4.** *Let $r=(mn+2)/(mn+1)$. Then there exists a constant $C>0$ such that*

$$\|\nabla_z K^{(k)}(\,\cdot\,,z)\|_{L^r(\Omega)}\le\frac{C}{d(z)},\qquad z\in\Omega,\ k=0,\dots,n-1.$$

Let us prove the estimate for $G=K^{(n-1)}$. We use the notation and estimates given in Section 4.2.2. Since $\delta(\zeta)\lesssim d(z)+|\varrho(\zeta)-\varrho(z)|$ and $\varepsilon\approx 2^l d(z)$ if $\zeta\in\mathcal{C}_l(z)$, we deduce from (4.14), (4.16), (4.19) and (3.5) the estimate

$$|\nabla_z G(\zeta,z)|\lesssim\frac{1}{2^l d(z)}\prod_{j\ne 2}\frac{1}{\tau_j^{(l)}(z)^2|\zeta-z|}\left(1+\frac{\delta(\zeta)}{|\zeta-z|}\right)\quad\text{for }\zeta\in\mathcal{C}_l(z).$$

The estimate on the $L^r$-norm of $|\nabla_z G(\,\cdot\,,z)|$ follows from the following lemma.

**Lemma 5.5.** *We have uniformly in $z$*

(a)

$$\mathcal{J}_l(z):=\int_{\mathcal{C}_l(z)}\frac{1}{|\zeta-z|^r}\prod_{j\ne 2}\frac{d\lambda(\zeta)}{\tau_j^{(l)}(z)^{2r}}=O(2^{nl/(mn+1)}),$$

(b)

$$\mathcal{J}_l'(z):=\int_{\mathcal{C}_l(z)}\frac{\delta(\zeta)^r}{|\zeta-z|^{2r}}\prod_{j\ne 2}\frac{d\lambda(\zeta)}{\tau_j^{(l)}(z)^{2r}}=O(2^{l(mn-1)/m(mn+1)}d(z)^{(m-1)/m(mn+1)}).$$

Part (a) is analogous to the result of Lemma 5.2 (mutatis mutandis).

In order to prove part (b) we remark first that $\tau_1^{(l)}(z) \approx 2^l d(z) \gtrsim \delta(\zeta)$ for $\zeta \in \mathcal{C}_l(z)$. Next, integrating $|\zeta - z|^{-2r} \tau_1^{(l)}(z)^{-r} \prod_{j=3}^n \tau_j^{(l)}(z)^{-2r}$ the same way as previously done in the proof of Lemma 5.2, we obtain the right estimate for $\mathcal{J}_l'(z)$.

**5.3.** In order to prove the remaining estimates of part (a) of Theorem 1.2, we can use a suitable version of Schur's lemma (cf. for instance [FR]). Thus we can prove directly—via computations similar to those in the proof of Lemmas 5.2 and 5.5—that for $0<\varepsilon<1$ there exists a constant $C_\varepsilon$ such that

$$\int_\Omega \frac{|K(\zeta,z)|^r \, d\lambda(z)}{|\varrho(z)|^\varepsilon} \le \frac{C_\varepsilon}{|\varrho(\zeta)|^\varepsilon},$$
$$\int_\Omega \frac{|K(\zeta,z)|^r \, d\lambda(\zeta)}{|\varrho(\zeta)|^\varepsilon} \le \frac{C_\varepsilon}{|\varrho(z)|^\varepsilon}.$$

The continuity of the operator $\Theta$ from $L^p_{(n,q)}(\Omega)$ to $L^s_{(n,q-1)}(\Omega)$, where $1<p<mn+2$, $r=(mn+2)/(mn+1)$ and $1/s=1/p+1/r-1=1/p-1/(mn+2)$, follows thus from the Hölder inequality.

The Lipschitz estimates in part (b) of Theorem 1.2 can be proved by using the Hardy–Littlewood Lemma 4.2 and by getting convenient estimates for

$$\int_\Omega |\nabla_z K|^{p'} \, d\lambda(\zeta),$$

where $1/p+1/p'=1$. □

*Addendum.* In the research announcement [C1] we gave the statement of Theorem 1.1 and a sketch of its proof (as is customary, the details of the proof were enclosed with the text of the note when it was submitted). The proof of the estimates in Theorem 1.1 relies on estimates of the Bergman kernel given by J. McNeal [M1], at the exclusion of other auxiliary results from the same article.

The estimates of the Bergman kernel given in [M1] are perfectly correct and can be proved by interpreting the article in an appropriate fashion. This was covered in an explanatory addendum [M3], to which we refer the reader. It is not necessary to call upon the support function recently constructed by K. Diederich and J. E. Fornæss [DF] to validate McNeal's estimates.

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