The Burgess Davis inequalities via Fefferman's inequality*

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Introduction

Let $(\Omega, \mathcal{F}, P)$ be a probability space and let $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_n \subset \ldots$ be a sequence of σ-fields such that $\mathcal{F} = \bigcup_{\nu=0}^{\infty} \mathcal{F}_\nu$.

For a random variable $f \in L_1(\Omega, \mathcal{F}, P)$ we shall set

$$f_n = E(f|\mathcal{F}_n), \quad \Delta f_n = f_n - f_{n-1},$$

$$f_n^* = \max_{\nu \leq n} |f_\nu|, \quad f^* = \sup_n |f_n|,$$

$$S_n(f) = \sqrt{\textstyle\sum_{\nu=1}^{n} \Delta f_\nu^2}, \quad S(f) = \sup S_n(f).$$

We also introduce the spaces

$$\mathcal{H}_p = \{f \colon E([S(f)]^p) < \infty\} \tag{I.1}$$

with norm

$$\|f\|_{\mathcal{H}_p} = [E([S(f)]^p)]^{1/p} \quad (p \geq 1).$$

Furthermore, we let

$$\mathrm{BMO} = \{f \colon \sup_{n \geq 1} \|E(|f - f_{n-1}|^2|\mathcal{F}_n)\|_\infty < \infty\} \tag{I.2}$$

with norm**

$$\|f\|_{\mathrm{BMO}} = \sup_{n \geq 1} \| \sqrt{E(|f - f_{n-1}|^2|\mathcal{F}_n)}\|_\infty.$$

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** Strictly speaking, these functionals are norms in the usual sense only when restricted to $\{f \colon E(f|\mathcal{F}_0) = 0\}$.

The reason for this notation lies in the fact that with an appropriate choice of $(\Omega, \mathcal{F}, P)$ and $\{\mathcal{F}_n\}$, the spaces in (I.1) can be identified with the classical $\mathcal{H}_p$ spaces of function theory, while at the same time the space defined in (I.2) can be identified with the class of functions of *Bounded Mean Oscillation* introduced by John and Nirenberg [6].

This given, Burgess Davis [3] proved the following remarkable inequalities

$$\frac{1}{C_1} E(f^*) \leq E(S(f)) \leq C_2 E(f^*) \quad (\forall f\colon E(f|\mathcal{F}_0) = 0), \tag{I.4}$$

where C_1 and C_2 are universal constants.

More recently, Charles Fefferman [4] showed that there is a universal constant C_3 such that for any two martingales $f_n = E(f|\mathcal{F}_n)$, $\varphi_n = E(\varphi|\mathcal{F}_n)$ with $f_0 = 0$, we have

$$\left| \int_\Omega f_n \varphi_n dP \right| \leq C_3 \|f\|_{\mathcal{H}_1} \|\varphi\|_{\mathrm{BMO}}. \tag{I.5}$$

Actually, in [4] Fefferman only states the corresponding inequality in a classical function theoretic setting, however, we understand that Fefferman had also proved (I.5).

The object of this paper is to show that both sides of (I.4) follow in a rather natural manner from (I.5).

We hope that our arguments here, in addition to throwing some new light on this matter, will actually provide what is perhaps one of the simplest ways of establishing the Burgess Davis inequalities.

For sake of completeness, at the end of this paper we shall include a proof of (I.5) with $C_3 = \sqrt{2}$, this will yield (I.4) with $C_1 = \sqrt{10}$ and $C_2 = 2 + \sqrt{5}$.

Perhaps it is worthwhile to include some historical remarks. First of all, it was Burkholder, Gundy and Silverstein [2] who made it apparent that the $\mathcal{H}_p$ spaces of classical function theory could be viewed in a most natural manner in a martingale setting.

This has become even more apparent now after some recent work of Getoor and Sharpe [5].

Burkholder and Gundy [1] conjectured the Burgess Davis inequalities for general martingales after proving them for the class of regular martingales*.

The space BMO as in (I.2) was introduced (after John-Nirenberg's paper) by R. Gundy a few years ago. We also understand that after learning of Fefferman's analogous functional theoretical result, R. Gundy and C. Herz worked together on a proof of (I.5). Indeed, the proof of (I.5) we shall present here is essentially a simplified version of Herz's proof we learned from Gundy.

* For the definition see [1].

In this connection, we wish to acknowledge here our gratefulness to the Mittag-Leffler institute for making possible during the summer of 1971 our mathematical exchanges with Burkholder, Fefferman and Gundy which provided us not only the stimulus but also the information without which this work could not have been carried out.

1. Construction of some BMO functions

Our arguments will be based upon two results that should be of independent interest.

LEMMA 1.1. *Let* $\{\theta_\nu\}$ *be a sequence of random variables satisfying*

$$\textstyle\sum_{\nu=1}^{\infty} |\theta_\nu| \leq 1. \tag{1.1}$$

Then, the function

$$\varphi = \textstyle\sum_{\nu=1}^{\infty} E(\theta_\nu|\mathcal{F}_\nu) \tag{1.2}$$

is BMO *and indeed*

$$E(|\varphi - \varphi_{n-1}|^2|\mathcal{F}_n) \leq 5, \quad \forall\, n \geq 1. \tag{1.3}$$

Proof. Clearly $\varphi \in L_1$ since

$$E(|\varphi|) \leq \textstyle\sum_{\nu=1}^{\infty} E(|\theta_\nu|) \leq 1.$$

To prove 1.3 set

$$\Phi_n = \textstyle\sum_{\nu=n}^{\infty} E(\theta_\nu|\mathcal{F}_\nu).$$

This given, from (1.2) we get

$$\varphi_{n-1} = E(\varphi|\mathcal{F}_{n-1}) = \textstyle\sum_{\nu=1}^{n-1} E(\theta_\nu|\mathcal{F}_\nu) + E(\Phi_n|\mathcal{F}_{n-1}).$$

So

$$\varphi - \varphi_{n-1} = \Phi_n - E(\Phi_n|\mathcal{F}_{n-1})$$

and, using (1.1), we obtain

$$\begin{aligned} E(|\varphi - \varphi_{n-1}|^2|\mathcal{F}_n) = E(\Phi_n^2|\mathcal{F}_n) - 2E(\Phi_n|\mathcal{F}_n)E(\Phi_n|\mathcal{F}_{n-1}) + \\ + [E(\Phi_n|\mathcal{F}_{n-1})]^2 \leq E(\Phi_n^2|\mathcal{F}_n) + 3. \end{aligned} \tag{1.4}$$

On the other hand, again using (1.1),

$$\begin{aligned} E(\Phi_n^2|\mathcal{F}_n) \leq 2 \textstyle\sum_{\nu=n}^{\infty} \sum_{\mu=\nu}^{\infty} E\{E(|\theta_\nu||\mathcal{F}_\nu)E(|\theta_\mu||\mathcal{F}_\mu)|\mathcal{F}_n\} = \\ = 2 \textstyle\sum_{\nu=n}^{\infty} \sum_{\mu=\nu}^{\infty} E\{E(|\theta_\nu||\mathcal{F}_\nu)E(|\theta_\mu||\mathcal{F}_\nu)|\mathcal{F}_n\} \leq 2 \sum_{\nu=n}^{\infty} E\{E(|\theta_\nu||\mathcal{F}_\nu)|\mathcal{F}_n\} \leq 2. \end{aligned} \tag{1.5}$$

Combining (1.4) and (1.5) we obtain (1.3) as asserted.

Our next result can be stated as follows:

LEMMA 1.2. *Let* $\{\gamma_\nu\}$ *be a sequence of random variables such that for each* $\nu \geq 0$, $\mathcal{F}(\gamma_\nu) \subset \mathcal{F}_\nu$ *and set* $\gamma^* = \sup_{\nu \geq 0} |\gamma_\nu|$. *Then the function*

$$\varphi = \sum_{\nu=1}^{\infty} \gamma_{\nu-1}\{E(1/\gamma^*|\mathcal{F}_\nu) - E(1/\gamma^*|\mathcal{F}_{\nu-1})\} \tag{1.6}$$

is BMO *and indeed*

$$E(|\varphi - \varphi_{n-1}|^2|\mathcal{F}_n) \leq 2. \tag{1.7}$$

Proof. Set for convenience $\psi_\nu = E(1/\gamma^*|\mathcal{F}_\nu)$, $\Delta\psi_\nu = \psi_\nu - \psi_{\nu-1}$.
We then have

$$E(|\varphi - \varphi_{n-1}|^2|\mathcal{F}_n) = \sum_{\nu=n}^{\infty} E(\gamma_{\nu-1}^2 \Delta\psi_\nu^2|\mathcal{F}_n) \leq \sum_{\nu=n}^{\infty} E(\gamma_{\nu-1}^{*2}\Delta\psi_\nu^2|\mathcal{F}_n), \tag{1.8}$$

where

$$\gamma_\nu^* = \max_{0 \leq \mu \leq \nu} |\gamma_\mu|.$$

Now, note that

$$-1 \leq -\gamma_{n-1}^* E(1/\gamma^*|\mathcal{F}_{n-1}) \leq \gamma_{n-1}^*\Delta\psi_n \leq \gamma_{n-1}^* E(1/\gamma^*|\mathcal{F}_n) \leq 1,$$

Thus

$$\gamma_{n-1}^{*2}\Delta\psi_n^2 \leq 1. \tag{1.9}$$

Finally

$$\sum_{\nu=n+1}^{N} E(\gamma_{\nu-1}^{*2}\Delta\psi_\nu^2|\mathcal{F}_n) = \sum_{\nu=n+1}^{N} E(\gamma_{\nu-1}^{*2}[\psi_\nu^2 - \psi_{\nu-1}^2]|\mathcal{F}_n) \leq$$
$$\leq \sum_{\nu=n+1}^{N} E(\gamma_\nu^{*2}\psi_\nu^2|\mathcal{F}_n) - \sum_{\nu=n}^{N-1} E(\gamma_\nu^{*2}\psi_\nu^2|\mathcal{F}_n) \leq E(\gamma_N^{*2}\psi_N^2|\mathcal{F}_n) \leq 1.$$

Letting $n \to \infty$ and combining with (1.8) and (1.9) our inequality (1.7) immediately follows.

These two lemmas provide some rather general methods of generating BMO functions. In fact, it can be shown that all BMO functions can be obtained in the manner given by Lemma 1.1.

Before closing this section, it is worthwhile to point out that these two lemmas are somewhat related.

Indeed, suppose we use Lemma 1.1 with $\theta_\nu = (\gamma_\nu^* - \gamma_{\nu-1}^*)/\gamma^*$ where γ_ν^*, γ^* are defined as in Lemma 1.2. Then we get that

$$\varphi = \sum_{\nu=1}^{\infty} E(\gamma_\nu^* - \gamma_{\nu-1}^*)/\gamma^*|\mathcal{F}_\nu)$$

is BMO. However, summation by parts yields

$$\sum_{\nu=1}^{n} \gamma_{\nu-1}^*\{E(1/\gamma^*|\mathcal{F}_\nu) - E(1/\gamma^*|\mathcal{F}_{\nu-1})\} =$$
$$= E(\gamma_n^*/\gamma^*|\mathcal{F}_n) - E(\gamma_0^*/\gamma^*|\mathcal{F}_0) - \sum_{\nu=1}^{n} E\left(\frac{\gamma_\nu^* - \gamma_{\nu-1}^*}{\gamma^*}\middle| \mathcal{F}_\nu\right). \tag{1.10}$$

So passing to the limit as $n \to \infty$, we obtain

$$\sum_{\nu=1}^{\infty} \gamma_{\nu-1}^{*}\{E(1/\gamma^{*}|\mathcal{F}_{\nu}) - E(1/\gamma^{*}|\mathcal{F}_{\nu-1})\} = 1 - E(\gamma_{0}^{*}/\gamma^{*}|\mathcal{F}_{0}) - \varphi. \tag{1.11}$$

Since any bounded function is clearly BMO, and the BMO norm of the left hand side of (1.11) is (as we have seen in the proof of Lemma 1.2) larger than that of the function in (1.6), we can see that »essentially» Lemma 1.1 implies Lemma 1.2. Unfortunately, it does not seem possible by this approach to derive as good an estimate as in (1.7).

Alternatively, suppose the functions θ_{ν} in Lemma 1.1 are of the form

$$\theta_{\nu} = (\gamma_{\nu} - \gamma_{\nu-1})/\gamma^{*} \quad (\gamma_{0} = 0, \mathcal{F}(\gamma_{\nu}) \subset \mathcal{F}_{\nu}).$$

Then the identity

$$\sum_{\nu=1}^{n} E(\theta_{\nu}|\mathcal{F}_{\nu}) = E(\gamma_{n}/\gamma^{*}|\mathcal{F}_{n}) - \sum_{\nu=1}^{n} \gamma_{\nu-1}\{E(1/\gamma^{*}|\mathcal{F}_{\nu}) - E(1/\gamma^{*}|\mathcal{F}_{\nu-1})\}$$

shows that in this case, Lemma 1.2 essentially implies Lemma 1.1 This seems to suggest that perhaps Lemma 1.1 might still remain true if the condition 1.1 is slackened to

$$\sup_{n} |\textstyle\sum_{\nu=1}^{n} \varphi_{\nu}| \leq 1.$$

However we shall have to leave this as an open question.

2. The »easy» side

Although in Burgess Davis paper the proof of the two inequalities is somewhat symmetrical in f^{*} and $S(f)$, historically the left hand side of (I.4) has been easier to prove.

Our proof here should be reminiscent of H. Weyl's »$n(x)$» method. Indeed, set for $\nu = 1, 2, \ldots, n$

$$E_{\nu} = \{\omega: f_{\nu-1}^{*} < f_{n}^{*}, f_{\nu}^{*} = f_{n}^{*}\}.$$

We then have, assuming $f_{0} = 0$

$$\int_{\Omega} f_{n}^{*} = \sum_{\nu=1}^{n} \int_{E_{\nu}} |f_{\nu}| = \sum_{\nu=1}^{n} \int_{\Omega} \theta_{\nu} f_{\nu} = \int_{\Omega} f_{n}[\textstyle\sum_{\nu=1}^{n} E(\theta_{\nu}|\mathcal{F}_{\nu})]dP, \tag{2.1}$$

where we have set

$$\theta_{\nu} = \chi_{E_{\nu}} \operatorname{sign}(f_{\nu}).$$

Since

$$\sum_{\nu=1}^{n} |\theta_{\nu}| = 1,$$

from Lemma 1.1 and (I.5) we get

$$\int_\Omega f_n^* \leq C_3 E(S_n(f)) \sqrt{5}.$$

3. The hard side

The idea here is to start from the estimate

$$E(S_n(f)) \leq \sqrt{E(f^*)} \sqrt{E(S_n^2(f)/f^*)} \tag{3.1}$$

which is a simple consequence of Schwarz's inequality. This given, we use the identity

$$S_n^2(f) = f_n^2 - 2 \textstyle\sum_{\nu=1}^n f_{\nu-1} \Delta f_\nu$$

and write the right hand most term of (3.1) in the form

$$E(S_n^2(f)/f^*) = E(f_n^2/f^*) - 2 \textstyle\sum_{\nu=1}^n E(\Delta f_\nu f_{\nu-1}/f^*).$$

We thus get

$$E(S_n(f)) \leq \sqrt{E(f^*)} \sqrt{E(f^*) + 2|Q|}. \tag{3.2}$$

where

$$Q = \textstyle\sum_{\nu=1}^n E(\Delta f_\nu f_{\nu-1}/f^*) = \sum_{\nu=1}^n E(\Delta f_\nu [E(f_{\nu-1}/f^* | \mathcal{F}_\nu) - E(f_{\nu-1}/f^* | \mathcal{F}_{\nu-1})]).$$

Since for $\mu \geq \nu + 1$ we have

$$E(\Delta f_\mu \cdot f_{\nu-1}[E(1/f^*)|\mathcal{F}_\nu) - E(1/f^* | \mathcal{F}_{\nu-1})]) = 0$$

we can write

$$Q = \int_\Omega f_n \varphi_n dP$$

where

$$\varphi_n = \textstyle\sum_{\nu=1}^n f_{\nu-1}[E(1/f^* | \mathcal{F}_\nu) - E(1/f^* | \mathcal{F}_{\nu-1})].$$

So by Lemma 1.2 with $\gamma_\nu = f_\nu$ and (I.5) we get

$$|Q| \leq C_3 E(S_n(f)) \sqrt{2}.$$

Substituting in 3.2 we get

$$E(S_n(f)) \leq \sqrt{E(f^*)} \sqrt{E(f^*) + 2\sqrt{2}\, C_3 E(S_n(f))}$$

and this is easily seen to yield

$$E(S_n(f)) \leq (\sqrt{2}\, C_3 + \sqrt{1 + 2C_3^2}) E(f^*).$$

4. A proof of Fefferman's inequality

Let $f_n = E(f|\mathcal{F}_n)$, $\varphi_n = E(\varphi|\mathcal{F}_n)$ with $f \in \mathcal{H}_1$ and $\varphi \in \mathrm{BMO}$. Let $f_0 \equiv \varphi_0 \equiv 0$ and set as before

$$\Delta f_n = f_n - f_{n-1}, \quad \Delta\varphi_n = \varphi_n - \varphi_{n-1}, \quad S_n = S_n(f) = \textstyle\sum_{\nu=1}^n [\Delta f_\nu]^2.$$

Note then that, since

$$[\Delta\varphi_n]^2 = |E(\varphi - \varphi_{n-1}|\mathcal{F}_n)|^2 \leq E(|\varphi - \varphi_{n-1}|^2|\mathcal{F}_n) \leq \|\varphi\|^2_{\mathrm{BMO}},$$

the product $f_n\varphi_n$ is integrable $\forall\, n \geq 1$ and we have

$$\int_\Omega f_n\varphi_n dP = \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega \Delta f_\nu \Delta\varphi_\nu dP. \tag{4.1}$$

A very clever idea due to C. Herz is to write the right hand side of 4.1 in the form

$$\textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega \frac{\Delta f_\nu}{\sqrt{S_\nu}} \Delta\varphi_\nu \sqrt{S_\nu} dP$$

and use Schwarz's inequality to obtain

$$\left|\int_\Omega f_n\varphi_n dP\right| \leq \sqrt{A}\,\sqrt{B} \tag{4.2}$$

where

$$A = \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega \frac{[\Delta f_\nu]^2}{S_\nu}\, dP, \quad B = \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega [\Delta\varphi_\nu]^2 S_\nu dP.$$

As we shall readily see this leads to a remarkably simple proof of Fefferman's inequality.

Indeed,

$$A = \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega \frac{S_\nu^2 - S_{\nu-1}^2}{S_\nu}\, dP \leq 2 \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega (S_\nu - S_{\nu-1}) dP = 2E(S_n(f)). \tag{4.3}$$

While, on the other hand we have

$$\begin{aligned} B &= \textstyle\sum_{\nu=1}^n \displaystyle\int_\Omega [\Delta\varphi_\nu]^2 \textstyle\sum_{\mu=1}^\nu (S_\mu - S_{\mu-1}) dP = \textstyle\sum_{\mu=1}^n \displaystyle\int_\Omega (S_\mu - S_{\mu-1}) \textstyle\sum_{\nu=\mu}^n [\Delta\varphi_\nu]^2 dP = \\ &= \textstyle\sum_{\mu=1}^n \displaystyle\int_\Omega (S_\mu - S_{\mu-1}) E(|\varphi_n - \varphi_{\mu-1}|^2|\mathcal{F}_\mu) dP \leq \|\varphi\|^2_{\mathrm{BMO}} E(S_n(f)). \end{aligned} \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) we obtain

$$\left| \int_\Omega f_n \varphi_n dP \right| \leq \sqrt{2} E(S_n(f)) \|\varphi\|_{\mathrm{BMO}}$$

which yields (I.5) with $C_3 = \sqrt{2}$ as asserted.

5. Further remarks

Before closing it migh be good to point out another interesting way of establishing the hard side of the Burgess Davis inequalities.

However, we shall only give an outline of the arguments here since some of the details are quite intricate and the resulting constant is not as good as that obtained in section 3.

The idea consists in establishing first the converse of Lemma 1.1, namely

LEMMA 5.1. *There is a universal constant* $c > 0$ *such that each* $\varphi \in \mathrm{BMO}$ *with* $\|\varphi\|_{\mathrm{BMO}} \leq 1$ *can be written in the form*

$$\varphi = \textstyle\sum_{\nu=1}^{\infty} E(\theta_\nu | \mathcal{F}_\nu)$$

with

$$\textstyle\sum_{\nu=1}^{\infty} |\theta_\nu| \leq c.$$

This given, when $E(f^*) < \infty$ and $\|\varphi\|_{\mathrm{BMO}} \leq 1$ we can write

$$\int_\Omega f_n \varphi_n dP = \int_\Omega f_n \textstyle\sum_{\nu=1}^{\infty} E(\theta_\nu | \mathcal{F}_\nu) dP = \textstyle\sum_{\nu=1}^{\infty} \displaystyle\int_\Omega f_{\nu \wedge n} \theta_\nu dP \leq$$

$$\leq \textstyle\sum_{\nu=1}^{\infty} \displaystyle\int_\Omega f_n^* |\theta_\nu| dP \leq c \int_\Omega f_n^* dP.$$

And this yields

$$\sup_{\|\varphi\|_{\mathrm{BMO}} \leq 1} \left| \int_\Omega f_n \varphi_n \right| \leq c \int_\Omega f_n^* dP. \tag{5.1}$$

On the other hand, we have

$$E(S_n(f)) = \textstyle\sum_{\nu=1}^{n} \displaystyle\int_\Omega [\Delta f_\nu]^2 / S_n \, dP =$$

$$= \textstyle\sum_{\nu=1}^{n} \displaystyle\int_\Omega \Delta f_\nu \{ E(\Delta f_\nu / S_n | \mathcal{F}_\nu) - E(\Delta f_\nu / S_n | \mathcal{F}_{\nu-1}) \} dP.$$

In other words

$$E(S_n(f)) = \int_\Omega f_n \varphi dP$$

with

$$\varphi = \sum_{\nu=1}^{n} \{E(\Delta f_\nu / S_n | \mathcal{F}_\nu) - E(\Delta f_\nu / S_n | \mathcal{F}_{\nu-1})\}$$

Now, it is easy to show that the following result, somewhat analogous to that of Lemma 1.1 holds.

LEMMA 5.2. *Let* $\{\theta_\nu\}$ *be a sequence of random variables satisfying*

$$\sum_{\nu=1}^{\infty} |\theta_\nu|^2 \leq 1$$

then the function

$$\varphi = \sum_{\nu=1}^{\infty} \{E(\theta_\nu | \mathcal{F}_\nu) - E(\theta_\nu | \mathcal{F}_{\nu-1})\}$$

is BMO *and indeed*

$$\|\varphi\|_{\text{BMO}} \leq \sqrt{5}.$$

This given we get

$$E(S_n(f)) \leq \sqrt{5} \sup_{\|\varphi\|_{\text{BMO}} \leq 1} \left| \int_\Omega f_n \varphi dP \right|, \tag{5.3}$$

and combining with (5.1) we finally obtain

$$E(S_n(f)) \leq \sqrt{5}\, c E(f_n^*).$$

Finally, we should mention that Lemma 5.2 like Lemma 1.1 has also a converse. But we hope to come back on this matter in a forthcoming publication.

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