

# The distribution of square-full integers

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## 1. Introduction

It is well-known that a positive integer  $n$  is called *square-full*, if in the canonical representation of  $n$  into prime powers each exponent is  $\geq 2$ ; or equivalently, if each prime factor of  $n$  occurs with multiplicity at least two. The integer 1 is also considered to be square-full. Let  $L$  denote the set of square-full integers and  $l(n)$  denote the characteristic function of the set  $L$ , that is,  $l(n) = 1$  or  $0$  according as  $n \in L$  or  $n \notin L$ . Let  $L(x)$  denote the enumerative function of the set  $L$ , that is,  $L(x) = \sum_{n \leq x} l(n)$ , where  $x$  is a real variable  $\geq 1$ .

In 1934, P. Erdős and G. Szekeres (cf. [7], § 2) proved the following asymptotic formula, using elementary methods:

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + O(x^{1/3}). \quad (1.1)$$

A simple proof of this result has been given later by A. Sklar [12]. In 1954, P. T. Bateman [1] improved the result (1.1) by means of the Euler Maclaurin sum formula to

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(x^{1/5}), \quad (1.2)$$

where

$$\zeta(s) = \frac{s}{s-1} - s \int_1^{\infty} \frac{(t-[t])}{t^{s+1}} dt, \quad (s > 0, s \neq 1); \quad (1.3)$$

and he remarked that, by more delicate methods, it is possible to sharpen the error term in (1.2) to  $O(x^{1/6} \log^2 x)$ .

Let

$$\Delta(x) = L(x) - \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} - \frac{\zeta(2/3)}{\zeta(2)} x^{1/3}. \quad (1.4)$$

In 1958, P. T. Bateman and E. Grosswald (cf. [2], § 5) improved the  $O$ -estimate of the error term in (1.2) to a considerable extent by proving that

$$\Delta(x) = O(x^{1/6} \exp \{-A \log^{4/7} x (\log \log x)^{-3/7}\}), \quad (1.5)$$

where  $A$  is an absolute positive constant. They also remarked (cf. [2], p. 95, lines 30–31) that one can expect to get an estimate of the form  $\Delta(x) = O(x^\alpha)$ ,  $\alpha$  fixed,  $\alpha < 1/6$  if and only if the least upper bound of the real parts of the zeros of the Riemann Zeta function is less than unity.

In this paper, we further improve the order estimate of  $\Delta(x)$ . In fact, we prove that  $\Delta(x) = O(x^{1/6} \delta(x))$ , where  $\delta(x) = \exp \{-A \log^{3/5} x (\log \log x)^{-1/5}\}$ ,  $A$  being a positive constant. Further, on the assumption of the Riemann hypothesis, we prove that  $\Delta(x) = O(x^{(1-\theta)/(7-12\theta)} \omega(x))$ , where  $\omega(x) = \exp \{A \log x (\log \log x)^{-1}\}$ ,  $A$  being a positive constant, and where  $\theta$  is the number which appears in the divisor problem, viz.,

$$\sum_{a^2 b^2 \leq x} 1 = \zeta(3/2) x^{1/2} + \zeta(2/3) x^{1/3} + O(x^\theta). \quad (1.6)$$

It is known that  $1/10 \leq \theta \leq 2/15$ . For the lower and upper bounds of  $\theta$ , we refer respectively to E. Krätzel (cf. [9], Satz 7) and H. E. Richert (cf. [10], Satz 2). We point out that whereas the proof of (1.5) given by P. T. Bateman and E. Grosswald [2] is somewhat complicated and many details are missing, the proofs of our results which will be given in § 3 are straightforward and elementary.

A brief historical account of the work done on  $L(x)$  by various earlier authors is given by E. Cohen (cf. [5], § 1). It should be mentioned that P. Erdős and G. Szekeres [7], B. Hronfeck [8], P. T. Bateman and E. Grosswald [2] actually considered  $k$ -full integers, that is, integers each of whose prime factors occur with multiplicity at least  $k$ , where  $k$  is any fixed integer  $\geq 2$ . For some other generalizations of the problem, we refer to A. Wintner (cf. [15], Section 62), and E. Cohen (cf. [3], [4] and [5]). For weaker order estimates of the error term of the enumerative function of  $k$ -full integers, using purely elementary methods, we refer to E. Cohen and K. J. Davis [6].

## 2. Preliminaries

In this section we recall some of the known results which are needed in our discussion. Throughout the following,  $x$  denotes a real variable  $\geq 3$ . We need the following best known estimate concerning the average of the Möbius function  $\mu(n)$  obtained by Arnold Walfisz [14]:

LEMMA 2.1 (cf. [14], Satz 3, p. 191).

$$M(x) = \sum_{n \leq x} \mu(n) = O(x\delta(x)), \tag{2.1}$$

where

$$\delta(x) = \exp \{ - A \log^{3/5} x (\log \log x)^{-1/5} \}, \tag{2.2}$$

$A$  being a positive constant.

LEMMA 2.2 (cf. [11], Lemma 2.2). For any  $s > 1$ ,

$$\sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O\left(\frac{\delta(x)}{x^{s-1}}\right). \tag{2.3}$$

LEMMA 2.3 (cf. [13], Theorem 14–26 (A), p. 316).

If the Riemann hypothesis is true, then

$$M(x) = \sum_{n \leq x} \mu(n) = O(x^{1/2}\omega(x)), \tag{2.4}$$

where

$$\omega(x) = \exp \{ A \log x (\log \log x)^{-1} \}, \tag{2.5}$$

$A$  being a positive constant.

LEMMA 2.4 (cf. [11], Lemma 2.6). For any  $s > 1$ ,

$$\sum_{n \leq x} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)} + O(x^{1/2-s}\omega(x)). \tag{2.6}$$

### 3. Main results

In this section, we prove the results mentioned in the introduction.

THEOREM 3.1. For  $x \geq 3$ ,

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + o(x^{1/6}\delta(x)), \tag{3.1}$$

where  $\delta(x)$  is given by (2.2).

*Proof.* We note that any square-full integer can be uniquely represented as  $n = d^2\delta^3$ , where  $\delta$  is square-free, that is, an integer which is not divisible by the square of any prime. Hence

$$l(n) = \sum_{d^2\delta^3=n} \mu^2(\delta) = \sum_{d^2\delta^3=n} \sum_{e^2f^3=\delta} \mu(e) = \sum_{d^2e^6f^3=n} \mu(e),$$

so that

$$L(x) = \sum_{n \leq x} l(n) = \sum_{d^2e^6f^3 \leq x} \mu(e).$$

Let  $z = x^{1/6}$ . Further, let  $0 < \varrho = \varrho(x) < 1$ , where the function  $\varrho$  will be suitably chosen later.

If  $d^2e^6f^3 \leq x$ , then both  $e > \varrho z$  and  $d^2f^3 > \varrho^{-6}$  cannot simultaneously hold good, and so we have

$$L(x) = \sum_{\substack{e \leq \varrho z \\ d^2e^6f^3 \leq x}} \mu(e) + \sum_{\substack{d^2f^3 \leq \varrho^{-6} \\ d^2e^6f^3 \leq x}} \mu(e) - \sum_{\substack{e \leq \varrho z \\ d^2f^3 \leq \varrho^{-6}}} \mu(e) = S_1 + S_2 - S_3, \text{ say.} \tag{3.3}$$

Now, by (1.6),

$$\begin{aligned} S_1 &= \sum_{\substack{e \leq \varrho z \\ d^2e^6f^3 \leq x}} \mu(e) = \sum_{e \leq \varrho z} \mu(e) \sum_{d^2f^3 \leq x/e^6} 1 \\ &= \sum_{e \leq \varrho z} \mu(e) \left\{ \zeta(3/2) \frac{x^{1/2}}{e^3} + \zeta(2/3) \frac{x^{1/3}}{e^2} + O\left(\frac{x^{\theta}}{e^{6\theta}}\right) \right\} \\ &= \zeta(3/2)x^{1/2} \sum_{e \leq \varrho z} \frac{\mu(e)}{e^3} + \zeta(2/3)x^{1/3} \sum_{e \leq \varrho z} \frac{\mu(e)}{e^2} + O\left(x^{\theta} \sum_{e \leq \varrho z} \frac{1}{e^{6\theta}}\right). \end{aligned}$$

Since,  $6\theta \leq 12/15 < 1$ , we have

$$x^{\theta} \sum_{e \leq \varrho z} \frac{1}{e^{6\theta}} = O(x^{\theta}(\varrho z)^{1-6\theta}) = O(\varrho^{1-6\theta}z).$$

Hence applying Lemma 2.2 for  $s = 2$  and  $3$ , we have

$$\begin{aligned} S_1 &= \zeta(3/2)x^{1/2} \left\{ \frac{1}{\zeta(3)} + O\left(\frac{\delta(\varrho z)}{(\varrho z)^2}\right) \right\} \\ &+ \zeta(2/3)x^{1/3} \left\{ \frac{1}{\zeta(2)} + O\left(\frac{\delta(\varrho z)}{\varrho z}\right) \right\} + O(\varrho^{1-6\theta}z) \\ &= \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(\varrho^{-2}z\delta(\varrho z)) + O(\varrho^{1-6\theta}z). \end{aligned} \tag{3.4}$$

We have

$$\begin{aligned} S_2 &= \sum_{\substack{d^2f^3 \leq \varrho^{-6} \\ d^2e^6f^3 \leq x}} \mu(e) = \sum_{d^2f^3 \leq \varrho^{-6}} \sum_{e \leq \left(\frac{x}{d^2f^3}\right)^{1/6}} \mu(e) = \sum_{d^2f^3 \leq \varrho^{-6}} M\left(\left(\frac{x}{d^2f^3}\right)^{1/6}\right) \\ &= O(x^{1/6}) \sum_{d^2f^3 \leq \varrho^{-6}} d^{-1/3}f^{-1/2} \delta\left(\left(\frac{x}{d^2f^3}\right)^{1/6}\right), \end{aligned}$$

by (2.1). Since  $\delta(x)$  is monotonic decreasing and  $(x/d^2f^3)^{1/6} \geq \varrho z$ , we have

$$\delta\left(\left(\frac{x}{d^2f^3}\right)^{1/6}\right) \leq \delta(\varrho z).$$

Also,

$$\begin{aligned} \sum_{d^2f^3 \leq \varrho^{-6}} d^{-1/3} f^{-1/2} &= \sum_{d \leq \varrho^{-3}} d^{-1/3} \sum_{f \leq \left(\frac{\varrho^{-6}}{d^2}\right)^{1/3}} f^{-1/2} \\ &= O\left(\sum_{d \leq \varrho^{-3}} d^{-1/3} \left(\frac{\varrho^{-6}}{d^2}\right)^{1/6}\right) = O(\varrho^{-1} \sum_{d \leq \varrho^{-3}} d^{-2/3}) = O(\varrho^{-1}(\varrho^{-3})^{1/3}) = O(\varrho^{-2}). \end{aligned}$$

Hence

$$S_2 = O(\varrho^{-2}z\delta(\varrho z)). \tag{3.5}$$

Also, we have by (2.1) and (1.6),

$$S_3 = \sum_{\substack{e \leq \varrho z \\ d^2f^3 \leq \varrho^{-4}}} \mu(e) = \sum_{e \leq \varrho z} \mu(e) \sum_{d^2f^3 \leq \varrho^{-4}} 1 = O(M(\varrho z)\varrho^{-3}) = O(\varrho^{-2}z\delta(\varrho z)). \tag{3.6}$$

Hence by (3.3), (3.4), (3.5) and (3.6),

$$L(x) = \frac{\zeta(3/2)}{\zeta(3)} x^{1/2} + \frac{\zeta(2/3)}{\zeta(2)} x^{1/3} + O(\varrho^{-2}z\delta(\varrho z)) + O(\varrho^{1-6\varrho}z). \tag{3.7}$$

Now, we choose

$$\varrho = \varrho(x) = \{\delta(x^{1/12})\}^{1/3}, \tag{3.8}$$

and write

$$f(x) = \log^{3/5}(x^{1/12}) \{\log \log(x^{1/12})\}^{-1/5} = (1/12)^{3/5} u^{3/5} (v - \log 12)^{-1/5}, \tag{3.9}$$

where  $u = \log x$  and  $v = \log \log x$ .

For  $v \geq 2 \log 12$ , that is,  $u \geq 144$ ,  $x \geq e^{144}$ , we have

$$v^{-1/5} \leq (u - \log 12)^{-1/5} \leq (v/2)^{-1/5} \tag{3.10}$$

so that

$$(1/12)^{3/5} u^{3/5} v^{-1/5} \leq f(x) \leq (1/12)^{3/5} (1/2)^{-1/5} u^{3/5} v^{-1/5}. \tag{3.11}$$

We assume without loss of generality that the constant  $A$  in (2.2) is less than unity. By (3.8), (2.2) and (3.9), we have

$$\varrho = \exp\{-Af(x)/3\}. \tag{3.12}$$

By (3.10), we have  $(1/3)(1/12)^{3/5}(1/2)^{-1/5}u^{3/5}v^{-1/5} \leq u/12$ .

Hence by (3.11), (3.12) and the above

$$\begin{aligned} \varrho &\geq \exp \left\{ -A(1/3)(1/12)^{3/5}(1/2)^{-1/5}u^{3/5}v^{-1/5} \right\} \\ &\geq \exp \left\{ -\frac{u}{12} \right\} = \exp \left\{ -\frac{\log x}{12} \right\}, \end{aligned}$$

so that  $\varrho \geq x^{-1/12}$ . Hence

$$\varrho z \geq x^{(1/6)-(1/12)} = x^{1/12}. \quad (3.13)$$

Since  $\delta(x)$  is monotonic decreasing,  $\delta(\varrho z) \leq \delta(x^{1/12}) = \varrho^3$ , by (3.8) and so, by (3.11) and (3.12), we have

$$\varrho^{-2}\delta(\varrho z) \leq \varrho \leq \exp \left\{ -\frac{A}{3}(1/12)^{3/5}u^{3/5}v^{-1/5} \right\}. \quad (3.14)$$

Hence the first  $O$ -term in (3.7) is equal to  $O(x^{1/6} \exp \{ - (A/3)(1/12)^{3/5}u^{3/5}v^{-1/5} \})$ , which is  $O(x^{1/6} \exp \{ - (A(1-6\theta)/3)(1/12)^{3/5}u^{3/5}v^{-1/5} \})$ , since  $0 < 1-6\theta < 1$ . By (3.12) and (3.11), we see that the second  $O$ -term in (3.7) is also of the above order. Thus if  $\Delta(x)$  denotes the sum of the two  $O$ -terms in (3.7), we have

$$\Delta(x) = O(x^{1/6} \exp \{ -B \log^{3/5} x (\log \log x^{-1/5}) \}), \quad (3.15)$$

where  $B = (A(1-6\theta)/3)(1/12)^{3/5}$ , a positive constant.

Hence Theorem 3.1 follows by (3.7) and (3.15).

**THEOREM 3.2.** *If the Riemann hypothesis is true, then the error term  $\Delta(x)$  in the asymptotic formula for  $L(x)$  is  $O(x^{(1-\theta)/(7-12\theta)}\omega(x))$ , where  $\theta$  is the number given by (1.6) and  $\omega(x)$  is given by (2.5).*

*Proof.* Following the procedure adopted in Theorem 3.1 using (2.6) instead of (2.3), we get that

$$\Delta(x) = O(\varrho^{-5/2}z^{1/2}\omega(\varrho z)) + O(\varrho^{1-6\theta}z). \quad (3.16)$$

Now, choosing  $\varrho = z^{-1/(7-12\theta)}$ , we see that  $0 < \varrho < 1$  and

$$\varrho^{-5/2}z^{1/2} = \varrho^{1-6\theta}z = x^{(1-\theta)/(7-12\theta)}.$$

Also, since  $\omega(x)$  is monotonic decreasing and  $\varrho z < z$ , we have  $\omega(\varrho z) < \omega(z) < \omega(x)$ . Hence by (3.16) and the above, we have

$$\Delta(x) = O(x^{(1-\theta)/(7-12\theta)}\omega(x)).$$

Hence Theorem 3.2 follows.

**COROLLARY 3.1.** *If the Riemann hypothesis is true, then*

$$\Delta(x) = O(x^{13/81}\omega(x)).$$

*Proof.* Since  $0 < \theta \leq 2/15$ , we have  $(1 - \theta)/(7 - 12\theta) \leq 13/81$ . Hence the corollary follows by Theorem 3.2.

COROLLARY 3.2. *If the Riemann hypothesis is true, then*

$$\Delta(x) = O(x^{(13/81)+\varepsilon}), \text{ for every } \varepsilon > 0.$$

*Proof.* This follows by Corollary 3.1, since  $\omega(x) = O(x^\varepsilon)$ , for every  $\varepsilon > 0$ .

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