The Cauchy problem for systems in L_p and $L_{p,\alpha}$

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0. Introduction

We shall consider the Cauchy problem

$$\begin{cases} \partial u/\partial t = P(D)u, & x \in R^n, \quad 0 \leq t \leq T, \\ u(x, 0) = u_0(x), \end{cases} \tag{0.1}$$

where u and u_0 are complex N-vector functions and where $P(D)$ is an $N \times N$ matrix of pseudo-differential operators, that is, $P(D)u$ is defined by

$$P(D)u(x) = \int \exp(-2\pi i\langle x, y\rangle) P(y)\hat{u}(y)dy$$

where $\hat{u}$ is the Fourier transform of u and where $P(y)$ is the symbol of $P = P(D)$. We assume that the operator P has order $d > 0$. The principal part P_d of P is defined by the nonvanishing symbol $P_d(y) = \lim_{\lambda\to\infty} \lambda^{-d}P(\lambda y)$. The operator $P(D)$ is a partial differential operator if and only if $P(y)$ is a polynomial. We define $\tilde{d} = \min\{d, (d - d_1)/(1 + \varepsilon)\}$, where d_1 is the order of $P - P_d$, and where $\varepsilon = 0$ if $P_dP = PP_d$, $\varepsilon = |1/2 - 1/p|$ otherwise. Then $\tilde{d} = d$ for homogeneous operators P. For details see [7] and section 5 below.

For $\alpha \geq 0$ we let $w_\alpha(y) = (1 + |y|^2)^{\alpha/2}$ and define for $1 \leq p \leq \infty$

$$\|u\|_{p,\alpha} = \|F^{-1}(w_\alpha \hat{u})\|_p,$$

where F^{-1} denotes the inverse Fourier transform.

We say, with some abuse of language, that (0.1) is well posed in $L_{p,\alpha}$, if there is some constant $C = C(T)$ such that for all $u_0 \in \mathscr{C}_0^\infty$ there exists a well defined solution u of (0.1) in L_p^N (cf. section 1) which satisfies

$$\|u(\cdot, t)\|_p \leq C\|u_0\|_{p,\alpha}, \quad 0 \leq t \leq T. \tag{0.2}$$

It is known that (0.1) is well posed in L_2 if and only if

$$\sup\{|\exp(tP(y))|; \ y \in R^n, \ 0 \leq t \leq T\} \leq C < \infty. \tag{0.3}$$

Kreiss [15] has given a complete characterization of the $N \times N$-matrices which satisfy (0.3). In particular, if the eigenvalues of P_d are imaginary, $y \in R^n$, then a necessary condition for (0.3) to hold is that $P_d(y)$ is uniformly diagonable on R^n. For $d = 1$, this is also sufficient.

Even if (0.1) is well posed in L_2, it need not be well posed in L_p for $p \neq 2$ (cf. Littman [19] for the wave-equation, and also the more general results below). However, provided that (0.1) is well posed in L_2, and under the assumption that $P(y)$ is in $\mathscr{C}^\infty$, the problem (0.1) is well posed in $L_{p,\alpha}$ for $\alpha > nd|1/2 - 1/p|$ (Theorem 5.5 below). In section 5 we will give necessary and sufficient conditions for (0.1) to be well posed in $L_{p,\alpha}$, assuming that (0.1) is well posed in L_2 and that the eigenvalues of $P_d(y)$ are imaginary, $y \in R^n$ (e.g. if $\partial/\partial t - P(D)$ is strongly hyperbolic; cf. [23] and section 6 below). The results are somewhat negative in character. Among the results proved in section 5 we mention the following for differential operators:

THEOREM 0.1. *Let* $0 \leq \alpha < \tilde{d}|1/2 - 1/p|$. *Assume that* P *is a differential operator and that the eigenvalues of* $P_d(y)$ *are imaginary for* $y \in R^n$. *Then* (0.1) *is well posed in* $L_{p,\alpha}$ *and in* L_2 *if and only if*

$$P_d(D) = \sum_{j=1}^{n} A_j \frac{\partial}{\partial x_j}, \tag{0.4}$$

where $A_1, \ldots, A_n$ *are commuting diagonable matrices with real eigenvalues.*

For symmetric hyperbolic systems and $\alpha = 0$, Theorem 0.1 was proved in [3] and later, still for $\alpha = 0$, for a larger class of hyperbolic systems by Kopáček [13]. See also the paper [14].

We will also give analogues to Theorem 0.1 for pseudo-differential operators in section 5.

As a consequence of Theorem 0.1 the Cauchy problem for the wave-equation is not well posed in $L_{p,\alpha}$, $0 \leq \alpha < |1/2 - 1/p|$. For $\alpha = 0$ this result is due to Littman [19]. In this particular case the bound $|1/2 - 1/p|$ can be improved. Using the methods of Littman, Muravei [22] proved that the Cauchy problem for the wave-equation is not well posed in $L_{p,\alpha}$ for $0 \leq \alpha < (n-1)|1/2 - 1/p|$. We generalize this as follows: Assume for simplicity that $N = 1$, and that $P_d(y)$ is imaginary. Define the rank r of P_d as the maximum rank of the $n \times n$-matrix $(\partial^2 P_d/\partial y_k \partial y_l)_{k,l}$. Then (0.1) is not well posed in $L_{p,\alpha}$ for $0 \leq \alpha < r\tilde{d}|1/2 - 1/p|$.

The corresponding result for systems is proved in Theorem 5.4. For $n > 1$ the rank of $|y|$ is $n - 1$, and this proves the above result of Muravei. For other examples, see section 5.

Let L_p^N denote the N-vector functions with components in L_p, and let $M_p^{N,N}$ denote the multipliers on FL_p^N. The natural norm in $M_p^{N,N}$ is denoted $M_p^{N,N}(\cdot)$.

To prove the results mentioned above, we notice that (0.1) is well posed in $L_{p,\alpha}$ if and only if

$$M_p^{N,N}(w_\alpha^{-1} \exp(tP)) \leq C(T), \quad 0 \leq t \leq T. \tag{0.5}$$

For $p = 2$, $\alpha = 0$, this reduces to (0.3) above. We shall prove that (0.5) implies at least locally on $R^n \setminus \{0\}$, that the m'th powers of $\exp(P_d)$ are $O(m^{\alpha/d})$ in $M_p^{N,N}$. For non-homogeneous operators, the lower order terms introduce an error-term for $\alpha > 0$ and d is replaced by $\tilde{d}$; for details see section 5. Under suitable smoothness assumptions, the growth of powers of elements in $M_p^{N,N}$ is studied in sections 2 (for $N = 1$) and 3. The methods used rely on the technique developed by Hörmander [6], and the methods used in [3], [4]. The results obtained will in particular imply theorems like Theorem 0.1 above.

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1. Multipliers on FL_p

For complex N-vectors u and v, $\langle u, v\rangle$ shall denote their scalar product and $|v|$ the Euclidean norm. The norm of an $N_1 \times N_2$-matrix A will be the operator norm $|A| = \sup\{|Av|; |v| \leq 1\}$. The transpose of A is denoted ${}^tA$.

By $\mathcal{C}^\nu(B)$ we will denote the set of N-vectors, and occasionally $N \times N$-matrices, with elements in $C^\nu(B)$. If $g \in C^\infty(R^n)$ and if

$$\sup\{|x|^m |D^k g(x)|; x \in R^n\} < +\infty$$

for $m = 0, 1, \ldots$ and for any multi-index $k = (k_1, \ldots, k_n)$, $|k| = k_1 + \ldots + k_n$, we say that $g \in S$. Here $D^k = (-2\pi i)^{-|k|}(\partial/\partial x_1)^{k_1} \ldots (\partial/\partial x_n)^{k_n}$. We give the linear space S the topology defined by the above family of semi-norms. The set of complex N-vectors with components in S is denoted S^N. The dual space S' of S is the space of tempered distributions.

The convolution $\mu * g$ between a $N_2 \times N_1$ matrix μ of tempered distributions and a function $g \in S^{N_1}$ is defined by $\mu(g(x - \cdot))$ (which has the obvious sense). The Fourier transform $\hat{\mu}$ of a tempered distribution μ is defined by $\hat{\mu}(f) = \mu(\hat{f})$, $f \in S$, where $\hat{f}$ is the function

$$\hat{f}(y) = Ff(y) = \int_{R^n} \exp(2\pi i\langle x, y\rangle) f(x)dx.$$

The Fourier transform is defined for matrix- and vector-valued tempered distributions by applying the transform elementwise. If $K \subseteq S'$, then FK denotes the corresponding set of Fourier transforms.

By L_p^N we mean the set of functions $v = (v_1, \ldots, v_N)$ with $v_j \in L_p$, $j = 1, \ldots, N$. For $p < \infty$ we let

$$\|v\|_p = \left(\int_{R^n} |v(x)|^p dx \right)^{1/p}$$

and for $p = \infty$

$$\|v\|_\infty = \text{ess sup} \{ |v(x)|; x \in R^n \}.$$

We shall in the following assume that $1 \leq p \leq \infty$.

Classically a multiplier on FL_p is a function λ such that for each $f \in L_p$, $1 \leq p \leq 2$, $\lambda \hat{f}$ is the Fourier transform of an L_p-function. By the closed graph theorem this defines a bounded operator on L_p. This operator is obviously translation invariant. Following Hörmander [6] we make the following definition: we say that an $N_2 \times N_1$ matrix μ, with elements in S', is a multiplier from $FL_p^{N_1}$ to $FL_p^{N_2}$, and write $\mu \in M_p^{N_1, N_2}$, if

$$M_p^{N_1, N_2}(\mu) = \sup \{ \|\hat{\mu} * f\|_p; f \in S^{N_1}, \|f\|_p \leq 1 \} < +\infty.$$

For $N_1 = N_2 = 1$, we also write M_p for $M_p^{1,1}$. We use the convention that $M_p^{N_1, N_2}(\mu) = +\infty$ if $\mu \notin M_p^{N_1, N_2}$. For $p = 1$ and ∞ we identify $M_p^{N_1, N_2}$ with a subset of $\mathscr{C}$, whenever convenient.

LEMMA 1.1.

(i) $M_p^{N_1, N_2} = M_{p'}^{N_1, N_2}$, $1/p + 1/p' = 1$, *and* $M_1^{N_1, N_2} \subseteq M_p^{N_1, N_2} \subseteq M_2^{N_1, N_2}$. *Further* $M_p(\cdot)$ *is a logarithmically convex function of* $1/p$.

(ii) $M_2^{N_1, N_2}$ *is the set of essentially bounded measurable* $N_2 \times N_1$ *matrix functions, and* $M_2^{N_1, N_2}(\cdot) = \text{ess sup } |\cdot|$. $M_1^{N_1, N_2}$ *is the set of* $N_2 \times N_1$ *matrices with elements that are Fourier-Stieltjes transforms of bounded measures.*

(iii) $M_p^{N, N}$ *is a Banach algebra under pointwise matrix multiplication and addition, with norm* $M_p^{N,N}(\cdot)$. *It is non-commutative for* $N > 1$. *Further, if* $\mu \in M_p^{N_1, N_2}$ *and* $\nu \in M_p^{N_2, N_3}$, *then* $\nu\mu \in M_p^{N_1, N_3}$ *and* $M_p^{N_1, N_3}(\nu\mu) \leq M_p^{N_1, N_2}(\mu) M_p^{N_2, N_3}(\nu)$.

(iv) *Let* $f_i \in M_p^{N_1, N_2}$ *and* $M_p^{N_1, N_2}(f_i) \leq C$, $i = 1, 2, \ldots$. *If* $f_i \to f$ *a.e. then* $M_p^{N_1, N_2}(f) \leq C$.

(v) *Let* $a: R^n \to R^m$, $m \leq n$ *be an affine and surjective transformation, and let* $a^*f(y) = f(ay)$. *Then* $M_p^{N_1, N_2}(a^*f) = M_p^{N_1, N_2}(f)$, *with norms in* R^n *and* R^m, *respectively.*

(vi) *If* $f \in M_p^{N_1, N_2} \cap \mathscr{C}$, *then the conclusion of* (v) *is valid for all affine maps.*

Proof. For the case $N_1 = N_2 = 1$, these statements can all be found in Chapter I in [6], and for (v) and (vi) in [18]. Most of the generalizations to N_1 and/or $N_2 > 1$ are obvious, and in those cases we only give the references to [6] and [18].

(i) For $N_1 = N_2 = 1$, this is Theorem 1.3 in [6].
(ii) Theorems 1.3 and 1.5 in [6].
(iii) Corollary 1.4 in [6].
(iv) This is a special case of Corollary 2.7 in [18]. We give here another proof. Let $u \in S^{N_1}$ and $v \in S^{N_2}$. By Parseval's formula and Hölder's inequality we have $(1/p + 1/p' = 1)$,

$$\left| \int_{R^n} \langle f_i(y)u(y), \ v(y)\rangle dy \right| \leq C\|\hat{u}\|_p\|\hat{v}\|_{p'}.$$

By dominated convergence (we use (i) and (ii)) then

$$\left| \int_{R^n} \langle f(y)u(y), \ v(y)\rangle dy \right| \leq C\|\hat{u}\|_p\|\hat{v}\|_{p'}.$$

By the converse of Hölder's inequality, this means that $f \in M_p^{N_1,N_2}$. Notice that since f is the limit of a sequence of measurable functions, f is certainly measurable.
(v) This is Theorem 1.13 in [6].
(vi) In view of (v) we may assume that $m \leq n$ and that a is the inclusion of R^m in R^n. The statement is then Proposition 3.2 in [18]. The following proof was suggested by Lars Hörmander: Write $R^n = R^m \oplus V$ and let $i: R^m \to R^n$ and $j: V \to R^n$ be the inclusions. For $\varepsilon > 0$ the mapping $a_\varepsilon = i \oplus \varepsilon j: R^n \to R^n$ is affine and surjective. Hence $M_p^{N_1,N_2}(a_\varepsilon^* f) = M_p^{N_1,N_2}(f)$, using (v), and since $a_\varepsilon^* f \to i^* f$ as $\varepsilon \to 0$, (iv) above applies and proves the statement.

In connection with (ii) we notice ($N_1 = N_2 = 1$ and with the convention that $M_1 \subseteq C$) that $\mu \in M_1$ if and only if

$$\mu(y) = \int_{R^n} \exp(-2\pi i\langle x, y\rangle)d\hat{\mu}(x)$$

with

$$M_1(\mu) = \int d|\hat{\mu}(x)| < \infty.$$

In particular: If $f \in L_1$ then $\hat{f} \in M_1$ and $M_1(\hat{f}) = \|f\|_1$.

Let B be a closed set in R^n, with positive measure. We say that an $N_2 \times N_1$ matrix function φ is a multiplier from $FL_p^{N_1}$ to $FL_p^{N_2}$ on B, $\varphi \in M_{p,B}^{N_1,N_2}$, if there is a $\mu \in M_p^{N_1,N_2}$ such that $\varphi = \mu$ on B. We use the quotient norm

$$M_{p,B}^{N_1,N_2}(\varphi) = \inf\{M_p^{N_1,N_2}(\mu); \mu = \varphi \text{ on } B\}.$$

For $N_1 = N_2 = 1$ we also write $M_{p,B}$ and $M_{p,B}(\cdot)$. We give some facts about $M_{p,B}$ in the following lemma.

LEMMA 1.2.

(i) *If* $B \subseteq B'$, *then* $M_{p,B}(\varphi) \leq M_{p,B'}(\varphi)$.

(ii) $M_{p,B}^{N_1,N_3}(\nu\varphi) \leq M_{p,B}^{N_1,N_2}(\varphi) M_{p,B}^{N_2,N_3}(\nu)$.

(iii) *If* $a \in R \setminus \{0\}$ *and* $y_0 \in R^n$, *then* $M_{p,B}(\varphi) = M_{p,a^{-1}B}(a^*\varphi) = M_{p,B-y_0}(\varphi_{y_0})$, *where* $\varphi_{y_0}(y) = \varphi(y + y_0)$.

(iv) *If* $k \in S$ *vanishes outside* B, *then*

$$M_p^{N_1,N_2}(\varphi k) \leq M_{p,B}^{N_1,N_2}(\varphi) \|\hat{k}\|_1.$$

(v) *If* $M_{p,B}^{N_1,N_2}(\varphi) \leq C$ *for all compact balls* $B \subseteq R^n$, *then* $M_p^{N_1,N_2}(\varphi) \leq C$.

Proof. The statements (i), (ii), (iii) and (iv) are all immediate consequences of the corresponding facts for $M_p^{N_1,N_2}$ and the definitions. The proof of (v) was given in Lemma 3 in [3].

We will later have use for the following version of the Hörmander-Mikhlin theorem (for a proof see Theorem 2.5 in [6]).

LEMMA 1.3. *Let* $\varphi \in C^\nu(R^n \setminus \{0\})$ *for some* $\nu > n/2$, *be homogeneous of degree zero on* R^n. *Then* $\varphi \in M_p$ *for* $1 < p < \infty$.

Under the assumptions in Lemma 1.3 we have for $|y| \geq 1$,

$$D^\alpha \varphi(y) = O(|y|^{-|\alpha|}), \quad |\alpha| \leq \nu. \tag{1.1}$$

In order to conclude that $\varphi \in M_1 = M_\infty$, we have to strengthen (1.1). We then get the following variant of Bernstein's theorem (cf. Proposition 2 and Lemma 3 in [24 — III]).

LEMMA 1.4. *Let* $r \in C^\nu(R^n)$ *and assume that*

$$D^\alpha r(y) = O(|y|^{-s-|\alpha|}), \quad |\alpha| \leq \nu, \tag{1.2}$$

for some $s > 0$, *as* $|y| \to \infty$. *If* $\nu > n/2$, *then* $r \in M_1 \subseteq M_p$.

Proof. (After Hörmander [6].) We will prove that $\hat{r} \in L_1$. Let $\psi_j(y) = \psi(2^{-j}y)$ where $\psi \in C^\infty$ has support in $1/2 < |y| < 2$ and where $\sum_{-\infty}^{\infty} \psi_j = 1$ for $y \neq 0$ (cf. Lemma 2.3 in [6]). Let $\psi_{-\infty} = 1 - \sum_{j=0}^{\infty} \psi_j$. From the classical Bernstein theorem $\psi_{-\infty} r \in FL_1$, since $\psi_{-\infty} r \in C_0^\nu$, $\nu > n/2$. It is left to estimate $(\psi_j r)^\wedge$ for $j \geq 0$ in L_1, and then add the results. Let $r_j = \psi_j r$. Then

$$\left(\int |\hat{r}_j(x)| dx \right)^2 \leq \int (1 + 2^{2j}|x|^2)^{-\nu} dx \int (1 + 2^{2j}|x|^2)^\nu |\hat{r}_j(x)|^2 dx. \tag{1.3}$$

But

$$\left(\int |(1 + 2^{2j}|x|^2)^{\nu/2} \hat{r}_j(x)|^2 dx \right) \leq C \sum_{|\alpha| \leq \nu} \left(\int |2^{j|\alpha|} D^\alpha \psi_j(y) r(y)|^2 dy \right)^{1/2}.$$

Using (1.2) and Leibnitz' formula we get

$$|2^{j|\alpha|}D^{\alpha}(\psi_j(y)r(y))| \leq C\,2^{-sj}, \quad |\alpha| \leq \nu,$$

and so by (1.3), since $\nu > n/2$, that

$$\int |\hat{r}_j(x)|dx \leq C\,2^{-sj}\left(\int \frac{2^{nj}dx}{(1+2^{2j}|x|^2)^{\nu}}\right)^{1/2} \leq C' 2^{-sj}. \tag{1.4}$$

Hence

$$\|\hat{r}\|_1 \leq \|\widehat{\psi_{-\infty} r}\|_1 + \sum_{j=0}^{\infty} \|\hat{r}_j\|_1 < +\infty,$$

which proves the lemma.

LEMMA 1.5. *Let $\varphi = r + q_0 \in C^{\nu}$, where $q_0 \in C^{\nu}(R^n \setminus \{0\})$ is homogeneous of degree zero. If r satisfies (1.2) for some $s > 0$ as $|y| \to \infty$, and if $\nu > n/2$, then $\varphi \in M_p$, $1 < p < \infty$. If q_0 is constant, then $\varphi \in M_1 \subseteq M_p$.*

It is clear that if $\varphi \in M_1$, then also $q_0 \in M_1 \subseteq C$, and so q_0 is constant.
Proof. The Lemma is an immediate consequence of Lemmas 1.3 and 1.4.
We will need the following version of the Wiener-Levy theorem in $M_{p,B}$.

PROPOSITION 1.1. *Assume that $|1/p - 1/2| > |1/q - 1/2|$ and that $B \subseteq R^n$ is compact. If $f_1, \ldots, f_N \in M_{p,B} \cap C$ and if F is analytic in a neighborhood of $\{(f_1(y), \ldots, f_N(y));\ y \in B\}$, then $F(f_1, \ldots, f_N) \in M_{q,B'}$, where B' is a closed ball contained in the interior of B.*

Proof. Let m_p denote the closure of S in M_p, and $m_{p,B}$ the corresponding restriction algebra on B. Since m_p has maximal ideal space R^n (Theorem 1.17 in [6]) and separates points on R^n, the maximal ideal space of $m_{p,B}$ is B. Further $M_{q,B'} \supseteq m_{q,B'} \supseteq M_{p,B} \cap C$ if B' is compact and contained in the interior of B, and with p, q as above (Theorem 1.16 in [6], where it is even proved that $m_q \supseteq M_p \cap C_0$). Standard results from Banach algebra theory now prove the proposition (cf. [2], Chapter 1, section 4).

Remark. For $p = 1$ or ∞ we can take $p = q$ in Proposition 1.1, since in this case $m_{1,B} = M_{1,B}$ for B compact ($m_1 = FL_1$). We do not know whether this is allowed in general.

A similar consequence of Theorems 2.6 and 2.7 in [6] is stated in the following proposition.

PROPOSITION 1.2. *Let $f \in C(R^n \setminus \{0\})$ be homogeneous of degree zero on R^n. Assume that $f \in M_p$ for all p with $1 < p < \infty$, and that F is analytic in a neighborhood of $\{f(y): y \in S^{n-1}\}$. Then $F(f) \in M_p$ for all p with $1 < p < \infty$.*

Here S^{n-1} denotes the unit sphere in R^n. We omit the details of the proof, which is close to that of Proposition 1.1.

2. Powers of multipliers on FL_p

In this section we shall give some results on the growth of powers of elements in M_p, which are modifications and localizations of results proved e.g. by Hörmander [6], Leblanc [17] and Kahane [9].

From now on we assume that $\{Q_m\}$ is a family of functions which is bounded in $C^1(R^n \setminus S_0)$ where S_0 is a closed nowhere dense set in R^n. When we discuss Cauchy problems below, such families will be generated by lower-order terms in the symbol of a pseudo-differential operator, and in that case $S_0 = \{0\}$.

The main results in this section are the following.

THEOREM 2.1. *Let* $0 \leq \alpha < |1/2 - 1/p|$. *Let* $B \subseteq R^n$ *be a closed ball. Assume that* $\varphi_m = \varphi + m^{-1}Q_m$ *and that* $\varphi \in C^2(B)$ *is real. If*

$$M_{p,B}(\exp(im\,\varphi_m)) \leq Cm^{\alpha}, \quad m = 1, 2, \ldots \tag{2.1}$$

then φ *is linear on* B.

For $p = 1, \infty$ and $\alpha = 0$, the C^2-condition is not necessary, as proved by Beurling and Helson [1]. For $\alpha = 0$ and $B = R^n$, the above result is due to Hörmander (Theorem 1.14 in [6]), and the local C^2-version, still for $\alpha = 0$, can be found in [3]. As we will see from the proof, Theorem 2.1 is essentially »one-dimensional». A more precise result in R^n is the following (cf. Leblanc [17]).

THEOREM 2.2. *Let* $B \subseteq R^n$ *be a closed ball. Assume that* $\varphi_m = \varphi + m^{-1}Q_m$, *that* Q_m *is uniformly in* $C^\infty(B \setminus S_0)$, $m \geq 1$, *and that* $\varphi \in C^\infty(B)$ *is real. If* r *is the maximum rank in* B *of the Hessian matrix* $J(y) = ((\partial^2/\partial y_k \partial y_l)\varphi(y))_{k,l}$ *of* φ *then there is a constant* $c > 0$ *such that*

$$M_{p,B}(\exp(im\,\varphi_m)) \geq cm^{r|1/2-1/p|}, \quad m = 1, 2, \ldots \tag{2.2}$$

We proceed to the proofs of the theorems above. We first state a version of van der Corput's lemma (see [26], p. 197).

LEMMA 2.1. *Assume that* $u \in C_0^1(R)$ *and that* $\varphi \in C^2$ *with* $|\varphi''(y)| \geq \delta > 0$ *on the support of* u. *Then*

$$\|F(\exp(i\varphi)u)\|_\infty \leq C\delta^{-1/2}\|Du\|_1 \tag{2.3}$$

where C *is independent of* u *and of* φ.

Using this we can prove the following:

LEMMA 2.2. *Assume that* $B_0 \subseteq R$ *is a closed interval, that* $\varphi \in C^2(B_0)$ *is real and that* $\varphi''(y_0) \neq 0$ *for some* $y_0 \in B_0$. *Then there is a constant* $c > 0$ *such that*

$$M_{p,B_0}(\exp(im\,\varphi)) \geq cm^{|1/2-1/p|}, \quad m = 1, 2, \ldots \tag{2.4}$$

Proof. We may assume that $\varphi'' \neq 0$ on B_0. Let $0 \neq u \in C_0^1(B_0)$. By Lemma 1.1(i) we may also assume that $1 \leq p \leq 2$. We let $1/p + 1/p' = 1$. Then, by Lemma 2.1, Parseval's formula and Hölder's inequality,

$$\begin{aligned} \|F(\exp(im\,\varphi)u)\|_{p'} &\leq \|F(\exp(im\,\varphi))\|_\infty^{1-2/p'} \|F(\exp(im\,\varphi)u)\|_2^{2/p'} \\ &\leq Cm^{-1/2|1-1/p'|} = Cm^{-|1/2-1/p|}. \end{aligned} \tag{2.5}$$

By (2.5) and Parseval's formula then

$$\begin{aligned} \int |u|^2 dy = \int \exp(im\,\varphi)u\,\overline{\exp(im\,\varphi)u}dy &\leq \|F(\exp(im\,\varphi)u)\|_p \|F(\exp(im\,\varphi)u)\|_{p'} \\ &\leq CM_{p,B_0}(\exp(im\,\varphi))m^{-|1/2-1/p|}, \end{aligned}$$

which proves (2.4).

Proof of Theorem 2.1. We may assume that B is compact. We will prove that all the second-order derivatives of φ vanish on $B \setminus S_0$, which by continuity implies that they vanish on all of B and so proves the theorem. Let $B_0 \subseteq B \setminus S_0$ be a ball. It is sufficient to prove that φ is linear on all lines through B_0. By Lemma 1.1(vi) we may then assume that $n = 1$, and so that B_0 is an interval. By assumption $Q_m \in C^1(B_0)$ uniformly in m, and so Q_m is uniformly bounded in $M_{p,B}$ (since $n = 1$). Hence

$$\begin{aligned} M_{p,B_0}(\exp(im\,\varphi)) &\leq M_{p,B_0}(\exp(im\,\varphi_m))M_{p,B_0}(\exp(-iQ_m)) \leq \\ &\leq CM_{p,B}(\exp(im\,\varphi_m)) \leq Cm^\alpha, \quad m = 1, 2, \ldots \end{aligned}$$

By Lemma 2.2 it then follows, since $\alpha < |1/2 - 1/p|$, that $\varphi'' = 0$ on B_0, and so that φ is linear on B_0. This completes the proof of Theorem 2.1.

Corresponding to Lemma 2.1 we have the following result in R^n (cf. Littman [20] and Leblanc [17]; the author is indebted to Yngve Domar and Noël Leblanc for these references).

LEMMA 2.3. *Assume that* $\varphi \in C^\infty$ *is real and that the Hessian* J *of* φ *has rank at least* r *on a ball* B, *and that* $u \in C_0^\infty(B)$. *Then there is a constant* $C = C(u, B)$ *such that*

$$\|F(\exp(im\,\varphi)u)\|_\infty \leq Cm^{-r/2}, \quad m = 1, 2, \ldots \tag{2.6}$$

From this Theorem 2.2 now follows as Theorem 2.1 followed from Lemma 2.1. We omit the details.

3. Powers of multipliers on FL_p^N

In this section we will first present some general multiplier theorems in $M_p^{N,N}$. The proofs of the main results will be postponed till the next section. Here we will only prove some corollaries, and also give the first step in the proofs, namely the characterization of the eigenvalues (Proposition 3.1 below).

THEOREM 3.1. *Let* $p \neq 2$ *and let* φ *be an* $N \times N$*-matrix in* $\mathcal{C}^{N+\nu}$, *for some* $\nu \geq 1$. *Assume that there is a constant* C *such that*

$$M_p^{N,N}(\exp(im\,\varphi)) \leq C, \quad m = \pm 1, \pm 2, \ldots \tag{3.1}$$

Then there exist mutually orthogonal idempotents $E_j \in M_p^{N,N} \cap \mathcal{C}^{\nu+1}$ *and real linear functions* α_j *such that* $\varphi = \sum_{j=1}^r \alpha_j E_j$.

As mentioned above, the proof is postponed till section 4.

COROLLARY 3.1. *Let* $p \neq 2$ *and let* P *be a homogeneous matrix polynomial of degree* $d > 0$ *with real eigenvalues. Then* $\exp(iP) \in M_p^{N,N}$ *if and only if*

$$P(y) = \sum_{j=1}^{n} A_j y_j, \tag{3.2}$$

where $A_1, \ldots, A_n$ *are diagonable, commuting matrices with real eigenvalues.*

Remark. In [4 — II] Theorem 3.1 was stated for $p = 1, \infty$ without any regularity assumptions. The proof of this result was not correct, however, since Proposition 1 in [4 — II] is only valid locally, not globally as incorrectly stated (cf. the discussion in the end of section 4 below). But the following result was proved:

THEOREM 3.1′. *Let* φ *be an* $N \times N$*-matrix such that for some constant* C,

$$M_\infty^{N,N}(\exp(im\,\varphi)) \leq C, \quad m = \pm 1, \pm 2, \ldots \tag{3.1'}$$

Then there exist mutually orthogonal idempotents $E_j \in M_\infty^{N,N}$ *and real linear functions* α_j *such that* $\varphi = \sum_{j=1}^r \alpha_j E_j$.

As Corollary 3.1 will follow from Theorem 3.1, Corollary 3 in [4 — II] follows from Theorem 3.1' (that is, Corollary 3.1 for $p = 1, \infty$, assuming that P is a homogeneous matrix *function*, not necessarily a polynomial).

Before we prove Corollary 3.1 we state and prove a lemma which may be of some independent interest.

LEMMA 3.1. *Let* φ *be an* $N \times N$*-matrix function. If the eigenvalues of* φ *are real on* R^n, *and if for some constant* C,

$$M_p^{N,N}(\exp(im\,\varphi)) \leq C, \quad m = 1, 2, \ldots \tag{3.3}$$

then (3.1) *holds. Conversely, if* (3.1) *holds then the eigenvalues of* φ *are real on* R^n.

Proof. Since $M_p^{N,N} \subseteq M_2^{N,N}$ and $M_2 = L_\infty$, (3.1) implies that the eigenvalues of $\exp(i\,\varphi)$ have bounded positive and negative powers on R^n, i.e. they have modulus 1 on R^n. This proves the last statement in Lemma 3.1.

If (3.3) holds, then also $\det(\exp(im\,\varphi))$ and so $\overline{\det(\exp(im\,\varphi))}$, is uniformly bounded in M_p for $m = 1, \ldots$ Since all the eigenvalues of $\exp(i\varphi)$ have modulus 1, we also have

$$\exp(-\,im\,\varphi) = \psi_m\, \overline{\det(\exp(im\,\varphi))}.$$

Here the elements of ψ_m are sums of products of elements of $\exp(im\,\varphi)$, with the number of terms and factors bounded independent of m. Since by (3.3) the elements of $\exp(im\,\varphi)$, and so ψ_m, are uniformly bounded in M_p for $m = 1, 2, \ldots$, this proves the lemma.

Proof of Corollary 3.1. Assume first that $\exp(iP) \in M_p^{N,N}$. We then prove that the powers of $\exp(iP)$ are bounded in $M_p^{N,N}$: For $m > 0$ we have $mP(y) = P(m^{1/d}y)$, and so the positive powers of $\exp(iP)$ are uniformly bounded in $M_p^{N,N}$, by Lemma 1.1(v). By Lemma 3.1 this means that (3.1) holds, with $\varphi = P$.

Now $P \in \mathscr{C}^\infty$, and so by Theorem 3.1 we can find continuous mutually orthogonal idempotents E_j and real linear functions α_j such that $P = \sum_{j=1}^r \alpha_j E_j$. This means that $d = 1$, the α_j's are homogeneous, and so for $t > 0$, that

$$P(y) = \sum_{j=1}^{r} \alpha_j(y) E_j(ty).$$

If we let $t \to 0$ we get by continuity that $P(y) = \sum_{j=1}^r \alpha_j(y) E_j(0)$, and so (3.2) follows. The converse follows as in [4] by the fact that we can find a common diagonalization of the A_j's.

If φ is homogeneous and satisfies the regularity assumptions in Theorem 3.1 only on $R^n \setminus \{0\}$, we have the following result.

THEOREM 3.2. *Let* $p \neq 2$, *and* $n > 1$. *Assume that* φ *is homogeneous of degree* $d > 0$, *that the eigenvalues of* $\varphi(y)$ *are real for* $y \in R^n$, *and that* $\varphi \in \mathscr{C}^{N+\nu}(R^n \setminus \{0\})$, *for some* $\nu \geq 1$. *If* $\exp(i\varphi) \in M_p^{N,N}$, *then there exist mutually orthogonal idempotents* $E_j \in M_p^{N,N} \cap \mathscr{C}^{\nu+1}(R^n \setminus \{0\})$, *homogeneous of degree* 0, *and real linear (homogeneous) functions* α_j *such that* $\varphi = \sum_{j=1}^r \alpha_j E_j$. *In particular* $d = 1$.

Remark. For $p = 1, \infty$ we have by convention that $M_p^{N,N} \subseteq \mathscr{C}$, and then Theorem 3.2 actually says that the E_j's are constant, and so φ can be written in the form (3.2).

We will now state analogues of Theorem 2.1 in $M_p^{N,N}$. In order to get useful results we have added the assumption that we have boundedness in $M_2^{N,N}$. For the applications we have in mind this is a quite natural assumption, and it is automatically satisfied (by $M_p^{N,N} \subseteq M_2^{N,N}$) in Theorems 3.1 and 3.2. We can then also replace the global condition corresponding to (3.1) by a local condition. This will be important in later applications.

As before we assume that $\{Q_m\}$ is bounded in $\mathscr{C}^1(R^n \setminus S_0)$, $m \geq 1$, where S_0 is a closed nowhere dense set. We let $\sigma = 1$ if $\varphi Q_m = Q_m \varphi$, and let $\sigma = 1 + \alpha$ otherwise.

THEOREM 3.3. *Let* $0 \leq \alpha < |1/2 - 1/p|$, *and let* $\varphi \in \mathscr{C}^{N+\nu}$, *for some* $\nu \geq 1$, *be an* $N \times N$*-matrix function with real eigenvalues on* R^n. *Let* $\varphi_m = \varphi + m^{-\sigma} Q_m$, *with* Q_m *as above. Assume that for each compact ball* $B \subseteq R^n \setminus S_0$ *there is a constant* C_B *such that*

$$M_{p,B}^{N,N}(\exp(ik\varphi_m)) \leq C_B m^{\alpha}, \quad 1 \leq k \leq m = 1, 2, \ldots \tag{3.4}$$

and further that for some constant C,

$$M_2(\exp(im\,\varphi)) \leq C, \quad m = 1, 2, \ldots \tag*{(3.4)'}$$

Then there exist mutually orthogonal idempotents $E_j \in M_2^{N,N} \cap \mathscr{C}^{\nu+1}$ *and real linear functions* α_j *such that* $\varphi = \sum_{j=1}^r \alpha_j E_j$.

COROLLARY 3.3. *Let* $0 \leq \alpha < |1/2 - 1/p|$, *and let* P *be a homogeneous* $N \times N$ *matrix polynomial of degree* $d > 0$ *with real eigenvalues on* R^n. *Let* $\varphi_m = P + m^{-\sigma} Q_m$, *with* Q_m *as above. If for each compact ball* $B \subseteq R^n \setminus S_0$ *there is a constant* C_B *such that* (3.4) *holds, and if* (3.4)' *holds for* $\varphi = P$, *then*

$$P(y) = \sum_{j=1}^{n} A_j y_j$$

where $A_1, \ldots, A_n$ *are commuting, diagonable matrices with real eigenvalues.*

The proof of Corollary 3.3 from Theorem 3.3 is the same as the proof of Corollary 3.1 from Theorem 3.1.

THEOREM 3.4. *Let* $0 \leq \alpha < |1/2 - 1/p|$, *and let* $n > 1$. *Assume that* φ *is a homogeneous* $N \times N$ *matrix function of degree* $d > 0$, *that the eigenvalues of* φ *are real on* R^n, *and that* $\varphi \in \mathscr{C}^{N+\nu}(R^n \setminus \{0\})$, *for some* $\nu \geq 1$. *Let, as above,* $\varphi_m = \varphi + m^{-\sigma} Q_m$. *Assume that* (3.4) *holds for each compact ball* $B \subseteq R^n \setminus S_0 \cup \{0\}$, *and that* (3.4)' *holds. Then there exist mutually orthogonal idempotents* $E_j \in \mathscr{C}^{\nu+1}(R^n \setminus \{0\})$, *homogeneous of degree zero, and real linear functions* α_j *such that* $\varphi = \sum_{j=1}^r \alpha_j E_j$. *In particular* $d = 1$.

Remark. By Mikhlin's theorem (Lemma 1.3) we get that if $\nu > n/2 - 1$, then $E_j \in M_p^{N,N}$ for $1 < p < \infty$.

In the remaining part of this section we will prove the parts of Theorems 3.1 through 3.4 which concern the eigenvalues.

LEMMA 3.2. *Let A be a Banach algebra with norm $\|\cdot\|$. If $a, b \in A$ then*

$$\|\exp(t(a+b))\| \leq M(t) \exp(M(t)t\|b\|),$$

where $M(t) = \sup\{\|\exp(sa)\|;\ 0 \leq s \leq t\}$.

Proof. Let $g(t) = \exp(t(a+b))$. Then

$$g'(t) = ag(t) + bg(t),\ \ g(0) = 1,$$

and so

$$g(t) = \exp(ta) + \int_0^t \exp((t-s)a)bg(s)ds.$$

This gives the estimate, $0 \leq \tau \leq t$,

$$\|g(\tau)\| \leq M(t) + M(t) \int_0^\tau \|b\|\|g(s)\|ds.$$

Gronwall's lemma then implies that

$$\|g(t)\| \leq M(t) \exp(M(t)t\|b\|),$$

which is the wanted inequality.

PROPOSITION 3.1. *Let $\Omega \subseteq R^n$ be open and connected. Let $0 \leq \alpha < |1/2 - 1/p|$. Assume that φ is a matrix function in $\mathscr{C}^{N+1}(\Omega)$ with real eigenvalues on Ω and that for each compact ball $B \subseteq \Omega \setminus S_0$ there is a constant C_B such that*

$$M_{p,B}^{N,N}(\exp(ik\varphi_m)) \leq C_B m^\alpha,\ \ 1 \leq k \leq m = 1, 2, \ldots \tag{3.5}$$

where as above $\varphi_m = \varphi + m^{-\sigma}Q_m$. Then there exist functions $\beta_1, \ldots, \beta_N$ of the form

$$\beta_j(y) = \beta_{j0} + \sum_{j=1}^n \beta_{jk}y_k,\ \ j = 1, \ldots, N, \tag{3.6}$$

where β_{jk} are real constants such that $\beta_1(y), \ldots, \beta_N(y)$ are the eigenvalues of $\varphi(y)$, counted with proper multiplicities, for $y \in \Omega$.

Proof. Let $B \subseteq \Omega \setminus S_0$ be a compact ball. Then there is a ball $B' \subseteq B$ such that the eigenvalues of φ have constant multiplicities on B'. The eigenvalues β_j

of φ can be chosen continuous on B' and such that $\exp(i\beta_j)$ is an analytic function of the elements of $\exp(i\varphi)$ on B'.

We will first prove that the functions β_j are linear on B'. It is sufficient to prove that the functions β_j are linear on each line through B'. In view of Lemma 1.1(vi) it is then no restriction to assume that B' is a compact interval and that $n = 1$ in this part of the proof.

In Lemma 3.2, let $A = M_{p,B}^{N,N}$ and $\|\cdot\|$ the corresponding norm. Let $a = im\varphi_m$ and $b = im(\varphi - \varphi_m)$, and introduce

$$C(m) = \sup\{\|\exp(ik\varphi_m)\|;\ 0 \leq k \leq m,\ k \text{ integer}\}.$$

Then, in the notation of Lemma 3.2,

$$M(1) = \sup\{\|\exp(is\varphi_m)\|;\ 0 \leq s \leq m\} \leq C(m)\exp(\|\varphi_m\|).$$

Lemma 3.2, with $t = 1$, then gives

$$\|\exp(im\varphi)\| \leq C(m)\exp(\|\varphi_m\| + C(m)m\exp(\|\varphi_m\|)\|\varphi - \varphi_m\|).$$

Since $\{Q_m\}$ is uniformly bounded in $\mathscr{C}^1(B')$, and $\mathscr{C}^1(B') \subseteq M_{p,B}^{N,N}$ for $n = 1$, formula (3.5) and the above estimate show that if $\varphi Q_m \neq Q_m\varphi$, then $M_{p,B}^{N,N}(\exp(im\varphi)) \leq Cm^\alpha \exp(Cm^{1+\alpha}m^{-\sigma}) \leq Cm^\alpha$. If $\varphi Q_m = Q_m\varphi$, we have trivially

$$M_{p,B'}^{N,N}(\exp(im\varphi)) \leq M_{p,B'}^{N,N}(\exp(im\varphi))\exp(M_{p,B'}^{N,N}(Q)),$$

and so we have in both cases that

$$M_{p,B'}^{N,N}(\exp(im\varphi)) \leq Cm^\alpha,\quad m = 1, 2, \ldots$$

By Proposition 1.1, $\exp(i\beta_j)$ belongs to $M_{q,B'}$, where we take

$$\alpha < |1/2 - 1/q| < |1/2 - 1/p|$$

(and where we may have to shrink the ball B' somewhat). In the same way we can find eigenvectors $v_j \in M_{q,B'}^{1,N}$ corresponding to the eigenvalues $\exp(i\beta_j)$ such that $|v_j| = 1$ on B'. On B'.

$$\exp(im\beta_j)v_j = \exp(im\varphi)v_j.$$

Multiplying by v_j^* from the left we get, still on B',

$$\exp(im\beta_j) = \langle\exp(im\varphi)v_j, v_j\rangle.$$

By Lemma 1.1(iii) and the above remarks, we then have

$$M_{q,B'}(\exp(im\beta_j)) \leq M_{q,B'}(v_j^*(\exp(im\varphi)v_j)) \leq$$
$$\leq M_{q,B'}^{N,N}(\exp(im\varphi))M_{q,B'}^{1,N}(v)M_{q,B'}^{N,1}(v^*) \leq CM_{q,B'}^{N,N}(\exp(im\varphi) \leq$$
$$\leq CM_{p,B'}^{N,N}(\exp(im\varphi)) \leq Cm^\alpha,\quad m = 1, 2, \ldots$$

Since $\varphi \in \mathscr{C}^{N+1}(\Omega) \subseteq \mathscr{C}^2(\Omega)$, we have that $\beta_j \in C^2(B')$. Theorem 2.1 then proves that the (by assumption real) eigenvalues β_j of φ on B' can be chosen to be linear functions. As mentioned above, we can then drop the assumption that $n = 1$.

Thus we have proved: for any open ball $B_0 \subseteq \Omega$, we can find a ball $B' \subseteq B_0$ such that φ has real linear eigenvalues on B', that is

$$\det(\varphi(y) - \beta) = \prod_{j=1}^{N} (\beta_j(y) - \beta), \quad y \in B', \tag{3.7}$$

where β_j are of the form (3.6). This means that the set S where (3.7) does not hold for some set of real linear functions β_j is nowhere dense in Ω. Since the $N+1$'st derivatives of $\det(\varphi(y) - \beta) \in \mathscr{C}^{N+1}(\Omega)$ vanish on $\Omega \setminus S$, by (3.7), it follows by continuity that the determinant is a polynomial so that (3.7) holds for all $y \in \Omega$, which proves the proposition.

Finally we give the counterpart of Theorem 2.2 for $N \times N$ matrix functions $\varphi \in \mathscr{C}^\infty$. Assume that an eigenvalue α_j of φ belongs to $C^2(B)$ on some ball B, and define then $r(\alpha_j; B) = \inf_{y \in B} \operatorname{rank} (\partial^2/\partial y_k \partial y_l) \alpha_j(y))_{k,l}$. We say that φ has *rank* r *on* B_0 if and only if r is the largest integer such that there is a ball $B \subseteq B_0$ and a real eigenvalue α_j of φ on B such that $r(\alpha_j; B) \geq r$. We let $\sigma = 1$ if $\varphi Q_m = Q_m \varphi$, and let $\sigma = 1 + r|1/2 - 1/p|$ otherwise.

PROPOSITION 3.2. *Assume that $\varphi \in \mathscr{C}^\infty$ is an $N \times N$ matrix function with rank at least r on some ball $B_0 \subseteq R^n$, and that $\varphi_m = \varphi + m^{-\sigma} Q_m \in \mathscr{C}^\infty(R^n \setminus S_0)$ uniformly for $m \geq 1$. Then there is a constant $c > 0$ such that*

$$\sup_{0 \leq k \leq m} M_{p,B_0}^{N,N}(\exp(ik\varphi_m)) \geq c m^{r|1/2 - 1/p|}, \quad m = 1, 2, \ldots \tag{3.8}$$

Since the proof of Proposition 3.2 is similar to that of Proposition 3.1, we omit the details. We only notice that we here use the fact that $\mathscr{C}^\infty(B) \subseteq M_{p,B}$ for B compact, instead of Proposition 1.1.

4. Proofs of Theorems 3.1 through 3.4

We begin with two lemmas, which are common to all the proofs.

LEMMA 4.1. *Let β be a real linear function on R^n and assume that*

$$\sup_{m \in Z} M_p^{N,N}(\exp(im\varphi)) \leq C. \tag{4.1}$$

Then

$$M_p^{N,N}((r-1)(r\exp(i\beta) - \exp(i\varphi))^{-1}) \leq C, \quad r > 1. \tag{4.2}$$

Proof. By (4.1) all the eigenvalues of $\varphi(y)$ are real, $y \in R^n$ (cf. Lemma 3.1). Hence for $r > 1$,

$$(r \exp(i\beta) - \exp(i\varphi))^{-1} = \sum_{j\geq 0} (r \exp(i\beta))^{-j-1} \exp(ij\varphi).$$

But, since $M_p(\exp(i\beta)) = 1$,

$$M_p^{N,N}((r \exp(i\beta))^{-j-1} \exp(ij\varphi)) \leq r^{-j-1} M_p(\exp(-(j+1)i\beta)) M_p^{N,N}(\exp(ij\varphi)) \leq C r^{-j-1}.$$

It follows that $(r \exp(i\beta) - \exp(i\varphi))^{-1} \in M_p^{N,N}$ for $r > 1$, and also that (4.2) holds.

LEMMA 4.2. *Let $\Omega \subseteq R^n$ be open and connected. Assume that φ is an $N \times N$-matrix function in $\mathscr{C}^{N+\nu}(\Omega)$, for some $\nu \geq 1$, which has a set $\beta_1, \ldots, \beta_r$ of real linear functions as all of its eigenvalues on Ω. If φ satisfies* (4.1) *for $p = 2$, then there exist mutually orthogonal idempotents $E_j \in M_2^{N,N} \cap \mathscr{C}^{\nu+1}(\Omega)$, with $\sum_{j=1}^r E_j = E$, such that*

$$\varphi(y) = \sum_{j=1}^{r} \beta_j(y) E_j(y), \quad y \in \Omega. \tag{4.3}$$

Proof. By Lemma 4.1, (4.2) holds with $\beta = \beta_j$, $j = 1, \ldots, r$, and for $p = 2$. Then (4.2) shows that φ (or, which is the same, $\exp(i\varphi)$) has only linear factors on Ω, i.e.

$$\varphi(y) = \sum_{j=1}^{r} \beta_j(y) E_j(y), \quad y \in \Omega,$$

for some set of mutually orthogonal idempotent matrices $E_1, \ldots, E_r$ with sum E. Following Strang [23], we have a.e.

$$\begin{aligned} &(r-1)(r \exp(i\beta_j) - \exp(i\varphi))^{-1} = \\ &= \textstyle\sum_k (r-1)(r \exp(i\beta_j) - \exp(i\beta_k))^{-1} E_k \to \exp(-i\beta_j) E_j, \quad \text{as } r \to 1+ \end{aligned} \tag{4.4}$$

and so by Lemma 4.1 that $E_j \in M_2^{N,N}$. Hence E_j is a bounded solution of $(\varphi - \beta_j) E_j = 0$ on Ω. It remains to prove that $E_j \in \mathscr{C}^{\nu+1}(\Omega)$. Since

$$E_j = \prod_{\neq k} \frac{\exp(i\varphi) - \exp(i\beta_k)}{\exp(i\beta_j) - \exp(i\beta_k)}$$

this follows from the fact that if $f, g \in C^{\mu+1}$ and if $g = 0 \Rightarrow f = 0$ and $dg \neq 0$, then $f/g \in C^{\mu}$. (This follows from Taylor's formula.)

COROLLARY 4.2. *Let Ω be open, connected, and dense in R^n. Assume that* (4.1.) *holds with $p \neq 2$ and that $\varphi \in \mathscr{C}^{N+\nu}(\Omega) \cap \mathscr{C}(R^n)$, for some $\nu \geq 1$. Then* (4.3) *holds for $y \in R^n$ with $E_j \in M_p^{N,N}$.*

Proof. By Proposition 3.1, (4.1) implies that, for $p \neq 2$, the conditions of Lemma 4.2 hold on Ω. As in the last part of the proof of Proposition 3.1, the β_j's are real linear functions on $\bar{\Omega} = R^n$, too. As above it follows that (4.3) holds on R^n, and by Lemma 1.1(iv), (4.4) and (4.2) that $E_j \in M_p^{N,N}$, and the corollary is proved.

Proof of Theorem 3.1. By Corollary 4.1, with $\Omega = R^n$, (4.3) holds on R^n with $E_j \in M_p^{N,N} \cap \mathscr{C}^{\nu+1}$ and with real linear functions β_j by Proposition 3.1. This proves the theorem.

Proof of Theorem 3.2. By homogeneity and Lemma 3.1 it follows that (4.1) holds. We can again apply Corollary 4.1 (now for $\Omega = R^n \setminus \{0\}$, which is connected since $n > 1$) and Lemma 4.2, together with Proposition 3.1, to prove that (4.3) holds on R^n, with real linear functions β_j and with $E_j \in M_p^{N,N} \cap \mathscr{C}^{\nu+1}(R^n \setminus \{0\})$. That E_j is homogeneous of degree zero follows from (4.3) and so Theorem 3.2 is proved.

Proof of Theorem 3.3. By assumption (4.1) and so (4.3) holds for $p = 2$, $\Omega = R^n$. It follows that $E_j \in M_2^{N,N} \cap \mathscr{C}^{\nu+1}$, by Lemma 4.2. That φ has linear eigenvalues on R^n, follows as above from Proposition 3.1.

The proof of Theorem 3.4 is similarly a modification of the proof of Theorem 3.2.

We take this opportunity to remark that the projections $E_j \in M \cap \mathscr{C}^\infty(R^n \setminus \{0\})$ in general cannot be diagonalized globally in $\mathscr{C}(R^n \setminus \{0\})$ (as was incorrectly stated in Proposition 1 in [4 — II] for $p = 1, \infty$; the proof there is actually only valid locally). As an example we notice that if $E_j \in \mathscr{C}^\infty(R^n \setminus \{0\})$ is homogeneous of degree zero, then $E_j \in M_p^{N,N} \cap \mathscr{C}^\infty(R^n \setminus \{0\})$, by Lemma 1.3. But as for odd n there is a nontrivial vectorbundle over S^{n-1}, the unit sphere in R^n, these E_j's can in general not be diagonalized globally in $\mathscr{C}^\infty(R^n \setminus \{0\})$, or even in $\mathscr{C}(R^n \setminus \{0\})$. Hence there seems to be little hope to go much further than in the statements of Theorems 3.1 through 3.4.

5. Initial value problems in L_p and $L_{p,\alpha}$

Let us consider the Cauchy problem

$$\begin{cases} \partial u / \partial t = P(D)u, \quad x \in R^n, \quad 0 \leq t \leq T, \\ u(x, 0) = u_0(x) \end{cases} \tag{5.1}$$

where P is an $N \times N$-matrix of pseudo-differential operators with constant coefficients, and where u and u_0 are complex N-vector functions. This means that we define $P(D)u$ for $u \in S^N$ by

$$P(D)u(x) = \int_{R^n} \exp\left(-2\pi i \langle x, y \rangle\right) P(y) \hat{u}(y) dy, \tag{5.2}$$

where $P(y)$, the symbol of $P(D)$, is an $N \times N$-matrix function, by our convention in $\mathscr{C}^1$, such that for a sequence P_{d-j} of homogeneous matrix functions in $\mathscr{C}^1(R^n \setminus \{0\})$ of degree $d - j$, $j \geq 0$, we have for each integer m

$$D^\alpha(P(y) - \sum_{j=0}^{m-1} P_{d-j}(y)) = O(|y|^{d-|\alpha|-m}), \quad |\alpha| \leq \mu, \tag{5.3}$$

as $|y| \to \infty$. We say that $P_d(y)$ is the principal part of $P(y)$, that $P_d(D)$ is the principal part of $P(D)$, and that d is the order of P (and P_d) provided $P_d \neq 0$. We will assume below that in (5.1) P has order $d > 0$.

Any constant coefficient partial differential operator $P(D)$ is of the above type. By (5.3) $P(y)$ is in general bounded by some polynomial, and so $P(y)\hat{u}(y)$ is in L_1^N for $u \in S^N$, and hence $P(D)$ is well defined by (5.2). For more details about pseudo-differential operators, generally with $\mu = \infty$, and for operators with variable coefficients, we refer to Hörmander [7], [8].

We say that the Cauchy problem (5.1) is well posed in L_p if $P(D)$ is the infinitesimal generator of a strongly continuous semi-group of operators $E(t)$ on L_p, that is: the family $E(t)$ (of solution operators of (5.1)) satisfies

$$E(0) = E = \text{identity}, \quad E(t+s) = E(t)E(s), \quad t, s \geq 0,$$

and

$$\|E(t)u_0\|_p \leq C(T)\|u_0\|_p, \quad 0 \leq t \leq T, \quad u_0 \in S^N, \tag{5.4}$$

and

$$\|(t^{-1}(E(s+t) - E(s)) - P(D)E(s))u_0\|_p \to 0, \quad t \to 0, \quad u_0 \in S^N. \tag{5.5}$$

For $1 \leq p < \infty$ this is the usual definition of a well posed problem (5.1) and for $p = \infty$, we say that (5.1) is well posed in L_∞, although the standard terminology should be »well posed in C_0».

Symbols for systems such that (5.1) is well posed in L_2 have been completely characterized by Kreiss [15]. In particular, if the eigenvalues of the principal part P_d are imaginary (e.g. when d is odd), then a necessary condition is that there exist uniformly bounded matrix functions S, S^{-1} on R^n such that $S^{-1}P_dS$ is diagonal. For $d = 1$, this is also a sufficient condition.

We will see below that in general systems that are well posed in L_2 are not well posed in L_p for $p \neq 2$ (cf. [3]). For a system such that (5.1) is well posed in L_2, one might try to replace the L_p-norm of u_0 in (5.4) by the norm

$$\|u_0\|_{p,\alpha} = \|F^{-1}(w_\alpha \hat{u})\|_p,$$

where $\alpha \geq 0$ and $w_\alpha(y) = (1 + |y|^2)^{\alpha/2}$. We denote the completion of S^N in this norm by $L_{p,\alpha}^N$ (and also write $L_{p,\alpha}$ for $N = 1$). Hence we replace the condition (5.4) by

$$\|E(t)u_0\|_p \leq C(T)\|u_0\|_{p,\alpha}, \quad 0 \leq t \leq T, \quad u_0 \in S^N. \tag{5.6}$$

If the solution of (5.1) exists in the sense of (5.5), and if (5.6) holds, we say for short that (5.1) is well posed in $L_{p,\alpha}$, although the standard notion should be »well posed $(L_{p,\alpha}, L_p)$».

We will now present the main results. The proofs will be postponed till the end of this section. First some more notations, however: If d_1 is the order of $P - P_d$, we define $\tilde{d} = \min\{d, (d - d_1)/(1 + \varepsilon)\}$, where $\varepsilon = 0$ if $P_d P = P P_d$, $\varepsilon = |1/2 - 1/p|$ otherwise. Then $\tilde{d} = d$ if the order of $P - P_d$ is $\leq -\varepsilon d$, e.g. if P is homogeneous.

THEOREM 5.1. *Let* $0 \leq \alpha < \tilde{d}|1/2 - 1/p|$. *Assume that* P *is a differential operator and that the eigenvalues of* $P_d(y)$ *are imaginary for* $y \in R^n$. *Then* (5.1) *is well posed in* $L_{p,\alpha}$ *and in* L_2 *if and only if*

$$P_d(y) = \sum_{j=1}^{n} A_j \partial/\partial x_j, \tag{5.7}$$

where $A_1, \ldots, A_n$ *are diagonable, commuting matrices with real eigenvalues. In particular* $d = 1$.

Remark. If (5.6) holds and if the order of P is odd, then it is well known that the eigenvalues of $P_d(y)$ are imaginary, $y \in R^n$ (cf. also Lemma 5.1 below).

If the problem (5.1) is well posed in L_p for some p, $1 \leq p \leq \infty$, then (5.1) is also well posed in L_2. Again we refer to Lemma 5.1 below.

THEOREM 5.2. *Let* $p \neq 2$, *and let* $n > 1$. *Assume that* $P_d \in \mathscr{C}^{N+\nu}(R^n \setminus \{0\})$, *for some* $\nu > 1$, *and that the eigenvalues of* $P_d(y)$ *are imaginary for* $y \in R^n$. *If* (5.1) *is well posed in* L_p, *then*

$$P_d(D) = \sum_{k,j} \alpha_{kj} (\partial/\partial x_j) E_k(D), \tag{5.8}$$

where α_{kj} *are real constants and where* $E_k(D)$ *are operators with symbols which are mutually orthogonal idempotents in* $M_p^{N,N} \cap \mathscr{C}^{\nu+1}(R^n \setminus \{0\})$ *and which are homogeneous of degree zero. In particular* $d = 1$. *Conversely, if* $P \in \mathscr{C}^\mu$ *for some* $\mu > n/2$, *then* (5.8) *implies that* (5.1) *is well posed in* L_p, $1 < p < \infty$.

Since $M_1^{N,N} = M_\infty^{N,N} \subseteq \mathscr{C}$, we have the following corollary for $p = 1, \infty$.

COROLLARY 5.1. *Let* $p = 1, \infty$, *and let* $n > 1$. *Under the assumptions in Theorem* 5.2 *it follows that* (5.7) *holds for a set of diagonable, commuting matrices* A_j *with real eigenvalues. In particular,* P_d *is a first order differential operator. Conversely, if* (5.7) *holds, with* $A_1, \ldots, A_n$ *as above, and if the term of degree zero in* (5.3) *is constant, then* (5.1) *is well posed in* L_1 (*and* L_∞), *provided* $P \in \mathscr{C}^\mu$ *for some* $\mu > n/2$.

Remark. As mentioned in section 3, although Theorem 1 in [4 — II] is incorrect as stated, the result of Corollary 3 in [4 — II], and so also Theorem 2 there holds. Hence Corollary 5.1 actually holds without any regularity assumptions on P_d.

THEOREM 5.3. *Let* $0 \leq \alpha < \tilde{d}|1/2 - 1/p|$, *and let* $n > 1$. *Assume that* $P_d \in \mathscr{C}^{N+\nu}$ *on* $R^n \setminus \{0\}$, *for some* $\nu \geq 1$, *and that the eigenvalues of* $P_d(y)$ *are imaginary for* $y \in R^n$. *If* (5.1) *is well posed in* $L_{p,\alpha}$ *and in* L_2, *then* (5.8) *holds where again* α_{kj} *are real constants, and where* $E_k(D)$ *are operators with symbols which are mutually orthogonal idempotent matrices, which are homogeneous of degree zero on* R^n, *and which belong to* $M_2^{N,N} \cap \mathscr{C}^{\nu+1}(R^n \setminus \{0\})$. *In particular* $d = 1$.

If $\nu > n/2 - 1$, *then* E_k *also belongs to* $M_p^{N,N}$ *for* $1 < p < \infty$. *Conversely, if* $P \in \mathscr{C}^{\mu}$ *for some* $\mu > n/2$, *then* (5.8) *implies that* (5.1) *is well posed in* L_p, $1 < p < \infty$.

In general the bound $\tilde{d}|1/2 - 1/p|$ above is not the best possible, in the sense that the problem (5.1) need not be well posed in $L_{p,\alpha}$ for $\alpha > \tilde{d}|1/2 - 1/p|$, even if it is well posed in L_2. However, to obtain results of the type (5.7) and (5.8), the bound $\tilde{d}|1/2 - 1/p|$ is essential. This will be clear from an example given in the end of this section, for $N = 1$.

If we merely want criteria for the non-existence of estimates of the form (5.5) we can use Proposition 3.2. Consider the system (5.1). We have defined the rank of P_d as the largest integer r such that there is some ball B and an imaginary eigenvalue α_j of P_d on B such that $\alpha_j \in C^2(B)$ and such that

$$\operatorname{rank}\left(\frac{\partial^2 \alpha_j}{\partial y_k \partial y_l}(y)\right)_{k,l} \geq r$$

on B. Also, let now $d = \min\{d, (d - d_1)/(1 + r\varepsilon)\}$, ε as above.

We then have the following result:

THEOREM 5.4. *Let* r *be the rank of* P_d. *Assume that* $P \in \mathscr{C}^{\infty}(R^n)$. *Then the Cauchy problem* (5.1) *is not well posed in* $L_{p,\alpha}$ *for* $0 \leq \alpha < r\tilde{d}|1/2 - 1/p|$.

Applications of Theorem 5.4 to specific cases, such as the wave- and the Schrödinger equation, will be given below.

A Cauchy problem (5.1) which is well posed in L_2, is also well posed in $L_{p,\alpha}$ for α large enough, provided the symbol is smooth. In view of Theorem 5.4 the following result is in a sense the best possible (cf. Lemma 1.4).

THEOREM 5.5. *Let* P *be an* $N \times N$*-matrix of pseudo-differential operators of order* $d > 0$ *on* R^n, *with symbol* $P \in \mathscr{C}^{\infty}$. *Assume that the Cauchy problem* (5.1) *is well posed in* L_2. *Then* (5.1) *is also well posed in* $L_{p,\alpha}$ *for* $\alpha > nd|1/2 - 1/p|$.

Before proving the theorems, we give some examples.

As a first example we consider the Cauchy problem for the wave equation

$$\begin{cases} \partial^2 u/\partial t^2 = \sum_{j=1}^n \partial^2 u/\partial x_j^2, & x \in R^n, \quad 0 \le t \le T, \\ u(x, 0) = u_{01}(x), \\ \partial u/\partial t(x, 0) = u_{02}(x). \end{cases} \tag{5.9}$$

It is easy to reformulate (5.9) as a Cauchy problem for a first order system of pseudo-differential operators, where the principal part P_1 of the symbol has eigenvalues of the form $\pm 2\pi i|y|$. The Hessian is $J_n(y) = \pm 2\pi i|y|^{-3}(|y|^2 E - (y_k y_l))$.

As a second example we take the Cauchy problem for a Schrödinger type equation. Let a_{kl} be real and symmetric and consider

$$\begin{cases} \partial u/\partial t = i \sum_{k,l=1}^n a_{kl}\partial^2 u/\partial x_k \partial x_l, & x \in R^n, \quad 0 \le t \le T, \\ u(x, 0) = u_0(x). \end{cases} \tag{5.10}$$

Here $P_d(y) = -(2\pi)^2 i \sum_{k.l=1}^n a_{kl}y_k y_l$. Hence $J(y) = 2(a_{kl})$ determines the rank of P_d.

There is a general problem, which in a sense includes (5.9) and (5.10). Let $\beta > 0$ and let $P_\beta(D)$ be the pseudo-differential operator which has the symbol

$$P_\beta(y) = i(\sum_{k,l=1}^n a_{kl}y_k y_l)^{\beta/2},$$

where $a_{kl} = a_{lk}$ and $\sum_{k,l=1}^n a_{kl}y_k y_l \ge 0$, $y \in R^n$. Consider the Cauchy problem

$$\begin{cases} \partial u/\partial t = P_\beta(D)u, & x \in R^n, \quad 0 \le t \le T, \\ u(x, 0) = u_0(x). \end{cases} \tag{5.11}$$

Since an orthogonal change of variables does not alter the property of being well posed in $L_{p.\alpha}$, and since (a_{kl}) is symmetric and non-negative, we may assume that $P_\beta(y) = i(\sum_{j=1}^r y_j^2)^{\beta/2}$, where $r = \text{rank}\,(a_{kl})$. The matrix $J_{\beta,r}(y)$ which determines the rank of P_β is in this case

$$J_{\beta,r}(y) = i\beta|y|^{\beta-4}(|y|^2 E + (\beta - 2)(y_k y_l)), \quad y \in R^r.$$

LEMMA. *Let $r \ge 1$, $\beta > 0$ and $J_{\beta,r}$ be as above. Then the rank of $J_{\beta,r}$ is for $y \ne 0$*

(i) $r - 1$ *for* $\beta = 1$
(ii) r *for* $\beta \ne 1$.

Proof. We may by homogenity assume that $|y| = 1$. Since the columns of $(y_k y_l)$ are of the form $y_k^t(y_1, \ldots, y_r)$, it follows that $(y_k y_l)$ has rank 1. Hence this symmetric matrix has only one non-zero eigenvalue. As the trace of $(y_k y_l)$ is $|y|^2 = 1$, the eigenvalue is $|y|^2 = 1$. Hence the symmetric matrix $J_{\beta,r}(y)$ has for $|y| = 1$ a diagonal form which is a multiple of

$$\begin{pmatrix} \beta - 1 & & 0 \\ & 1 \dots & \\ 0 & & 1 \end{pmatrix},$$

which proves the lemma.

From this lemma and Theorem 5.4 we then have the following result:

PROPOSITION 5.1. *Let* $n > 1$, $n = \operatorname{rank}(a_{kl})$. *Then the Cauchy problem* (5.11) *is not well posed in* $L_{p,\alpha}$ *for*

$$0 \leq \alpha < \begin{cases} n\beta|1/2 - 1/p|, & \beta \neq 1, \\ (n-1)|1/2 - 1/p|, & \beta = 1. \end{cases}$$

For the wave-equation (5.9) this result was obtained by other methods by Mauravei [22], who also proved that (5.9) was well posed in $L_{p,\alpha}$ for $\alpha > (n-1)|1/2 - 1/p|$. For the Schrödinger equation cf. Lanconelli [16]. By Theorem 5.5, (5.11) is well posed in $L_{p,\alpha}$ for $\alpha > n\beta|1/2 - 1/p|$ for $\beta \neq 1$, and so Proposition 5.1 is in a sense best possible also in this case. Cf. also [27].

We now proceed to the proofs of the above theorems. We first transform (5.6) to multiplier form (cf. Theorem 2 in [3]). As above $w_\alpha(y) = (1 + |y|^2)^{\alpha/2}$.

LEMMA 5.1. *If the Cauchy problem* (5.1) *is well posed in* $L_{p,\alpha}$ *then*

$$M_p^{N,N}(w_\alpha^{-1} e^{tP}) \leq C(T), \quad 0 \leq t \leq T. \tag{5.12}$$

Conversely, if $P \in \mathscr{C}^\mu$ *for* $\mu > n/2$, *then* (5.12) *implies that* (5.1) *is well posed in* $L_{p,\alpha}$.

Proof. Assume first that (5.1) is well posed in $L_{p,\alpha}$. Since by (5.6) $u(t, x) = E(t)u_0(x)$ belongs to L_p^N, we can take Fourier transforms in the distribution sense, of the elements of (5.1) with respect to x (t fixed) and get by the definition of $P(D)$ that

$$\begin{cases} \partial\hat{u}(y, t)/\partial t = P(y)\hat{u}(y, t), & y \in R, \quad 0 \leq t \leq T, \\ \hat{u}(y, 0) = \hat{u}_0(y) \end{cases}$$

and so with

$$\varphi_t(y) = \exp(tP(y)), \quad \hat{u}(y, t) = \varphi_t(y)\hat{u}_0(y).$$

Then (5.6) implies by definition (5.12).

On the other hand, assume that (5.12) holds. With $\hat{\mu}_t = \varphi_t$, we have

$$u(x, t) = \mu_t * u_0(x)$$

and so $u(., t) \in \mathscr{C}^\infty$ since the elements of $\mu_t \in S'$ and since differentiation is continuous in S. From

$$\frac{\partial \varphi_t}{\partial t} = P\varphi_t, \quad \varphi_0 = E,$$

it follows that

$$\varphi_{t+h} = \varphi_t + hP\varphi_t + \int_t^{t+h} (\tau - t)P^2\varphi_\tau d\tau.$$

Hence, using (5.12) we have

$$\|h^{-1}(u(.,t+h) - u(.,t)) - P(D)u(.,t)\|_p \leq |h| \sup_{t\leq\tau\leq t+h} \|P(D)^2u(.,\tau)\|_p \leq C|h|\|P(D)^2u_0\|_{p,\alpha}.$$

If $P(D)^2u_0 \in L^N_{p,\alpha}$ for fixed $u_0 \in S^N$, this will prove (5.5).

For $2 \leq p \leq \infty$, $P(D)^2u_0 \in L^N_{p,\sigma}$ for any $\sigma \geq 0$ by the Hausdorff-Young inequality. For $1 \leq p \leq 2$ this is certainly the case if $P(y) \in \mathscr{C}^\mu$ for some $\mu > n/2$, by Lemma 1.4 (Bernstein's theorem). Together these results prove Lemma 5.1.

Remark. If $2 \leq p \leq \infty$, then (5.12) alone implies that (5.1) is well posed in $L_{p,\alpha}$, by the above proof. For $\alpha = 0$ this is also the case for $1 \leq p \leq \infty$ if $\varphi_t^{-1} \in M_p^{N,N}$, $0 \leq t \leq T$, e.g. if P is homogeneous and has imaginary eigenvalues, cf. Lemma 3.1.

We use Lemma 5.1 to obtain necessary conditions for the Cauchy problem (5.1) to be well posed in L_p and in $L_{p,\alpha}$.

LEMMA 5.2. *Assume that* (5.1) *is well posed in* L_p. *Then* $\exp(P_d) \in M_p^{N,N}$.

Proof. If (5.1) is well posed in L_p, then (5.12) holds for $\alpha = 0$. Let $t = s^d$ and $\psi_s(y) = e^{tP(s^{-1}y)}$. By Lemma 1.1(v) then

$$M_p^{N,N}(\psi_s) \leq C(T), \quad 0 < s \leq T^{1/d},$$

and since by (5.4), $\psi_s(y) \to \exp(P_d(y))$, at least for $y \neq 0$, and since P_d is continuous $(d > 0)$ we have by Lemma 1.1(iv) that $\exp(P_d) \in M_p^{N,N}$.

For $\alpha > 0$ we have the following local result as a consequence of (5.12). Here $\varepsilon = |1/2 - 1/p|$, and $\tilde{d}(r) = \min(d, (d - d_1)/(1 + r\varepsilon))$, $r \geq 1$.

LEMMA 5.3. *Let* $\alpha \geq 0$, $d > 0$. *Let* $B \subseteq R^n\setminus\{0\}$ *be a compact ball. Assume that* (5.12) *holds. Then*

$$M^{N,N}_{p,B}(\exp(j(P_d + m^{-(1+r\varepsilon)}Q_m))) \leq C_B m^{\alpha/\tilde{d}(r)}, \quad 1 \leq j \leq m, \tag{5.13}$$

where $\{Q_m\}$ *is bounded in* $\mathscr{C}^1(R^n\setminus\{0\})$ *and where* $Q_m \to P_{d_1}$ *or* 0 *as* $m \to \infty$, *on* $R^n\setminus\{0\}$.

Proof. Write $P = P_d + Q$ where (the principal part of) Q has degree $d_1 \leq d - 1$. By (5.12)

$$M_p^{N,N}(w_\alpha^{-1}\exp(t(P_d+Q))) \le C, \quad 0 \le t \le T.$$

Put $t = Tj\mu^{-1}$. From (5.3) we have after a change of variables that

$$M_p^{N,N}(w_\alpha^{-1}((T^{-1}\mu)^{1/d}.)\exp(j(P_d+\mu^{-1+d_1/d}Q_\mu)) \le C, \quad 1 \le j \le \mu,$$

where $Q_\mu(y) = T\mu^{-d/d_1}Q(T^{-1/d}\mu^{1/d}y)$ is bounded in $\mathscr{C}^1(R^n\setminus\{0\})$. Let B be a closed ball, with $0 \notin B$. Then

$$M_{p,B}^{N,N}(\exp(j(P_d+\mu^{-1+d_1/d}Q_\mu)) \le CM_{p,B}(w_\alpha(\mu^{1/d}.)) \le C_B\mu^{\alpha/d}, \quad 1 \le j \le \mu.$$

Let now $m = \mu^{\tilde{d}(r)/d}$, and let $Q_m = m^{(1+\varepsilon r)}\mu^{-1+d_1/d}Q_\mu$. Then

$$\tilde{d}(r)(1+\varepsilon r) - (d-d_1) = (1+\varepsilon r)\left(\tilde{d}(r) - \frac{d-d_1}{1+\varepsilon r}\right) \le 0,$$

and hence Q_m has the properties stated in the lemma. Further, $\mu^{\alpha/d} = m^{\alpha/\tilde{d}(r)}$ and $m \le \mu$, and so (5.13) is proved.

Lemma 5.4. *Let $P = P' + P''$ where the degree of $P'' \in \mathscr{C}^\mu$ is ≤ 0, $\mu > n/2$. If $p = 1$ or ∞, we also assume that the zero-order term of P'' is constant. If*

$$M_p^{N,N}(\exp(tP')) \le C', \quad 0 \le t \le T,$$

then

$$M_p^{N,N}(\exp(tP)) \le C(T), \quad 0 \le t \le T.$$

Proof. Immediate from Lemma 1.5 and the formula proved in Lemma 3.2.

Proof of Theorem 5.1. By Lemma 5.1 and 5.2, or 5.3 and 5.2 (for $p = 2$), if $\alpha > 0$, we see that the conditions of Corollary 3.1 and 3.3, respectively, are satisfied. This proves (5.7). Here we used $r = 1$ in (5.13).

To prove the converse, notice that (5.7) implies that $\exp(tP_d) \in M_p^{N,N}$ uniformly for $0 \le t \le T$ (e.g. by Corollary 3.1). Since $d = 1$ and P is a polynomial, Lemma 5.4 implies that (5.12) holds; by Lemma 5.1 and the regularity of P, (5.1) is then well posed in L_p.

Proof of Theorem 5.2. The proof of (5.8) is similar to the proof of (5.7), now using Theorem 3.2. Also the proof of the converse of Theorem 5.2 is similar to the above proof of Theorem 5.1.

The proof of Corollary 5.1 is evident from the fact that $M_p^{N,N} \subseteq \mathscr{C}$ for $p = 1, \infty$, and so the projections E_j are constants.

Theorem 5.3 follows from Theorem 3.4 and the converse from Lemma 1.3, in analogy with the above proofs.

Proof of Theorem 5.4. By Lemma 5.3, if (5.1) is well posed in $L_{p,\alpha}$ then for each compact ball $B_0 \subseteq R^n\setminus\{0\}$, we have

$$M_{p,B_0}^{N,N}(\exp(j(P_d+m^{-(1+r\varepsilon)}Q_m))) \le C_{B_0}m^{\alpha/\tilde{d}}, \quad 1 \le j \le m.$$

If $\alpha < r\tilde{d}|1/2 - 1/p|$, this contradicts the assumptions of Theorem 5.4 and Proposition 3.2. Hence (5.1) is not well posed in $L_{p,\alpha}$ for $0 \leq \alpha < r\tilde{d}|1/2 - 1/p|$ in this case.

To prove Theorem 5.5 finally, it is sufficient to prove the following result.

PROPOSITION 5.2. *Let* $\varphi \in \mathcal{C}^\infty$ *be the symbol of an* $N \times N$ *matrix of pseudo-differential operators of order* $d > 0$. *Assume that* $\exp(i\varphi) \in M_2^{N,N}$. *Then*

$$M_p^{N,N}(w_\alpha^{-1} e^{it\varphi}) \leq C_T, \quad 0 \leq t \leq T, \tag{5.14}$$

for $\alpha > nd|1/2 - 1/p|$.

Proof. We will provide bounds for $p = \infty$, $\alpha > nd/2$, and then interpolate with the known bound for $p = 2$. To handle the non-commutativity we proceed as follows. Let $\varphi_t = \exp(it\varphi)$. Then

$$\frac{d\varphi_t}{dt} = i\varphi\varphi_t.$$

Multiplying with w_α^{-1} and then differentiating we get that

$$\frac{\partial}{\partial t} D^\gamma(w_\alpha^{-1}\varphi_t) = i\varphi D^\gamma(w_\alpha^{-1}\varphi_t) + \sum_{\gamma' \neq 0} D^{\gamma'}(i\varphi) D^{\gamma''}(w_\alpha^{-1}\varphi_t).$$

The summation is over all γ', γ'' such that $\gamma' + \gamma'' = \gamma$, with $\gamma' \neq 0$. Solving this we have

$$D^\gamma(w_\alpha^{-1}\varphi_t) = \varphi_t D^\gamma w_\alpha^{-1} + \int_0^t \varphi_{t-\tau} \sum_{\gamma' \neq 0} D^{\gamma'}(i\varphi) D^{\gamma''}(w_\alpha^{-1}\varphi_t) d\tau, \quad \gamma' + \gamma'' = \gamma.$$

Using (5.3) we get for $|y| \geq 1$ that

$$\sup_{0 \leq t \leq T} |D^\gamma(w_\alpha^{-1}\varphi_t)| \leq C_T(|y|^{-\alpha-|\gamma|}) + C \sum_{\gamma' \neq 0} |y|^{(d-|\gamma'|)} \sup |D^{\gamma''}(w_\alpha^{-1}\varphi_t)|, \quad \gamma' + \gamma'' = \gamma.$$

After $|\gamma|$ steps we have for $0 \leq t \leq T$, $|y| \geq 1$,

$$|D^\gamma(w_\alpha^{-1}\varphi_t)| \leq C_T |y|^{(d-1)|\gamma| - \alpha}. \tag{5.15}$$

Hence we may apply Lemma 1.4 as soon as $(d-1)|\gamma| - \alpha < -|\gamma|$ for $|\gamma| \leq \nu$, some $\nu > n/2$, i.e. for $\alpha > nd/2$. Hence (5.14) is proved for $p = 1, \infty$ and also holds for $p = 2$ $(\alpha = 0)$ by assumption. The general case then follows by an interpolation argument (see e.g. [21]). Another proof is based on the Carlson-Beurling inequality

$$M_\infty(f) \leq C(\sum \|D^{\gamma'} f\|_2 \|D^{\gamma''} f\|_2)^{1/2},$$

with summation over multi-indices γ', γ'' with sum $(1, \ldots, 1)$. From (5.15) and a simple computation we then have that (with ψ_j as in Lemma 1.4, now writing ψ_0 for $\psi_0 + \psi_{-\infty}$)

$$M_\infty^{N,N}(w_\alpha^{-1}\psi_j\varphi_t) \leq C_T 2^{-j\{-n/2+(d-1)n/2\}}, \quad j \geq 0,$$

and hence by Lemma 1.1(i) and our assumptions that

$$M_p^{N,N}(w_\alpha^{-1}\psi_j\varphi_t) \leq C_T 2^{-j\{\alpha-dn|1/2-1/p|\}}, \quad j \geq 0.$$

Adding these inequalities, (5.14) follows for $\alpha > nd|1/2 - 1/p|$.

By the method used, we can also prove that $w_\alpha^{-1}\exp(ity^3 - ty^2) \in M_p$ for $0 \leq t \leq T$, if $\alpha > |1/2 - 1/p| = \tilde{d}|1/2 - 1/p|$. Since $d = 3$ in this example, it follows that the factor $\tilde{d}$ cannot be replaced by d in Theorem 5.1, even for $N = 1$.

6. Corrections to the papers »Power bounded matrices of Fourier-Stieltjes transforms I, II»

We give here a short list of corrections of some of the incorrect statements in [4], most of which were pointed out for the author by Lars Hörmander.

[4 — I]: Theorem 1 is only proved for $\Gamma = R^n$, and as stated does not even hold for $\Gamma = T^n$. The error occurs in the last sentence in the proof on top of page 120.

Theorem 2, which was proved by a similar argument, is for the same reasons only proved for $\Gamma' = R^n$ or T^n.

The example on p. 125 is correct, in spite of the erroneous proof of the fact that $\chi(1 - \lambda) \in B$. Using the method of the stationary phase, one can however prove this in a straightforward way, as was suggested by Lars Hörmander.

[4 — II]: Proposition 1 in section 2, p. 41, is proved only locally, not globally as incorrectly stated (cf. the discussion in section 4 above). Hence neither Corollary 1 nor Theorem 1 are proved. But from the proof of formula (4) on p. 44 we have the result stated as Theorem 3.1′ of the present paper (see section 3 above). Hence, as mentioned in connection with Theorem 3.1′, both Corollary 2 and Theorem 2 are correct. Finally, Corollary 1 on p. 44 is not correct, and the statement »H in β» should be replaced by »H in β^{loc}». The error here comes from not taking in account the well-known fact that B is not symmetric on its maximal ideal space (the Wiener-Pitt phenomenon).

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