Spaces of distributions of Besov type on Euclidean n-space. Duality, interpolation

Hans Triebel

Universität Jena, Jena, DDR

Contents

1. Introduction

The object of this paper is to give a self-contained treatment for some aspects of the theory of Banach spaces of distributions. The basis is on one side a generalization of the well-known multiplier theorem in L_p-spaces of Michlin-Hörmander [13, 8], and on the other side the general interpolation theory for Banach spaces. We use the real interpolation method developed by Lions-Peetre [10, 15, 19, 7], and the complex method developed by Lions, Calderón [4], and S. G. Krejn [9]. Further we need some facts from the theory of the vector-valued L_p-spaces $L_p(B)$, where B is a Banach space [5a, 6].

In 2 we describe some results about distributions and interpolation without proofs.

In 3 we consider the operator $\mathcal{K}$,

$$(\mathcal{K}f)(x) = \int_{R_n} K(x-y)f(y)dy, \; f(y) = (f_j(y))_{-\infty<j<\infty},$$

$$K(x) = (K_{ij}(x))_{-\infty<i,j<\infty}, \; K_{ij}(x) \in L_1^{\mathrm{loc}}(R_n),$$

and give sufficient conditions for acting from $L_p(l_2)$ into itself and for acting from $L_p(l_r)$ into itself, $1 < p, r < \infty$. The result is a generalization of the multiplier theorem proved by Hörmander [8].

In the following parts we consider special spaces of distributions. Let

$$l_p^\sigma = \{\xi | \xi = (\xi_j)_{j=0,1,2,\ldots}, \; \|\xi\|_{l_p^\sigma} = (\sum_{j=0}^{\infty} (2^{\sigma j}|\xi_j|)^p)^{\frac{1}{p}} < \infty\}, \; 1 < p < \infty; \quad -\infty < \sigma < \infty. \tag{1.1}$$

Then we define for $-\infty < s < \infty$, $1 < p < \infty$, $1 < q < \infty$,

$$F_{pq}^s = \Big\{ f | f \in S'(R_n), \; f = \sum_{j=0}^{\infty} a_j(x) \text{ (convergence in } S'), \tag{1.2}$$
$$\|\{a_j(x)\}\|_{L_p(l_q^s)} < \infty, \; \mathrm{supp}\, Fa_j \subset \{\xi | 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\}, \; (j = 1, 2, \ldots),$$
$$\mathrm{supp}\, Fa_0 \subset \{\xi | |\xi| \leqq 2\} \Big\},$$

$$B_{pq}^s = \Big\{ f | f \in S'(R_n), \; f = \sum_{j=0}^{\infty} a_j(x) \text{ (convergence in } S'), \tag{1.3}$$
$$\|\{a_j(x)\}\|_{l_q^s(L_p)} < \infty, \; \mathrm{supp}\, Fa_j \subset \{\xi | 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\}, \; (j = 1, 2, \ldots),$$
$$\mathrm{supp}\, Fa_0 \subset \{\xi | |\xi| \leqq 2\} \Big\},$$

Ff being the Fourier transform of $f \in S'$. (We also consider B_{pq}^s with $q = 1$ or $q = \infty$.) We set

$$\|f\|_{F_{pq}^s} = \inf_{\Sigma a_j = f} \|\{a_j\}\|_{L_p(l_q^s)}, \; \|f\|_{B_{pq}^s} = \inf_{\Sigma a_j = f} \|\{a_j\}\|_{l_q^s(L_p)}.$$

B_{pq}^s are the Besov spaces, and the definition (1.3) is essentialy the same as by Nikol'skij [14], p. 256. H_p^s are the well-known Bessel potential spaces defined by

$$H_p^s = \{f | f \in S'(R_n), \; F^{-1}(1 + |\xi|^2)^{\frac{s}{2}} Ff \in L_p(R_n)\}.$$

Here F^{-1} denotes the inverse Fourier transform.

Here are some results proved in 4—10[1]). F_{pq}^s and B_{pq}^s are Banach spaces, $C_0^\infty(R_n)$ is dense for $q < \infty$ (theorems 6.1.1 and 6.2.1). We have

[1]) The exact formulation of the assumptions is given in the theorems.

$$B^s_{pq} \subset F^s_{pq} \subset B^s_{pp} \text{ for } q \leqq p\,,$$
$$B^s_{pp} \subset F^s_{pq} \subset B^s_{pq} \text{ for } q \geqq p\,,$$

in the sense of continuous embedding, $F^s_{p2} = H^s_p$ (theorem 5.2.3), $F^s_{pp} = B^s_{pp}$ (theorem 5.2.3). If we set

$$I_s f = F^{-1}(1 + |x|^2)^{\frac{s}{2}} Ff, \; f \in S', \; -\infty < s < \infty\,,$$

then holds

$$I_s F^\sigma_{pq} = F^{\sigma-s}_{pq}, \; I_s B^\sigma_{pq} = B^{\sigma-s}_{pq}, \; -\infty < \sigma, s < \infty\,,$$

(theorem 5.1.1). Further we show (with the usual interpretation)

$$(B^s_{pq})' = B^{-s}_{p'q'}, \; (F^s_{pq})' = F^{-s}_{p'q'};$$

where B' is the dual of B, and $1/p + 1/p' = 1/q + 1/q' = 1$; $1 < p, q < \infty$; (theorems 7.1.7 and 7.2.2). We also prove the interpolation theorems

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta, q} = B^s_{pq}, \; s = (1-\theta)s_0 + \theta s_1; \; s_0 \neq s_1$$

(theorem 8.1.3), and

$$(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta, p} = B^s_{pp} \tag{1.4}$$

for

$$s = (1-\theta)s_0 + \theta s_1, \; \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}; \; s_0 \neq s_1 \tag{1.5}$$

(theorem 8.3.3). Special cases of (1.4) and (1.5) are

$$(B^{s_0}_{p_0p_0}, B^{s_1}_{p_1p_1})_{\theta, p} = (H^{s_0}_{p_0}, H^{s_1}_{p_1})_{\theta, p} = (H^{s_0}_{p_0}, B^{s_1}_{p_1p_1})_{\theta, p} = B^s_{pp} \tag{1.6}$$

Further interpolation results are

$$[B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1}]_\theta = B^s_{pq}\,, \tag{1.7}$$

$$s = (1-\theta)s_0 + \theta s_1; \; \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}; \; \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}; \tag{1.8}$$

(theorem 10.1.1), and

$$[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta = F^s_{pq} \tag{1.9}$$

with (1.8) (theorem 10.2.1). Special cases of (1.9) are

$$[H^{s_0}_{p_0}, H^{s_1}_{p_1}]_\theta = H^s_p \tag{1.10}$$

and

$$[H^{s_0}_{p_0}, B^{s_1}_{p_1p_1}]_\theta = F^s_{pq} \tag{1.11}$$

with (1.8) and $q_0 = 2$, $q_1 = p_1$. We show that the definition (1.3) coincides with the usual definition of the spaces B^s_{pq} (theorems 9.2.2, 9.2.5, and 9.2.9).

2. Basic facts about distributions, interpolation, and function spaces

2.1. *Distributions* [21]

$S = S(R_n)$ is the space of rapidly decreasing functions, $S' = S'(R_n)$ the dual space of tempered distributions with the usual topologies. We use in $S'(R_n)$ the strong topology. By

$$(F\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R_n} e^{-i\langle x,\xi\rangle}\varphi(x)dx, \quad \varphi \in S(R_n)\,, \tag{2.1a}$$

with $\langle x, \xi\rangle = \sum_{j=1}^{n} x_j\xi_j$, we denote the Fourier transformation. Then holds

$$(F^{-1}\varphi)(\xi) = (2\pi)^{-\frac{n}{2}} \int_{R_n} e^{i\langle x,\xi\rangle}\varphi(x)dx\,. \tag{2.1b}$$

We extend F and F^{-1} on S' in the usual way. The Fourier transformation is a continuous operation in S and S'. That means if

$$\varphi_j \underset{S}{\to} \varphi, \quad f_j \underset{S'}{\to} f$$

then

$$F\varphi_j \underset{S}{\to} F\varphi, \quad Ff_j \underset{S'}{\to} Ff\,. \tag{2.2}$$

Is $f \in S'$ and Ff has compact support then f is a continuous function (regular distribution). This follows from the Paley-Wiener-Schwartz theorem [21, II, p. 127].

We define the convolution

$$(f * \varphi)(x) = f_y(\varphi(x-y)), \quad f \in S', \quad \varphi \in S\,. \tag{2.3}$$

Then holds $f * \varphi \in C^\infty(R_n) \cap S'(R_n)$. The convolution is a continuous operation in S and in S'. That means if

$$\varphi_j \underset{S}{\to} \varphi, \quad \psi \in S, \quad g_j \underset{S'}{\to} g, \quad f \in S'\,, \tag{2.4}$$

then holds

$$S \ni \psi * \varphi_j \underset{S}{\to} \psi * \varphi, \quad f * \varphi_j \underset{S'}{\to} f * \varphi, \quad g_j * \psi \underset{S'}{\to} g * \psi \text{ [2)]} \tag{2.5}$$

For $\varphi \in S$, $f \in S'$ we have

$$F(f * \varphi) = (2\pi)^{\frac{n}{2}} F\varphi \cdot Ff, \quad F^{-1}(f * \varphi) = (2\pi)^{\frac{n}{2}} F^{-1}\varphi \cdot F^{-1}f\,. \tag{2.6}$$

[2)] This follows from (2.6) and the continuity of F and F^{-1} in S and S'.

2.2. *Interpolation theory*

We give a brief sketch of the real interpolation method developed by Lions and Peetre [10], and describe the K-method [15], and the L-method [19] given by Peetre. Finally we describe the complex method [4].

2.2.1. *The K-method* [15]. Let A_0 and A_1 be a couple of (real or complex) Banach spaces, continuously embedded into a linear Hausdorff space. Then

$$A_0 + A_1 = \{a \mid \exists\, a_0 \in A_0,\ \ a_1 \in A_1,\ \ a = a_0 + a_1\}$$

with the norm

$$K(t, a) = K(t, a, A_0, A_1) = \inf_{\substack{a = a_0 + a_1 \\ a_0 \in A_0,\ a_1 \in A_1}} (\|a_0\|_{A_0} + t\|a_1\|_{A_1}), \quad \infty > t > 0\,,$$

and $A_0 \cap A_1$ with

$$\|a\|_{A_0 \cap A_1} = \|a\|_{A_0} + \|a\|_{A_1}$$

are Banach spaces, see [4]. Now we define the interpolation spaces $(A_0, A_1)_{\theta, p}$, $0 < \theta < 1$, by

$$(A_0, A_1)_{\theta, p} = \left\{a \mid a \in A_0 + A_1, \|a\|_{\theta,p} = \left[\int_0^\infty (t^{-\theta}K(t, a))^p \frac{dt}{t}\right]^{\frac{1}{p}} < \infty\right\}, \quad 1 \leqq p < \infty\,, \tag{2.7a}$$

$$(A_0, A_1)_{\theta, \infty} = \{a \mid a \in A_0 + A_1, \|a\|_{\theta, \infty} = \sup_{t > 0} t^{-\theta}K(t, a) < \infty\}, \quad p = \infty\,. \tag{2.7b}$$

$(A_0, A_1)_{\theta, p}$ are Banach spaces. For abbreviation we write $A_{\theta, p} = (A_0, A_1)_{\theta, p}$.

2.2.2. *Interpolation property*, [10][3]. Let (A_0, A_1) and (B_0, B_1) two couples of Banach spaces with the embedding property of 2.2.1. Let T an operator acting from $A_0 + A_1$ into $B_0 + B_1$. Further we suppose that the restriction of T on A_i is a bounded linear operator into B_i with the norm $\|T\|_i$, $(i = 0, 1)$. Then the restriction of T on $A_{\theta, p}$ is a bounded linear operator into $B_{\theta, p}$, $0 < \theta < 1$; $1 \leqq p \leqq \infty$, and we can estimate its norm $\|T\|_{\theta, p}$ by

$$\|T\|_{\theta, p} \leqq \|T\|_0^{1-\theta}\|T\|_1^{\theta}\,. \tag{2.8}$$

2.2.3. *Reiteration theorem (stability theorem)*, [10]. Let be (A_0, A_1) the couple from 2.2.1 and $0 < \theta < 1$. Then we define the class $K(\theta) = K(\theta, A_0, A_1)$ of Banach spaces by

$$E \in K(\theta) \Leftrightarrow A_{\theta, 1} \subset E \subset A_{\theta, \infty}\,. \tag{2.9}$$

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[3]) In [15] PEETRE showed that the K-method and the methods given in [10] are equivalent. So we can quote the results from [10] and formulate them in the sense of the K-method.

$\subset$ always means that the embedding is continuous. We note that always

$$A_0 \cap A_1 \subset A_{\theta,1} \subset A_{\theta,p} \subset A_{\theta,\infty} \subset A_0 + A_1; \quad 0 < \theta < 1; \quad 1 \leqq p \leqq \infty \tag{2.10}$$

If $0 < \theta_0 < \theta_1 < 1$ and $E_i \in K(\theta_i)$, $(i = 0, 1)$, then holds

$$(E_0, E_1)_{\eta,p} = (A_0, A_1)_{\theta,p} \tag{2.11}$$

with $0 < \eta < 1$; $\theta = \theta_0(1 - \eta) + \theta_1\eta$; and $1 \leqq p \leqq \infty$.

2.2.4. *Duality*. Let (A_0, A_1) be the couple from 2.2.1, and $A_0 \cap A_1$ dense both in A_0 and A_1. If B is a Banach space we denote its dual by B'. Then follows from (2.10) with the usual interpretation

$$A_i' \subset (A_0 \cap A_1)'; \quad i = 0, 1; \quad (A_{\theta,p})' \subset (A_0 \cap A_1)',$$

and, [10],

$$(A_{\theta,p})' = (A_0, A_1)'_{\theta,p} = (A_0', A_1')_{\theta,p'} \tag{2.12}$$

with $1 \leqq p < \infty$, $1/p + 1/p' = 1$. We remark that $A_0 \cap A_1$ is always dense in $A_{\theta,p}$ for $1 \leqq p < \infty$, also without the assumption $A_0 \cap A_1$ dense both in A_0 and A_1 [10].

2.2.5. *The case of one semi-group*, [10]. Let B be a Banach space and $G(t)$, $0 \leqq t < \infty$, a strongly continuous semi-group of bounded operators, acting in B, $G(0) = I$ (identity operator), $\|G(t)\| \leqq M$. By A we denote the infinitesimal generator of $\{G(t)_{0 \leqq t < \infty}\}$. A^m is the iteration, and $D(A^m)$ its domain of definition with $\|b\|_{D(A^m)} = \|A^m b\| + \|b\|$, $(m = 1, 2, \ldots)$. If $0 < \theta < 1$ we write $m\theta = j + \varkappa$, j integer, $0 < \varkappa \leqq 1$. Then holds

$$(B, D(A^m))_{\theta,p} = \left\{ b \,|\, b \in D(A^j),\ \|b\| + \left[\int_0^\infty (t^{-\varkappa} \|\{G(t) - I\}^2 A^j b\|)^p \frac{dt}{t} \right]^{\frac{1}{p}} < \infty \right\} \tag{2.13}$$

for $1 \leqq p < \infty$ and an analogous formula for the case $p = \infty$. For $0 < \varkappa < 1$ (2.13) simplifies to

$$(B, D(A^m))_{\theta,p} = \left\{ b \,|\, b \in D(A^j),\ \|b\| + \left[\int_0^\infty (t^{-\varkappa} \|(G(t) - I) A^j b\|)^p \frac{dt}{t} \right]^{\frac{1}{p}} < \infty \right\} \tag{2.14}$$

(also with the usual modification for $p = \infty$). Here the norm $\|b\|_{(B, D(A^m))_{\theta,p}}$ and the norms in (2.13) and (2.14) are equivalent.

2.2.6. *The case of several commutative semi-groups*, [7, p. 189]. Let B be a Banach space and $G_k(t)$, $k = 1, 2, \ldots, N$; $0 \leqq t < \infty$; N strong continuous commutative semi-groups of bounded operators, acting in B. That means

$$G_j(t) G_k(s) = G_k(s) G_j(t), \quad 0 \leqq t, s < \infty, \quad k \neq j. \tag{2.15}$$

We denote with A_k the infinitesimal generators of $\{G_k(t)\}_{0\leqq t<\infty}$. Then holds

$$(B, \bigcap_{k=1}^{N} D(A_k^{m_k}))_{\theta,p} = \bigcap_{k=1}^{N} (B, D(A_k^{m_k}))_{\theta,p} \tag{2.16}$$

$0<\theta<1$; $1\leqq p\leqq\infty$. $m_k>0$ are integers. The norms of the intersection spaces are constructed in the usual way. Using (2.13), (2.14), and (2.16) we can describe these norms explicitly.

2.2.7. *The L-method*, [19]. Peetre proved in [19], theorem 2.2, the following result. Let (A_0, A_1) be a couple of Banach spaces in the sense of 2.2.1. We construct for $1\leqq p_0,\ p_1<\infty$ and $a\in A_0+A_1$

$$L(t, a) = L(t, a, A_0, A_1) = \inf_{\substack{a=a_0+a_1\\ a_0\in A_0,\ a_1\in A_1}} (\|a_0\|_{A_0}^{p_0} + t\|a_1\|_{A_1}^{p_1}) , \tag{2.17}$$

$\infty>t>0$. Then holds

$$\|a\|_{A_{\theta,p}}^{p} \sim \int_0^\infty t^{-\eta} L(t, a)\, \frac{dt}{t} \;{}^{4)} \tag{2.18}$$

with

$$0<\theta<1,\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1},\quad \eta = \frac{\theta p}{p_1}$$

These interpolation theorems are sufficient for our purpose. For further considerations of real interpolation methods see [10, 7, 2].

2.2.8. *The complex method*, [4]. The complex method is developed by J. L. Lions, A. P. Calderón, and S. G. Krejn. Let (A_0, A_1) be a couple of complex Banach spaces in the sense of 2.2.1. We consider functions $F(z)$, $z=\tau+it$, in the strip $0\leqq\tau\leqq 1$ with values in A_0+A_1 continuous and bounded with respect to the norm of A_0+A_1 in $0\leqq\tau\leqq1$ and analytic in $0<\tau<1$, and such that $F(it)\in A_0$ is A_0-continuous and bounded, $F(1+it)\in A_1$ is A_1-continuous and bounded. The set of these functions we denote by $\mathcal{F}=\mathcal{F}[A_0, A_1]$,

$$\|F\|_{\mathcal{F}} = \max\,[\sup_t \|F(it)\|_{A_0},\ \sup_t \|F(1+it)\|_{A_1}]\,.$$

The interpolation space $[A_0, A_1]_\theta$, $0<\theta<1$, is defined by

$$[A_0, A_1]_\theta = \{a \mid a\in A_0+A_1,\ \exists\, F\in\mathcal{F} \text{ with } F(\theta)=a\},\quad \|a\|_{[A_0,A_1]_\theta} = \inf_{F(\theta)=a} \|F\|_{\mathcal{F}}. \tag{2.19}$$

$[A_0, A_1]_\theta$ is a Banach space. $A_0\cap A_1$ is dense in $[A_0, A_1]_\theta$. Two couples (A_0, A_1) and (B_0, B_1) have the interpolation property analogous to 2.2.2.

[4]) $\sim$ means that we can estimate the left side by the right side with help of a positive constant and vice versa.

2.2.9. *Duality for the complex method*, [4]. Let (A_0, A_1) be a couple of Banach spaces in the sense of 2.2.1, and let $A_0 \cap A_1$ be dense both in A_0 and A_1. Further we assume that at least one of the spaces A_0 or A_1 is reflexive. Then holds (in the sense of 2.2.4)

$$([A_0, A_1]_\theta)' = [A_0', A_1']_\theta \tag{2.20}$$

2.3. *Function spaces*

We need a few results about vector-valued function spaces.

2.3.1. *Definition*. Let B be a Banach space and $1 \leqq p < \infty$. We consider a function $f(x)$, $x \in R_n$, with values in B, and put

$$L_p(B) = \left\{ f(x) \,|\, \|f(x)\| \text{ Lebesgue-measurable, } \|f\|_{L_p(B)} = \left(\int_{R_n} \|f(x)\|^p dx \right)^{\frac{1}{p}} < \infty \right\}.$$

$L_p(B)$ is a Banach space [5a, 6].

2.3.2. *Duality*. By B' we denote the dual of the Banach space B. $\langle f, g \rangle$, $f \in B$, $g \in B'$ is the symbol for linear continuous functionals. Let B be a reflexive Banach space, $1 < p < \infty$ and $1/p + 1/p' = 1$. Then holds

$$(L_p(B))' = L_{p'}(B') \tag{2.21}$$

in the sense

$$\int_{R_n} \langle h(x), l(x) \rangle dx, \quad h(x) \in L_p(B), \quad l(x) \in L_{p'}(B'), \tag{2.22}$$

as representation of the general linear continuous functional over $L_p(B)$, [6, theorem 8.20.5].

3. A generalization of the Michlin-Hörmander multiplier theorem

3.1. *Hörmander's multiplier theorem*

Let $L_p = L_p(C)$, $1 \leqq p \leqq \infty$ be the usual space of complex Lebesgue-measurable functions with

$$\|\varphi\|_{L_p} = \left(\int_{R_n} |\varphi(x)|^p dx \right)^{\frac{1}{p}} < \infty \tag{3.1}$$

(and the usual modification for $p = \infty$). In [8, theorem 2.5] Hörmander proved the following result. Let $f \in S'$, $Ff \in L_\infty$, and

$$\int_{\frac{R}{2} \leqq |\xi| \leqq 2R} |R^{|\alpha|} D^\alpha Ff|^2 \frac{d\xi}{R^n} \leqq B^2 \, ^{5)} \tag{3.2}$$

for all $R > 0$, and all α with $0 \leqq |\alpha| \leqq [n/2] + 1$, and a suitable positive number B. Then for all p, $1 < p < \infty$, there exists a constant $c = c(n, p)$, which depends only on n and p, with

$$\|f * \varphi\|_{L_p} \leqq cB\|\varphi\|_{L_p}, \quad \varphi \in S . \tag{3.3}$$

(2.6) and

$$f * \varphi = F^{-1}F(f * \varphi) = F^{-1}((2\pi)^{\frac{n}{2}} F\varphi \cdot Ff)$$

show that (3.3) is equivalent to

$$\|F^{-1}(F\varphi \cdot Ff)\|_{L_p} \leqq (2\pi)^{-\frac{n}{2}} cB\|\varphi\|_{L_p}, \quad \varphi \in S . \tag{3.3'}$$

(3.2) and (3.3) (or (3.3′)) is a generalization of Michlin's multiplier theorem [13].

3.2. *Modification of a theorem of J. Schwartz*, [20]

For our purpose we have to generalize the result of 3.1. First we extend a theorem of J. Schwartz [20, theorem 2, p. 788]. If B is a Banach space we denote by $L_0(B)$ the set of all Lebesgue-measurable bounded functions vanishing outside of a compact set in R_n, having values in B.

THEOREM 3.2. *Let B_0 and B_1 be two reflexive Banach spaces, and let $K(x)$ be a function of $x \in R_n$ having values in the Banach space of bounded linear mappings of B_0 into B_1 for almost all $x \in R_n$. Suppose that $K(x)$ is integrable over every finite region. Let $\infty > q \geqq 1$ and $A > 0$ be given; and suppose that there exists a constant $C < \infty$ such that for each $t > 0$ we have*

$$\left(\int_{|x| \geqq A} \|K(t(x - y)) - K(tx)\|^q dx \right)^{\frac{1}{q}} \leqq Ct^{-\frac{n}{q}} \tag{3.4}$$

for all y such that $\|y\| \leqq A^{-1}$. Put

$$\mathcal{K}(f)(x) = \int_{R_n} K(x - y) f(y) dy, \quad f \in L_0(B_0) . \tag{3.5}$$

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[5]) We use $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \ldots \partial x_n^{\alpha_n}}$, $\alpha = (\alpha_1 \ldots . \alpha_n)$, $|\alpha| = \sum_{j=1}^n \alpha_j$.

Suppose also that for some p *and* r *satisfying*

$$\infty > p > 1; \quad \infty > r > 1; \quad 1/p - 1/r = 1 - 1/q \tag{3.6}$$

we have

$$\Big(\int_{R_n} \|(\mathcal{K}f)(x)\|_{B_1}^r dx\Big)^{\frac{1}{r}} \leqq C\Big(\int_{R_n} \|f(x)\|_{B_0}^p dx\Big)^{\frac{1}{p}} \tag{3.7}$$

with the same constant C *as in* (3.4). *Then we can extend the mapping* $\mathcal{K}$ *for all* s, t *with*

$$1 < s \leqq t < \infty, \quad 1/s - 1/t = 1 - 1/q \tag{3.8}$$

in a unique way to a linear continuous operator acting from $L_s(B_0)$ *into* $L_t(B_1)$. *For the norm of the extended operator (also denoted with* $\mathcal{K}$*) holds*

$$\|\mathcal{K}\|_{L_s(B_0) \to L_t(B_1)} \leqq \alpha \cdot C\,, \tag{3.9}$$

where C *is the same constant as in* (3.4) *and* (3.7), *and* α *depends only on* n, A, q, r, p, s, t.

3.3. *Proof of theorem* 3.2

Step 1. In theorem 2 and corollary 4 of his paper [20] J. Schwartz proved

$$\|\mathcal{K}f\|_{L_t(B_1)} \leqq C'\|f\|_{L_s(B_0)}, \quad f \in L_0(B_0)\,, \tag{3.10a}$$

for s, t with

$$1/s - 1/t = 1 - 1/q, \quad 1 < s \leqq p, \quad (q < t \leqq r)\,. \tag{3.10b}$$

A homogeneity argument shows that

$$C' = \alpha \cdot C\,, \tag{3.11}$$

where C is the constant from (3.4) and (3.7), and α depends only on n, A, q, r, p, s, t. (We replace K in (3.4) and (3.7) by $C^{-1}K$ and prove (3.10) for this modificated operator.)

Step 2. By $\mathcal{K}'$ we denote the dual operator to $\mathcal{K}$, acting from $L_{t'}(B_1')$ into $L_{s'}(B_0')$ with

$$1/t' - 1/s' = 1 - 1/q, \quad r' \leqq t' < q', \quad (p' \leqq s' < \infty)\,. \tag{3.12}$$

This follows from (3.10). An estimate of type (3.10), (3.11) is true. Now we can determine $\mathcal{K}'$ explicitly. Let $K'(x)$ be the dual operator of $K(x)$ acting from B_1' into B_0'. Because $\|K'\| = \|K\|$, (3.4) is true for $K'(x)$. (3.7) holds for $\mathcal{K}'$ and the couples (t', s') with (3.12). Let $f(x) \in L_0(B_0)$ and $g(x) \in L_0(B_1')$. With the aid of (2.22) and the integration theory in Banach spaces (see [6] or [5a, especially III, 2.19]) follows

$$\int_{R_n} \langle f(x), \mathcal{K}' g(x) \rangle dx = \int_{R_n} \langle (\mathcal{K} f)(x), g(x) \rangle dx$$
$$= \int_{R_n}\int_{R_n} \langle K(x-y) f(y), g(x) \rangle dy dx = \int_{R_n} \Big\langle f(y), \int_{R_n} K'(x-y) g(x) dx \Big\rangle dy . \tag{3.13}$$

Because $L_0(B_0)$ is dense in $L_s(B_0)$, and 2.3.2 shows

$$(L_{s'}(B_0'))' = L_s(B_0'') = L_s(B_0) ,$$

we find

$$(\mathcal{K}' g)(y) = \int_{R_n} K(x-y) g(x) dx . \tag{3.14}$$

If we use the operator $(Eh)(x) = h(-x)$, we see that $E\mathcal{K}'E$ is an operator of the type (3.5), for which (3.4) holds. (3.6) and (3.7) we have to replace by

$$\left(\int_{R_n} \|\mathcal{K}' f(x)\|_{B_0'}^{p'} dx\right)^{\frac{1}{p'}} \leqq C \left(\int_{R_n} \|f(x)\|_{B_1'}^{r'} dx\right)^{\frac{1}{r'}} , \quad \frac{1}{r'} - \frac{1}{p'} = 1 - \frac{1}{q} .$$

(3.10) shows that $\mathcal{K}'$ is a linear bounded operator, acting from $L_{t'}(B_1')$ into $L_{s'}(B_0')$, with

$$1/t' - 1/s' = 1 - 1/q; \quad 1 < t' \leqq r', \quad (q < s' \leqq p') .$$

Together with (3.12) we find that $\mathcal{K}'$ is a linear bounded operator from $L_{t'}(B_1')$ into $L_{s'}(B_0')$ for all t', s' with

$$1/t' - 1/s' = 1 - 1/q; \quad 1 < t' \leqq s' < \infty \tag{3.15}$$

Step 3. Because B_0 and B_1 are reflexive Banach spaces we have from 2.3.2 $\mathcal{K}'' = \mathcal{K}$. The theorem follows by a duality argument from the second step.

3.4. *Remark*

The proof shows that the dual operator $\mathcal{K}'$, (3.14), has essentially the same structure as the operator $\mathcal{K}$ (3.5). After replacing $\mathcal{K}$ by $\mathcal{K}'$ (3.9) with (3.8) is true. The second step shows we needed for this statement only the reflexivity of B_0, but not of B_1. This was necessary in the last step.

3.5. *A multiplier theorem*

Now we are able to extend the multiplier theorem given by Hörmander [8, theorem 2.5]. Let l_r be the sequence space

$$l_r = \{\xi \mid \xi = (\xi_j)_{-\infty<j<\infty},\ \xi_j \text{ complex},\ \|\xi\|_{l_r} = (\sum_{j=-\infty}^{\infty} |\xi_j|^r)^{\frac{1}{r}} < \infty\}, \qquad (3.16)$$

$1 \leqq r < \infty$, and $L_p(l_r)$ the space in the sense of 2.3.1. We consider the matrix

$$K(x) = (K_{ij}(x))_{-\infty<i,j<\infty},\ K_{ij}(x) \in L_1^{\text{loc}}(R_n), \qquad (3.17)$$

with complex coefficients. Now we construct the operator

$$(\mathcal{K}f)(x) = \int_{R_n} K(x-y)f(y)dy \qquad (3.18)$$

with

$$f = (f_j)_{-\infty<j<\infty},\ f_j(x) \equiv 0 \text{ for } |j| \geqq N,\ f_j(x) \in C_0^\infty(R_n). \qquad (3.19)$$

THEOREM 3.5. *Let* $K(x)$ *be the matrix* (3.17). *Further we suppose* 1. $K_{ij}(x) \in S'(R_n)$ *(regular tempered distribution)*, 2. *the Fourier transform* $(FK_{ij})(\xi)$ *is a regular distribution with classical derivates* $D^\alpha(FK_{ij})(\xi)$ *for all* $\alpha, 0 \leqq |\alpha| \leqq [n/2]+1$, 3. *the existence of a positive number* B *with*

$$\int_{R/2 \leqq |\xi| \leqq 2R} \sum_{i,j=-\infty}^{\infty} |D^\alpha(FK_{ij})(\xi)|^2 d\xi \leqq B^2 R^{n-2|\alpha|}, \qquad (3.20)$$

for all $R > 0$ *and all* $\alpha, 0 \leqq |\alpha| \leqq [n/2]+1$.

(a) *We can extend the operator* $\mathcal{K}$, (3.18), *to a linear bounded operator from* $L_p(l_2)$ *into itself*, $1 < p < \infty$. *We have* $\|\mathcal{K}\| \leqq \alpha B$, *where* α *depends only on* p.

(b) *If we suppose additionally* $K_{ij} = 0$ *for* $i \neq j$, *we can extend the operator* $\mathcal{K}$ *to a linear bounded operator from* $L_p(l_r)$ *into itself*, $1 < r < \infty$; $1 < p < \infty$. *We have* $\|\mathcal{K}\| \leqq \alpha B$, *where* α *depends only on* p *and* r.

This theorem is an extension of Hörmander's multiplier theorem described in 3.1.

3.6. *Proof of theorem* 3.5 (a)

The proof follows the lines given by Hörmander [8, theorem 2.5].

Step 1. In [8, lemma 2.3] Hörmander proved the existence of a non-negative function $\varphi(x) \in C_0^\infty(R_n)$ with support in $\{\xi \mid \frac{1}{2} < |\xi| < 2\}$ and

$$\sum_{l=-\infty}^{\infty} \varphi(2^{-l}\xi) = 1,\ \xi \neq 0.$$

We set

$$(FK_{ij})^l(\xi) = (FK_{ij})(\xi)\varphi(2^{-l}\xi), \quad -\infty < i, j, l < \infty . \tag{3.21}$$

From (3.20) and

$$|D^\alpha(FK_{ij})^l| \leqq c \sum_{|\beta| \leqq |\alpha|} |D^\beta FK_{ij}| 2^{-l(|\alpha|-|\beta|)} \text{ [6]}, \quad 0 \leqq |\alpha| \leqq [n/2] + 1 ,$$

follows for $R = 2^l$

$$\int_{R_n} \sum_{i,j=-\infty}^{\infty} |D^\alpha(FK_{ij})^l|^2 d\xi \leqq cB^2 2^{l(n-2|\alpha|)} . \tag{3.22}$$

We set

$$g_{ij}^l(x) = F^{-1}(FK_{ij})^l, \quad -\infty < i, j, l < \infty . \tag{3.23}$$

A remark in 2.1 shows that $g_{ij}^l(x)$ is a continuous function. By Parseval's formula we get with $\varkappa = [n/2] + 1$

$$\int_{R_n} (1 + |x|^2 2^{2l})^\varkappa \sum_{i,j=-\infty}^{\infty} |g_{ij}^l(x)|^2 dx \leqq cB^2 2^{ln} . \tag{3.24}$$

Because $\varkappa > n/2$ follows with the help of Minkowski's inequality for integrals

$$\left[\sum_{i,j=-\infty}^{\infty} \left(\int_{R_n} |g_{ij}^l| dx\right)^2\right]^{\frac{1}{2}} \leqq \int_{R_n} \left[\sum_{i,j=-\infty}^{\infty} |g_{ij}^l|^2\right]^{\frac{1}{2}} dx \leqq cB\left[2^{nl} \int_{R_n} \frac{dx}{(1 + 2^{2l}|x|^2)^\varkappa}\right]^{\frac{1}{2}} = c'B . \tag{3.25}$$

Formula (3.23) shows

$$|(FK_{ij})^l(x)| \leqq (2\pi)^{-\frac{n}{2}} \left| \int_{R_n} e^{-i\langle x, \xi\rangle} g_{ij}^l(\xi) d\xi \right| \leqq c \int_{R_n} |g_{ij}^l(\xi)| d\xi . \tag{3.26}$$

From (3.25), (3.26), and the construction of $(FK_{ij})^l(x)$ we get

$$\sum_{i,j=-\infty}^{\infty} |(FK_{ij})(x)|^2 \leqq cB^2 . \tag{3.27}$$

Step 2. We set

$$G_{ij}^N(x) = \sum_{l=-N}^{N} g_{ij}^l(x), \quad G^N(x) = (G_{ij}^N(x)))_{-\infty < i,j < \infty} . \tag{3.28}$$

(3.25) shows $G_{ij}^N \in L_1(R_n) \subset L_1^{\text{loc}}(R_n) \cap S'(R_n)$. We consider the operator

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[6]) For several constants we use the same letters c, c', c''. The constants c and c' depend *only on* n.

$$(\mathcal{G}^N f)(x) = \int_{R_n} G^N(x-y) f(y) dy , \tag{3.29}$$

where f has the properties (3.19). We have

$$\|\mathcal{G}^N f\|^2_{L_2(l_2)} = \sum_{i=-\infty}^{\infty} \left\| \sum_{j=-\infty}^{\infty} \int_{R_n} G^N_{ij}(x-y) f_j(y) dy \right\|^2_{L_2} = c \sum_{i=-\infty}^{\infty} \| \sum_{j=-\infty}^{\infty} F G^N_{ij} F f_j \|^2_{L_2}$$

$$\leqq c \int_{R_n} \sum_{i,j=-\infty}^{\infty} |FG^N_{ij}|^2 \sum_{j=-\infty}^{\infty} |Ff_j|^2 dx .$$

We used (2.6). From (3.23), (3.27), and (3.28) follows

$$\|\mathcal{G}^N f\|^2_{L_2(l_2)} \leqq c B^2 \|f\|^2_{L_2(l_2)} . \tag{3.30}$$

We can extend $\mathcal{G}^N$ to a bounded linear operator from $L_2(l_2)$ into itself. The relation (3.7) with $\mathcal{K} = \mathcal{G}^N$, $B_0 = B_1 = l_2$, $p = r = 2$, $q = 1$, is true.

Step 3. For the application of theorem 3.2 we need an estimate of

$$\int_{|x| \geqq 2} \|G^N(t(x-y)) - G^N(tx)\| dx \quad \text{for} \quad |y| \leqq 1 . \tag{3.31}$$

Here is $\|\cdot\|$ the norm of the operator $G^N(t(x-y)) - G^N(tx)$ acting from l_2 into itself. In the same way as in (3.25) omitting the term 1 we find

$$\int_{|x| \geqq t} [\sum_{i,j=-\infty}^{\infty} |g^l_{ij}|^2]^{\frac{1}{2}} dx \leqq cB(2^l t)^{\frac{n}{2} - \varkappa} . \tag{3.32}$$

It follows for $|y| \leqq t$

$$\int_{|x| \geqq 2t} [\sum_{i,j=-\infty}^{\infty} |g^l_{ij}(x-y) - g^l_{ij}(x)|^2]^{\frac{1}{2}} dx \leqq cB(2^l t)^{\frac{n}{2} - \varkappa} . \tag{3.33}$$

For $2^l t \leqq 1$ we need another estimate. We have

$$g^l_{ij}(x) - g^l_{ij}(x-y) = F^{-1}[(1 - e^{-i\langle y, \xi \rangle})(FK_{ij})^l(\xi)] .$$

Using (3.22), $2^l t \leqq 1$, and $|y| \leqq t$ we find for $0 \leqq |\alpha| \leqq \varkappa$

$$\int_{R_n} \sum_{i,j=-\infty}^{\infty} |D^\alpha (1 - e^{-i\langle y, \xi \rangle})(FK_{ij})^l(\xi)|^2 d\xi \leqq cB^2 2^{l(n-2|\alpha|)} 2^{2l} t^2 .$$

The estimate analogous to (3.25) leads us now to

$$\int_{R_n} [\sum_{i,j=-\infty}^{\infty} |g^l_{ij}(x-y) - g^l_{ij}(x)|^2]^{\frac{1}{2}} dx \leqq cB 2^l t . \tag{3.34}$$

Together with (3.33) and the definition of G^N we get

$$\int_{|x|\geq 2t} \|G^N(x-y) - G^N(x)\|dx \leqq \sum_{l=-N}^{N} \int_{|x|\geq 2t} [\sum_{i,j=-\infty}^{\infty} |g^l_{ij}(x-y) - g_{ij}(x)|^2]^{\frac{1}{2}} dx \leqq$$
$$\leqq cB \sum_{l=-\infty}^{\infty} \min(2^l t, (2^l t)^{\frac{n}{2}-\varkappa}) \leqq c'B, \quad |y| \leqq t \tag{3.35}$$

We write the last estimate in the form

$$\int_{|x|\geq 2} \|G^N(t(x-y)) - G^N(tx)\|dx \leqq cBt^{-n}, \quad |y| \leqq 1\,. \tag{3.36}$$

This is the desired estimate of (3.31).

Step 4. Now we are able to apply theorem 3.2. We set $B_0 = B_1 = l_2$, $K(x) = G^N(x)$, $q = 1$, $A = 2$, $\mathcal{K} = \mathcal{G}^N$, and $r = p = 2$. (3.25) shows that $G^N(x)$ is a bounded operator from l_2 into l_2, and $\|G^N(x)\| \in L_1^{\mathrm{loc}}(R_n)$. (3.4) and (3.7) follow from (3.36) and (3.30). An application of theorem 3.2 gives now

$$\left(\int_{R_n} \|(\mathcal{G}^N f)(x)\|^p_{l_2} dx\right)^{\frac{1}{p}} \leqq cB \left(\int_{R_n} \|f(x)\|^p_{l_2} dx\right)^{\frac{1}{p}}; \quad 1 < p < \infty\,, \tag{3.37}$$

where c depends only on n and p. f has the property (3.19).

Step 5. We consider the limit $N \to \infty$. Because $(L_p(l_2))' = L_{p'}(l_2)$ and with the aid of (2.6) we see that (3.37) is equivalent to

$$\sum_{i,j=-\infty}^{\infty} \int_{R_n} \int_{R_n} G^N_{ij}(x-y) f_j(y) h_i(x) dy dx \leqq cB\|f\|_{L_p(l_2)} \|h\|_{L_{p'}(l_2)} \tag{3.37'}$$

and to

$$\sum_{i,j=-\infty}^{\infty} \int_{R_n} (FG_{ij})^N(\xi)(Ff_j)(\xi) F^{-1}h_i(\xi) d\xi \leqq cB\|f\|_{L_p(l_2)} \|h\|_{L_{p'}(l_2)}\,. \tag{3.37''}$$

f and h have the property (3.19). (Elements with (3.19) are dense in $L_p(l_2)$). (3.23), (3.28), and (3.27) lead us for $N \to \infty$ to

$$\sum_{i,j=-\infty}^{\infty} \int_{R_n} FK_{ij}(\xi) \cdot Ff_j(\xi) \cdot F^{-1}h_i(\xi) d\xi \leqq cB\|f\|_{L_p(l_2)} \|h\|_{L_{p'}(l_2)}$$

and

$$\sum_{i,j=-\infty}^{\infty} \int_{R_n} \left(\int_{R_n} K_{ij}(x-y) f_j(y) dy\right) h_i(x) dx \leqq cB\|f\|_{L_p(l_2)} \|h\|_{L_{p'}(l_2)}\,.$$

From this follows

$$\|\mathcal{K}f\|_{L_p(l_2)} \leqq cB\|f\|_{L_p(l_2)} . \tag{3.38}$$

This completes the proof of theorem 3.5 (a).

3.7. *Proof of theorem* 3.5 (b)

We consider again the operator $\mathcal{G}^N$ from (3.29). Now we have $G^N_{ij} = 0$ for $i \neq j$. The estimate after (3.29) we change into

$$\|\mathcal{G}^N f\|^r_{L_r(l_r)} = \sum_{i=-\infty}^{\infty} \left\| \int_{R_n} G^N_{ii}(x-y) f_i(y) dy \right\|^r_{L_r} .$$

Now we use (3.37) with $p = r$ and

$$f^{(i)} = (\ldots, 0, f_i(x), 0 \ldots,) .$$

Then holds

$$\|\mathcal{G}^N f\|^r_{L_r(l_r)} \leqq c^r B^r \sum_{i=-\infty}^{\infty} \|f_i\|^r_{L_r} = c^r B^r \|f\|^r_{L_r(l_r)} . \tag{3.39}$$

This formula we have to take instead of (3.30). The estimate (3.35) we can take without any change because

$$\|G^N(x-y) - G^N(x)\|_{l_r \to l_r} = \sup_{-\infty < i < \infty} |G_{ii}(x-y) - G_{ii}(x)|$$
$$\leqq \sum_{l=-N}^{N} [\sum_{i=-\infty}^{\infty} |g^l_{ii}(x-y) - g^l_{ii}(x)|^2]^{\frac{1}{2}} .$$

Then follows (3.37) with l_r instead of l_2. The fifth step we can take over with suitable changes. This completes the proof.

3.8. *Remark*

The assumption (3.20) is realized if we find a positive number $\tilde{B}$ with

$$[\sum_{i,j=-\infty}^{\infty} |D^\alpha(FK_{ij})(\xi)|^2]^{\frac{1}{2}} \leqq \frac{\tilde{B}}{|\xi|^{|\alpha|}} , \quad 0 \leqq |\alpha| \leqq [n/2] + 1 . \tag{3.40}$$

4. The spaces B^s_{pq}, F^s_{pq}, and H^s_p. Equivalent norms

4.1. *Definitions*

4.1.1. *The spaces* l^σ_p. First we define the sequence space l^σ_p. Let $-\infty < \sigma < \infty$. For $1 \leqq p < \infty$ we set

$$l_p^\sigma = \{\xi | \xi = (\xi_j)_{j=0,1,2,\dots},\ \xi_j \text{ complex},\ \|\xi\|_{l_p^\sigma} = (\sum_{j=0}^{\infty} (2^{j\sigma} |\xi_j|)^p)^{\frac{1}{p}} < \infty\}. \quad (4.1)$$

For $p = \infty$ we set

$$l_\infty^\sigma = \{\xi | \xi = (\xi_j)_{j=0,1,2,\dots},\ \xi_j \text{ complex},\ \|\xi\|_{l_\infty^\sigma} = \sup 2^{j\sigma} |\xi_j| < \infty\}. \quad (4.2)$$

l_p^σ are Banach spaces.

4.1.2. *The spaces* F_{pq}^s. For $-\infty < s < \infty$; $1 < p, q < \infty$ we set

$$F_{pq}^s = F_{pq}^s(R_n) = \Big\{ f | f \in S'(R_n),\ f \underset{S'}{=} \sum_{j=0}^{\infty} a_j(x),$$

$$\|\{a_j\}\|_{L_p(l_q^s)} = \left(\int_{R_n} [\sum_{j=0}^{\infty} (2^{sj} |a_j(x)|)^q]^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty, \quad (4.3)$$

$$\operatorname{supp} Fa_j \subset \{\xi | 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\} \text{ for } j = 1, 2, \dots;$$

$$\operatorname{supp} Fa_0 \subset \{\xi | |\xi| \leqq 2\} \Big\}.$$

$\sum_{j=0}^{\infty} a_j(x) =_{S'} f$ means that $\sum_{j=0}^{N} a_j(x)$ converges in $S'(R_n)$ to f. $\operatorname{supp} g$ denotes the support of the distribution g. A remark in 2.1 shows that $a_j(x)$ are continuous functions. We set

$$\|f\|_{F_{pq}^s} = \inf_{f = \Sigma a_j} \|\{a_j\}\|_{L_p(l_q^s)}. \quad (4.4)$$

4.1.3. *The spaces* B_{pq}^s. For $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqq q < \infty$ we set

$$B_{pq}^s = B_{pq}^s(R_n) = \Big\{ f | f \in S'(R_n),\ f \underset{S'}{=} \sum_{j=0}^{\infty} a_j(x),$$

$$\|\{a_j\}\|_{l_q^s(L_p)} = (\sum_{j=0}^{\infty} (2^{sj} \|a_j(x)\|_{L_p})^q)^{\frac{1}{q}} < \infty, \quad (4.5a)$$

$$\operatorname{supp} Fa_j \subset \{\xi | 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\} \text{ for } j = 1, 2, \dots;$$

$$\operatorname{supp} Fa_0 \subset \{\xi | |\xi| \leqq 2\} \Big\}.$$

For $-\infty < s < \infty$; $1 < p < \infty$; $q = \infty$, we set

$$B_{p\infty}^s = B_{p\infty}^s(R_n) = \Big\{ f | f \in S'(R_n),\ f \underset{S'}{=} \sum_{j=0}^{\infty} a_j(x),$$

$$\|\{a_j\}\|_{l_\infty^s(L_p)} = \sup_j 2^{sj} \|a_j(x)\|_{L_p} < \infty, \quad (4.5b)$$

$$\operatorname{supp} Fa_j \subset \{\xi | 2^{j-1} \leqq |\xi| \leqq 2^{j+1}\} \text{ for } j = 1, 2, \dots;$$

$$\operatorname{supp} Fa_0 \subset \{\xi | |\xi| \leqq 2\} \Big\}.$$

We set

$$\|f\|_{B^s_{pq}} = \inf_{f=\Sigma a_j} \|\{a_j\}\|_{l^s_q(L_p)} . \tag{4.6}$$

These are the well-known Besov spaces. The definition is similar to the definition given by Nikol'skij [14], p. 256.

4.1.4. *The spaces* H^s_p. For $-\infty < s < \infty$; $1 < p < \infty$, we set

$$H^s_p = H^s_p(R_n) = \{f | f \in S'(R_n),\ F^{-1}(1 + |x|^2)^{\frac{s}{2}} Ff \in L_p(R_n)\} \tag{4.7}$$

and

$$\|f\|_{H^s_p} = \|F^{-1}(1 + |x|)^{\frac{s}{2}} Ff\|_{L_p} . \tag{4.8}$$

H^s_p are the well-known spaces of Bessel potentials, Lebesgue spaces, or Liouville spaces [14, p. 379 ff.] or [25] (where there are further references). Sometimes the spaces are denoted by L^s_p.

4.2. *Equivalent norms*

For the further considerations we need some equivalent norms.

4.2.1. *A special system of functions.* We consider functions with

1. $\varphi_k(x) \in S(R_n)$, $F\varphi_k(\xi) \geqq 0$, $(k = 0, 1, 2, \ldots)$;
2. $\exists\, N$, $(N = 1, 2, \ldots)$, with $\operatorname{supp} F\varphi_k \subset \{\xi | 2^{k-N} \leqq |\xi| \leqq 2^{k+N}\}$, $(k = 1, 2, \ldots)$; $\operatorname{supp} F\varphi_0 \subset \{\xi | |\xi| \leqq 2^N\}$. (4.9)
3. $\exists\, c_1 > 0$ with $c_1 \leqq (\sum_{j=0}^{\infty} F\varphi_j)(\xi)$; (4.10)
4. $\exists\, c_2 > 0$ with

$$|(D_\alpha F\varphi_k)(\xi)| \leqq \frac{c_2}{|\xi|^{|\alpha|}} \quad \text{for} \quad 0 \leqq |\alpha| \leqq \left[\frac{n}{2}\right] + 1, \quad (k = 1, 2, \ldots) . \tag{4.11}$$

The most important system of functions of this type is the following. We consider a function $\varphi(x) \in S(R_n)$, $F\varphi(\xi) \geqq 0$ with

$$\operatorname{supp} F\varphi \subset \{\xi | 2^{-N} \leqq |\xi| \leqq 2^N\}, \quad (F\varphi)(\xi) > 0 \quad \text{for} \quad \frac{1}{\sqrt{2}} \leqq |\xi| \leqq \sqrt{2} . \tag{4.12}$$

It is not difficult to see that the functions $\varphi_k(x)$ with

$$(F\varphi_k)(\xi) = (F\varphi)(2^{-k}\xi), \quad k = 1, 2, \ldots \tag{4.13}$$

by suitable choice of $\varphi_0(x)$ are a system of above type.

4.2.2. *Equivalent norms in* F^s_{pq} *and* B^s_{pq}.

THEOREM 4.2.2. *Let* $\{\varphi_k\}_{k=0,1,2,\dots}$ *a system of type* 4.2.1, $-\infty < s < \infty$, *and* $1 < p < \infty$. *Then we have*

(a) $\|f\|_{F^s_{pq}} \sim \|\{f * \varphi_k\}\|_{L_p(l^s_q)}$ [4]), $f \in F^s_{pq}$; $1 < q < \infty$;

(b) $\|f\|_{B^s_{pq}} \sim \|\{f * \varphi_k\}\|_{l^s_q(L_p)}$ [4]), $f \in B^s_{pq}$; $1 \leqq q \leqq \infty$.

4.2.3. *Proof of theorem* 4.2.2. (a).

Step 1. Let $f \in F^s_{pq}$ and $f = \sum_{j=0}^{\infty} a_j(x)$ in the sense of (4.3). (2.5) and (2.6) show

$$f * \varphi_k \underset{S'}{=} \sum_{j=0}^{\infty} a_j * \varphi_k = \sum_{j=k-N-1}^{k+N+1} (a_j * \varphi_k)(x)$$

because

$$a_j * \varphi_k = F^{-1}F(a_j * \varphi_k) = F^{-1}(2\pi)^{\frac{n}{2}} F\varphi_k \cdot Fa_j = 0$$

for $j < k - N - 1$ and $j > k + N + 1$. Further we set $\varphi_k \equiv a_k \equiv 0$ for $k < 0$. It follows that

$$\|\{f * \varphi_k\}\|_{L_p(l^s_q)} \leqq \sum_{r=-N-1}^{N+1} \|\{a_{k+r} * \varphi_k\}\|_{L_p(l^q_s)} . \tag{4.14}$$

a_k belongs to L_p; we approximate a_k,

$$C_0^\infty(R_n) \ni a_{k,\varepsilon} \underset{L_p}{\to} a_k \quad \text{for} \quad \varepsilon \downarrow 0 . \tag{4.15}$$

For fixed numbers ε, r and M we can apply theorem 3.5 (b) with $K_{ii} = \varphi_i$ [7]). ((3.40) is fulfilled because (4.9) and (4.11) hold.) Then follows

$$\|\{a_{k+r,\varepsilon} * \varphi_k\}_{k=0}^M\|_{L_p(l^s_q)} \leqq c\|\{a_{k+r,\varepsilon}\}_{k=0}^M\|_{L_p(l^s_q)} .$$

Here c is independent of ε, r, and M. $\varepsilon \to 0$ shows that the last estimate is true for $\varepsilon = 0$. Setting $M = \infty$ on the right hand side of the last inequality we find that

$$\|\{f * \varphi_k\}\|_{L_p(l^s_q)} \leqq c\|\{a_k\}\|_{L_p(l^s_q)} .$$

Taking the infimum on the right hand side we get

$$\|\{f * \varphi_k\}\|_{L_p(l^s_q)} \leqq c\|f\|_{F^s_{pq}} . \tag{4.16}$$

Step 2. We prove the opposite inequality. We need an auxiliary system of functions. Let $\varrho(x)$ be a real function with

$$\varrho(x) = 1 \quad \text{for} \quad 2^{-N} \leqq |x| \leqq 2^N, \ \varrho(x) \in C_0^\infty(\{\xi | 2^{-N-1} < |\xi| < 2^{N+1}\}) .$$

We set

$$\varrho_k(x) = \varrho(2^{-k}\xi), \quad (k = 1, 2, \ldots) .$$

Further let $\varrho_0(x)$ be a real function with

$$\varrho_0(x) = 1 \quad \text{for} \quad |\xi| \leqq 2^N, \quad \varrho_0(x) \in C_0^\infty(\{\xi \,|\, |\xi| < 2^{N+1}\}) .$$

Then the assumptions of theorem 3.5 (b) for K_{ii},

$$(FK_{ii})(x) = (\sum_{l=0}^{\infty} F\varphi_l)^{-1} \varrho_i(x)(2\pi)^{-n}, \quad i = 0, 1, 2, \ldots \text{ [7]}$$

are fulfilled. We construct

$$\psi_k = \varphi_k * K_{kk}, \quad (F\psi_k = (2\pi)^{\frac{n}{2}} F\varphi_k \cdot FK_{kk}) ,$$

and find by application of theorem 3.5 (b) with the aid of an analogous continuity argument as in the first step for the function $\varphi_k * f \in L_p$

$$\|\{f * \psi_k\}\|_{L_p(l_q^s)} \leqq c \|\{f * \varphi_k\}\|_{L_p(l_q^s)} . \tag{4.17}$$

Now we have, see (2.6),

$$(\sum_{k=0}^{\infty} F\psi_k)(\xi) = (2\pi)^{-\frac{n}{2}} .$$

Therefore

$$f = F^{-1}Ff = F^{-1}(\sum_{k=0}^{\infty} F\psi_k \cdot Ff)(2\pi)^{\frac{n}{2}} \underset{S'}{=} \sum_{k=0}^{\infty} F^{-1}(F\psi_k \cdot Ff)(2\pi)^{\frac{n}{2}} = \sum_{k=0}^{\infty} f * \psi_k .$$

For the case $N = 1$ with $a_k = f * \psi_k$ follows now

$$\|f\|_{F_{pq}^s} \leqq \|\{a_k\}\|_{L_p(l_q^s)} \leqq c \|\{f * \varphi_k\}\|_{L_p(l_q^s)} . \tag{4.18}$$

Step 3. Let $N > 1$. Then we have to modify the last part of the second step. We need an auxiliary system of functions. Put

$$\chi_k(x) \in S(R_n), \quad (k = 0, 1, 2, \ldots)$$
$$\operatorname{supp} F\chi_k \subset \{\xi \,|\, 2^{k-1} \leqq |\xi| \leqq 2^{k+1}\}, \quad (k = 1, 2, \ldots) ,$$
$$\operatorname{supp} F\chi_0 \subset \{\xi \,|\, |\xi| \leqq 2\} ,$$
$$\sum_{k=0}^{\infty} F\chi_k = (2\pi)^{-\frac{n}{2}} , \quad |(D^\alpha F\chi_k)(\xi)| \leqq \frac{c}{|\xi|^{|\alpha|}} .$$

The existence of such a system follows from the beginning of the first step of 3.6. For $K_{kk} = \chi_k$ the assumptions of theorem 3.5 (b) are fulfilled[7]. We find that

[7]) $K_{ii} = 0$ for $i < 0$.

$$\|\{f * \psi_k * \chi_{k+r}\}\|_{L_p(l_q^s)} \leqq c\|\{\psi_k * f\}\|_{L_p(l_q^s)}, \quad (r = -N-1, \ldots, N+1),$$

and

$$f * \psi_k = \sum_{l=k-N-1}^{k+N+1} f * \psi_k * \chi_l$$

(ψ_k is the same function as in the second step, $\psi_k \equiv 0$ for $k < 0$). Setting

$$a_k(x) = \sum_{r=-N-1}^{N+1} f * \psi_{k+r} * \chi_k, \quad (k = 0, 1, 2, \ldots)$$

we find

$$\|\{a_k\}\|_{L_p(l_q^s)} \leqq c\|\{f * \psi_k\}\|_{L_p(l_q^s)}, \ \operatorname{supp} a_k \subset \{\xi | 2^{k-1} \leqq |\xi| \leqq 2^{k+1}\}, \ (k = 1, 2, \ldots) \quad (4.19)$$

(with the usual modification for $k = 0$), and by using (2.6)

$$\sum_{k=0}^{\infty} a_k(x) = \sum_{k=0}^{\infty} \Big(\sum_{r=-N-1}^{N+1} f * \psi_k * \chi_{k+r} \Big) = \sum_{k=0}^{\infty} (f * \psi_k) \underset{S'}{=} f.$$

With the aid of (4.17) follows from (4.19) the estimate (4.18). This proves the theorem 4.2.2. (a).

4.2.4. *Proof of theorem* 4.2.2 (b). The proof is the same as in 4.2.3. We have to change $\|\cdot\|_{L_p(l_q^s)}$ into $\|\cdot\|_{l_q^s(L_p)}$. Further we need only the scalar case of theorem 3.5 (b) with

$$\|\varphi_k * a_{k+r}\|_{L_p} \leqq c\|a_{k+r}\|_{L_p},$$

where c is independent of k. The limit cases $q = 1$ and $q = \infty$ do not disturb the considerations. (The scalar case of theorem 3.5 is the usual multiplier theorem.)

4.2.5. *Remark.* The definition of the Besov spaces in the sense of theorem 4.2.2 (b) is given by Peetre [17]. (See also [16]) For similar constructions see [14, 8.8—8.10].

4.2.6. *An equivalent norm in* H_p^s.

THEOREM 4.2.6. *Let* $-\infty < s < \infty$; $1 < p < \infty$. *Then holds*

$$H_p^s = F_{p2}^s.$$

For the proof it is sufficient to show that

$$\|\{f * \varphi_k\}\|_{L_p(l_2^s)} \sim \|f\|_{H_p^s} = \|F^{-1}(1 + |x|^2)^{\frac{s}{2}} Ff\|_{L_p} \quad (4.20)$$

where $\{\varphi_k\}$ is a system of functions of type 4.2.1. In this sense theorem 4.2.6 is a result about equivalent norms. First we prove the theorem for $s = 0$, that means (with $l_2 = l_2^0$)

$$\|\{f * \varphi_k\}\|_{L_p(l_2)} \sim \|f\|_{L_p} . \tag{4.21}$$

This is a theorem of Paley-Littlewood type. A systematic treatment of Paley-Littlewood theorems is given by Littmann, McCarthy and Riviere in [11]. There are also further references. The proof of the general case of theorem 4.2.6 for an arbitrary s we give in 5.1.4.

4.2.7. *Proof of* (4.21). We consider a system of functions of type 4.2.1, and set $\varphi_k = 0$ for $k < 0$.

Step 1. Let $f \in L_p$. With $f_0 = f$, $f_j = 0$ for $j \neq 0$,

$$K_{k0}(x) = \varphi_k(x), \quad -\infty < k < \infty; \quad K_{kj} = 0 \quad \text{for} \quad j \neq 0 ,$$

we apply theorem 3.5 (a). With the aid of a continuity argument as in the first step of 4.2.3 we get

$$\|\{f * \varphi_k\}\|_{L_p(l_2)} \leqq c\|f\|_{L_p} \tag{4.22}$$

Step 2. Let $\{f * \varphi_k\} \in L_p(l_2)$. Then we set $f_k = f * \varphi_k$, and

$$K_{0k}(x) = \psi_k(x), \quad (k = 0, 1, 2, \ldots), \quad K_{jk} = 0 \quad \text{otherwise.}$$

$\{\psi_k\}$ is also a system of functions of type 4.2.1 with

$$(F\psi_k)(\xi) = 1 \quad \text{for} \quad \xi \in \operatorname{supp} \varphi_k . \tag{4.23}$$

Then we apply theorem 3.5 (a), and find with the aid of a continuity argument

$$\Big\|\sum_{k=0}^{\infty} f * \varphi_k * \psi_k\Big\|_{L_p} \leqq c\|\{f * \varphi_k\}\|_{L_p(l_2)} . \tag{4.24}$$

Now we have from (4.23) and (2.6)

$$\sum_{k=0}^{\infty} f * \varphi_k * \psi_k = F^{-1}(2\pi)^n \sum_{k=0}^{\infty} Ff \cdot F\varphi_k \underset{S'}{=} (2\pi)^n F^{-1}\Big(\sum_{k=0}^{\infty} F\varphi_k \cdot Ff\Big) \tag{4.25}$$

(convergence in S'). Without loss of generality we may assume

$$(2\pi)^n \sum_{k=0}^{\infty} F\varphi_k = 1 . \tag{4.26}$$

Then (4.24) and (4.25) show

$$\|f\|_{L_p} \leqq c\|\{f * \varphi_k\}\|_{L_p(l_2)} . \tag{4.27}$$

(4.22) and (4.27) prove (4.21), and theorem 4.2.6 with $s = 0$.

4.2.8. *Remark.* Is (4.26) not true we prove (4.21) for an other system $\{\psi_k\}$ for which (4.26) holds, and use then the equivalence

$$\|f\|_{F^0_{p2}} \sim \|\{f * \psi_k\}\|_{L_p(l_2)} \sim \|\{f * \varphi_k\}\|_{L_p(l_2)} \,.$$

This shows that (4.21) is always true.

5. Relations between the spaces B^s_{pq}, F^s_{pq}, and H^s_p. Lifting property

5.1. *Lifting property*

We consider the operation

$$I_s f = F^{-1}(1 + |x|^2)^{\frac{s}{2}} Ff, \quad -\infty < s < \infty \,. \tag{5.1}$$

It is easy to see that I_s is a linear continuous one-to-one mapping from $S(R_n)$ onto $S(R_n)$, and from $S'(R_n)$ onto $S'(R_n)$,

$$I_s^{-1} = I_{-s} \tag{5.2}$$

5.1.1. *Lifting property.*

THEOREM 5.1.1. *Let* $-\infty < s, \sigma < \infty$ *and* $1 < p < \infty$. I_s *is a linear bounded one-to-one operator from* H^σ_p *onto* $H^{\sigma-s}_p$, *from* F^σ_{pq} *onto* $F^{\sigma-s}_{pq}$, $(1 < q < \infty)$, *and from* B^σ_{pq} *onto* $B^{\sigma-s}_{pq}$, $(1 \leqq q \leqq \infty)$.

5.1.2. *Proof of theorem* 5.1.1.

Step 1. The formula

$$\|I_s f\|_{H^{\sigma-s}_p} = \|f\|_{H^\sigma_p} \tag{5.3}$$

is clear. (5.2) shows that the theorem is true for the spaces H^σ_p.

Step 2. Let $\{\varphi_k\}$ be a system of type 4.2.1. Then $\{\psi_k\}_{k=0,1,2,\ldots}$,

$$\psi_k = \varphi_k * F^{-1}\left(\frac{2^{ks}}{(1 + |x|^2)^{s/2}}\right), \tag{5.4}$$

is also a system of such a type. This follows from (2.6). For $f \in S'(R_n)$ holds

$$\begin{aligned} I_s f * \psi_k &= F^{-1}((2\pi)^{\frac{n}{2}} F\psi_k \cdot FI_s f) \\ &= F^{-1}((2\pi)^{\frac{n}{2}} 2^{ks} F\varphi_k \cdot Ff) = 2^{ks} f * \varphi_k \,. \end{aligned} \tag{5.5}$$

Theorem 4.2.2 shows now

$$\|I_s f\|_{F^{\sigma-s}_{pq}} \sim \|f\|_{F^{\sigma}_{pq}}, \quad \|I_s f\|_{B^{\sigma-s}_{pq}} \sim \|f\|_{B^{\sigma}_{pq}} . \tag{5.6}$$

From this and from (5.2) the theorem follows.

5.1.3 *Remark.* For the spaces H^s_p and B^s_{pq} these lifting properties are well-known, see for instance [14, p. 370].

5.1.4. *Proof of theorem* 4.2.6. We know that (4.20) holds for $s = 0$ (this is (4.21)). This means that $L_p = H^0_p = F^0_{p2}$. The general case follows now from theorem 5.1.1.

5.2. *Relations between the spaces* B^s_{pq}, F^s_{pq}, *and* H^s_p

5.2.1. *Relations between the spaces* B^s_{pq}.

THEOREM 5.2.1. *Let* $-\infty < s < \infty$; $\varepsilon > 0$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. *Then holds*

(a)
$$B^s_{p1} \subset B^s_{pq} \subset B^s_{p\infty} \text{ [8]} \tag{5.7}$$

and

(b)
$$B^{s+\varepsilon}_{p\infty} \subset B^s_{pq} \subset B^{s-\varepsilon}_{p1} \text{ [8]} \tag{5.8}$$

5.2.2. *Proof of theorem* 5.2.1. (5.7) follows immediately from the monotony property of the spaces l_q.

For the proof of (5.8) we suppose in the sense of 4.1.3.

$$f = \sum_{j=0}^{\infty} a_j(x) \in B^{s+\varepsilon}_{p\infty}, \quad \sup_j 2^{(s+\varepsilon)j}\|a_j(x)\|_{L_p} \leqq 2\|f\|_{B^{s+\varepsilon}_{p\infty}} .$$

Then holds

$$\|f\|_{B^s_{pq}} \leqq \|\{2^{sj}\|a_j\|_{L_p}\}\|_{l_q} \leqq 2\|f\|_{B^{s+\varepsilon}_{p\infty}}\|\{2^{-\varepsilon j}\}\|_{l_q} \leqq c\|f\|_{B^{s+\varepsilon}_{p\infty}} .$$

This proves the left side of (5.8). The right side follows in the same way. If $f \in B^s_{pq} \subset B^s_{p\infty}$ we replace in the last inequality s by $s - \varepsilon$, q by 1, $s + \varepsilon$ by s.

5.2.3. *Further relations.*

THEOREM 5.2.3. (a) *Let* $-\infty < s < \infty$ *and* $1 < p < \infty$. *Then holds*

[8] The sign $\subset$ always means continuous embedding.

$$F^s_{p2} = H^s_p \quad and \quad F^s_{pp} = B^s_{pp} \tag{5.9}$$

(b) *For* $-\infty < s < \infty$ *holds*

$$B^s_{pq} \subset F^s_{pq} \subset B^s_{pp}\,{}^{8}),\quad 1 < q \leqq p < \infty\,; \tag{5.10a}$$

$$B^s_{pp} \subset F^s_{pq} \subset B^s_{pq}\,{}^{8}),\quad 1 < p \leqq q < \infty\,. \tag{5.10b}$$

5.2.4. *Proof of theorem* 5.2.3.

Step 1. (5.9) follows from theorem 4.2.6 and the definition of the spaces B^s_{pp} and F^s_{pp}.

Step 2. The right side of (5.10 a) and the left side of (5.10 b) follow from the monotony property of the spaces l_q.

Step 3. We prove the left side of (5.10 a). Let $\{\varphi_k\}$ be a system of functions of type 4.2.1. With the aid of the triangle inequality follows for $f \in B^s_{pq}$

$$\|f\|_{F^s_{pq}} \leqq c\|\{f * \varphi_j\}\|_{L_p(l^s_q)} = c\|\sum_{j=0}^{\infty} (2^{sj}|f * \varphi_j|)^q\|_{L_{\frac{p}{q}}}^{\frac{1}{q}}$$
$$\leqq c[\sum_{j=0}^{\infty} 2^{sjq}\||f * \varphi_j|^q\|_{L_{\frac{p}{q}}}]^{\frac{1}{q}} = c\|\{f * \varphi_j\}\|_{l^s_q(L_p)} \leqq c'\|f\|_{B^s_{pq}}\,.$$

We prove the right side of (5.10 b). Let be $f \in F^s_{pq}$. Then holds

$$\|f\|_{B^s_{pq}} \leqq c\|\{f * \varphi_j\}\|_{l^s_q(L_p)} = c\left\|\left\{\int_{R_n} 2^{sjp}|f * \varphi_j|^p dx\right\}\right\|_{l_{q/p}}^{\frac{1}{p}}$$
$$\leqq c\left[\int_{R_n} \|\{2^{sjp}|f * \varphi_j|^p\}\|_{l_{q/p}} dx\right]^{\frac{1}{p}} = c\|\{f * \varphi_j\}\|_{L_p(l^s_q)} \leqq c'\|f\|_{F^s_{pq}}$$

This proves the theorem.

5.2.5. *Remark*. From (5.9) and (5.10) follows for $-\infty < s < \infty$

$$B^s_{p2} \subset H^s_p \subset B^s_{pp},\quad 2 \leqq p < \infty\,; \tag{5.11a}$$

$$B^s_{pp} \subset H^s_p \subset B^s_{p2},\quad 1 < p \leqq 2\,. \tag{5.11b}$$

We have also

$$H^s_2 = B^s_{22} \tag{5.12}$$

The relations (5.11) and (5.12) are well-known. See for instance [14], [25, theorem 15] or [12].

6. *B*-space property. Density

Until now we have not shown the completeness of the spaces B^s_{pq} and F^s_{pq}. The reason is that we want to use the lifting property, theorem 5.1.1, and (5.6). This is not necessary, but convenient.

6.1. *The spaces* F^s_{pq}

6.1.1. *B-space property, density.*

THEOREM 6.1.1. *Let* $-\infty < s < \infty$; $1 < p, q < \infty$. F^s_{pq} *is a Banach space, and* $C_0^\infty(R_n)$ *is dense in it.*

6.1.2. *Proof of theorem* 6.1.1.

Step 1. Density. Let in the sense of definition 4.1.2

$$F^s_{pq} \ni f \underset{S'}{=} \sum_{k=0}^{\infty} a_k(x) .$$

Then $\|\{a_k\}\|^p_{l^s_q}$ is an integrable function with

$$\lim_{N\to\infty} \|\{a_k\}^N_{k=0}\|^p_{l^s_q} = \|\{a_k\}\|^p_{l^s_q} .$$

From the Lebesgue dominated convergence theorem follows

$$\lim_{N\to\infty} \int_{R_n} \|\{a_k\}^\infty_{k=N}\|^p_{l^s_q}\, dx = \int_{R_n} \lim_{N\to\infty} \|\{a_k\}^\infty_{k=N}\|^p_{l^s_q}\, dx = 0 .$$

This shows

$$\|f - f_N\|_{F^s_{pq}} \to 0 \quad \text{for} \quad N \to \infty, \; f_N = \sum_{k=0}^{N} a_k(x) .$$

$a_k(x) \in L_p(R_n)$ we approximate in L_p with functions in $C_0^\infty(R_n)$. From this follows the possibility of approximation of f_N with functions in $C_0^\infty(R_n)$ in F^s_{pq}. This proves the density property.

Step 2. B-space property. Theorem 5.1.1, especially (5.6), shows that we can restrict the considerations without loss of generality to the case $s > 0$. From the theorems 5.2.1 and 5.23 follows

$$F^s_{pq} \subset F^0_{pp} = L_p . \tag{6.1}$$

It is clear that F^s_{pq} is a linear normed space. We consider a sequence $\{f^{(j)}\}_{j=1}^\infty \subset F^s_{pq}$ with

$$\|f^{(j)}\|_{F^s_{pq}} \leq 2^{-j}, \; j = 1, 2, \ldots \tag{6.2}$$

We want to show the existence of an element $f \in F^s_{pq}$ with

$$\sum_{j=1}^{N} f^{(j)} \underset{F^s_{pq}}{\to} f \quad \text{for} \quad N \to \infty\,. \tag{6.3}$$

This is sufficient for the proof of the completeness. From (6.1) follows the existence of an element $f \in L_p$ with

$$\sum_{j=1}^{N} f^{(j)} \underset{L_p}{\to} f \quad \text{for} \quad N \to \infty\,. \tag{6.4}$$

We set in the sense of 4.1.2

$$f^{(j)} \underset{S'}{=} \sum_{k=0}^{\infty} a_k^{(j)}(x) \quad \text{with} \quad \|\{a_k^{(j)}\}_k\|_{L_p(l_q^s)} \leqq 2^{-j+1}\,. \tag{6.5}$$

Then holds

$$\sum_{j=1}^{N} a_k^{(j)}(x) \underset{L_p}{\to} a_k(x) \in L_p \quad (k = 0, 1, 2, \ldots)\,. \tag{6.6}$$

In particular we have

$$\sum_{j=1}^{N} a_k^{(j)}(x) \underset{S'}{\to} a_k(x), \quad \sum_{j=1}^{N} Fa_k^{(j)} \underset{S'}{\to} Fa_k \quad (k = 0, 1, 2, \ldots)\,.$$

The properties of the functions $a_k(x)$, $(k = 1, 2, \ldots)$, show that for

$$\varphi \in S(R_n), \quad \operatorname{supp} \varphi \cap \{\xi \,|\, 2^{k-1} \leqq |\xi| \leqq 2^{k+1}\} = \emptyset$$

we have

$$(Fa_k)(\varphi) = \lim_{N\to\infty} \sum_{j=1}^{N} (Fa_k^{(j)})(\varphi) = 0\,.$$

That means

$$\operatorname{supp} Fa_k \subset \{\xi \,|\, 2^{k-1} \leqq |\xi| \leqq 2^{k+1}\}\,. \tag{6.7}$$

Of course we have an analogous formula for $\operatorname{supp} Fa_0$. Now we consider for $N = 0, 1, 2, \ldots$; $K = 0, 1, 2, \ldots$; $K < L$; the estimate

$$\|\{a_k - \sum_{j=1}^{N} a_k^{(j)}\}_{k=K}^{L}\|_{L_p(l_q^s)} \leqq \|\{a_k - \sum_{j=1}^{M} a_k^{(j)}\}_{k=K}^{L}\|_{L_p(l_q^s)} + \|\{\sum_{j=N+1}^{M} a_k^{(j)}\}_{k=K}^{L}\|_{L_p(l_q^s)}\,, \tag{6.8}$$

$M > N$. For $N = 0$ the sum $\sum_{k=1}^{N}$ is zero. First we set $N = 0$. Then the considerations of the first step and (6.5) show that the second term on the right hand of (6.8) is smaller than $\varepsilon/2$ for $K \geqq K_0(\varepsilon)$ independently of M and L $(> K)$. For M sufficiently large the first term is also smaller than $\varepsilon/2$. The inclusion (6.1) leads us to

$$\|\sum_{k=K}^{L} a_k\|_{L_p} \leqq \varepsilon \quad \text{for} \quad L > K \geqq K_0(\varepsilon)\,.$$

That means

$$\sum_{k=0}^{L} a_k \xrightarrow[L_p]{} g \in L_p, \quad \sum_{k=0}^{L} Fa_k \xrightarrow[S']{} Fg . \tag{6.9}$$

We show $f = g$. If $\chi \in S$ with $F\chi \in C_0^\infty(R_u)$ then follows from (6.4), (6.6), (6.7), (6.9) for a suitable number L

$$F\chi \cdot Fg = \sum_{k=0}^{L} F\chi \cdot Fa_k \underset{S'}{=} \sum_{k=0}^{L} F\chi \cdot \sum_{j=1}^{\infty} Fa_k^{(j)} = \sum_{j=1}^{\infty} F\chi \cdot Ff^{(j)} \underset{S'}{=} F\chi \cdot Ff .$$

From this follows $f = g$ and

$$f \underset{S'}{=} \sum_{k=0}^{\infty} a_k . \tag{6.10}$$

Now we set $K = 0$ in (6.8). Then we get in the same manner as above for all L

$$\|\{a_k - \sum_{j=1}^{N} a_k^{(j)}\}_{k=0}^{L}\|_{L_p(l_q^s)} \leqq \varepsilon \quad \text{for} \quad N \geqq N_0(\varepsilon) .$$

For $L \to \infty$ we get with the aid of (6.10) and (6.7) $f \in F_{pq}^s$ and

$$\|f - \sum_{j=1}^{N} f^{(j)}\|_{F_{pq}^s} \to 0 \quad \text{for} \quad N \to \infty .$$

This proves the completeness.

6.1.3. *Remark.* We know that $H_p^s = F_{p2}^s$, theorem 4.2.6. The last theorem shows that $C_0^\infty(R_n)$ is a dense subset of H_p^s. The B-space property is clear without the last theorem, because L_p is a Banach space, and we have the lifting property 5.1.1.

6.2. *The spaces* B_{pq}^s

6.2.1. *B-space property, density.*

THEOREM 6.2.1. *Let* $-\infty < s < \infty$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. B_{pq}^s *is a Banach space. For* $q < \infty$ *is* $C_0^\infty(R_n)$ *a dense subset.*

6.2.2. *Proof of theorem* 6.2.1.

Step 1. Density. It is immediate to see that elements of the form

$$f = \sum_{k=0}^{N} a_k(x) \quad \text{(in the sense of 4.1.3)}$$

are dense in B_{pq}^s for $q < \infty$. Then follows the density property in the same way as in the first step of 6.1.2.

Step 2. The proof of the B-space property is the same as in the second step of 6.1.2.

6.3. *Remark*

It is well-known, and easy to see that

$$S \subset L_p \subset S' \,[9] \tag{6.11}$$

where the sign $\subset$ always means continuous embedding. The lifting operator I. from (5.1) is an one-to-one continuous mapping from S onto S, and from S' onto S'. Together with the lifting property 5.1.1 and (6.11) follows

$$S \subset F^s_{pq}, H^s_p, B^s_{pq} \subset S' \tag{6.12}$$

Here s, p, q have values for which the spaces are defined.

7. The dual of F^s_{pq} and B^s_{pq}

7.1. *The dual of* F^s_{pq}

7.1.1. *Embedding in* S'. (6.12) and theorem 6.1.1 show that S is a dense subset of F^s_{pq} for all values for which F^s_{pq} is defined. So we have by the usual interpretation the possibility to write

$$(F^s_{pq})' \subset S' . \tag{7.1}$$

In this sense we consider the dual space $(F^s_{pq})'$ of F^s_{pq}. If $g \in (F^s_{pq})'$ and $\varphi \in S$ we write $g(\varphi)$ (interpreting $g \in S'$). But if $f \in F^s_{pq}$ we write $\langle g, f \rangle$. For convenience we start with a few lemmas.

7.1.2. *Systems of functions*. We need two systems of functions. Let

$$\varphi \in S(R_n), \quad (F\varphi)(\xi) = 1 \quad \text{for} \quad 1/\sqrt{2} \leqq |\xi| \leqq \sqrt{2},$$
$$0 \leqq F\varphi \in C_0^\infty(\{\xi \mid 1/\sqrt{2} - \varepsilon < |\xi| < \sqrt{2} + \varepsilon\})$$

and

$$\varrho \in S(R_n), \quad 0 \leqq F\varrho \in C_0^\infty(\{\xi \mid 1/\sqrt{2} + \delta < |\xi| < \sqrt{2} - \delta\}) .$$

ε and δ are positive numbers. We construct $\{\varphi_k\}_{k=0}^\infty$ and $\{\varrho_k\}_{k=0}^\infty$ by

$$(F\varphi_k)(\xi) = (F\varphi)(2^{-k}\xi), \quad (F\varrho_k)(\xi) = (F\varrho)(2^{-k}\xi) . \tag{7.2}$$

We choose ε and δ sufficient small and so that

$$F\varrho_k \cdot F\varphi_l \equiv 0 \quad \text{for} \quad k \neq l . \tag{7.3}$$

[9]) We recall that we took the strong topology in S'.

7.1.3. *Lemma*

LEMMA 7.1.3. *Let* $g \in (F^s_{pq})'$ *and*

$$\tilde{g} \underset{S'}{=} \sum_{k=0}^{\infty} F^{-1}(F\varrho_k Fg) . \tag{7.4}$$

($\{\varrho_k\}$ *is the system of* 7.1.2*). Then* $\tilde{g} \in (F^s_{pq})'$ *and*

$$\|\tilde{g}\|_{(F^s_{pq})'} \leqq c\|g\|_{(F^s_{pq})'} , \tag{7.5}$$

where c *does not depend on* g.

7.1.4. *Proof of lemma* 7.1.3. Let $f \in S(R_n)$. Then holds

$$\tilde{g}(f) = (F\tilde{g})(F^{-1}f) \underset{S'}{=} \sum_{k=0}^{\infty} Fg(F\varrho_k F^{-1}f)$$

$$= (2\pi)^{-\frac{n}{2}} \sum_{k=0}^{\infty} g(\check{\varrho}_k * f) = (2\pi)^{-\frac{n}{2}} g\Big(\sum_{k=0}^{\infty} \check{\varrho}_k * \varphi_k * f\Big) \text{ [10]} .$$

We use $g \in (F^s_{pq})'$ and theorem 3.5 (b) with $K_{kk} = \check{\varrho}_k$. Then follows

$$|\tilde{g}(f)| \leqq \|g\| \, \|\{\check{\varrho}_k * \varphi_k * f\}\|_{L_p(l^s_q)} \leqq c\|g\| \, \|\{f * \varphi_k\}\|_{L_p(l^s_q)} ,$$

if $\{\varphi_k\}$ is a system of type 4.2.1 (without the function φ_0). We get from theorem 4.2.2

$$|\tilde{g}(f)| \leqq c\|g\| \cdot \|f\|_{F^s_{pq}} .$$

This completes the proof.

7.1.5. *Lemma.* We need another lemma.

LEMMA 7.1.5. *Let* $\{\varrho_k\}_{k=0}^{\infty}$ *and* $\{\varphi_k\}_{k=0}^{\infty}$ *be the systems of* 7.1.2, $g \in (F^s_{pq})'$, *and* $\tilde{g}$ *the functional of lemma* 7.1.3. *Let* $\{b_k\}_{k=0}^{N}$ *be a system of functions with* $b_k \in L_p(R_n)$. *We set*

$$a_k = F^{-1}(F\varrho_k \cdot Fg), \quad (k = 0, 1, \dots, N), \quad c_k = b_k * \varphi_k \tag{7.6}$$

and

$$f = \sum_{k=0}^{N} c_k \tag{7.7}$$

Then holds

$$\langle \tilde{g}, f \rangle = (2\pi)^{\frac{n}{2}} \int_{R_n} \sum_{k=0}^{N} a_k b_k \, dx . \tag{7.8}$$

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[10]) $\check{\varrho}(x) = \varrho(-x)$. We have $(F^2\varrho_k)(x) = \varrho_k(-x) = \check{\varrho}_k(x)$.

7.1.6. *Proof of Lemma* 7.1.5.

Step 1. We prove

$$a_k \in L_{p'}, \quad 1/p + 1/p' = 1 \,. \tag{7.9}$$

We repeat the consideration of 7.1.4 and find for $f \in S(R_n)$

$$|a_k(f)| \leqq c\|g\| \|\{\delta_{jk}\varphi_k * f\}_j\|_{L_p(l_q^s)} \leqq c'\|f\|_{L_p} \,.$$

This proves (7.9), and shows that the right side of (7.8) has a sense.

Step 2. First we assume $b_k \in S(R_n)$. Then holds also $c_k \in S(R_n)$ and $f \in S(R_n)$. We have (see (7.4))

$$\langle \tilde{g}, f \rangle = \tilde{g}(f) \underset{S'}{=} \sum_{k=0}^{\infty} a_k(f) \,,$$

and

$$a_k(f) = \sum_{l=0}^{N} a_k(c_l) = \sum_{l=0}^{N} Fa_k(F^{-1}c_l) = \sum_{l=0}^{N} (2\pi)^{\frac{n}{2}} (Fa_k \cdot F^{-1}\varphi_l)(F^{-1}b_l) \,.$$

$(F^{-1}h)(\xi) = (Fh)(-\xi)$, (7.3), and (7.6) show that

$$a_k(f) = (2\pi)^{\frac{n}{2}} Fa_k(F^{-1}b_k) = (2\pi)^{\frac{n}{2}} \int_{R_n} a_k b_k dx \,.$$

From this follows (7.8) for $b_k \in S(R_n)$.

Step 3. If $b_k \in L_p(R_n)$ we approximate

$$S(R_n) \ni b_{k,j} \underset{L_p}{\rightarrow} b_k$$

and set

$$c_{k,j} = b_{k,j} * \varphi_k, \quad f_j = \sum_{k=0}^{N} c_{k,j} \,.$$

(7.8) is true for f_j and $b_{k,j}$. We have

$$c_{k,j} \underset{L_p}{\rightarrow} c_k, \quad f_j \underset{F_{pq}^s}{\rightarrow} f \,.$$

The proof of (7.8) follows now from the last relations and (7.9).

7.1.7. *The dual of* F_{pq}^s.

THEOREM 7.1.7. *Let* $-\infty < s < \infty$; $1 < p, q < \infty$; $1/p + 1/p' = 1/q + 1/q' = 1$. *Then holds*

$$(F_{pq}^s)' = F_{p'q'}^{-s} \,.$$

7.1.8. *Proof of theorem* 7.1.7.

Step 1. Let $f \in S$, $g \in F^{-s}_{p'q'}$, and $\{\psi_k\}_{k=0}^{\infty}$ be a system of functions of type 4.2.1 with $\sum_{k=0}^{\infty} F\psi_k \equiv 1$. The existence of such a system follows from the beginning of the first step of 3.6. Then

$$f \underset{S}{=} \sum_{k=0}^{\infty} f * \psi_k, \quad f * \psi_k \in S, \quad g \underset{S'}{=} \sum_{k=0}^{\infty} g * \psi_k .$$

It follows from (4.9)

$$g(f) \underset{S'}{=} \sum_{k=0}^{\infty} (g * \psi_k)(f) \underset{S'}{=} \sum_{k=0}^{\infty} [\sum_{l=0}^{\infty} (g * \psi_k)(f * \psi_l)]$$
$$\underset{S'}{=} \sum_{k=0}^{\infty} [\sum_{r=-2N}^{2N} (g * \psi_k)(f * \psi_{k+r})] \quad (\psi_j = 0 \text{ for } j < 0) .$$

Using Hölder's inequality we get

$$|g(f)| \leqq c \int_{R_n} \|\{g * \psi_k\}\|_{l^{-s}_{q'}} \|\{f * \psi_k\}\|_{l^s_q} dx$$
$$\leqq c\|\{g * \psi_k\}\|_{L_p(l^{-s}_{q'})} \|\{f * \psi_k\}\| \leqq c'\|g\|_{F^{-s}_{p'q'}} \|f\|_{F^s_{pq}} .$$

S is dense in F^s_{pq}, theorem 6.1.1. Together with the last estimate this shows that

$$g \in (F^s_{pq})', \quad \|g\|_{(F^s_{pq})'} \leqq c\|g\|_{F^{-s}_{p'q'}} . \tag{7.10}$$

Step 2. Let be $g \in (F^s_{pq})'$. We assume that $\tilde{g}$, and the functions φ_k and ϱ_k have the same sense as in 7.1.2, lemma 7.1.3, and lemma 7.1.5. We show $\tilde{g} \in F^{-s}_{p'q'}$. For this purpose we construct the functions a_k, (7.6), and set

$$b_k = \operatorname{sgn} a_k \cdot |a_k|^{q'-1} 2^{-skq'} \|\{a_j\}_{j=0}^N\|^{p'-q'}_{l^{-s}_{q'}}, \quad (k = 0, 1, \ldots, N) . \tag{7.11}$$

(If $a_k = 0$ we set $b_k = 0$.) We have

$$\|\{b_k\}_0^N\|_{L_p(l^s_q)} = \left(\int_{R_n} \left[\sum_{k=0}^{N} 2^{ksq(1-q')} |a_k|^{q(q'-1)} \right]^{\frac{p}{q}} \|\{a_k\}_0^N\|^{p(p'-q')}_{l^{-s}_{q'}} dx \right)^{\frac{1}{p}} .$$

With

$$(q' - 1)q = q', \quad ksq(1 - q') = - ksq', \quad \frac{q'}{q} p + p(p' - q') = p' \tag{7.12}$$

follows

$$\|\{b_k\}_0^N\|_{L_p(l^s_q)} = \|\{a_k\}_0^N\|^{\frac{p'}{p}}_{L_{p'}(l^{-s}_{q'})} . \tag{7.13}$$

In particular $b_k \in L_p$. Now we construct the function f from (7.7). Using lemma 7.1.5 and (7.5) we find

$$\int_{R_n} \sum_{k=0}^{N} 2^{-skq'} |a_k|^{q'} \|\{a_j\}_0^N\|_{l_{q'}^{-s}}^{p'-q'} dx = \int_{R_n} \sum_{k=0}^{N} a_k b_k dx$$
$$= \langle \tilde{g}, f \rangle \leqq c \|g\|_{(F_{pq}^s)'} \|\{b_k * \varphi_k\}_0^N\|_{L_p(l_q^s)} .$$

We use again theorem 3.5 (b) with $\varphi_k = K_{kk}$. Then follows from the last estimate and (7.13)

$$\|\{a_k\}_0^N\|_{L_{p'}(l_{q'}^{-s})}^{p'} \leqq c \|g\|_{(F_{pq}^s)'} \|\{a_k\}_0^N\|_{L_{p'}(l_{q'}^{-s})}^{\frac{p'}{p}} ,$$

where c is independent of g and N. This shows

$$\|\{a_k\}_0^N\|_{L_{p'}(l_{q'}^{-s})} \leqq c \|g\|_{(F_{pq}^s)'} .$$

The last estimate, (7.4), and (7.6) lead us to

$$\tilde{g} \in F_{p'q'}^{-s}, \quad \|\tilde{g}\|_{F_{p'q'}^{-s}} \leqq c \|g\|_{(F_{pq}^s)'} . \tag{7.14}$$

Step 3. We specialize the function $\varrho(x)$ from 7.1.2 setting

$$(F\varrho)(\xi) = 1 \quad \text{for} \quad \frac{1}{\sqrt{2}} + 2\delta \leqq |\xi| \leqq \sqrt{2} - 2\delta .$$

(δ sufficiently small). Then we find a function $\tilde{\varrho}(x) \in S(R_n)$ with

$$0 \leqq F\tilde{\varrho} \in C_0^\infty(\{\xi | \sqrt{2} - 2\delta < |\xi| < \sqrt{2} + 5\delta\}) ,$$

$$(F\varrho + F\tilde{\varrho} + F\varrho_1)(\xi) = 1 \quad \text{for} \quad \frac{1}{\sqrt{2}} + 2\delta < |\xi| \leqq 2(\sqrt{2} - 2\delta) .$$

As in (7.2) we construct $\tilde{\varrho}_k(x)$, and as in (7.4) we put

$$\tilde{\tilde{g}} \underset{S'}{=} \sum_{k=0}^{\infty} F^{-1}(F\tilde{\varrho}_k F g) .$$

$\tilde{\tilde{g}}$ has similar properties, notably (7.14) holds. Changing the function $\tilde{\varrho}_0 = \tilde{\varrho}$ (which is not important for the considerations) we can obtain $g = \tilde{g} + \tilde{\tilde{g}}$. Then (7.14) shows

$$g \in F_{p'q'}^{-s}, \quad \|g\|_{F_{p'q'}^{-s}} \leqq c \|g\|_{(F_{pq}^s)'} . \tag{7.15}$$

(7.10) and (7.15) complete the proof.

7.1.9. *Remark.* From $H_p^s = F_{p2}^s$ follows

$$(H_p^s)' = H_{p'}^{-s}, \quad -\infty < s < \infty; \quad 1 < p < \infty; \quad 1/p + 1/p' = 1 . \tag{7.16}$$

7.2. *The dual of* B^s_{pq}

7.2.1. *Embedding in* S'. We consider the spaces B^s_{pq} with $-\infty < s < \infty$; $1 < p < \infty$; $1 < q < \infty$. Theorem 6.2.1 shows that S is a dense subset in B^s_{pq}. In the same manner as in 7.1.1 we interpret the dual of B^s_{pq} in the sense

$$(B^s_{pq})' \subset S' . \tag{7.17}$$

7.2.2. *The dual of* B^s_{pq}

THEOREM 7.2.2. *If* $-\infty < s < \infty$; $1 < p, q < \infty$; $1/p + 1/p' = 1/q + 1/q' = 1$, *then holds*

$$(B^s_{pq})' = B^{-s}_{p'q'} . \tag{7.18}$$

7.2.3. *Proof of theorem* 7.2.2. The proof follows the same line as the proof of theorem 7.1.7.

Step 1. $f \in S$, $g \in B^{-s}_{p'q'}$. In the same manner as in the first step of 7.1.8 we find

$$|g(f)| \leqq c' \|g\|_{B^{-s}_{p'q'}} \|f\|_{B^s_{pq}}$$

and

$$g \in (B^s_{pq})', \quad \|g\|_{(B^s_{pq})'} \leqq c \|g\|_{B^{-s}_{p'q'}} . \tag{7.19}$$

Step 2. We start with $g \in (B^s_{pq})'$ and construct $\tilde{g}$, (7.4). The proof of lemma 7.1.3. shows

$$\tilde{g} \in (B^s_{pq})', \quad \|\tilde{g}\|_{(B^s_{pq})'} \leqq c \|g\|_{(B^s_{pq})'} .$$

Here we use the scalar case of theorem 3.5. After constructing a_k in the sense of (7.6) we set

$$b_k = \operatorname{sgn} a_k |a_k|^{p'-1} 2^{-skq'} \|a_k\|^{q'-p'}_{L_{p'}}, \quad (k = 0, 1, 2, \ldots) .$$

We remark that $a_k \in L_{p'}$. This follows from

$$(B^s_{pq})' \subset (F^{s+\varepsilon}_{pq})' = F^{-s-\varepsilon}_{p'q'}, \quad \varepsilon > 0$$

(see theorem 5.2.1, 5.2.3, 7.1.7). The formula analogous to (7.13) is

$$\|\{b_k\}_0^N\|_{l^s_q(L_p)} = \|\{a_k\}_0^N\|^{\frac{q'}{q}}_{l^{-s}_{q'}(L_{p'})} . \tag{7.20}$$

With the aid of the function f, (7.7), we find in the same manner as in the second step of 7.1.8

$$\|\{a_k\}_0^N\|_{l^{-s}_{q'}(L_{p'})} \leqq c \|g\|_{(B^s_{pq})'} .$$

From this follows as in the first step of 7.1.8

$$g \in B^{-s}_{p'q'}, \quad \|g\|_{B^{-s}_{p'q'}} \leqq c\|g\|_{(B^s_{pq})'} .$$

This completes the proof.

7.2.4. *Remark.* Theorem 7.2.2 is well-known, see Taibleson [26, theorem 5].

7.2.5. *Remark.* The theorems 7.1.7 and 7.2.2 show that the spaces $F^s_{pq}, H^s_p, B^s_{pq}$ $(-\infty < s < \infty;\ 1 < p, q < \infty)$ are reflexive.

8. Interpolation of the spaces B^s_{pq} and F^s_{pq}, real method

In this part we use the general interpolation theory, described in 2.2.

8.1. *Interpolation of the spaces* B^s_{pq}

8.1.1. *Lemma.* We consider a system of functions $\{\varphi_k\}_{k=0}^{\infty}$ of type 4.2.1. For simplicity we suppose $N = 1$. We use theorem 4.2.2 (b). K is the functional defined in 2.2.1.

LEMMA 8.1.1. *Let* $-\infty < s_0, s_1 < \infty;\ s_0 \neq s_1;\ 1 < p < \infty;\ 1 \leqq r < \infty$. *Then holds*

$$K^r(t, f, B^{s_0}_{pr}, B^{s_1}_{pr}) \sim \sum_{k=0}^{\infty} \min\,(2^{ks_0r}, t^r 2^{ks_1r})\|f * \varphi_k\|^r_{L_p}, \quad f \in B^{\min(s_0, s_1)}_{pr} . \tag{8.1}$$

8.1.2. *Proof of lemma* 8.1.1. Theorem 5.2.1 shows

$$B^{\min(s_0, s_1)}_{pr} = B^{s_0}_{pr} + B^{s_1}_{pr} . \tag{8.2}$$

Step 1. For $f \in B^{\min(s_0, s_1)}_{pr}$ follows

$$\begin{aligned} K^r(t, f, B^{s_0}_{pr}, B^{s_1}_{pr}) &\sim \inf_{\substack{f=f_0+f_1\\ f_i \in B^{s_i}_{pr}}} \sum_{k=0}^{\infty} (2^{ks_0r}\|f_0 * \varphi_k\|^r_{L_p} + t^r 2^{ks_1r}\|f_1 * \varphi_k\|^r_{L_p}) \\ &\geqq \sum_{k=0}^{\infty} \inf_{\substack{g_k+h_k=f*\varphi_k\\ g_k, h_k \in L_p}} (2^{ks_0r}\|g_k\|^r_{L_p} + t^r 2^{ks_1r}\|h_k\|^r_{L_p}) \sim \sum_{k=0}^{\infty} \min\,(2^{ks_0r}, 2^{ks_1r}t^r)\|f * \varphi_k\|^r_{L_p} . \end{aligned} \tag{8.3}$$

Step 2. We assume $s_0 > s_1$. For a given $t \geqq 1$ we choose the integer $k(t)$ by

$$2^{k(t)(s_0-s_1)} \leqq t < 2^{[k(t)+1](s_0-s_1)} .$$

Further we choose a function $\psi_t(x) \in S(R_n)$ with

$$0 \leqq F\psi_t \leqq 1, \quad F\psi_t \in C_0^\infty(\{\xi \,|\, |\xi| < 2^{k(t)+1}\}), \quad (F\psi_t)(\xi) = 1 \quad \text{for} \quad |\xi| \leqq 2^{k(t)+1/2}. \tag{8.4}$$

With the aid of this function ψ we consider the special decomposition of a given function $f \in B^{s_0}_{pr} + B^{s_1}_{pr} = B^{s_1}_{pr}$

$$f = f_0 + f_1, \quad Ff_0 = F\psi_t \cdot Ff, \quad Ff_1 = (1 - F\psi_t)Ff. \tag{8.5}$$

It is clear that $f_0 \in B^{s_0}_{pr}$, $f_1 \in B^{s_1}_{pr}$. Using the scalar case of theorem 3.5, and the properties of the system of functions $\{\varphi_k\}$ we find for $t \geqq 1$

$$\begin{aligned} K^r(t, f, B^{s_0}_{pr}, B^{s_1}_{pr}) &\leqq c(\|f_0\|^r_{B^{s_0}_{pr}} + t^r\|f_1\|^r_{B^{s_1}_{pr}}) \\ &\leqq c' \sum_{k=0}^{k(t)+1} 2^{s_0kr}\|f_0 * \varphi_k\|^r_{L_p} + c' \sum_{k=k(t)}^{\infty} 2^{s_1kr}t^r\|f_1 * \varphi_k\|^r_{L_p} \qquad (8.6a) \\ &\leqq c'' \sum_{k=0}^{\infty} \min(2^{s_0kr}, t^r 2^{s_1kr})\|f * \varphi_k\|^r_{L_p}. \end{aligned}$$

We choose the function ψ_t in such a way that by applying theorem 3.5 the constant c'' is independent of t. For $0 < t \leqq 1$ we have

$$K^r(t, f, B^{s_0}_{pr}, B^{s_1}_{pr}) \leqq t^r\|f\|^r_{B^{s_1}_{pr}}. \tag{8.6b}$$

(8.3) and (8.6) prove the lemma.

Step 3. If $s_0 < s_1$ we have to change only the cases $t \geqq 1$ and $t \leqq 1$ in the second step.

8.1.3. *Interpolation theorem.*

THEOREM 8.1.3. *Let* $-\infty < s_0, s_1 < \infty$; $s_0 \neq s_1$; $1 < p < \infty$; $1 \leqq q_0, q_1, q \leqq \infty$. *Then holds for* $0 < \theta < 1$

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta, q} = B^s_{pq} \quad \textit{with} \quad s = (1-\theta)s_0 + \theta s_1.$$

8.1.4. *Proof of theorem* 8.1.3.

Step 1. First we show for $1 \leqq r < \infty$

$$(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta, r} = B^s_{pr}. \tag{8.7}$$

From lemma 8.1.1 follows

$$\begin{aligned} \|f\|^r_{(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta, r}} &\sim \int_0^\infty t^{-\theta r - 1} \sum_{k=0}^{\infty} \min(2^{ks_0r}, 2^{ks_1r}t^r)\|f * \varphi_k\|^r_{L_p} dt \\ &= \sum_{k=0}^{\infty} \|f * \varphi_k\|^r_{L_p} \left(\int_{t < 2^{k(s_0 - s_1)}} 2^{ks_1r} t^{r(1-\theta)-1} dt + \int_{t \geqq 2^{k(s_0 - s_1)}} 2^{ks_0r} t^{-\theta r - 1} dt \right) \\ &= c \sum_{k=0}^{\infty} \|f * \varphi_k\|^r_{L_p} 2^{krs} \sim \|f\|^r_{B^s_{pq}}. \end{aligned}$$

This proves (8.7).

Step 2. We show for $1 \leqq r < \infty$

$$(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta,\infty} = B^s_{p\infty} . \tag{8.8}$$

We have

$$\|f\|^r_{(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta,\infty}} \sim \sup_{t>0} \Big(\sum_{k=0}^{\infty} \min (2^{ks_0 r}, 2^{ks_1 r} t^r) \|f * \varphi_k\|^r_{L_p} \Big) t^{-\theta r} .$$

Choosing the sequence $\{t_l = 2^{l(s_0-s_1)}\}_{l=0}^{\infty}$ we find

$$\|f\|^r_{(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta,\infty}} \geqq c \sup_l 2^{lr[s_0-\theta(s_0-s_1)]} \|f * \varphi_l\|^r_{L_p} \sim \|f\|^r_{B^s_{p\infty}} . \tag{8.9}$$

On the other hand we have

$$\|f\|^r_{B(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta,\infty}} \leqq c \sup_{t>0} \Big(\sum_{2^{k(s_0-s_1)} \leq t} 2^{kr(s_0-s)} t^{-\theta r} + \sum_{2^{k(s_0-s_1)} > t} 2^{kr(s_1-s)} t^{r(1-\theta)} \Big) \|f\|^r_{B^r_{p\infty}} . \tag{8.10}$$

With

$$s_0 - s = \theta(s_0 - s_1), \quad s_1 - s = -(1-\theta)(s_0 - s_1) ,$$

we estimate the expression in the brackets by

$$\sum_{2^{k(s_0-s_1)} t^{-1} \leq 1} (2^{k(s_0-s_1)} t^{-1})^{\theta r} + \sum_{t 2^{-k(s_0-s_1)} \leq 1} (2^{-k(s_0-s_1)} t)^{r(1-\theta)} \leqq \sum_{k=0}^{\infty} (2^{-k|s_0-s_1|\theta r} + 2^{-k|s_0-s_1|(1-\theta) r})$$

independent of t. Then (8.10) shows

$$\|f\|_{(B^{s_0}_{pr}, B^{s_1}_{pr})_{\theta,\infty}} \leqq c \|f\|_{B^s_{p\infty}} . \tag{8.11}$$

Now (8.8) follows from (8.9) and (8.11).

Step 3. We have, (8.7) and (8.8) with $r = 1$,

$$(B^{s_0}_{p1}, B^{s_1}_{p1})_{\theta,1} = B^s_{p1}, \quad (B^{s_0}_{p1}, B^{s_1}_{p1})_{\theta,\infty} = B^s_{p\infty} . \tag{8.12}$$

With the aid of theorem 5.2.1 (a) and (2.9) follows

$$B^s_{pr} \in K(\theta, B^{s_0}_{p1}, B^{s_1}_{p1}); \quad 1 \leqq r \leqq \infty . \tag{8.13}$$

In the sense of (2.11) we find for $1 \leqq q < \infty$ using (8.7)

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,q} = (B^{s_0}_{pq}, B^{s_1}_{pq})_{\theta,q} = B^s_{pq} . \tag{8.14}$$

If $q = \infty$ we use (8.12),

$$(B^{s_0}_{pq_0}, B^{s_1}_{pq_1})_{\theta,\infty} = (B^{s_0}_{p1}, B^{s_1}_{p1})_{\theta,\infty} = B^s_{p\infty} .$$

This proves the theorem.

8.1.5. *Remark.* Theorem 8.1.3 is well-known [18], [7], [10].

8.2. *Interpolation of the sequence spaces* l_p^σ

In theorem 8.1.3 we considered the interpolation of $B^{s_0}_{pq_0}$ and $B^{s_1}_{pq_1}$ with the same p. In 8.3 we interpolate the sapces F^s_{pq} and B^s_{pq} with different values of p. For this purpose we need an interpolation result about the sequence spaces l_p^σ from 4.1.1 which is given without proof by Peetre [17].

8.2.1. *Lemma*

LEMMA 8.2.1. *Let* $-\infty < \sigma_0, \sigma_1 < \infty$; $\sigma_0 \neq \sigma_1$; $1 < p, q < \infty$. *Then holds for* $0 < \theta < 1$

$$(l_q^{\sigma_0}, l_q^{\sigma_1})_{\theta, p} = l_p^\sigma \quad \textit{with} \quad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1 \,. \tag{8.15}$$

8.2.2. *Proof of lemma* 8.2.1.

Step 1. For $\xi = (\xi_j)_{j=0}^\infty \in l_q^{\min(\sigma_0, \sigma_1)}$ we have

$$K^q(t, \xi, l_q^{\sigma_0}, l_q^{\sigma_1}) \sim \inf_{\substack{\xi=\xi^0+\xi^1 \\ \xi^i \in l_q^{\sigma_i}}} \Big(\sum_{j=0}^\infty 2^{\sigma_0 jq}|\xi_j^0|^q + t^q \sum_{j=0}^\infty 2^{\sigma_1 jq}|\xi_j^1|^q\Big) = \sum_{j=0}^\infty \min\,(2^{\sigma_0 jq}, t^q 2^{\sigma_1 jq})|\xi_j|^q \tag{8.16}$$

Step 2. Because $\sigma_0 - \sigma_1 \neq 0$ we can divide the interval $(0, \infty)$ in parts $2^{(k-1)|\sigma_0-\sigma_1|} \leqq t < 2^{k|\sigma_0-\sigma_1|}$; $-\infty < k < \infty$. We find for $\xi = (\xi_j)_{j=0}^\infty$ (and $\xi_j = 0$ for $j < 0$)

$$\begin{aligned}\|\xi\|^p_{(l_q^{\sigma_0}, l_q^{\sigma_1})_{\theta,p}} &= \int_0^\infty t^{-\theta p} K^p(t, \xi) \,\frac{dt}{t} \\ &\sim \sum_{k=-\infty}^\infty 2^{-\theta pk(\sigma_0-\sigma_1)}\Big[\sum_{j=-\infty}^\infty \min\,(2^{\sigma_0 jq}, 2^{\sigma_1 jq + k(\sigma_0-\sigma_1)q})|\xi_j|^q\Big]^{\frac{p}{q}} \\ &\geqq c \sum_{k=-\infty}^\infty 2^{pk[-\theta(\sigma_0-\sigma_1)+\sigma_0]}|\xi_k|^p = c\|\xi\|^p_{l_p^\sigma}\,.\end{aligned}$$

We estimated the sum over j by the term with $j = k$. This shows

$$(l_q^{\sigma_0}, l_q^{\sigma_1})_{\theta, p} \subset l_p^\sigma\,. \tag{8.17}$$

Step 3. With the usual interpretation we have for the dual of l_q^σ

$$(l_q^\sigma)' = l_{q'}^{-\sigma}, \quad 1/q + 1/q' = 1\,. \tag{8.18}$$

Using (2.12) and (8.17) we find

$$l_p^\sigma = (l_{p'}^{-\sigma})' \subset [(l_{q'}^{-\sigma_0}, l_{q'}^{-\sigma_1})_{\theta, p'}]' = (l_q^{\sigma_0}, l_q^{\sigma_1})_{\theta, p}\,. \tag{8.19}$$

The lemma follows from (8.17) and (8.19).

8.2.3. *Theorem.*

THEOREM 8.2.3. *Let* $-\infty < \sigma_0, \sigma_1 < \infty$; $\sigma_0 \neq \sigma_1$; $1 < p, q_0, q_1 < \infty$. *Then holds for* $0 < \theta < 1$

$$(l_{q_0}^{\sigma_0}, l_{q_1}^{\sigma_1})_{\theta, p} = l_p^\sigma \quad \textit{with} \quad \sigma = (1-\theta)\sigma_0 + \theta\sigma_1 . \tag{8.20}$$

8.2.4. *Proof of theorem* 8.2.3. Let $q_0 \leqq q_1$. Then holds

$$l_p^\sigma = (l_{q_0}^{\sigma_0}, l_{q_0}^{\sigma_1})_{\theta, p} \subset (l_{q_0}^{\sigma_0}, l_{q_1}^{\sigma_1})_{\theta, p} \subset (l_{q_1}^{\sigma_0}, l_{q_1}^{\sigma_1'})_{\theta, p} = l_p^\sigma .$$

This proves the theorem.

8.2.5. *Remark.* Sequence spaces of type l_p with weights show an interesting behaviour by interpolation. For the simplest case, the ordinary l_p-spaces, holds for $0 < \theta < 1$; $1 \leqq p, q_0, q_1 \leqq \infty$; $q_0 \neq q_1$,

$$(l_{q_0}, l_{q_1})_{\theta, p} = l_{q, p}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} . \tag{8.21}$$

$l_{q, p}$ are the »Lorentz sequence spaces», defined by

$$\begin{aligned} l_{qp} &= \{\xi \mid \xi = (\xi_j)_{j=0}^\infty, \ \|\xi\|_{lqp} = [\sum_{j=0}^\infty |\xi_j^*|^p j^{\frac{p}{q}-1}]^{\frac{1}{p}} < \infty\}, \ 1 \leqq p < \infty; \ 1 < q < \infty ; \\ l_{q\infty} &= \{\xi \mid \xi = (\xi_j)_{j=0}^\infty, \ \|\xi\|_{lq\infty} = \sup_j |\xi_j^*| j^{\frac{1}{q}} < \infty\}, \ 1 < q < \infty . \end{aligned} \tag{8.22}$$

$(\xi_j^*)_{j=0}^\infty$ is the rearrangement sequence, $\xi_j^* = \xi_{k(j)}$, $|\xi_0^*| \geqq |\xi_1^*| \geqq |\xi_2^*| \geqq \ldots$ Here $\|\xi\|_{lqp}$ is only a quasinorm [27]. With the aid of the reiteration theorem we formulate interpolation theorems for the l_{qp} spaces. For $0 < \theta < 1$; $1 < q_0, q_1 < \infty$; $1 \leqq p_0, p_1 \leqq \infty$, holds

$$(l_{q_0 p_0}, l_{q_1 p_1})_{\theta, p} = l_{qp}, \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} . \tag{8.23}$$

(8.20) and (8.23) show similar behaviour. It would be interesting to give a systematic treatment of interpolation of general sequence spaces with weights.

8.2.6. *Remark.* The real interpolation methods in particular the L-method [19] are applicable also for quasi-Banach spaces. So we can extend the parameters p, q for the spaces l_q^σ and l_{qp} to the interval $(0, \infty]$.

8.3. *Interpolation of the spaces* F_{pq}^s

With the aid of 8.2 we are able to interpolate the spaces F_{pq}^s.

8.3.1. *Lemma.*

LEMMA 8.3.1. *Let* $-\infty < s_0, s_1 < \infty$; $1 < p_0, p_1, q_0, q_1 < \infty$; $0 < \theta < 1$. *If* $\{\varphi_k\}_{k=0}^{\infty}$ *is a system of functions of type* 4.2.1 *then there exists a positive number* c *with*

$$\|f\|^p_{(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p}} \geqq c \int_{R_n} \|\{f * \varphi_k\}\|^p_{(l^{s_0}_{q_0}, l^{s_1}_{q_0})_{\theta,p}} dx \,,$$
$$f \in (F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p}, \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \,. \tag{8.24}$$

8.3.2. *Proof of lemma* 8.3.1. We use the functional L (2.17) with $A_0 = F^{s_0}_{p_0q_0}$ and $A_1 = F^{s_1}_{p_1q_1}$. With the aid of (2.18) and theorem 4.2.2 we get

$$\|f\|^p_{(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p}} \geqq c \int_0^\infty t^{-\eta-1} \left[\int_{R_n} \inf_{\substack{a_{k,0}(x)+a_{k,1}(x) \\ =(f*\varphi_k)(x)}} [\|\{a_{k,0}\}\|^{p_0}_{l^{s_0}_{q_0}} + t\|\{a_{k,1}\}\|^{p_1}_{l^{s_1}_{q_1}}] dx \right] dt \,.$$

Changing the order of integration and using again (2.18) we find (8.24). This proves the lemma.

8.3.3. *Theorem.*

THEOREM 8.3.3. *Let* $-\infty < s_0, s_1 < \infty$; $1 < p_0, p_1, q_0, q_1 < \infty$; $0 < \theta < 1$.
(a) *For* $s_0 \neq s_1$, $s = (1-\theta)s_0 + \theta s_1$; $1/p = (1-\theta)/p_0 + \theta/p_1$; *holds*

$$(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p} = B^s_{pp} \,. \tag{8.25}$$

(b) *For* $s_0 = s_1 = s$; $q_0 \neq q_1$; $(1-\theta)/p_0 + \theta/p_1 = 1/p = (1-\theta)/q_0 + \theta/q_1$ *holds*

$$(F^{s}_{p_0q_0}, F^{s}_{p_1q_1})_{\theta,p} = B^s_{pp} \,. \tag{8.26}$$

(c) *For* $s_0 = s_1 = s$; $q_0 = q_1 = q$; $1/p = (1-\theta)/p_0 + \theta/p_1$ *holds*

$$(F^{s}_{p_0q}, F^{s}_{p_1q})_{\theta,p} = F^s_{pq} \,. \tag{8.27}$$

8.3.4. *Proof of theorem* 8.3.3.

Step 1. Proof of (a). Let $f \in (F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p}$. Then follows from lemma 8.3.1 and theorem 8.2.3.

$$\|f\|^p_{(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p}} \geqq c \int_{R_n} \|\{f * \varphi_k\}\|^p_{l^s_p} dx \sim \|f\|^p_{B^s_{pp}} \,.$$

This shows

$$(F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta,p} \subset B^s_{pp} \,. \tag{8.28}$$

Now the theorems 6.1.1 and 6.2.1 show that $C_0^\infty(R_n)$ is a dense subset in the considered spaces. So we are able to apply (2.12). We find from (8.28) and the theorems 7.1.7 and 7.2.2

$$B^s_{pp} = (B^{-s}_{p'p'})' \supset [(F^{-s_0}_{p_0'q_0'}, F^{-s_1}_{p_1'q_1'})_{\theta, p'}]' = (F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1})_{\theta, p} .$$

Together with (8.28) (a) follows.

Step 2. Proof of (b). We have

$$(l^s_{q_0}, l^s_{q_1})_{\theta, p} = l^s_p \text{ [11]} . \tag{8.29}$$

Using again lemma 8.3.1 we get (8.28). A duality argument shows that (8.26) is true.

Step 3. Proof of (c). We have

$$(l^s_q, l^s_q)_{\theta, p} = l^s_q .$$

We find then the relation (8.27) in the same manner as in the first and second step. This proves the theorem.

8.4. *Special cases of theorem* 8.3.3

8.4.1. Let $-\infty < s_0, s_1 < \infty$; $1 < p_0, p_1 < \infty$; $0 < \theta < 1$; and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1 :$$

Then holds

$$(B^{s_0}_{p_0p_0}, B^{s_1}_{p_1p_1})_{\theta, p} = B^s_{pp} . \tag{8.30}$$

This follows form $B^s_{pp} = F^s_{pp}$, and theorem 8.3.3 (a), (b). (8.30) due to Grisvard [7], see also Peetre [17].

8.4.2. Let $-\infty < s_0, s_1 < \infty$; $s_0 \neq s_1$; $1 < p_0, p_1 < \infty$; $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad s = (1-\theta)s_0 + \theta s_1 .$$

---

[11]) See (8.23). (8.29) is well known [10]. We give a short proof using the L-method [19] (see (2.18)). We may set $s = 0$. For $\xi = (\xi_j)_{j=0}^\infty$ holds

$$\|\xi\|^p_{(l_{q_0}, l_{q_1})} \sim \sum_{k=0}^\infty \int_0^\infty t^{-\eta} \inf_{\xi_k^0 + \xi_k^1 = \xi_k} [|\xi_k^0|^{q_0} + t|\xi_k^1|^{q_1}] \frac{dt}{t}$$

$$\leq \sum_{k=0}^\infty \int_0^\infty t^{-\eta-1} \min (|\xi_k|^{q_0}, t|\xi_k|^{q_1}) dt = c \sum_{k=0}^\infty |\xi_k|^p = c\|\xi\|^p_{l_p} .$$

Using a duality argument we get (8.29).

Then holds

$$(H^{s_0}_{p_0}, H^{s_1}_{p_1})_{\theta, p} = B^s_{pp} \, . \tag{8.31}$$

This follows form $H^s_p = F_{p2}$, theorem 4.2.6, and theorem 8.3.3 (a). This result is stated by Peetre [17] without proof. For the special case $p_0 = p_1 = p$ (8.31) follows also from 5.2.5, theorem 8.1.3, and the reiteration theorem 2.2.3.

8.4.3. Let $-\infty < s < \infty$; $1 < p_0, p_1 < \infty$; $0 < \theta < 1$; and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \, .$$

Then holds

$$(H^s_{p_0}, H^s_{p_1})_{\theta, p} = H^s_p \, . \tag{8.32}$$

This follows from theorem 8.3.3 (c) with $q_0 = q_1 = 2$. (We used again $H^s_p = F^s_{p2}$). We remark that (8.32) follows immediately from the lifting theorem 5.1.1, $H^0_p = L_p$, and $(L_{p_0}, L_{p_1})_{\theta, p} = L_p$.

8.4.4. Let $-\infty < s_0, s_1 < \infty$; $s_0 \neq s_1$; $1 < p_0, p_1 < \infty$; $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \, ; \quad s = (1-\theta)s_0 + \theta s_1 \, .$$

Then holds

$$(H^{s_0}_{p_0}, B^{s_1}_{p_1 p_1})_{\theta, p} = B^s_{pp} \, . \tag{8.33}$$

This follows form theorem 8.3.3 (a), and $H^{s_0}_{p_0} = F^{s_0}_{p_0 2}$, $B^{s_1}_{p_1 p_1} = F^{s_1}_{p_1 p_1}$.

8.4.5. Let $-\infty < s < \infty$; $1 < p_1 < \infty$; $0 < \theta < 1$, and

$$\frac{1}{p} = \frac{1-\theta}{2} + \frac{\theta}{p_1} \, .$$

Then holds

$$(H^s_2, B^s_{p_1 p_1})_{\theta, p} = B^s_{pp} \, . \tag{8.34}$$

This is a special case of (8.30), $B^s_{22} = F^s_{22} = H^s_2$.

9. Equivalent norms

For applications of the spaces B^s_{pq} to differential equations some equivalent norms are convenient. Using the general interpolation theory, 2.2.5 and 2.2.6, and theorem 8.1.3 it is not difficult to give suitable equivalent norms.

9.1. *The spaces* H_p^s

9.1.1. *Theorem.*

THEOREM 9.1.1. *Let* $1 < p < \infty$; $s \geqq 0$, *integer. Then holds*

$$H_p^s = \left\{f | f \in S',\ \frac{\partial^s f}{\partial x_j^s} \in L_p,\ j = 1, \ldots, n\right\} = \{f | f \in S',\ D^\alpha f \in L_p,\ |\alpha| \leqq s\}, \tag{9.1}$$

and

$$\|f\|_{H_p^s} \sim \left(\|f\|_{L_p}^p + \sum_{j=1}^n \left\|\frac{\partial^s f}{\partial x_j^s}\right\|_{L_p}^p\right)^{\frac{1}{p}} \sim \Big(\sum_{|\alpha| \leqq s} \|D^\alpha f\|_{L_p}^p\Big)^{\frac{1}{p}}. \tag{9.2}$$

9.1.2. *Proof of theorem* 9.1.1.

Step 1. We start with a remark to formula (3.3′). Let $g \in S'(R_n)$, $Fg \in C^\infty(R_n)$, and

$$|(D^\beta Fg)(\xi)| \leqq \frac{c^\beta}{|\xi|^{|\beta|}} \tag{9.3}$$

for all β. Then (3.2) is true (this is a special case of theorem 3.5), and we have, see (3.3′)

$$\|F^{-1}(Ff \cdot Fg)\|_{L_p} \leqq c\|f\|_{L_p}. \tag{9.4}$$

$f \in S(R_n)$. If $f \in L_p$, we approximate $S \ni f_j \underset{L_p}{\rightarrow} f$. The assumption (9.3) shows tha $\{F^{-1}(Ff_j \cdot Fg)\}_j$ is a Cauchy sequence in L_p, and

$$F^{-1}(Ff_j \cdot Fg) \underset{S'}{\rightarrow} F^{-1}(Fg \cdot Ff).$$

For $j \to \infty$ follows (9.4) for arbitrary $f \in L_p$ (we replace $Ff \cdot Fg$ by $Fg \cdot Ff$ in the sense of multiplication of $Ff \in S'$ with Fg).

Step 2. We set $\xi^\alpha = \xi_1^{\alpha_1} \ldots \xi_\alpha^{\alpha_n}$, $\alpha = (\alpha_1, \ldots, \alpha_n)$. For $g_\alpha \in S'$

$$Fg_\alpha = \xi^\alpha (1 + |\xi|^2)^{-\frac{s}{2}},\quad |\alpha| \leqq s,$$

the assumption (9.3) is true. From (9.4) and the definition (4.8) follows

$$\|D^\alpha f\|_{L_p} = c\|F^{-1}\xi^\alpha Ff\|_{L_p} \leqq c'\|F^{-1}(1 + |\xi|^2)^{\frac{s}{2}} Ff\|_{L_p} = c'\|f\|_{H_p^s},\ f \in H_p^s. \tag{9.5}$$

Step 3. We denote the Fourier transformation in R_n with F, and in R_1 with F_1. Further we consider a function $\varrho(t) \in C^\infty(R_1)$,

$$\varrho(t) = 0 \text{ for } |t| \leqq \tfrac{1}{2},\ \varrho(t) = 1 \text{ for } t \geqq 1;\ \varrho(t) = -1 \text{ for } t \leqq -1.$$

For $g \in S'$,

$$Fg = (1 + |\xi|^2)^{\frac{s}{2}}(1 + \sum_{j=1}^{n} \varrho^s(\xi_j)\xi_j^s)^{-1} \text{ [12]}) ,$$

the assumption (9.3) is true. For a function $f \in L_p$, $\partial^s f/\partial x_j^s \in L_p$, $(j = 1, 2, \ldots, n)$ follows from (9.4)

$$\|f\|_{H_p^s} \leqq c\|F^{-1}(1 + \sum_{j=1}^{n} \varrho^s(\xi_j)\xi_j^s)Ff\|_{L_p} \leqq \|f\|_{L_p} + \sum_{j=1}^{n} \| \|F_1^{-1}\varrho^s(\xi_j)\xi_j^s F_1 f\|_{L_p(R_1)}\|_{L_p(R_{n-1})} .$$

For the function $\varrho^s(t)$ is the one dimensional case of (9.3) true. We apply (9.4). In view of $F_1^{-1}\xi_j^s F_1 f = c\partial^s f/\partial x_j^s$ it follows that

$$\|f\|_{H_p^s} \leqq c \left(\|f\|_{L_p}^p + \sum_{j=1}^{n} \left\| \frac{\partial^s f}{\partial x_j^s} \right\|_{L_p}^p \right)^{\frac{1}{p}} . \tag{9.6}$$

(9.5) and (9.6) prove the theorem.

9.1.3. *Remark.* $H_p^s = W_p^s$, $s \geqq 0$, s an integer, are the Sobolev spaces, introduced by Sobolev [23, 24].

9.2. *The spaces* B_{pq}^s

9.2.1. *Semi-groups in* L_p. We consider the strong continuous semi-groups of isometric operators in L_p

$$[G_j(t)f](x) = f(x_1, \ldots, x_{j-1}, x_j + t, x_{j+1}, \ldots, x_n), \; f \in L_p , \tag{9.7}$$

$0 \leqq t < \infty$; $j = 1, 2, \ldots, n$. For these groups the assumptions of 2.2.5 and 2.2.6 are true. It is easy to prove that the infinitesimal generators A_j are

$$(A_j f)(x) = \frac{\partial f}{\partial x_j}(x), \; D(A_j) = \left\{ f | f \in S', \; f \in L_p, \; \frac{\partial f}{\partial x_j} \in L_p \right\} . \tag{9.8}$$

Iteration shows

$$D(A_j^k) = \left\{ f | f \in S', \; \frac{\partial^l f}{\partial x_j^l} \in L_p; \; l = 0, 1, \ldots, k \right\} . \tag{9.9}$$

9.2.2. *Theorem.* We write

$$(\Delta_{h,k} f)(x) = f(x_1, \ldots, x_{k-1}, \; x_k + h, \; x_{k+1}, \ldots, x_n) - f(x), \; \Delta_{h,k}^2 f = \Delta_{h,k}(\Delta_{h,k} f) .$$

THEOREM 9.2.2. *Let* $0 < s < \infty$ *with the decomposition* $s = j + \varkappa$, j *integer,* $0 < \varkappa \leqq 1$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. *Then for* $q < \infty$ *holds*

[12]) For $s \equiv 0(2)$ the function $\varrho(t)$ is not necessary.

$$B^s_{pq} = \left\{ f | f \in S',\ \|f\|^*_{B^s_{pq}} = \left[\int_0^\infty h^{-\varkappa q} \sum_{k=1}^n \left\| \Delta^{1+[\varkappa]}_{h,k} \frac{\partial^j f}{\partial x_k^j} \right\|^q_{L_p} \frac{dh}{h} \right]^{\frac{1}{q}} + \|f\|_{L_p} < \infty \right\} \tag{9.10a}$$

and for $q = \infty$

$$B^s_{p\infty} = \left\{ f | f \in S',\ \|f\|^*_{B^s_{p\infty}} = \sup_{h>0} h^{-\varkappa} \sum_{k=1}^n \left\| \Delta^{1+[\varkappa]}_{h,k} \frac{\partial^j f}{\partial x_k^j} \right\|_{L_p} + \|f\|_{L_p} < \infty \right\}. \tag{9.10b}$$

Further holds

$$\|f\|_{B^s_{pq}} \sim \|f\|^*_{B^s_{pq}} .$$

($[\varkappa] = 0$ *for* $0 < \varkappa < 1$, $[\varkappa] = 1$ *for* $\varkappa = 1$).

9.2.3. *Proof of theorem* 9.2.2. Theorem 5.2.1 and theorem 5.2.3 show $B^s_{p1} \subset H^s_p \subset B^s_{p\infty}$. Then follows form the reiteration theorem 2.2.3 and from theorem 8.1.3

$$(L_p, H^k_p)_{\Theta, q} = B^{\Theta k}_{pq};\quad 0 < \theta < 1;\quad 1 \leqq q \leqq \infty\,, \tag{9.11}$$

$k > 0$, integer. On the other hand (9.9) and theorem 9.1.1 show $H^k_p = \bigcap_{j=1}^n D(A^k_j)$. (9.10) follows now from (2.16), (2.13) and (2.14). This completes the proof.

9.2.4. *Remark.* (9.10) is the definition of the spaces B^s_{pq} given by Besov [1]. There are many other equivalent norms for the spaces B^s_{pq}, see also Nikol'skij [14], and Taibleson [25]. Grisvard considered Besov spaces for vector-valued functions [7].

9.2.5. *Theorem.* We consider an other equivalent norm.

THEOREM 9.2.5. *Let* $0 < s < \infty$ *with the decomposition* $s = j + \varkappa$, j *integer,* $0 < \varkappa \leqq 1$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. *Then*

$$B^s_{pq} = \left\{ f | f \in S',\ \|f\|^+_{B^s_{pq}} = \left[\int_0^\infty h^{-\varkappa q} \sum_{k=1}^n \sum_{|\alpha| \leqq j} \|\Delta^{1+[\varkappa]}_{h,k} D^\alpha f\|^q_{L_p} \frac{dh}{h} \right]^{\frac{1}{q}} + \|f\|_{L_p} < \infty \right\} \tag{9.12}$$

(and the usual modification for $q = \infty$*),*

$$\|f\|_{B^s_{pq}} \sim \|f\|^+_{B^s_{pq}} . \tag{9.13}$$

9.2.6. *Proof of theorem* 9.2.5. Let $0 < \varkappa < 1$ and $|\alpha| \leqq j$. Then $D^\alpha f$ is a linear continuous mapping from H^j_p into L_p and from H^{j+1}_p into H^1_p, theorem 9.1.1. The interpolation property 2.2.2 and (9.11) show that $D^\alpha f$ is a linear continuous mapping from $B^{j+\varkappa}_{pq} = (H^j_p, H^{j+1}_p)_{\varkappa, q}$ into $(L_p, H^1_p)_{\varkappa, q} = B^\varkappa_{pq}$, i.e.

$$\|D^\alpha f\|_{B^{\varkappa}_{pq}} \leqq c\|f\|_{B^{j+\varkappa}_{pq}} .$$

Now the theorem follows from (9.10). If $\varkappa = 1$ we consider $D^\alpha f$ as a mapping from H_p^j into L_p, and from H_p^{j+2} into H_p^2. The interpolation property shows that $D^\alpha f$ is a continuous mapping from $B_{pq}^{j+1} = (H_p^j, H_p^{j+2})_{\frac{1}{2}, q}$ into $(L_p, H_p^2)_{\frac{1}{2}, q} = B_{pq}^1$, i.e.

$$\|D^\alpha f\|_{B^1_{pq}} \leqq c\|f\|_{B^{j+1}_{pq}} .$$

The theorem follows again from (9.10).

9.2.7. *Lemma.* We want to simplify the norm $\|f\|^+_{B^s_{pq}}$. For this purpose we need the following lemma.

LEMMA 9.2.7. *Let* $1 < p < \infty$; $1 \leqq q \leqq \infty$.
(a) *For* $0 < \varkappa < 1$ *holds (with the necessary modification for* $q = \infty$*)*

$$\sum_{j=1}^n \int_0^\infty h^{-\varkappa q} \|\Delta_{h,j} f\|^q_{L_p} \frac{dh}{h} \sim \int_{R_n} |h|^{-\varkappa q} \|f(x+h) - f(x)\|^q_{L_p} \frac{dh}{|h|^n} \tag{9.14}$$

(b) *For* $\varkappa = 1$ *holds (with the necessary modification for* $q = \infty$*).*

$$\sum_{j=1}^n \int_0^\infty h^{-q} \|\Delta^2_{h,j} f\|^q_{L_p} \frac{dh}{h} \sim \int_{R_n} |h|^{-q} \|f(x+2h) - 2f(x+h) + f(x)\|^q_{L_p} \frac{dh}{|h|^n} . \tag{9.15}$$

9.2.8. *Proof of lemma* 9.2.7.
Step 1. Let $\lambda > 0$. Then for $q < \infty$ holds

$$\int_{R_{n-1}} \frac{dx_2 \ldots dx_n}{(\lambda^2 + x_2^2 + \ldots + x_n^2)^{\frac{n}{2} + \frac{\varkappa q}{2}}} = \frac{c}{\lambda^{1+\varkappa q}} , \tag{9.16}$$

where c is independent of λ. First we estimate the right hand side of (9.14) by the left side. We set

$$R_n \ni h = (h_1, \ldots, h_n), \quad \bar{h}_j = (0, \ldots, 0, h_j, 0, \ldots, 0) \quad (\text{place } j) .$$

With the aid of (9.16) follows

$$\int_{R_n} |h|^{-\varkappa q} \|f(x+h) - f(x)\|^q_{L_p} \frac{dh}{|h|^n} \leqq c \sum_{j=1}^n \int_{R_n} |h|^{-\varkappa q} \|f(x+\bar{h}_j) - f(x)\|^q_{L_p} \frac{dh}{|h|^n} =$$

$$= c' \sum_{j=1}^n \int_{R_1} |h_j|^{-\varkappa q} \|\Delta_{h_j, j} f\|^q_{L_p} \frac{dh_j}{|h_j|} = 2c' \sum_{j=1}^n \int_0^\infty h^{-\varkappa q} \|\Delta_{h,j} f\|^q_{L_p} \frac{dh}{h} .$$

For $q = \infty$ we have an analogous estimate. In the same way we show that the right side of (9.15) is smaller than the left side.

Step 2. We prove the other inequality of (9.14). We choose a system of orthogonal coordinates represented by the orthonormal vectors ν_j

$$\nu_j \cdot \nu_k = \delta_{jk} . \tag{9.17}$$

Then holds for $h > 0$

$$\|f(x_1 + h, x_2, \ldots, x_n) - f(x)\|_{L_p} \leqq \sum_{j=1}^{n} \|f(x + \nu_j h \cos(\nu_j, x_1)) - f(x)\|_{L_p} .$$

With the aid of the transformation $\lambda = h|\cos(\nu_j, x_1)|$ we find

$$\int_0^\infty h^{-\varkappa q}\|\Delta_{h,1} f\|_{L_p}^q \frac{dh}{h} \leqq c \sum_{j=1}^{n} \int_0^\infty \lambda^{-\varkappa q}|\cos(\nu_j, x_1)|^{\varkappa q}\|f(x + \nu_j\lambda) - f(x)\|_{L_p}^q \frac{d\lambda}{\lambda} . \tag{9.18}$$

This is also true for $\cos(\nu_j, x_1) < 0$ after the transformation $y = x - \nu_j\lambda$. We consider now all systems of orthogonal coordinates, represented by (9.17). If $\omega_n = \{x \mid |x| = 1\}$ is the unit sphere in R_n with the usual measure and topology, then the set of all systems of orthogonal coordinates (represented by (9.17)) is a closed subset of $\omega_n \times \ldots \times \omega_n$ (n times). In $\omega_n \times \ldots \times \omega_n$ we introduce the product measure and product topology. We integrate (9.18) over all systems of orthogonal coordinates. It follows

$$\int_0^\infty h^{-\varkappa q}\|\Delta_{h,1} f\|_{L_p}^q \frac{dh}{h} \leqq c \sum_{j=1}^{n} \int_0^\infty \lambda^{-\varkappa q} \int_{\omega_n} \|f(x + \nu_j\lambda) - f(x)\|_{L_p}^q d\nu_j \frac{d\lambda}{\lambda}$$

$$= c \int_{R_n} |\mu|^{-\varkappa q}\|f(x + \mu) - f(x)\|_{L_p}^q \frac{d\mu}{|\mu|^n} .$$

For $q = \infty$ we have an analogous estimate. This proves (9.14). In the same way we prove (9.15), see [28].

9.2.9. *Theorem.* We write $(\Delta_h f)(x) = f(x + h) - f(x)$, $h \in R_n$, and $\Delta_h^2 f = \Delta_h(\Delta_h f)$.

THEOREM 9.2.9. *Let* $0 < s < \infty$ *with the decomposition* $s = j + \varkappa$, j *integer*, $0 < \varkappa \leqq 1$; $1 < p < \infty$; $1 \leqq q \leqq \infty$. *Then holds*

$$B_{pq}^s = \left\{ f \mid f \in S', \ \|f\|_{B_{pq}^s}^0 = \left[\int_{R_n} |h|^{-\varkappa q} \sum_{|\alpha| \leqq j} \|\Delta_h^{1+[\varkappa]} D^\alpha f\|_{L_p}^q \frac{dh}{|h|^n} \right]^{\frac{1}{q}} + \|f\|_{L_p} < \infty \right\} \tag{9.19}$$

(and the usual modification for $q = \infty$*),*

$$\|f\|_{B_{pq}^s} \sim \|f\|_{B_{pq}^s}^0 . \tag{9.20}$$

9.2.10. *Proof of theorem* 9.2.9. The proof follows immediately from lemma 9.2.7 and theorem 9.2.5.

9.2.11. *Remark.* If $p = q$ and $0 < \varkappa < 1$ we have

$$\|f\|^0_{B^s_{pp}} \sim \left[\int_{R_n}\int_{R_n} \sum_{|\alpha| \leqq j} \frac{|D^\alpha f(y) - D^\alpha f(x)|^p}{|x - y|^{n+\varkappa p}} \, dxdy \right]^{\frac{1}{p}} + \|f\|_{L_p} \,. \tag{9.21}$$

The spaces $B^s_{pp} = W^s_p$ are introduced by Slobodeckij [22]. We can give an analogous formula for $\varkappa = 1$.

9.2.12. *Remark.* The considerations show that

$$\left[\int_{R_n} |h|^{-\varkappa q} \sum_{|\alpha| = j} \|\Delta_h^{1+[\varkappa]} D^\alpha f\|^q_{L_p} \frac{dh}{|h|^n} \right]^{\frac{1}{q}} + \|f\|_{L_p} \tag{9.22}$$

is also a norm equivalent to $\|f\|_{B^s_{pq}}$, $s = j + \varkappa$, j integer, $0 < \varkappa \leqq 1$.

10. Interpolation of the spaces B^s_{pq} and F^s_{pq}, complex method

10.1. *The spaces* B^s_{pq}

10.1.1. *Theorem.*

THEOREM 10.1.1. *Let* $-\infty < s_0, s_1 < \infty$; $1 < p_0, p_1, q_0, q_1 < \infty$, *and* $0 < \theta < 1$. *Then holds*

$$[B^{s_0}_{p_0 q_0}, B^{s_1}_{p_1 q_1}]_\theta = B^s_{pq} \,, \tag{10.1}$$

$$s = (1 - \theta)s_0 + \theta s_1; \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \,; \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1} \,. \tag{10.2}$$

10.1.2. *Proof of theorem* 10.1.1.

Step 1. We consider a function $f \in B^s_{pq}$ with

$$f(x) = \sum_{j=0}^{N} a_j(x) \tag{10.3}$$

in the sense of definition (4.5). Let $\{\varphi_k\}_{k=0}^\infty$ be a system of functions of type 4.2.1 with $N = 1$ and

$$\sum_{k=0}^{\infty} (F\varphi_k)(\xi) = 1 \,. \tag{10.4}$$

For a complex number z, $z = x + iy$, $0 \leqq x \leqq 1$, we construct

$$a_k(x, z) = 2^{\varrho_1(z)k} \|a_k\|_{L_p}^{\varrho_2(z)} \|\{a_j\}_{j=0}^N\|_{l_q^s(L_p)}^{\varrho_3(z)} a_k^{\varrho_4(z)}(x) , \tag{10.5}$$

$$\varrho_1(z) = sq\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right) - (1-z)s_0 - zs_1 ,$$

$$\varrho_2(z) = q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right) - p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) ,$$

$$\varrho_3(z) = 1 - q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right) , \quad \varrho_4(z) = p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) .$$

(If $a_k(x) \equiv 0$ we set $a_k(x, z) \equiv 0$), and

$$f(x, z) = \sum_{k=0}^{N} \Big(\sum_{j=-2}^{2} \varphi_{k+j}\Big) * a_k(x, z) = \sum_{k=0}^{N+2} \varphi_k * \Big(\sum_{j=-2}^{2} a_{k+j}(x, z)\Big) , \tag{10.6}$$

($\varphi_j \equiv 0$ for $j < 0$, $a_j = 0$ for $j < 0$ and $j > N$). This is a representation of $f(x, z)$ in the sense of 4.1.3. It is not difficult to see that

$$a_k^{\varrho_4(z)} \in \mathcal{F}[L_{p_0}, L_{p_1}] \tag{10.7}$$

in the sense of 2.2.8. It follows that

$$a_k(x, z) \in \mathcal{F}[L_{p_0}, L_{p_1}] \tag{10.8}$$

and

$$f(x, z) \in \mathcal{F}[B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1}] .$$

With the aid of theorem 3.5 (scalar case) follows

$$\|f(x, it)\|_{B_{p_0q_0}^{s_0}} \leqq \sum_{j=-2}^{2} \|\{\varphi_{k+j} * a_k(x, it)\}_0^N\|_{l_{q_0}^{s_0}(L_{p_0})} \leqq c\|\{a_k(x, it)\}_0^N\|_{l_{q_0}^{s_0}(L_{p_0})} \leqq c'\|\{a_k\}_0^N\|_{l_q^s(L_p)}$$

and

$$\|f(x, 1 + it)\|_{B_{p_1q_1}^{s_1}} \leqq c'\|\{a_k\}_0^N\|_{l_q^s(L_p)}$$

Further we have $f(x, \theta) = f(x)$. Then follows from 2.2.8 and an infimum construction in the sense of (4.6)[13]) that

$$\|f\|_{[B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1}]_\theta} \leqq c\|f\|_{B_{pq}^s} .$$

Functions of type (10.3) are dense in B_{pq}^s. This and the last inequality show that

$$B_{pq}^s \subset [B_{p_0q_0}^{s_0}, B_{p_1q_1}^{s_1}]_\theta . \tag{10.9}$$

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[13]) By the infimum in (4.6) we may assume that for the functions $f(x)$ from (10.3) all representations of the type (10.3).

Step 2. We use the duality for the complex method described in 2.2.9. The spaces B^s_{pq} are reflexive, see theorem 7.2.2. The same theorem, (10.9) and 2.2.9 lead us to

$$[B^{s_0}_{p_0q_0}, B^{s_1}_{p_1q_1}]_\theta = ([B^{-s_0}_{p_0'q_0'}, B^{-s_1}_{p_1'q_1'}]_\theta)' \subset (B^{-s}_{p'q'})' = B^s_{pq}\,. \tag{10.10}$$

(10.9) and (10.10) prove the theorem.

10.1.3. *Remark.* Theorem 10.1.1 is well-known and proved by Grisvard [7] and Taibleson [26].

10.2. *The spaces* F^s_{pq}

10.2.1. *Theorem.*

THEOREM 10.2.1. *Let* $-\infty < s_0, s_1 < \infty$; $1 < p_0, p_1, q_0, q_1 < \infty$; *and* $0 < \theta < 1$. *Then*

$$[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta = F^s_{pq},$$

$$s = (1-\theta)s_0 + \theta s_1; \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}\,; \quad \frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}\,. \tag{10.11}$$

10.2.2. *Proof of theorem* 10.2.1. The proof is almost the same as in 10.1.2. First we consider a function $f(x) \in F^s_{pq}$ with (10.3) and a system $\{\varphi_k\}$ of type 4.2.1 with $N = 1$ and (10.4). We replace (10.5) by

$$a_k(x, z) = 2^{\varrho_1(z)} \|\{a_j\}_0^N\|^{\varrho_2(z)}_{l^s_q} \|\{a_j\}_0^N\|^{\varrho_3(z)}_{L_p(l^s_q)} a_k^{\varrho_4(z)}(x)\,, \tag{10.12}$$

$$\varrho_1(z) = sq\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right) - (1-z)s_0 - zs_1\,,$$

$$\varrho_2(z) = p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right) - q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right),$$

$$\varrho_3(z) = 1 - p\left(\frac{1-z}{p_0} + \frac{z}{p_1}\right), \quad \varrho_4(z) = q\left(\frac{1-z}{q_0} + \frac{z}{q_1}\right).$$

($a_k(x, z) \equiv 0$ for $a_k(x) \equiv 0$). We construct $f(x, z)$ in (10.6) and find that $f(x, z) \in \mathcal{F}[F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]$, and with the aid of theorem 3.5 (b)

$$\|f(x, it)\|_{F^{s_0}_{p_0q_0}} \leq c\|\{a_k\}_0^N\|_{L_p(l^s_q)}, \quad \|f(x, 1+it)\|_{F^{s_1}_{p_1q_1}} \leq c\|\{a_k\}_0^N\|_{L_p(l^s_q)}\,.$$

In the same manner as in 10.1.2 we get

$$F^s_{pq} \subset [F^{s_0}_{p_0q_0}, F^{s_1}_{p_1q_1}]_\theta\,.$$

The theorem follows now by a duality argument (see theorem 7.1.7) as in 10.1.2.

10.2.3. *Remark.* We note some special cases. We have $H_p^s = F_{p2}^s$, theorem 4.2.6. We find that

$$[H_{p_0}^{s_0}, H_{p_1}^{s_1}]_\theta = H_p^s$$
$$-\infty < s_0, s_1 < \infty;\ 1 < p_0, p_1 < \infty;\ 0 < \theta < 1; \tag{10.13}$$
$$s = (1-\theta)s_0 + \theta s_1;\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

This result due to Calderón [3]. An another special case is $F_{pp}^s = B_{pp}^s$. We find

$$[H_{p_0}^{s_0}, B_{p_1 p_1}^{s_1}]_\theta = F_{pq}^s,$$
$$-\infty < s_0, s_1 < \infty,\ 1 < p_0, p_1 < \infty;\ 0 < \theta < 1; \tag{10.14}$$
$$s = (1-\theta)s_0 + \theta s_1;\ \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1};\ \frac{1}{q} = \frac{1-\theta}{2} + \frac{\theta}{p_1}.$$

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Received February 2, 1972.

Hans Triebel
Sektion Mathematik der Universität
69 Jena
Universitätshochhaus
DDR