Triple covers in positive characteristic

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Introduction

The theory of triple covers in algebraic geometry has been developed by R. Miranda in his paper [3], in which he establishes a 1—1 correspondence between triple covers of varieties over fields of characteristic not equal to 2 or 3 and sections of certain vector bundles.

The purpose of this work was to analyse in the same spirit the characteristic 3 case, since this has some special features, e.g. the existence of inseparable triple covers.

Actually, it turned out that it is possible to extend Miranda's theory in such a way to describe triple covers of schemes of finite type over any noetherian domain R with the property that 2 is invertible in R. In doing so, we think we have reached a more conceptual view of the problem and we hope this may also lead to applications in number theory.

Sections 1, 2, and 3 contain the general description of triple covers in terms of sections of vector bundles; the example of Section 4 shows that the general situation is indeed more complex than the case of varieties over a field of characteristic different from 2, 3. Ramification, branch locus and local structure of triple covers are described in Section 5, while inseparable triple covers are analysed in Section 6. In Section 7, we set up the problem of lifting a triple cover in characteristic 3 to characteristic 0 and provide a couple of examples to clarify the matter. Section 8 is devoted to computing the invariants of triple covers of surfaces: these are expressed by the same formulas both in the separable and inseparable case, although the usual computational methods cannot be applied in the latter situation. Finally, an appendix takes care of the characteristic 2 case.

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1. Preliminary facts

Let R be a noetherian domain such that 2 is invertible in R and let Y be an integral separated scheme of finite type over $\operatorname{spec} R$.

Definition 1.1. *A triple cover of Y consists of the data (X, φ) of a scheme X over* spec R *and of a flat finite R-morphism $\varphi\colon X \to Y$ such that $\varphi_* \mathcal{O}_X$ is a rank* 3 *$\mathcal{O}_Y$-bundle. Triple covers (X, φ) and (X', φ') are isomorphic iff there exists an isomorphism $\psi\colon X \to X'$ such that the following diagram commutes:*

$$\begin{tikzcd} X \arrow[rr, "\psi"] \arrow[dr, "\Phi"'] & & X \arrow[dl, leftarrow, "\varphi'"] \\ & Y. & \end{tikzcd}$$

To a triple cover (X, φ) there corresponds a short exact sequence of locally free sheaves on Y:

$$0 \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_X \to E \to 0$$

E is locally free of rank 2, since the map: $\mathcal{O}_Y \to \varphi_* \mathcal{O}_X$ has no zeros on Y. In what follows E will be called the associated vector bundle of the cover (X, φ). $\varphi_* \mathcal{O}_X$ is a rank 3 $\mathcal{O}_Y$-algebra and the natural projection $p\colon \operatorname{spec} \varphi_* \mathcal{O}_X \to Y$ gives a triple cover isomorphic to (X, φ).

So, if we are given a scheme Y and a rank 2 vector bundle E on Y, the problem of describing the triple covers of Y whose associated bundle is E is completely equivalent to that of determining the pairs (V, μ), where V is an extension $0 \to \mathcal{O}_Y \to V \xrightarrow{\pi} E \to 0$ and $\mu\colon S^2 V \to V$ is a $\mathcal{O}_Y$-linear map defining a commutative ring structure on V compatible with the $\mathcal{O}_Y$-module structure. It may be worthwhile remarking that compatibility with the $\mathcal{O}_Y$-module structure is equivalent the to the requirement that μ split the following short exact sequence:

$$0 \to V \xrightarrow{i} S^2 V \to S^2 E \to 0$$

where $i\colon V \to S^2 V$ is the natural inclusion.

Since V is locally free, given $L \in \operatorname{Hom}(V, V)$, the trace of L is a well defined element of $\mathcal{O}_Y$. Given μ, $\forall z \in V$ $L_z\colon \begin{array}{c} V \to V \\ y \to \mu(zy) \end{array}$ is a $\mathcal{O}_Y$-linear map whose trace we will call the trace of z and denote by $\operatorname{Tr}(z)$. $\operatorname{Tr}\colon V \to \mathcal{O}_Y$ is $\mathcal{O}_Y$-linear, i.e. it is a section of $V^{\check{}}$.

Proposition 1.2. i) *If $3 \in R$ is invertible, then $\frac{1}{3}\operatorname{Tr}\colon V \to \mathcal{O}_Y$ is a splitting of the sequence $0 \to \mathcal{O}_Y \to V \to E \to 0$ and therefore V is the trivial extension;*

ii) *if the characteristic of R equals* 3, *then the trace map gives a well defined map on the quotient,* $\operatorname{Tr}\colon E \to \mathcal{O}_Y$*;*

iii) *let* $\xi\in Y$ *be the generic point and assume* V_ξ *is an integral domain (and therefore a field). Then* $\mathrm{Tr}\colon V\to\mathcal{O}_Y$ *vanishes identically iff the characteristic of* R *is equal to* 3 *and* $\mathcal{O}_{Y,\xi}\subseteqq V_\xi$ *is an inseparable field extension.*

Proof. To prove i) and ii) it is sufficient to remark that $\forall y\in\mathcal{O}_Y$ one has $\mathrm{Tr}(y)=3y$. To prove iii) observe that Y is integral and V is locally free and therefore $\mathrm{Tr}\colon V\to\mathcal{O}_Y$ vanishes identically on Y iff $\mathrm{Tr}_\xi\colon V_\xi\to\mathcal{O}_{Y,\xi}$ is the zero map. Hence (see [4] page 93—94), the trace map of V_ξ over $\mathcal{O}_{Y,\xi}$ vanishes identically iff the extension is purely inseparable. ∎

We will now introduce another global section of a locally free sheaf related to (V,μ): $\forall x,y\in V$ define: $Q(x,y)=\mathrm{Tr}(xy)$. Q is a symmetric bilinear form on V and therefore it induces a linear map $L\colon V\to V^\vee$. Taking exterior powers, we get a map: $\wedge^3 L\colon \wedge^3 V\to\wedge^3 V^\vee$. Since $\wedge^3 V$ is isomorphic to $\wedge^2 E$, $\wedge^3 L$ can be identified with a section B of $\mathrm{Hom}(\wedge^2 E,\wedge^2 E^\vee)\cong(\wedge^2 E)^{-2}$.

Corollary 1.4. *Assume* V_ξ *is a domain. Then* $B\in H^0\big(Y,(\wedge^2 E)^{-2}\big)$ *is the zero section iff the characteristic of* R *is* 3 *and* $\mathcal{O}_{Y,\xi}\subseteqq V_\xi$ *is an inseparable extension.*

Proof. B is the zero section iff the form Q is everywhere degenerate iff Q is degenerate at the generic point ξ of Y. In turn, this means that there exists $z\in V_\xi\setminus\{0\}$ such that $\forall y\in V_\xi\ \mathrm{Tr}(zy)=0$, i.e. $\forall w\in V_\xi\ \mathrm{Tr}(w)=0$, since V_ξ is a field. The corollary now follows from Proposition 1.2.

2. Triple covers and sections of vector bundles

The purpose of this section is to establish a 1—1 correspondence between the pairs (V,μ) described above and the elements of $H^0(Y,S^3E\otimes(\wedge^2 E)^{-2})$. Before we can do this, we have to introduce two short exact sequences of locally free sheaves on Y.

Proposition 2.1. i) *For every rank* 2 *vector bundle* E *on* Y *one has the following short exact sequence:*

$$\text{(A)}\qquad 0\to E\otimes\wedge^2 E\xrightarrow{\ i\ } S^2E\otimes E\xrightarrow{\ \sigma\ } S^3E\to 0.$$

If 3 *is invertible in* R, *then the sequence is split exact.*

ii) *If* R *is a field of characteristic* 3 *and* $F\colon Y\to Y$ *is the Frobenius morphism one has also:*

$$\text{(B)}\qquad 0\to F^*E\xrightarrow{\ j\ } S^3E\xrightarrow{\ \tau\ } E\otimes\wedge^2 E\to 0.$$

Proof. i) The map σ is just symmetrization, defined by $\sigma(x_1x_2\otimes x_3)=x_1x_2x_3$ for simple tensors and extended linearly. To define the map i, we set $i\big(x_1\otimes(x_2\wedge x_3)\big)=$

$x_1x_2\otimes x_3-x_1x_3\otimes x_2$. It is easy to see that this is a good definition and that $\sigma\circ i=0$. The fact that the image of i and the kernel of σ coincide can be verified using local coordinates for the sheaf E. In case 3 is invertible, the map

$$\varrho\colon S^3E \to S^2E\otimes E$$

$$x_1x_2x_3 \to \tfrac{1}{3}(x_1x_2\otimes x_3+x_1x_3\otimes x_2+x_3x_2\otimes x_1)$$

splits the sequence.

ii) The map j: $F^*E\to S^3E$ is just the natural immersion. The map τ can be defined by $\tau(xyz)=y\otimes(z\wedge x)+x\otimes(z\wedge y)$. To check that τ is well defined one uses characteristic 3 and the identity $\alpha\otimes(\beta\wedge\gamma)+\beta\otimes(\gamma\wedge\alpha)+\gamma\otimes(\alpha\wedge\beta)$ for α, β, γ in $E\otimes\Lambda^2E$. With the aid of local coordinates as in case i) one verifies that the sequence is exact. Tensoring with $(\Lambda^2E)^{-2}$ and using the isomorphism $E^\vee\cong E\otimes(\Lambda^2E)^{-1}$, we get two more sequences:

$$\text{(A}'\text{)}\qquad 0\to E^\vee \xrightarrow{i'} S^2E\otimes E\otimes(\Lambda^2E)^{-2}\xrightarrow{\sigma'} S^3E\otimes(\Lambda^2E)^{-2}\to 0$$

$$\text{(B}'\text{)}\qquad 0\to F^*E\otimes(\Lambda^2E)^{-2}\xrightarrow{j'} S^3E\otimes(\Lambda^2E)^{-2}\xrightarrow{\tau'} E^\vee\to 0.$$

The next step is to describe μ locally.

Proposition 2.2. *Let $\{U_i\}_{i\in I}$ be an affine open covering of Y such that $E|_{U_i}$ is trivial $\forall i\in I$. Let $\{1, z_i, w_i\}$ be a base for $V|_{U_i}$. Then μ: $S^2V\to V$ turns the $\mathcal{O}_Y$-module V into an associative $\mathcal{O}_Y$-algebra iff it has the following form on U_i $\forall i\in I$ (we omit the index i to simplify notation):*

$$\mu(1)=1;\quad \mu(z)=z;\quad \mu(w)=w;$$

$$\mu(z^2)=az+bw+be+f^2-af-bd;\quad \mu(zw)=ez+fw+bc-ef;$$

$$\mu(w^2)=cz+dw+e^2+cf-ac-de$$

where a, b, c, d, e, f are in $\mathcal{O}_Y$.

Proof. See [3], Lemma (2.4).

Corollary 2.2. *Let $U\subseteq Y$ be such that $E|_U$ is trivial and let $\{z', w'\}$ be a base for $E|_U$. Then there exists a unique base of the form $\{1, z, w\}$ for $V|_U$ such that z, w lift z', w' and such that μ has the following local form:*

$$\mu(z^2)=bw+be+f^2$$

$$\mu(zw)=ez+fw+bc-ef$$

$$\mu(w^2)=cz+e^2+cf.$$

Proof. Let z_0, w_0 be any two elements of $V|_U$ that lift z' and w': by Proposition 2.2, it is enough to set $z=z_0-\frac{a}{2}$, $w=w_0-\frac{d}{2}$. ∎

Definition 2.3. *The local form for μ described in Corollary 2.2 will be called a "normal local form" for V with respect to μ and with respect to the local coordinates z, w.*

Remark 2.3b. Triple covers are locally determinantal varieties. In fact, the normal local form of Corollary 2.2 is given by:

$$\operatorname{rank}\begin{bmatrix} z-f & w+e & c \\ b & z+f & w-e \end{bmatrix} = 1.$$

We are now in a position to state and prove:

Theorem 2.4. a) *Assume we are given an integral separated scheme of finite type Y over* $\operatorname{spec} R$, *a rank 2 vector bundle E on Y and a pair (V,μ), where V is an extension of $\mathcal{O}_Y$ by E and μ defines an associative and commutative $\mathcal{O}_Y$-algebra structure on V.*

Then there exists an element $\sigma(V,\mu)\in H^0(Y, S^3E\otimes(\wedge^2E)^{-2})$ such that:

i) *If ∂: $H^0(S^3E\otimes(\wedge^2E)^{-2})\to H^1(Y, E^\vee)$ denotes the coboundary map in the cohomology long exact sequence associated to the sequence* (A'), *then $\frac{1}{4}\partial\sigma(V,\mu)$ represents the isomorphism class of the extension $0\to\mathcal{O}_Y\to V\to E\to 0$.*

ii) *If R is a field of characteristic* 3, *then $\tau'\sigma(V,\mu)$, where*

$$\tau'\colon H^0(Y, S^3E\otimes(\wedge^2E)^{-2}) \to H^0(Y, E^\vee)$$

is the map on global sections induced by the map τ' in sequence (B'), *is the trace of μ* (see Prop. 1.2, ii)).

iii) *$(V,\mu)\cong(V',\mu')$ as extensions and as $\mathcal{O}_Y$-algebras iff $\sigma(V,\mu)=\sigma(V',\mu')$.*

b) *Conversely, given Y, E as above and $\sigma\in H^0(Y, S^3E\otimes(\wedge^2E)^{-2})$ there is (V,μ) as above such that $\sigma(V,\mu)=\sigma$. In particular, if R is a field of characteristic* 3, *there is a natural* 1—1 *correspondence between $H^0(Y, F^*E\otimes(\wedge^2E)^{-2})$ and isomorphism classes (V,μ) with zero trace.*

Proof. Assume we are given (V,μ) as in the hypotheses. Define $\mu'\colon S^2V\to E$ by composing μ with the projection $\pi\colon V\to E$. Consider the following map:

$$\Phi(\mu)\colon V^{\otimes 6} \longrightarrow S^3E$$

$$(x_1\otimes x_2\otimes x_3)\otimes(y_1\otimes y_2\otimes y_3) \to -\sum_{\sigma\in S_3}(-1)^{\varepsilon(\sigma)}\prod_{i=1,2,3}\mu'(x_iy_{\sigma(i)})$$

where $\varepsilon(\sigma)$ is the sign of the permutation $\sigma\in S_3$.

Let $q\colon V\otimes V\otimes V\to\wedge^3V\cong\wedge^2E$ be the alternation map. One has the following diagram:

$$\begin{tikzcd} V^{\otimes 6} \arrow[r, "\Phi(\mu)"] \arrow[d, "q\otimes q"'] & S^3E \\ (\wedge^2E)^{-2} \arrow[ur, dashed, "\sigma(V,\mu)"'] & \end{tikzcd}$$

It is easy to check that the dotted arrow is in fact a well defined map, i.e. an element $\sigma(V,\mu)\in H^0(Y, S^3E\otimes(\wedge^2E)^{-2})$. Using normal local coordinates for V with respect to μ we get the following local expression for $\sigma(V,\mu)$:

$$(z'\wedge w')^2\to(1\wedge z\wedge w)^2\to cz'^3-2ez'^2w'-2fz'w'^2+bw'^3$$

where z', w' are the images of z, w in E.

From this and from the normal local form for μ in Corollary 2.2, it is immediate to deduce the statement in the theorem about the uniqueness of (V,μ) up to isomorphism of extensions preserving multiplication. Using again normal local coordinates, we get the following expression for the trace map of (V,μ):

$$1\xrightarrow{\mathrm{Tr}}3;\quad z\xrightarrow{\mathrm{Tr}}f;\quad w\xrightarrow{\mathrm{Tr}}e.$$

The statement about the trace map in characteristic 3 follows at once. To determine the class of the extension V we shall use a trivialization $\{\{U_i\},\{z_i',w_i'\}|i\in I\}$ of the bundle E and denote by $\{1,z_i,w_i\}$ the corresponding local normal base for V on U_i. Assume the following relation holds for the bundle E on $U_i\cap U_j$:

$$\begin{bmatrix} z_j'\\ w_j'\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix} z_i'\\ w_i'\end{bmatrix}.$$

The corresponding relation for the bundle V on $U_i\cap U_j$ will have the form:

$$\begin{bmatrix} z_j\\ w_j\end{bmatrix}=\begin{bmatrix}\alpha & \beta\\ \gamma & \delta\end{bmatrix}\begin{bmatrix} z_i\\ w_i\end{bmatrix}+\begin{bmatrix} t_{ji}\\ s_{ji}\end{bmatrix}$$

(t_{ji},s_{ji}) represents on $U_i\cap U_j$ the element of $H^1(Y,E^{\check{}})$ corresponding to the extension V. By imposing the condition that $\{1,z_j,w_j\}$ and $\{1,z_i,w_i\}$ be normal local coordinates, we get the following equalities:

$$t_{ji}=\frac{1}{2(\alpha\delta-\beta\gamma)}(-\beta^2\delta c_i+\gamma\alpha^2b_i-\alpha\beta\gamma(-2f_i)+\alpha\beta\delta(-2e_i))$$

$$s_{ji}=\frac{1}{2(\alpha\delta-\beta\gamma)}(\beta\delta^2c_i-\gamma^2\alpha b_i-\delta\beta\gamma(-2e_i)+\alpha\delta\gamma(-2f_i)).$$

A computation with transition matrices for the bundles involved shows that (t_{ji},s_{ji})

represents on $U_j \cap U_i$ the element $\frac{1}{4}\partial\sigma(V,\mu) \in H^1(Y, E^\vee)$. The converse part of the theorem now follows by remarking that the local formulas computed above allow one to construct explicitly a pair (V,μ) whose associated section is any chosen element of $H^0(Y, S^3E \otimes (\wedge^2 E)^{-2})$. The last statement of the theorem is just a consequence of the exactness of

$$0 \to H^0(Y, F^*E \otimes (\wedge^2 E)^{-2}) \to H^0(Y, S^3E \otimes (\wedge^2 E)^{-2}) \to H^0(Y, E^\vee). \quad \blacksquare$$

Remark 2.5. Theorem 2.4 also gives another proof of Proposition 1.1, i). In fact Proposition 2.1, i) implies that the map $\partial\colon H^0(Y, S^3E \otimes (\wedge^2 E)^{-2}) \to H^1(Y, E^\vee)$ is zero and so we conclude that V is always the trivial extension.

Corollary 2.6. Assume $E = \mathscr{L}^{-1} \oplus \mathscr{M}^{-1}$, $\mathscr{L}$, $\mathscr{M}$ invertible sheaves on Y. If (V,μ) is an extension of $\mathcal{O}_Y$ by E that is also an $\mathcal{O}_Y$-algebra, then $V \cong \mathcal{O}_Y \oplus E$.

Proof. One has:

$$H^0(Y, S^3E \otimes (\wedge^2 E)^{-2}) = H^0(Y, \mathscr{M}^2\mathscr{L}^{-1}) \oplus H^0(Y, \mathscr{M}) \oplus H^0(Y, \mathscr{L}) \oplus H^0(Y, \mathscr{L}^2\mathscr{M}^{-1}),$$

$$H^0(Y, E^\vee) = H^0(Y, \mathscr{M}) \oplus H^0(Y, \mathscr{L}),$$

$$H^0(Y, S^2E \otimes E \otimes (\wedge^2 E)^{-2})$$

$$= 2H^0(Y, \mathscr{L}) \oplus H^0(Y, \mathscr{L}^2\mathscr{M}^{-1}) \oplus 2H^0(Y, \mathscr{M}) \oplus H^0(Y, \mathscr{M}^2\mathscr{L}^{-1}).$$

Looking at the addenda involved, one sees immediately that the sequence of global sections associated to sequence (A′) is exact. The result then follows from Theorem 2.4. $\blacksquare$

Remark 2.7. By what we have observed in Section 1, all the statements in this section can be reformulated as statements on triple covers. For instance, Theorem 2.4 gives a 1—1 correspondence between triple covers of Y whose associated module is E and elements of $H^0(Y, S^3E \otimes (\wedge^2 E)^{-2})$ modulo the natural action of $\mathrm{Aut}(E)$. Following the terminology of [3] we will say that a section $\sigma \in H^0(Y, S^3E \otimes (\wedge^2 E)^{-2})$ "builds" a triple cover (X, φ) and we will call the section of $H^0(Y, S^3E \otimes (\wedge^2 E)^{-2})$ corresponding to a given triple cover the "building map" of that cover. Moreover, given a triple cover (X, φ) with building map σ, we will denote by $B(\sigma)$ or by $B(X, \varphi)$ the element of $H^0(Y, (\wedge^2 E)^{-2})$ that corresponds to σ as described in Section 1, Proposition 1.4.

3. Conditions for the triple cover to be reduced

A triple cover map $\varphi\colon X \to Y$, as we have defined it, is closed and affine, and therefore the set $\{\varphi^{-1}(U) | U \subseteq Y \text{ affine}\}$ is a basis of affine open sets for X. So X is reduced (integral) iff $\varphi_*\mathcal{O}_{X,\xi}$ has no nilpotents (is an integral domain). The problem of determining whether X is integral amounts then to deciding when a given rank 3 algebra over the field of rational functions of Y is an integral domain. Up to a linear change of coordinates in $\varphi_* \mathcal{O}_{X,\xi}$, it is always possible to solve for, say, z in the expression of Corollary 2.2 at the generic point, obtaining a relation of the form $z^3+tz^2+sz+q=0$. Then $\varphi_* \mathcal{O}_{X,\xi}$ is an integral domain iff the polynomial above has no root in $\mathcal{O}_{Y,\xi}$. This shows that, although it may be possible to get an answer in special cases, there does not seem to be a general solution in terms of the properties of the building section of the cover. However, we end this section with a "geometrical" criterion of irreducibility suggested by F. Catanese.

Lemma 3.1. *Let* $z \in \varphi_* \mathcal{O}_{X,\xi}$ *be nilpotent; then* $z^3=0$.

Proof. $\forall z \in \varphi_* \mathcal{O}_{X,\xi}$, define $L_z\colon \underset{x \to zx}{\varphi_* \mathcal{O}_{X,\xi} \to \varphi_* \mathcal{O}_{X,\xi}}$. L_z is of course linear and one has: $(L_z)^n = L_{z^n} = 0 \Leftrightarrow z^n = 0$, since $z^n = L_{z^n}(1)$. On the other hand, $\varphi_* \mathcal{O}_{X,\xi}$ is a vector space of dimention 3 over $\mathcal{O}_{Y,\xi}$ and so any of its endomorphisms A is nilpotent iff $A^3=0$. ∎

Lemma 3.2. *If the characteristic of* R *is equal to* 3, *then to every element* $\tau \in H^0(Y, F^* E \otimes (\wedge^2 E)^{-2})$ *there corresponds an element* $\Lambda(\tau) \in H^0(Y, \operatorname{Hom}(\wedge^2 E, \Omega^1_X))$ *in a natural way. If* τ *builds a triple cover* (X, φ) *we will also write* $\Lambda(X, \varphi)$ *for* $\Lambda(\tau)$.

Proof. Let $\{U_i\}_{i \in I}$ be an affine open covering of Y such that $E|_{U_i}$ is trivial with basis $\{z_i, w_i\}$ $\forall i \in I$. Let τ be represented by (b_i, c_i) on U_i with respect to the given trivialization. Assume the following relation holds for the bundle E on $U_i \cap U_j$:

$$\begin{bmatrix} z_j \\ w_j \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} \begin{bmatrix} z_i \\ w_i \end{bmatrix}.$$

If we set $\Lambda(\tau)_i = b_i\, dc_i - c_i\, db_i$, a computation shows:

$$\Lambda(\tau)_j = (\alpha\delta - \beta\gamma)\, \Lambda(\tau)_i.$$

Therefore the map $\Lambda(\tau)\colon \wedge^2 E \to \Omega^1_Y$ defined by $z_i \wedge w_i \to \Lambda(\tau)_i$ on U_i is well defined on Y. ∎

Proposition 3.3. *Let* $\varphi\colon X \to Y$ *be a triple cover with zero trace. Then* X *is reduced iff* $\Lambda(X, \varphi)$ *does not vanish at the generic point of* Y.

Proof. By Proposition 1.2, iii), the characteristic of R is equal to 3. Assume X is not reduced. We can choose a normal local base for $\varphi_* \mathcal{O}_{X,\xi}$ such that $z+t$ is nilpotent for some $t \in \mathcal{O}_{Y,\xi}$. By Corollary 2.2 and by $\operatorname{tr}(z)=f$, $\operatorname{tr}(w)=e$, the following relations hold in $\varphi_* \mathcal{O}_{X,\xi}$:

$$\begin{cases} z^2 = bw \\ zw = bc \\ w^2 = cz. \end{cases}$$

By Lemma 3.1 we have: $0=(z+t)^3=z^3+t^3=b^2c+t^3$. Taking differentials we get: $b(bdc-cdb)=0$ and therefore $\Delta(X,\varphi)=0$ at the generic point. Conversely, if $bdc-cdb=0$, then $b^2c=t^3$ for some $t\in\mathcal{O}_{Y,\xi}$ and $z-t$ is nilpotent. ∎

Proposition 3.4. *Assume (X,φ) is a triple cover with nonzero trace. Then X is reduced iff $B(X,\varphi)$ does not vanish identically.*

Proof. Recall that $B(X,\varphi)$ vanishes identically iff the bilinear form Q on $\varphi_*\mathcal{O}_X$ (see Section 1) is degenerate at the generic point ξ. By Lemma 3.1, $z\in\varphi_*\mathcal{O}_{X,\xi}$ is nilpotent iff for the characteristic polynomial $p_z(t)$ of the linear map $L_z\colon \underset{x\to zx}{\varphi_*\mathcal{O}_{X,\xi}\to\varphi_*\mathcal{O}_{X,\xi}}$ we have: $p_z(t)=t^3$. In particular, $\mathrm{Tr}(z)=0$. Assume now $z\in\varphi_*\mathcal{O}_{X,\xi}$ is nilpotent. $\forall x\in\varphi_*\mathcal{O}_{X,\xi}$, zx is nilpotent too, and therefore $Q(z,x)=\mathrm{Tr}(zx)=0\ \forall x$ and Q is degenerate. Conversely, assume Q is degenerate. Then there exists $z\in\varphi_*\mathcal{O}_{X,\xi}\setminus\{0\}$ such that $\mathrm{Tr}(zx)=0\ \forall x\in\varphi_*\mathcal{O}_{X,\xi}$. This implies that the image of L_z is contained in the kernel of the trace map, which is a 2-dimensional subspace by the assumptions. So $\det(L_z)=\mathrm{Tr}(L_z)=0$ and the characteristic polynomial of L_z has the form: $p_z(t)=t^3+\lambda t$. If $\lambda=0$, we are set. So assume $\lambda\neq 0$: considering if necessary an algebraic extension of $\mathcal{O}_{Y,\xi}$ containing $\alpha=\sqrt{-\lambda}$, we see that the matrix of L_z can be put in diagonal form:

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & -\alpha \end{bmatrix}.$$

This implies $\mathrm{Tr}(z^2)=2\alpha^2=-2\lambda\neq 0$, contradicting the assumption that $\mathrm{Tr}(zx)=0$ $\forall x\in\varphi_*\mathcal{O}_{X,\xi}$. So we must have $\lambda=0$ and z is nilpotent. ∎

Criterion 3.5. *Assume that X, Y are varieties over an algebraically closed field* **K**, *that $f\colon X\to Y$ is a triple cover map, that the branch locus of f is reduced and that the set of points of Y over which f is totally ramified is nonempty. Then X is irreducible.*

Proof. X is Cohen—Macaulay by Remark 2.3b, and therefore it is nonsingular in codimension 1. X is also connected, since there is at least a total ramification point. So we conclude that X is irreducible. ∎

4. An example of triple cover with non trivial associated extension

Now that we have characterized the building maps such that the corresponding cover is reduced, we are able to show that there exist triple covers such that the extension $0\to\mathcal{O}_Y\to\varphi_*\mathcal{O}_X\to E\to 0$ is not split. The following example was pointed out to the author by T. Ekedahl.

Example 4.1. Let Y be a smooth complete variety over an algebraically closed field of characteristic 3. Let $\mathscr{L}$ be an invertible sheaf on Y such that the following conditions hold:

$$\mathscr{L} \geqq 0; \quad H^1(Y, \mathscr{L}) \neq \{0\}; \quad H^1(Y, \mathscr{L}^3) = \{0\}; \quad H^0(Y, \mathscr{L}^3) \supsetneqq H^0(Y, \mathscr{L})^3.$$

For instance, one may take Y a nonsingular plane quartic and $\mathscr{L} = \mathcal{O}_Y(\theta)$, where θ is an effective half-canonical divisor. Let $0 \to \mathcal{O}_Y \to W \to \mathscr{L}^{-1} \to 0$ be a non trivial extension and let E be the rank 2 vector bundle defined by:

$$(*) \qquad 0 \to \mathcal{O}_Y \to S^2 W \to E \to 0.$$

We wish to show that there exists a triple cover (X, φ) of Y such that the associated module is E and such that the extension $0 \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_X \to E \to 0$ is isomorphic to $(*)$. We remark that E is an extension, too:

$$0 \to \mathscr{L}^{-1} \to E \to \mathscr{L}^{-2} \to 0,$$

and therefore one has:

$$0 \to \mathscr{L}^3 \to F^* E \otimes (\wedge^2 E)^{-2} \to \mathcal{O}_Y \to 0.$$

Let $\{U_i, \{1, z_i\}\}_{i \in I}$ be a trivialization of the bundle W over Y such that the following relation holds over $U_j \cap U_i$:

$$\begin{bmatrix} 1 \\ z_j \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \varepsilon_{ji} & g_{ji} \end{bmatrix} \begin{bmatrix} 1 \\ z_i \end{bmatrix}.$$

Then the transition relations for the bundle E on $U_j \cap U_i$ are:

$$\begin{bmatrix} z_j \\ z_j^2 \end{bmatrix} = \begin{bmatrix} g_{ji} & 0 \\ -\varepsilon_{ji} g_{ji} & g_{ji}^2 \end{bmatrix} \begin{bmatrix} z_i \\ z_i^2 \end{bmatrix}.$$

Since we have assumed $H^1(Y, \mathscr{L}^3) = \{0\}$, there exists $t \in H^0(Y, F^* E \otimes (\wedge^2 E)^{-2})$ that lifts the section $1 \in H^0(Y, \mathcal{O}_Y)$. Assume t is represented by $(t_i, 1)$ on U_i with respect to the given trivializations of the bundles involved. By means of the formulae computed in the proof of Theorem 2.4 it can be checked easily that $\Delta t \in H^1(Y, E^\vee)$, where Δ is the coboundary map for (A′), is represented on $U_j \cap U_i$ by $(\Delta t)_{ji} = (\varepsilon_{ji}, \varepsilon_{ji}^2)$. This amounts to saying that the extension associated with the inseparable triple cover corresponding to the building map t is isomorphic to $(*)$. If the cover corresponding to t is not reduced we consider on U the building map t' obtained by adding to t an element $\gamma \in H^0(Y, \mathscr{L}^3) \setminus H^0(Y, \mathscr{L})^3$. The cover (X, φ) corresponding to t' will be reduced and the extension $0 \to \mathcal{O}_Y \to \varphi_* \mathcal{O}_X \to E \to 0$ will again be isomorphic to $(*)$. Since the covering map φ is inseparable, X and Y are homeomorphic and therefore we can conclude that X is integral. ∎

5. Branch locus and singularities

Throughout this section we shall assume $R=\mathbf{K}$, where $\mathbf{K}$ is an algebraically closed field. Proposition 1.2, ii) can be reformulated as follows:

Proposition 5.1. *Assume that the characteristic of* $\mathbf{K}$ *is* 3 *and that* $\varphi\colon X\to Y$ *is a triple cover with* X *integral. Then* φ *is a separable morphism iff the trace of* (X, φ) *is not zero.*

Proposition 5.2. *Let* (X, φ) *be an integral triple cover. Then the branch locus of* φ *is the zero locus of the section* $B(X, \varphi)\in H^0(Y, (\wedge^2 E)^{-2})$. *If the characteristic of* $\mathbf{K}$ *is* 3, *then the locus of points of* Y *over which there is total ramification is defined by the vanishing of the trace; if the characteristic of* $\mathbf{K}$ *is different from* 3, *it is defined as the locus where the form* Q *has rank* 1.

Proof. We refer to [1, Proposition 6.6, page 124] for the proof of the first statement and we just prove the last part of the proposition here. Let $y_0\in Y$ be a closed point and let $A=\varphi_*\mathcal{O}_{X,y_0}\otimes k(y_0)$. A is a 3-dimensional $\mathbf{K}$-algebra, and therefore it decomposes as a direct sum of local algebras. If φ is not totally ramified over y_0, then there exists an isomorphism $A\cong\mathbf{K}\oplus A_2$, where A_2 is a 2-dimensional $\mathbf{K}$-algebra. In this case, $\mathrm{Tr}_{A/\mathbf{K}}=\mathrm{Id}_{\mathbf{K}}\oplus\mathrm{Tr}_{A_2/\mathbf{K}}$ and rank $Q\geqq 2$. Conversely, assume φ is totally ramified over y_0. Then A is a local algebra with maximal ideal M. We may choose $t, s\in M$ in such a way that $\{1, t, s\}$ is a basis for A over $\mathbf{K}$. The corresponding matrix representation for Q is:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The conclusion is now evident. ∎

Corollary 5.3. *Assume the characteristic of* $\mathbf{K}$ *is different from* 3. *The set of points of* Y *over which there is total ramification is the zero set of a symmetric map* $\zeta\colon E\to E^\vee$ *such that* $\det(\zeta)=3B$.

Proof. Taking the second exterior power of the exact sequence $0\to\mathcal{O}_Y\to\varphi_*\mathcal{O}_X\to E\to 0$ one obtains:

$$0\to E\xrightarrow{i}\wedge^2\varphi_*\mathcal{O}_X\to\wedge^2 E\to 0$$

and dualizing:

$$0\to(\wedge^2 E)^{-1}\to\wedge^2\varphi_*\mathcal{O}_X^\vee\xrightarrow{\pi}E^\vee\to 0.$$

Also the bilinear map Q defines a linear map $L\colon \varphi_*\mathcal{O}_X\to\varphi_*\mathcal{O}_X^\vee$. Define ζ as follows:

$$\begin{tikzcd} E \arrow[r, "\zeta"] \arrow[d, "i"'] & E^{\vee} \\ \Lambda^2 \varphi_* \mathcal{O}_X \arrow[r, "\Lambda^2 L"] & \Lambda^2 \varphi_* \mathcal{O}_X^{\vee} \arrow[u, "\pi"'] \end{tikzcd} .$$

Choosing normal local coordinates $\{1, z, w\}$ for $\varphi_* \mathcal{O}_X$ over an open set $U \subseteqq Y$, one gets:

$$Q = \begin{bmatrix} 3 & f & e \\ f & 4be+3f^2 & 3bc-ef \\ e & 3bc-ef & 4cf+3e^2 \end{bmatrix}; \quad \zeta = \begin{bmatrix} 4(3be+2f^2) & 9bc-4ef \\ 9bc-4ef & 4(3cf+2e^2) \end{bmatrix}$$

and $\det(\zeta) = 3(72bcef + 32be^3 + 32cf^3 + 16e^2f^2 - 27b^2c^2)$.

By Proposition 5.2, there is total ramification over a point $y_0 \in Y$ iff the rank of Q is 1. It is immediate to check that this is true iff ζ vanishes at y_0. ∎

Proposition 5.4. *Let T denote the set of points $y \in Y$ such that the triple cover (X, φ) is totally ramified over y. If φ is separable, T is in general a codimension* 2 *subset of Y and the branch locus B is singular at the points of T. Assume Y is smooth. Then X is not smooth over a point $y \in Y$ iff:*

a) *$y \in B$, $y \notin T$ and B is singular at y.*
b) char $(\mathbf{K}) = 3$, *$y \in T$ and*:
 i) $b = c = 0$ at y.
 ii) $b \neq 0$, $-z^2df + bzde + b(cdb - bdc) = 0$ at y.
 iii) $c \neq 0$, $-w^2de + cwdf + c(bdc - cdb) = 0$ at y.
In particular, if φ is inseparable then X is singular over y iff $\Delta(y) = 0$.
c) char $(\mathbf{K}) \neq 3$, *$y \in T$ is a point of B of multiplicity > 2.*

Proof. Assume first char $(\mathbf{K}) \neq 3$. Let U be an open affine neighbourhood of y such that $E|_U$ is trivial and choose normal local coordinates $\{1, z, w\}$ for $\varphi_* \mathcal{O}_X|_U$. We have two cases to consider:

i) $b = c = 0$ at y.

If $y \notin B$, then X is smooth over y, so we may assume $y \in B$. Using the formulas computed in Corollary 5.3, one sees that this implies $ef = 0$ at y. If $e = f = 0$ at y, then it is easy to check directly, using the local equations for the triple cover, the branch locus and T, that $y \in T$ it is a point of multiplicity $\geqq 4$ on B and that X is singular over y. If, say, $e \neq 0$, then $y \notin T$ and B is smooth at y iff $db \neq 0$. The fibre of φ at y consists of the points $x_1 = (0, e)$ and $x_2 = (0, -e)$. It is easy to verify that x_2 is always a smooth point of X, while x_1 is smooth iff $db \neq 0$. The case $f \neq 0$ follows by symmetry.

ii) $b \neq 0$ or $c \neq 0$ at y.

Say $b \neq 0$ at y. Then it is possible to solve for w in the equations of the triple cover and x can be described locally around y as the zero locus of a polynomial

$P(z)+z^3+tz^2+pz+q$, $t, p, q\in \mathcal{O}_Y$. By a change of coordinates of the form $z'=z+\frac{t}{3}$ it is possible to eliminate the coefficient of z^2 from P. So we may assume that X is defined locally by: $z^3+pz+q=0$. Then one has:

$$Q=\begin{bmatrix} 3 & 0 & -2p \\ 0 & -2p & -3q \\ -2p & -3q & 2p^2 \end{bmatrix}; \quad B=\{27q^2+4p^3=0\}; \quad T=\{p=q=0\}.$$

If $y\in T$, then X is smooth over y iff $dq\neq 0$ at y iff B has a double point at y. If $y\in B\setminus T$, X is simply ramified at the point $z'=-\frac{3q}{2p}$ over y. z' is a singular point of X iff $0=z\,dp+dq|_{z'}=-\frac{3q}{2p}dp+dq|_y$ iff, using the relation: $4p^3+27q^2|_y=0$, iff $2p^2\,dp+9q\,dq|_y=0$ iff B is singular at y.

Assume now char (K)$=3$. We distinguish again between two cases:

i) $b=c=0$ at y. Exactly the same argument as in characteristic different from 3, case i), applies.

ii) $b\neq 0$ or $c\neq 0$ at y. Say $b\neq 0$; again it is possible to solve for w and get a local equation for X of the form $z^3+tz^2+pz+q=0$, $p, t, q\in \mathcal{O}_Y$. Assume $t\neq 0$ at y. Changing coordinates to $z'=z+\frac{p}{t}$ one gets the following equation for X in a neighbourhood of y: $z^3+tz^2+r=0$. Then one has the following local formulas:

$$Q=\begin{bmatrix} 0 & -t & t^2 \\ -t & t^2 & -t^3 \\ t^2 & -t^3 & t^4+rt \end{bmatrix}; \quad B=\{t^3r=0\}$$

$y\in B$ iff $r=0$ at y. Then X is ramified at the point $z'=0$ over y and it is singular there iff $dr=0$ at y iff B is singular at y. Assume now $t=0$ at y and $y\in B$. In this case $y\in T$ and B is singular at y. The equation for X around y is: $z^3-fz^2+(be-f^2)z-bef+f^3-b^2c=0$. Since φ is totally ramified over y, by Proposition 5.2 we have $e=f=0$ and X is smooth over y iff b) ii) holds. The case $c\neq 0$ follows by symmetry. ∎

Corollary 5.5. i) *If the dimension of Y is $\geqq 4$, then the general separable triple cover of Y is singular.*

ii) *If the dimension of Y is $\geqq 2$, then the general inseparable triple cover of Y is singular.*

Proof. i) (See [3, Corollary 5.3].) If the dimension of Y is $\geqq 4$, the building map will have in general at least a zero $y\in Y$ and by Proposition 5.4 X will be singular over y.

ii) Same argument as in the proof of i), remarking that the building map is in this case a section of a rank 2 bundle. ∎

6. Inseparable triple covers

In this section we will study in greater detail inseparable triple covers. We will assume that Y is a smooth variety of dimension n over an algebraically closed field $\mathbf{K}$ of characteristic 3 and that (X, φ) is a smooth inseparable triple cover of Y whose associated module is E.

Proposition 6.1. *In the hypotheses above, Ω_Y^1 is an extension of $\wedge^2 E$ by a locally free sheaf $\mathscr{F}$ of rank $n-1$. The map $\wedge^2 E \to \Omega_Y^1$ is given by the section*

$$\Lambda \in H^0(Y, \operatorname{Hom}(\wedge^2 E, \Omega_Y^1))$$

defined in Lemma 3.2. *Moreover, the image of the tangent map $\varphi_*\colon \mathscr{T}_X^1 \to \varphi^* \mathscr{T}_Y^1$ is isomorphic to $\varphi^* \mathscr{F}^\vee$.*

Proof. Since X is smooth, it is in particular reduced and so, by Proposition 3.3, $\Lambda(X, \varphi)$ is not the zero map. We have a short exact sequence of sheaves on Y: $0 \to \wedge^2 E \xrightarrow{\Lambda} \Omega_Y^1 \to \mathscr{F} \to 0$, where $\mathscr{F} = \operatorname{coker} \Lambda$, $\mathscr{F}$ is locally free iff Λ does not vanish anywhere on Y iff, by Proposition 5.4, b), X is nonsingular. Now let $y \in Y$: since X is nonsingular by assumption, we have $b \neq 0$ or $c \neq 0$ at y. Say $b \neq 0$; then we can solve for w and X is defined locally around y by $z^3 - b^2 c = 0$. So the annihilator of the image of φ_* is generated by $bdc - cdb$ around y and it is therefore the pullback via φ^* of the image of Λ. So $\operatorname{Im} \varphi_*$ is isomorphic to $\varphi^* \mathscr{F}^\vee$. ∎

Corollary 6.2. *Let Y be a nonsingular curve and let (X, φ) be a nonsingular inseparable triple cover of Y with associated bundle E. Then: $\omega_Y \cong \wedge^2 E$.*

Proof. Immediate by Proposition 6.1. ∎

Corollary 6.3. *Let Y be a nonsingular surface and let (X, φ) be a nonsingular inseparable triple cover of Y with associated module E. Then we have an exact sequence of sheaves on Y:*

$$0 \to \wedge^2 E \xrightarrow{\Lambda} \Omega_Y^1 \to \omega_Y \otimes (\wedge^2 E)^{-1} \to 0.$$

In particular, if Y is complete, then the second Chern class $c_2(Y)$ is even.

Proof. The first statement is just Proposition 6.1. If D is the divisor on Y corresponding to $\wedge^2 E$ and K is the canonical divisor of Y, then we have $c_2 = D(K-D)$ and so, by Riemann—Roch, c_2 is even. ∎

Proposition 6.4. *Let (X, φ) be a smooth inseparable triple cover of Y with associated module E. Then we have the following short exact sequence of locally free sheaves on Y:*

$$0 \to S^2 E^\vee \otimes \wedge^2 E \to \varphi_* \mathscr{T}_X^1 \to \mathscr{F}^\vee \otimes \varphi^* \mathcal{O}_X \to 0.$$

Proof. By Proposition 6.1 we have an exact sequence of sheaves on Y: $0\to\mathscr{L}\to\mathscr{T}_X^1\xrightarrow{\varphi_*}\varphi^*\mathscr{F}^\vee\to 0$, where $\mathscr{L}$ is a line bundle. Taking the direct image of this sequence, one gets: $0\to\varphi_*\mathscr{L}\to\varphi_*\mathscr{T}_X^1\to\mathscr{F}^\vee\otimes\varphi_*\mathcal{O}_X\to 0$, since $\mathscr{R}^1\varphi_*\mathscr{L}=0$, the map φ being finite. So, to prove the proposition we must show that $\varphi_*\mathscr{L}$ is isomorphic to $S^2E^\vee\otimes\Lambda^2E$. We will do this by means of an explicit computation. Assume that $\{U_i\}_{i\in I}$ is an affine open cover of Y, that $\{1, z_i, w_i\}_{i\in I}$ are normal local coordinates for $\varphi_*\mathcal{O}_X|_{U_i}\,\forall i\in I$ and that the following relations hold on $U_i\cap U_j$:

$$\begin{bmatrix} z_j \\ w_j \end{bmatrix} = \begin{bmatrix} \alpha & \gamma \\ \beta & \delta \end{bmatrix}\begin{bmatrix} z_i \\ w_i \end{bmatrix} + \begin{bmatrix} t_{ji} \\ s_{ji} \end{bmatrix}$$

where t_{ji} and s_{ji} are given by the formulas computed in the proof of Theorem 2.4. Then a local basis for $\varphi_*\mathscr{L}$ over an open set U_j is the following:

$$v_j^1 = w_j\frac{\partial}{\partial z_j} - c_j\frac{\partial}{\partial w_j};\quad v_j^2 = z_j\frac{\partial}{\partial z_j} - w_j\frac{\partial}{\partial w_j};\quad v_j^3 = b_j\frac{\partial}{\partial z_j} - z_j\frac{\partial}{\partial w_j}.$$

An explicit computation of transition formulas now yields:

$$\begin{bmatrix} v_j^1 \\ v_j^2 \\ v_j^3 \end{bmatrix} = \frac{1}{\alpha\delta-\beta\gamma}\begin{bmatrix} \delta^2 & -2\gamma\delta & \gamma^2 \\ -\beta\delta & \alpha\delta+\beta\gamma & -\alpha\gamma \\ \beta^2 & -2\alpha\beta & \alpha^2 \end{bmatrix}\begin{bmatrix} v_j^1 \\ v_j^2 \\ v_j^3 \end{bmatrix}.$$

Then one concludes by remarking that the matrix above is the transition matrix for $S^2E^\vee\otimes\Lambda^2E$ on $U_j\cap U_i$. ∎

Corollary 6.5. *Let Y be a smooth curve and let (X, φ) be a smooth inseparable triple cover of Y with associated module E. Then $\varphi_*\mathscr{T}_X^1\cong S^2E^\vee\otimes\Lambda^2E$.*

Corollary 6.6. *Let Y be a smooth surface and let (X, φ) be a smooth inseparable triple cover of Y with associated module E. Then one has the following short exact sequence of locally free sheaves on Y:*

$$0\to S^2E^\vee\otimes\Lambda^2E\to\varphi_*\mathscr{T}_X^1\to\omega_Y^{-1}\otimes\Lambda^2E\otimes\varphi_*\mathcal{O}_X\to 0.$$

Examples 6.6. i) *Curves.* All curves have a nonsingular inseparable triple cover, the Frobenius morphism. A necessary condition for the existence of a reduced inseparable triple cover of a curve C with associated module E is that $(\Lambda^2E)^{-1}\otimes\omega_C$ is a nonnegative line bundle (see Proposition 3.3).

ii) *Nonsingular inseparable triple covers of surfaces.* From Corollary 5.5, i), we know that the "general" inseparable triple cover of a surface is singular. Corollary 6.2 adds more: there are surfaces, e.g. $\mathbf{P}^2(\mathbf{K})$, that do not have any smooth inseparable triple cover. On the other hand, if there exists a curve C and a smooth morphism ψ: $Y\to C$, it is not difficult to check that the pullback of the Frobenius morphism of C is a smooth inseparable triple cover of Y. If E is a vector bundle associated to such a cover, then from Corollary 6.2 we have $\psi^*\omega_C \cong \Lambda^2 E$. Trivial examples of surfaces admitting a smooth morphism onto a curve are products of curves and ruled surfaces.

iii) *Inseparable triple covers of surfaces with "nice" singularities.*

From the local equations for an inseparable triple cover, one sees that at points where the building map $\sigma \in H^0(Y, F^*E \otimes (\Lambda^2 E)^{-2})$ vanishes the triple cover has a singularity with embedding dimension 4. Therefore a necessary condition for a triple cover of a surface Y to have only hypersurface singularities is that the bundle $F^*E \otimes (\Lambda^2 E)^{-2}$ has a nowhere vanishing section. We propose here a method for constructing inseparable triple covers of surfaces with singularities of type $z^3 = xy$, that are obviously the best one can get in this case. Take $\mathscr{L}$ a line bundle on Y such that (see Section 4):

1) $H^0(Y, \mathscr{L}^3) \setminus H^0(Y, \mathscr{L})^3 \neq 0$

2) $H^1(Y, \mathscr{L}^3) = 0.$

Take E an extension $0 \to \mathscr{L}^{-1} \to E \to \mathscr{L}^{-2} \to 0$. Then one gets $F^*E \otimes (\Lambda^2 E)^{-2} \cong \mathcal{O}_Y \oplus \mathscr{L}^3$. Normalizing the building map $\sigma \in H^0(Y, \mathcal{O}_Y) \oplus H^0(Y, \mathscr{L}^3)$, we may assume it is of the form $\sigma = (1, \tau)$ and we may identify it with $\tau \in H^0(Y, \mathscr{L}^3)$. (Sections of type $(0, \tau)$ do not build reduced covers!) The local description of the cover is the following:

$$\begin{cases} z^2 = w \\ zw = c \\ w^2 = cz \end{cases}$$

where c is a local representation for τ. Equivalently, the cover can be described locally by $z^3 = c$. So the cover corresponding to a section τ is singular over a point $y \in Y$ iff $dc = 0$ at y. The singularity is of type $z^3 = xy$ iff c has a nondegenerate critical point at y. Note that both these conditions are in fact conditions on the section τ, i.e. they do not depend on the trivialization chosen for $\mathscr{L}$. Set $W = H^0(Y, \mathscr{L}^3) \setminus H^0(Y, \mathscr{L})^3$. Define $V \subseteqq W \times Y$ as follows: $V = \{(\tau, y) | d\tau = 0 \text{ at } y\}$. Consider the following properties:

a) the codimension of V is exactly 2 and V is irreducible.

b) there exists $(\tau, y) \in V$ such that τ has a nondegenerate critical point at y.

The general section of $H^0(Y, \mathscr{L}^3)$ yields a cover with isolated singularities iff a) holds. There exists an open set of $H^0(Y, \mathscr{L}^3)$ such that the corresponding covers have singularities of type $z^3 = xy$ iff both a) and b) hold. For instance, if $Y = \mathbf{P}^2(\mathbf{K})$ and $\mathscr{L} = \mathcal{O}_Y(m)$, $m \geqq 1$, both a) and b) are verified and the covers one gets in this case are the well known Zariski surfaces.

7. Lifting triple covers to characteristic zero

Assume R is a discrete valuation ring of characteristic 0, $\mathscr{M}\subseteq R$ is the maximal ideal and $3\in\mathscr{M}$. Denote by K the residue field $R/\mathscr{M}$. Let Y be a scheme over $\operatorname{spec} R$ and let Y_0 be the fibre of Y over the closed point of of $\operatorname{spec} R$:

$$\begin{array}{ccc} Y_0 & \longrightarrow & Y \\ \downarrow & & \downarrow \\ \operatorname{spec} K & \rightarrow & \operatorname{spec} R \end{array}$$

Definition 7.1. *Given a triple cover (X_0, φ_0) of Y_0, we say that (X_0, φ_0) "lifts to characteristic zero" if there exists a triple cover (X, φ) of Y such that (X_0, φ_0) is obtained by (X, φ) by base change.*

Assume now that E_0 is the vector bundle on Y_0 associated to (X_0, φ_0) and that E is a rank 2 vector bundle on Y such that $E_0=E|_Y$. Then the problem of determining whether there exists a cover (X, φ) of Y such that the associated module is E and such that its reduction $\operatorname{mod}\mathscr{M}$ *is (X, φ) is equivalent to determining whether a given section of $S^3E_0\otimes(\wedge^2E_0)^{-2}$ can be lifted to a section of $S^3E\otimes(\wedge^2E)^{-2}$. Let $t\in\mathscr{M}$ be a uniformizing parameter, let $\mathscr{F}$ be a coherent sheaf on Y and let $\mathscr{F}_0=\mathscr{F}|_{Y_0}$. Then we have the following short exact sequence of sheaves on Y: $0\to\mathscr{F}\xrightarrow{t}\mathscr{F}\to\mathscr{F}_0\to 0$ and the corresponding cohomology long exact sequence:*

$$0\to H^0(Y,\mathscr{F})\xrightarrow{t}H^0(Y,\mathscr{F})\to H^0(Y,\mathscr{F}_0)\xrightarrow{\partial}H^1(Y,\mathscr{F})\to\dots.$$

So $\sigma\in H^0(Y,\mathscr{F}_0)$ lifts to characteristic zero iff $\partial\sigma=0$.

Example 7.2. $Y=\mathbf{P}^n_R$, $Y_0=\mathbf{P}^n_K$, $E_0=\mathcal{O}_{\mathbf{P}^n_K}(m)\oplus\mathcal{O}_{\mathbf{P}^n}(1)$, m, $l\in\mathbf{Z}$. Then every triple cover of Y lifts to characteristic zero. In fact one just takes $E=\mathcal{O}_{\mathbf{P}^n_R}(m)\oplus\mathcal{O}_{\mathbf{P}^n_R}(l)$ and observes that the map: $H^0(Y,\mathcal{O}_Y(m))\to H^0(Y_0,\mathcal{O}_{Y_0}(m))$ is a surjection $\forall m\in\mathbf{Z}$.

Proposition 7.3. *Let Y be a projective R-scheme such that $Y_0=Y\times_{\operatorname{spec} R}\operatorname{spec} K$ be a curve. Denote by $\bar{K}$ the algebraic closure of K and assume $\bar{Y}=Y_0\times_{\operatorname{spec} K}\operatorname{spec}\bar{K}$ is irreducible of genus $\geqq 2$. Let $F_0\colon Y_0^{(3)}\to Y_0$ be the Frobenius morphism. Then F_0 cannot be lifted to characteristic zero.*

Proof. We wish to prove the proposition by contradiction. So assume $F\colon X\to Y$ is a triple cover that lifts F_0 and assume E is the vector bundle associated to (X, F). By Proposition 3.4, $H^0(Y,(\wedge^2E)^{-2})\neq\{0\}$. Since Y is projective, $H^0(Y,(\wedge^2E)^{-2})$ is a finitely generated R-module. Then the map $H^0(Y,(\wedge^2E)^{-2})\xrightarrow{t}H^0(Y,(\wedge^2E)^{-2})$ is not surjective by Nakayama's lemma. If $E_0=E|_{Y_0}$, it follows that $H^0(Y_0,(\wedge^2E_0)^{-2})\neq\{0\}$. Let $\bar{E}$ be the pullback of E_0 to $\bar{Y}$, then one has $H^0(\bar{Y},(\wedge^2E)^{-2})=H^0(Y,(\wedge^2E)^{-2})\otimes_K\bar{K}\neq\{0\}$. Let $\nu\colon Y'\to\bar{Y}$ be the normalization map. Then

$H^0(\bar{Y}, (\wedge^2\bar{E})^{-2})$ injects in $H^0(Y', v^*(\wedge^2\bar{E})^{-2})$ and so the latter is a nonzero module. On the other hand, $v^* \bar{E}$ is the module associated to the Frobenius map on Y' and so, by Corollary 6.2, there is an isomorphism $\wedge^2 E \cong \omega_Y$, contradicting the existence of a nonzero section of $(\wedge^2 E)^{-2}$. ∎

Remark 7.4. Proposition 7.3 can be proven exactly in the same way in the more general case of a curve defined over any field of positive characteristic. Moreover, the proposition is almost trivial if Y_0 is a smooth curve. In this case Y is a regular scheme and the fibre Y_ξ over the generic point of $\operatorname{spec} R$ is a curve of the same genus as Y_0, by flatness. Assume that X is a scheme over $\operatorname{spec} R$ with closed fibre Y and assume that $G\colon X \to Y$ lifts $F\colon Y_0 \to Y_0$. Then the corresponding morphism $G_\xi\colon X_\xi \to Y_\xi$ of the generic fibres is an isomorphism, since X_ξ and Y_ξ are smooth curves of the same (positive) genus. It follows that the degree of G is 1 and therefore that the degree of F is also 1, which is a contradiction.

8. Invariants of triple covers of surfaces

In this section we assume that X, Y are smooth complete surfaces over an algebraically closed field and that $\varphi\colon X \to Y$ is a triple cover. We wish to compute the invariant K_X^2 and $\chi(\mathcal{O}_X)$ for the surface X in terms of K_Y^2, $\chi(\mathcal{O}_Y)$ and of the Chern classes b_1 and b_2 of the bundle E.

Proposition 8.1. *Let X, Y be smooth complete surfaces and let $\varphi\colon X \to Y$ be a triple cover, then:*

$$\chi(\mathcal{O}_X) = 3\chi(\mathcal{O}_Y) + \tfrac{1}{2} b_1^2 - \tfrac{1}{2} b_1 K_Y - b_2.$$

Proof. Since φ is an affine morphism, $\chi(\mathcal{O}_X) = \chi(\varphi_*\mathcal{O}_X) = \chi(\mathcal{O}_Y) + \chi(E)$. The result now follows from the Riemann—Roch theorem. ∎

To compute K_X^2 we need the following:

Lemma 8.2. *Let X, Y be smooth varieties and let $\varphi\colon X \to Y$ be a triple cover, then:*

$$\varphi_*\omega_X^2 = S^2 E^\vee \otimes \omega_Y^2.$$

Proof. Consider the following short exact sequence of locally free sheaves on Y:

$$0 \to \mathcal{R} \to \varphi_*\omega_X \otimes_{\mathcal{O}_Y} \varphi_*\omega_X \to \varphi_*\omega_X^2 \to 0$$

$\mathcal{R}$ denotes the rank 3 subbundle of $\varphi_*\omega_X \otimes_{\mathcal{O}_Y} \varphi_*\omega_X$ generated by

$$\{sa_1 \otimes a_2 - a_1 \otimes sa_2 \mid s \in \varphi_*\mathcal{O}_X,\ a_1, a_2 \in \varphi_*\omega_X\}.$$

Duality for finite flat morphisms implies: $\varphi_*\omega_X \cong \varphi_*\mathcal{O}_X^\vee \otimes \omega_Y$. Let $U \subseteq Y$ be an affine open set, let $\{1, z, w\}$ be a normal local base for $\varphi_*\mathcal{O}_X$ on U and let σ be a generator for $\omega_Y|_U$. X is described over U by the relations in Corollary 2.2. Denote by $\{\varphi^1, \varphi^2, \varphi^3\}$ the base of $\varphi_*\mathcal{O}_X^\vee$ dual to $\{1, z, w\}$. The following is a local basis for $\varphi_*\mathcal{O}_X^\vee \otimes \omega_Y$ on U: $\psi^1=\varphi^1\otimes\sigma$; $\psi^2=\varphi^2\otimes\sigma$; $\psi^3=\varphi^3\otimes\sigma$. The structure of $\varphi_*\mathcal{O}_X$-module of $\varphi_*\mathcal{O}_X^\vee \otimes \omega_Y$ is given by:

$$z\psi^1 = (be+f^2)\psi^2+(bc-ef)\psi^3; \quad w\psi^1 = (bc-ef)\psi^2+(cf+e^2)\psi^3$$
$$z\psi^2 = \psi^1+e\psi^3; \quad w\psi^2 = e\psi^2+c\psi^3$$
$$z\psi^3 = b\psi^2+f\psi^3; \quad w\psi^3 = \psi^1+f\psi^2$$

$\mathcal{R}$ is generated by $\{z\psi^i\otimes\psi^j-\psi^i\otimes z^j, w\psi^i\otimes\psi^j-\psi^i\otimes w\psi^j | i, j=1, 2, 3,\}$. It is easy to chzck that the submodule of $\mathcal{R}$ generated by $\{z\psi^i\otimes\psi^i-\psi^i\otimes z\psi^i, w\psi^i\otimes\psi^i-\psi^i\otimes w\psi^i, z\psi^i\otimes\psi^j-\psi^i\otimes z\psi^j+(z\psi^j\otimes\psi^i-\psi^i\otimes z\psi^j), w\psi^i\otimes\psi^j-\psi^j\otimes w\psi^i+(w\psi^j\otimes\psi^i-\psi^j\otimes w\psi^i) | i, j=1, 2, 3\}$ coincides with the submodule A generated by:

$$\{\psi^1\otimes\psi^2-\psi^2\otimes\psi^1, \psi^1\otimes\psi^3-\psi^3\otimes\psi^1, b(\psi^2\otimes\psi^3-\psi^3\otimes\psi^2), e(\psi^2\otimes\psi^3-\psi^3\otimes\psi^2),$$
$$f(\psi^2\otimes\psi^3-\psi^3\otimes\psi^2), c(\psi^2\otimes\psi^3-\psi^3\otimes\psi^2)\}.$$

Since X is nonsingular by assumption, c, b, e, f cannot vanish simultaneously and so A is just the submodule of alternating elements, generated by $\{\psi^1\otimes\psi^2-\psi^2\otimes\psi^1, \psi^1\otimes\psi^3-\psi^3\otimes\psi^1, \psi^2\otimes\psi^3-\psi^3\otimes\psi^2\}$. Then it follows $\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y\otimes\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y/\mathrm{A} \cong S^2\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2$ and one has the following short exact sequence on Y:

$$0 \to \mathcal{R}/A \to S^2\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2 \to \varphi_*\omega_X^2 \to 0.$$

By abuse of notation we will denote the elements in $\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y\otimes\varphi_*\mathcal{O}_X^\vee\omega_Y$ and their classes in $S^2\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2$ by the same symbols. $\mathcal{R}/A$ is generated by $\{z\psi^i\otimes\psi^j-\psi^i\otimes z\psi^j, w\psi^i\otimes\psi^j-\psi^i\otimes w\psi^j | i, j=1, 2, 3, i<j\}$. A computation shows that a basis for $\mathcal{R}/A$ the following:

$$z\psi^1\otimes\psi^2-\psi^1\otimes z\psi^2 = -\psi^1\otimes\psi^1-e\psi^1\otimes\psi^3+(be+f^2)\psi^2\otimes\psi^2+(bc-ef)\psi^2\otimes\psi^3$$
$$z\psi^2\otimes\psi^3-\psi^2\otimes z\psi^3 = \psi^1\otimes\psi^3+e\psi^3\otimes\psi^3-b\psi^2\otimes\psi^2-f\psi^2\otimes\psi^3$$
$$w\psi^2\otimes\psi^3-\psi^2\otimes w\psi^3 = -\psi^2\otimes\psi^1-f\psi^2\otimes\psi^2+e\psi^2\otimes\psi^3+c\psi^3\otimes\psi^3.$$

Now consider the following short exact sequence on Y:

$$0 \to \varphi_*\mathcal{O}_X \to S^2\varphi_*\mathcal{O}_X \to S^2E \to 0.$$

Taking duals and tensoring with ω_Y^2, one gets:

$$0 \to S^2E^\vee\otimes\omega_Y^2 \to S^2\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2 \to \varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2 \to 0.$$

We will prove the statement of the lemma by showing that the subbundle $S^2E^\vee\otimes\omega_Y^2$ of $S^2\varphi_*\mathcal{O}_X^\vee\otimes\omega_Y^2$ maps isomorphically onto $\varphi_*\omega_X^2$. A local basis for $S^2E^\vee\otimes\omega_Y^2$

on U is: $\{\psi^2\otimes\psi^3, \psi^3\otimes\psi^3, \psi^2\otimes\psi^2\}$. Then it is evident that this basis together with the basis of $\mathscr{R}/A$ written above form a basis of $S^2\varphi_*\mathcal{O}_X^{\vee}\otimes\omega_Y^2$ and therefore that map $S^2E^{\vee}\otimes\omega_Y^2\to\varphi_*\omega_X^2$ is an isomorphism. ∎

Corollary 8.3. *Let X, Y be smooth complete surfaces and let $\varphi\colon X\to Y$ be a triple cover, then:*

$$K_X^2 = 3K_Y^2-4b_1K_Y+2b_1^2-3b_2.$$

Proof. Applying the Riemann—Roch theorem on X yields:

$$K_X^2 = \chi(\omega_X^2)-\chi(\mathcal{O}_X).$$

Since φ is affine, one has: $\chi(\omega_X^2)=\chi(\varphi_*\omega_X^2)$ and, by Lemma 8.2, $\chi(\varphi_*\omega_X^2)=\chi(S^2E^{\vee}\otimes\omega_Y^2)$. A Chern class computation, together with Proposition 8.1, now gives the desired result. ∎

Remark 8.4. The invariants of a triple cover of a surface are computed in [3], Section 10, with a different method.

Appendix: triple covers in characteristic 2

For the sake of completeness we describe here in short the structure of triple covers in characteristic 2. So in this section we assume $\mathbf{K}$ is a field of characteristic 2, Y is an integral separated scheme of finite type over spec $\mathbf{K}$, E is a rank 2 vector bundle on Y and (X, φ) is a triple cover of Y with associated module E as defined in Section 1. The approach to the problem presented in Section 1 applies here too, and therefore the trace map $\mathrm{Tr}\colon \varphi_*\mathcal{O}_X\to\mathcal{O}_Y$, the bilinear form Q and the section $B(X, \varphi)\in H^0(Y, (\wedge^2E)^{-2})$ are defined exactly in the same way. In particular, it is easy to see that Proposition 1.2, i) holds.

Proposition A.1. *Let V be a rank 3 $\mathcal{O}_Y$-algebra such that V is an extension of $\mathcal{O}_Y$ by E, when we regard it as an $\mathcal{O}_Y$-module. Then the trace map $\mathrm{Tr}\colon V\to\mathcal{O}_Y$ gives a splitting of the sequence $0\to\mathcal{O}_Y\to V\to E\to 0$.*

Proof. Just remark that $\forall t\in\mathcal{O}_Y$ $\mathrm{Tr}(t)=t$. ∎

Therefore it is possible to choose local coordinates $\{1, z, w\}$ for V such that $\{z, w\}$ is a basis of the trace zero submodule of V.

Definition A.2. *Such coordinates will be call "special local coordinates".*

Proposition A.3. *Assume V is as in Proposition A.1. Let $\mu\colon S^2V\to V$ be the map defining the multiplication on V and let $\{1, z, w\}$ be special local coordinates for*

$V|_U$, *U an open affine subset of Y. Then μ has the following local expression on U:*

$$\mu(1) = 1; \quad \mu(z) = z; \quad \mu(w) = w;$$

$$\mu(z^2) = az+bw; \quad \mu(w^2) = cz+dw;$$

$$\mu(zw) = dz+aw+ad+bc.$$

Proof. Follows from Proposition 2.2, whose proof does not require any assumption on the characteristic of $\mathbf{K}$. ∎

As in the general case, we have:

Theorem A.4. *Let $\mathbf{K}$ be a field of characteristic 2. Let Y be an integral separated scheme of finite type over* spec $\mathbf{K}$ *and let E be a rank 2 vector bundle on Y. Assume V is a rank 3 vector bundle that is an extension of $\mathcal{O}_Y$ by E and μ: $S^2V \to V$ defines on V an associative multiplication compatible with the $\mathcal{O}_Y$-module structure. Then it is possible to associate with the pairs (V, μ) an element $\sigma(V, \mu) \in H^0(Y, S^3E(\wedge^2E)^{-2})$ in a natural way. Given two such pair $(V, \mu), (V', \mu')$, we have $\sigma(V, \mu) = \sigma(V', \mu')$ iff there exists an isomorphism of extensions ψ: $V \to V'$ which is also an isomorphism of the algebras corresponding to μ and μ'. All such extensions are trivial. Conversely, given $\tau \in H^0(Y, S^3E \otimes (\wedge^2E)^{-2})$, it is possible to define on the trivial extension $0 \to \mathcal{O}_Y \to \mathcal{O}_Y \oplus E \to E \to 0$ a multiplication μ such that $\tau = \sigma(\mathcal{O}_Y \oplus E, \mu)$.*

Proof. The theorem can be proven by using special local coordinates and arguing as in Theorem 2.4. ∎

The results of Section 3 hold in a slightly different form.

Proposition A.5. *Assume* char $\mathbf{K} = 2$. *Let (X, φ) be a triple cover of Y. Then $B(X, \varphi)$ vanishes identically on Y iff either X is not reduced or X consists of two irreducible components, one of which maps inseparably onto Y.*

Proof. First of all remark that the vanishing of B implies that $\varphi_*\mathcal{O}_{X,\xi}$ is not a domain. In fact, assume B vanishes identically: then the form Q is degenerate at the generic point $\xi \in Y$. So there exist $z \in \varphi_*\mathcal{O}_{X,\xi} \setminus \{0\}$ such that $Q(z, x) = \mathrm{Tr}\,(zx) = 0$ $\forall x \in \varphi_*\mathcal{O}_{X,\xi}$. If $\varphi_*\mathcal{O}_{X,\xi}$ were a field, then it would follow $\mathrm{Tr}\,(x) = 0$ $\forall x \in \varphi_*\mathcal{O}_{X,\xi}$, contradicting Proposition 1.2, iii). So we may assume $\varphi_*\mathcal{O}_{X,\xi}$ is not a field. Since an artinian algebra can be decomposed as a direct sum of local algebras, we will prove the proposition by examining all the possible cases:

a) $\varphi_*\mathcal{O}_{X,\xi}$ is a local algebra. Then it has nilpotent elements and Q is degenerate.

b) $\varphi_*\mathcal{O}_{X,\xi} \cong A_1 \oplus A_2 \oplus A_3$, A_i a 1-dimensional $\mathcal{O}_{Y,\xi}$-algebra. In this case X is reduced and consists of three irreducible components and Q is nondegenerate.

c) $\varphi_* \mathcal{O}_{X,\xi} \cong A_1 \oplus A_2$, where A_1, A_2 are local algebras of dimensions 1 and 2 respectively. We have two subcases to consider:

c′) A_2 is not a field. Then it is easy to check that Q is degenerate and X is not reduced.

c″) A_2 is a field. Then Q is degenerate iff A_2 is an inseparable extension of $\mathcal{O}_{Y,\xi}$ iff X consists of 2 components, one of which maps inseparably onto Y. ∎

The branch locus and the singularities of triple covers are described in the next two propositions.

Proposition A.6. *Let Y be a variety over an algebraically closed field $\mathbf{K}$ of characteristic 2. Let (X, φ) be a reduced triple cover of Y with associated module E. Then the branch locus of φ is the zero locus of $B(X, \varphi) \in H^0(Y, (\wedge^2 E)^{-2})$. The branch locus is not reduced and the form Q has rank 1 at branch points. The set of points over which φ is totally ramified is the zero set of a section of $S^2 E \otimes (\wedge^2 E)^{-2}$.*

Proof. For the first statement see Proposition 5.2. To prove the rest of the proposition, take special local coordinates $\{1, z, w\}$ for $\varphi_* \mathcal{O}_X$ on an open affine set $U \subseteq Y$. Then one gets:

$$Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & ad+bc \\ 0 & ad+bc & 0 \end{bmatrix}; \quad B = (ad+bc)^2$$

and the second statement of the proposition is now evident. There is total ramification over a point $y \in Y$ iff $A = \varphi_* \mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathbf{K}(y)$ is a local algebra. Denote by x' the image in A of an element $x \in \varphi_* \mathcal{O}_{X,y}$. At a branch point, one has:

$$\begin{cases} z'^2 = a'z' + b'w' \\ z'w' = d'z' + a'w' \\ w'^2 = c'z' + d'w'. \end{cases}$$

The ideal generated by z' and w' is the only maximal ideal iff it is nilpotent iff $z'^3 = w'^3 = 0$. One has: $z'^3 = (a'^2 + b'd')z'$; $w'^3 = (d'^2 + d'c')w'$. So φ is totally ramified over a point $y \in Y$ iff $a^2 + bd = ad + bc = 0$ at y. To finish the proof we show that these expressions represent locally on U a section of $S^2 E \otimes (\wedge^2 E)^{-2}$. $\varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{K}$, where $\mathcal{K} \cong E$ is the kernel of the trace map. Define ψ as follows:

$$\begin{tikzcd} S^2 \varphi_* \mathcal{O}_X = \mathcal{O}_Y \oplus \mathcal{K} \oplus S^2 \mathcal{K} \arrow[r, "\mu"] & \mathcal{O}_Y \oplus \mathcal{K} = \varphi_* \mathcal{O}_X \\ S^2 E \cong S^2 \mathcal{K} \arrow[u] \arrow[r] & \mathcal{K} \cong E \arrow[u] \end{tikzcd}$$

$\wedge^2\psi\colon \wedge^2 S^2\mathscr{K} \to \wedge^2\mathscr{K}$ has the following local form:

$$z^2\wedge w^2 \to (ac+bd)z\wedge w; \quad z^2\wedge zw \to (a^2+bd)z\wedge w;$$

$w^2\wedge zw \to (c^2+ad)z\wedge w$. We can regard $\wedge^2\psi$ as global section of

$$(\wedge^2 S^2 E)^{\check{}}\otimes\wedge^2 E \cong S^2E\otimes(\wedge^3 S^2 E)^{-1}\otimes\wedge^2 E \cong S^2E\otimes(\wedge^2 E)^{-2}. \quad \blacksquare$$

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