Approximation of plurisubharmonic functions

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Introduction

Let $PSH(X)$, $C(X)$, $C^\infty(X)$ denote respectively the plurisubharmonic functions, the continuous functions and the smooth functions defined on a neighborhood of a set $X\subset \mathbf{C}^n$. Let Ω be a domain in $\mathbf{C}^n$. How are $PSH(\Omega)$, $PSH(\Omega)\cap C(\Omega)$, $PSH(\Omega)\cap C(\bar\Omega)$ and $PSH(\bar\Omega)\cap C(\bar\Omega)$ related? What if we replace $C(\Omega)$ or $C(\bar\Omega)$ by $C^\infty(\Omega)$, $C^\infty(\bar\Omega)$. More specifically, is it possible to approximate elements of one of these classes with elements of a smaller one.

Richberg [4] showed that for every strictly plurisubharmonic $f\in C(\Omega)$ and every $\varepsilon(z)\in C(\Omega)$, $\varepsilon(z)>0$, there exists a strictly plurisubharmonic $\Phi\in C^\infty(\Omega)$ such that $0<\Phi(z)-f(z)<\varepsilon(z)$. The first author, [2], exhibited a smooth Hartogs domain D in $\mathbf{C}^2$ and a plurisubharmonic function f on it, so that f cannot be approximated from above with functions in $PSH(D)\cap C(D)$. Sibony [5] showed that if Ω is a pseudoconvex domain with C^∞-boundary, then $f\in PSH(\Omega)\cap C(\bar\Omega)$ can be approximated uniformly on $\bar\Omega$ with $\Phi\in PSH(\bar\Omega)\cap C^\infty(\bar\Omega)$. He asked if this were true for pseudoconvex domains with C^1 boundary also.

Section 1 deals with Sibony's question. In Theorem 1 it is answered positively for *arbitrary* bounded domains with C^1-boundary. Our proof is entirely different from Sibony's. Assuming in addition that Ω is pseudoconvex, we also show that every $f\in PSH(\Omega)\cap C(\Omega)$ can be approximated uniformly on compact sets with $\Phi\in PSH(\bar\Omega)\cap C^\infty(\bar\Omega)$, Theorem 2. Section 2 contains an example that Theorem 2 without the pseudoconvexity assumption is false. In Section 3 and 4 we show that if Ω is a Reinhardt or a tube domain in $\mathbf{C}^n$, then $f\in PSH(\Omega)$ is pointwise the limit of a monotonically decreasing sequence $\Phi_j\in PSH(\Omega)\cap C^\infty(\Omega)$.

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1. Approximation of continuous plurisubharmonic functions

Theorem 1. *Let Ω be a bounded domain in $\mathbf{C}^n$ with C^1-boundary. Every function in $PSH(\Omega)\cap C(\bar{\Omega})$ can be approximated with functions in $PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$, uniformly on $\bar{\Omega}$.*

Proof. Using compactness of $\bar{\Omega}$ and smoothness of $\partial\Omega$, we can find a finite open cover $B_0, \dots, B_m$ of $\bar{\Omega}$ with the following properties: $B_0 \subset\subset \Omega$; for every j, $1 \leqq j \leqq m$ there exists $x_j \in \partial\Omega \cap B_j$ such that for all small enough $\nu > 0$:

$$\Omega \subset\subset \bigcup_{j=0}^m \tilde{B}_{j,\nu}, \tag{1}$$

where

$$\tilde{B}_{j,\nu} = \{z = \zeta + \nu n_j,\ \zeta \in B_j \cap \Omega\},$$

with n_j, $j \geqq 1$, the unit outward normal to $\partial\Omega$ at x_j, while $n_0 = 0$; and finally,

$$\operatorname{distance}(\partial\Omega, (\partial\Omega\cap B_j) + \nu n_j) > \tfrac{1}{2}\nu. \tag{2}$$

Then for every $0 \leqq j \leqq m$ we can find possibly empty compact sets $K_{j,k}$, $k \neq j$ such that $K_{j,k} \subset B_k$ and $\partial B_j \cap \bar{\Omega} \subset \bigcup_{\substack{k=0 \\ k\neq j}}^m K_{j,k}$. Put $K_k = \bigcup_j K_{j,k}$; $K_k \subset\subset B_k$. Let

$$d = \min_k \operatorname{distance}(K_k, \partial B_k).$$

For every k there exists $\chi_k \in C^\infty(\bar{B}_k)$ with $-1 \leqq \chi_k \leqq 0$, $\chi_k(z) \equiv 0$ for distance $(z, K_k) < \frac{1}{2}d$, while $\chi_k \equiv -1$ on ∂B_k.

Now let $f \in PSH(\Omega)\cap C(\bar{\Omega})$ and $\varepsilon > 0$. Put $\tilde{f}(z) = f(z) + \varepsilon|z|^2$. There exists $\eta_0(\varepsilon) > 0$ such that for $0 < \eta < \eta_0$ the function

$$f_{k,\eta} = \tilde{f} + \eta\chi_k$$

is continuous and plurisubharmonic on $\bar{B}_k \cap \Omega$. We define $f_{k,\eta} = -\infty$ outside $\bar{B}_k \cap \Omega$. Now we set

$$g_\nu(z) = \max_k f_{k,\eta}(z - \nu n_k). \tag{3}$$

Of course, g_ν also depends on ε, η. We will choose ν later, but at least $0 < \nu < d/4$ and so small that (1) is satisfied. Note that (1) implies that for $z \in \bar{\Omega}$ there is at least one k such that $z - \nu n_k \in B_k \cap \Omega$. Now g_ν will approximate f. For $z \in \bar{\Omega}$

$$|f(z) - g_\nu(z)| = |f(z) - \max_k f_{k,\eta}(z - \nu n_k)|$$

$$= |f(z) - \max_{k,(z-\nu n_k)\in\Omega\cap B_k} f_{k,\eta}(z - \nu n_k)| \leqq \eta + \varepsilon M + |f(z) - \max_{z-\nu n_k\in\Omega} f(z - \nu n_k)|,$$

where $M = \max_{z\in\bar{\Omega}} |z|^2$. In view of the uniform continuity of f on $\bar{\Omega}$, the last expression can be made arbitrarily small for η, ε, ν small enough, independently of $z \in \bar{\Omega}$.

We will now show that given $\varepsilon>0$, $0<\eta<\eta_0(\varepsilon)$ the function g_ν will be plurisubharmonic and continuous on a neighborhood of $\bar{\Omega}$ if ν is small enough. By (1) and (2) we can find neighborhoods Ω_ν, $0<\nu<\nu_0$, of $\bar{\Omega}$ such that $\Omega_\nu\subset\bigcup_{j=0}^m B_j$ and $\Omega_\nu\cap((\partial\Omega\cap B_j)+\nu n_j)=\emptyset$ for every j.

Any $z\in\Omega_\nu$ belongs to at least one $\tilde{B}_{k,\nu}$ and may be in the boundary of some other $\tilde{B}_{j,\nu}$. Observe that $z\in\Omega_\nu\cap\partial\tilde{B}_{j,\nu}$ implies that there exists k such that $z-\nu n_j\in K_k$, hence $B(z,\nu)\subset B_k$, as $\nu<d/4$. Therefore

$$f_{j,\eta}(z-\nu n_j)=\tilde{f}(z-\nu n_j)-\eta=f_{k,\eta}(z-\nu n_j)-\eta\leqq f_{k,\eta}(z-\nu n_k)-\eta$$
$$+|\tilde{f}(z-\nu n_j)-\tilde{f}(z-\nu n_k)|\leqq f_{k,\eta}(z-\nu n_k)-\eta/2,$$

if ν is small enough, by uniform continuity of $\tilde{f}$. We conclude that for $z\in\Omega_\nu$, ν small enough, the maximum in (3) is already obtained by taking only those k into account, for which $(z-\nu n_k)$ is an interior point of B_k. Hence g_ν will be continuous and plurisubharmonic on Ω_ν. Finally, approximate g_ν with $g'_\nu\in PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$, by convolving g_ν with a suitable approximate identity. □

Remark. N. Sibony observed that the proof of Theorem 1 also gives the following result.

If a continuous function f on a compact set $K\subset\mathbf{C}^n$ has the property that for every $\varepsilon>0$, $z\in K$ there exists a neighborhood U of z and a plurisubharmonic function h on U such that $|h-f|<\varepsilon$ on $K\cap U$, then f is the uniform limit on K of smooth plurisubharmonic functions defined on a neighborhood of K.

A continuous plurisubharmonic functions φ on a domain $\Omega\subset\mathbf{C}^n$ is called a bounded plurisubharmonic exhaustion function if

a) $\varphi(z)\leqq 0\quad\forall z\in\Omega$,

b) $\Omega_c=\{z\in\Omega\colon\ \varphi(z)<c\}\subset\subset\Omega$ if $c<0$,

c) $\Omega=\bigcup_{c<0}\Omega_c$.

K. Diederich and J. E. Fornaess [1] and later N. Kerzman and J. P. Rosay [3] showed that such functions exist for bounded pseudoconvex domains with C^2, respectively C^1 boundary.

Theorem 2. *Let Ω be a bounded pseudoconvex domain in $\mathbf{C}^n$ with C^1 boundary. Every function in $PSH(\Omega)\cap C(\Omega)$ can be approximated, uniformly on compact sets in Ω, by functions in $PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$.*

Proof. Let φ be a bounded plurisubharmonic exhaustion function for Ω. Let $K_\varepsilon=\{z\in\Omega;\ \varphi(z)\leqq-\varepsilon\}$. It will be enough to show that the approximation is possible on every K_ε, $\varepsilon>0$. Let $f\in PSH(\Omega)\cap C(\Omega)$. Let $\varphi_\varepsilon=\varphi+3\varepsilon/4$, then $\varphi_\varepsilon\leqq-\varepsilon/4$ on K_ε and $\varphi_\varepsilon\geqq\varepsilon/4$ on $L_\varepsilon=K_{\varepsilon/4}-K_{\varepsilon/2}$. We can find $N>0$ such that

$$f_{\varepsilon,N}=\mathrm{Max}\,(f,N\varphi_\varepsilon)\quad\text{on }K_{\varepsilon/4}$$

satisfies $f_{\varepsilon,N}=f$ on K_ε, $f_{\varepsilon,N}=N\varphi_\varepsilon$ on L_ε. Then $f_{\varepsilon,N}$ extends to a continuous plurisubharmonic function on $\bar{\Omega}$ by setting $f_{\varepsilon,N}=N\varphi_\varepsilon$ outside $K_{\varepsilon/4}$. By Theorem 1 we can approximate $f_{\varepsilon,N}$ uniformly on $\bar{\Omega}$ by functions in $PSH(\bar{\Omega})\cap C^\infty(\bar{\Omega})$. Hence f can be approximated by such functions on K_ε. □

2. An example of non-approximability

We next give an example of a bounded domain with C^∞ boundary in $\mathbf{C}^2$, which shows that Theorem 2 without the pseudoconvexity condition is false.

Example. Let

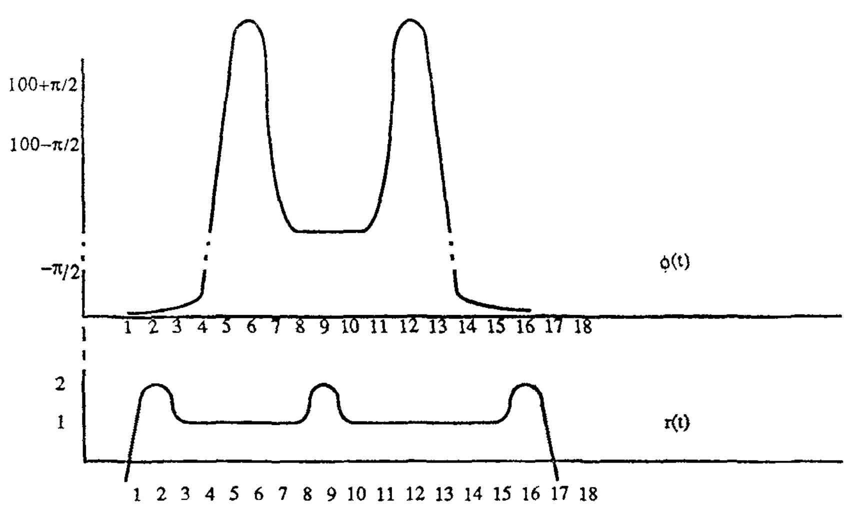
$$\Omega = \{(z, w)\in\mathbf{C}^2;\ |w-e^{i\varphi(|z|)}|^2 < r(|z|)\},$$

where r and φ are as in the figure, that is $r\in C^\infty(\mathbf{R})$, $-1\leqq r\leqq 2$ such that $r(t)\leqq 0$ if and only if $t\leqq 1$ or $t\geqq 17$, $r(t)=2$ for $t=2, 9, 16$, $r(t)=1$ for $3\leqq t\leqq 8$, $10\leqq t\leqq 15$, $r'(1)=2$, $r'(17)=-2$ and $\varphi\in C^\infty(\mathbf{R})$ such that

$$\varphi(t) < -\pi/2 \qquad \text{for} \quad t\leqq 4, \quad t\geqq 14$$

$$\varphi(t) > +\pi/2+100 \qquad 5\leqq t\leqq 6, \quad 12\leqq t\leqq 13$$

$$\varphi(t) < -\pi/2+100 \quad \text{for} \quad 7 < t < 10,$$



Clearly Ω is invariant under $(z, w)\to(e^{i\theta}z, w)$, i.e. Ω is a Hartogs domain, Ω has a smooth boundary, as the gradient of the defining function is non-vanishing at the boundary.

For fixed z_0, $\Omega\cap\{z=z_0\}$ is a disc with radius $(r(|z_0|))^{1/2}$ and centered at $e^{i\varphi(|z_0|)}$. When we vary $|z_0|$ the disc will spin around $w=0$ with varying radius. The annulus

$$A = \{(z, w)\colon\ w = 0,\ 2 \leqq |z| \leqq 15\}$$

is contained in $\bar{\Omega}$. The circles $C_1=\{w=0,\ |z|=2\}$, $C_2=\{w=0,\ |z|=9\}$, $C_3=\{w=0, |z|=16\}$ lie compactly in Ω.

On $\Omega\cap\{3<|z|<8 \text{ or } 10<|z|<15\}$ there exists $h(z, w)$, a continuous branch of $\arg w$, such that

$$\varphi(z)-\pi/2 \leqq h(z, w) \leqq \varphi(z)+\pi/2.$$

Define

$$f_1(z, w) = 0 \quad \text{on } \Omega \quad \{|z| < 4 \text{ or } |z| > 14\} = \Omega_1$$

$$f_2(z, w) = \max\{0, h(z, w)\} \quad \text{on } \Omega \quad \{3 < |z| < 6 \text{ or } 12 < |z| < 14\} = \Omega_2$$

$$f_3(z, w) = \max\{100, h(z, w)\} \quad \text{on } \Omega \quad \{5 < |z| < 8 \text{ or } 10 < |z| < 13\} = \Omega_3$$

$$f_4(z, w) = 100 \quad \text{on} \quad 7 < |z| < 11 = \Omega_4.$$

The functions f_i are plurisubharmonic on Ω_i and $f_i=f_j$ on $\Omega_i\cap\Omega_j$ so $f(z, w):= f_i(z, w)$ for $(z, w)\in\Omega_i$ is plurisubharmonic on Ω.

If g is plurisubharmonic on a neighborhood of $\bar{\Omega}$, then the restriction of g to the annulus A is subharmonic. Now $|g-f|<1$ on C_1, C_2 and C_3 is impossible, because then g would violate the maximum principle on A. Hence f cannot be approximated on compact sets by functions in $PSH(\bar{\Omega})$.

3. Smoothing of families of subharmonic functions

In this section we study the behavior of families of subharmonic functions on domains in **C** when convolved with Lebesgue measure of a finite interval.

For Ω a domain in **C**, let $SH(\Omega)$ denote the set of subharmonic functions on Ω. Let $0<R<1/4$, $I=(-1/4, 1/4)$, $c=\dfrac{1}{2\pi i}$. Let

$$\mathcal{F} = \{\varphi\in SH(I^2)\cap C^\infty(I^2)\colon\ -1 < \varphi < 0$$

and

$$\varphi(z) = c\int \Delta\varphi(\zeta)\log|z-\zeta|\, d\zeta\wedge d\bar{\zeta},\ \operatorname{supp}\Delta\varphi \subset -(R, R)^2\}.$$

Fix $0<\varepsilon<1/4-R$. Let

$$\mathcal{F}_\varepsilon = \{\varphi_\varepsilon\colon\ \varphi\in\mathcal{F}\},$$

where $\varphi_\varepsilon(z)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\varphi(z+t)\,dt$, and let

$$f_\varepsilon(z)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\log|z+t|\,dt.$$

Lemma 1. *For $\varepsilon>0$, $f_\varepsilon(z)$ is a continuous real valued function on* $\mathbf{C}$. *Moreover $f_\varepsilon(z)-\log|z|$ is bounded from below independently of ε and tends to 0 as $z\to\infty$.*

Proof. One computes directly, putting $z=x+iy$,

$$f_\varepsilon(z)=\frac{1}{4\varepsilon}\Big[(x+\varepsilon)\log\big((x+\varepsilon)^2+y^2\big)-2(x+\varepsilon)+2y\arctan\frac{x+\varepsilon}{y}$$

$$-(x-\varepsilon)\log\big((x-\varepsilon)^2+y^2\big)+2(x-\varepsilon)-2y\arctan\frac{x-\varepsilon}{y}\Big].$$

The arctan terms are defined as 0 if $y=0$. Now continuity of $f_\varepsilon(z)$ is obvious. Note that $f_\varepsilon(\varepsilon z)-\log|\varepsilon z|=f_1(z)-\log|z|$. Hence for the second part of the lemma we only have to consider $f_1(z)-\log|z|$ and the result is immediate. □

Lemma 2. *For every $\varepsilon>0$ $\mathscr{F}_\varepsilon$ is equicontinuous on I^2.*

Proof. Pick $\varphi\in\mathscr{F}$, $z, z'\in I^2$, then

$$\varphi_\varepsilon(z)-\varphi_\varepsilon(z')=\frac{1}{2\varepsilon}c\int_{|\zeta|<1}\Delta\varphi(\zeta)\,d\zeta\wedge d\bar\zeta\int_{-\varepsilon}^{\varepsilon}\log\left(\frac{|z-\zeta+t|}{|z'-\zeta+t|}\right)dt$$

$$=c\int_{|\zeta|<1}\Delta\varphi(\zeta)\big(f_\varepsilon(z-\zeta)-f_\varepsilon(z'-\zeta)\big)\,d\zeta\wedge d\bar\zeta.$$

We have

$$0\leqq c\int_{|\zeta|<1}\Delta\varphi(\zeta)\,d\zeta\wedge d\bar\zeta\leqq 1/\log(1/4\sqrt{2})\,c\int\Delta\varphi(\zeta)\log|\zeta|\,d\zeta\wedge d\bar\zeta$$

$$=\varphi(0)/\log(1/4\sqrt{2})<2/\log 8,\quad \text{independently of } \varphi, \text{ because } \varphi(0)>-1.$$

Now the result follows from Lemma 1. □

Lemma 3. *For every $\eta>0$ there exists $\varepsilon_0>0$ so that if $0<\varepsilon<\varepsilon_0$, $\varphi\in\mathscr{F}$ and $z\in I^2$, then $\varphi_\varepsilon(z)>\varphi(z)-\eta$.*

Proof. Computation gives

$$\varphi_\varepsilon(z)-\varphi(z)=c\int_{|\zeta|<1}\Delta\varphi(\zeta)\big(f_\varepsilon(z-\zeta)-\log|z-\zeta|\big)\,d\zeta\wedge d\bar\zeta.$$

Fix $\delta>0$. For some $\varrho>1$, $|f_1(z)-\log|z||<\delta$ if $|z|\geqq\varrho$ and also $|f_\varepsilon(z)-\log|z||<\delta$ if $|z|\geqq\varepsilon\varrho$, by Lemma 1 and the note in its proof. Hence,

$$\varphi_\varepsilon(z)-\varphi(z)\geqq c\int_{|z-\zeta|>\varrho\varepsilon}\Delta\varphi(\zeta)(-\delta)+c\int_{|z-\zeta|<\varrho\varepsilon}\Delta\varphi(\zeta)(-C),$$

where $-C$ is the lower bound given by Lemma 1. We choose $\delta>0$ so small that

$$c\int_{\zeta\in I^2}\Delta\varphi(\zeta)\,\delta<\eta/2, \quad \text{uniformly in } \mathscr{F}.$$

Next choose $\varepsilon>0$ so small that $\log|z-\zeta|\leqq -2C/\eta$ if $|z-\zeta|\leqq\varrho\varepsilon$. Then

$$\varphi_\varepsilon(z)-\varphi(z)>-\eta/2+\eta/2\,c\int_{\zeta\in I^2}\Delta\varphi(\zeta)\log|z-\zeta|\,d\zeta\wedge d\bar\zeta\geqq -\eta. \quad \square$$

Let $A=\{\varrho_1<|z|<\varrho_2\}\subset\mathbf{C}$ and $S=\{z\colon \eta_1<\operatorname{Im} z<\eta_2\}$. Let $\mathscr{F}_A$, $\mathscr{F}_S$ be the subharmonic functions on A, respectively S with values in $(-1,0)$. For $\varepsilon>0$ put

$$\mathscr{F}_A^\varepsilon=\{\varphi_\varepsilon,\ \varphi\in\mathscr{F}_A\},$$

where $\varphi_\varepsilon(z)=\dfrac{1}{2\varepsilon}\displaystyle\int_{-\varepsilon}^{\varepsilon}\varphi(ze^{i\theta})\,d\theta$ and

$$\mathscr{F}_S^\varepsilon=\{\varphi_\varepsilon,\ \varphi\in\mathscr{F}_S\}$$

where $\varphi_\varepsilon=\dfrac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\varphi(z+t)\,dt$. Let K_1 be compact in $A\setminus\{0\}$, K_2 be compact in A, K_3 be compact in S.

Proposition 1.

(i) *The families* $\mathscr{F}_A^\varepsilon|_{K_1}$, $\mathscr{F}_S^\varepsilon|_{K_3}$ *are equicontinuous.*

(ii) *If* $\varepsilon>0$ *is small enough,* $\varphi\in\mathscr{F}_A$ *respectively* $\mathscr{F}_S$ *and* $z\in K_2$ *respectively* $z\in K_3$, *then* $\varphi_\varepsilon(z)-\varphi(z)>-\eta$.

Proof. First we reduce the case of the (degenerated) annulus to that of the strip. Consider the holomorphic map $F(w)=e^{-iw}=z$, $w=u+iv$, $z=x+iy$ on $\tilde S=\{\ln\varrho_1<v<\ln\varrho_2\}$. Here $\ln\varrho_1$ is interpreted as $-\infty$ if $\varrho_1\leqq 0$. Let $G_{\tilde S}$ be the family of subharmonic functions on $\tilde S$ of the form $\psi=\varphi\circ F$, $\varphi\in\mathscr{F}_A$. If $\varphi\circ F\in G_{\tilde S}$ then

$$(\varphi_\varepsilon\circ F)(w)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}\varphi(e^{-iw+i\theta})\,d\theta=(\varphi\circ F)_\varepsilon(w).$$

In view of the periodicity of functions in $G_{\tilde S}$ equicontinuity of $\mathscr{F}_A^\varepsilon|_{K_1}$ will follow from equicontinuity of $\mathscr{F}_S^\varepsilon|_{K_3}$. Let R be an open rectangle such that $K_3\subset R\subset S$ and let g be Green's function for R with pole in $w_0\in K_3$. Let $M=\max_{w\in K_3}(-g(w))<0$ and $\tilde{\mathscr{F}}_R=\left\{\max\left(\dfrac{1}{M}g,\ \dfrac{1}{2}(\varphi-1)\right)\colon \varphi\in\mathscr{F}_S\right\}$. It will be enough to show that $\tilde{\mathscr{F}}_R^\varepsilon|_{K_3}$ is equicontinuous, since the functions in $\tilde{\mathscr{F}}_R$ are harmonic on $R\setminus\{-g\leqq M/2\}$, this follows from Lemma 2 applied to scaled smoothings of elements of $\tilde{\mathscr{F}}_R$.

Next (ii) follows in the same way from Lemma 3. $\square$

4. Smoothing on tubes and Reinhardt domains

We will show that on tube domains and on Reinhardt domains every plurisubharmonic function can be approximated pointwise and monotonically from above by C^∞-strictly plurisubharmonic functions.

For φ a plurisubharmonic function defined on a neighborhood of $z\in\mathbf{C}^n$ and $\varepsilon=(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i>0$ small enough we put

$$\varphi_\varepsilon(z) = \frac{1}{2^n\varepsilon_1\dots\varepsilon_n}\int_{-\varepsilon_1}^{\varepsilon_1}\dots\int_{-\varepsilon_n}^{\varepsilon_n}\varphi(z_1e^{i\theta_1}, \dots, z_ne^{i\theta_n})\,d\theta_n\dots d\theta_1$$

and

$$\varphi^\varepsilon(z) = \frac{1}{2^n\varepsilon_1\dots\varepsilon_n}\int_{-\varepsilon_1}^{\varepsilon_1}\dots\int_{-\varepsilon_n}^{\varepsilon_n}\varphi(z_1+t_1, \dots, z_n+t_n)\,dt_n\dots dt_1.$$

Lemma 4. *Let ψ be plurisubharmonic in a neighborhood of 0 in $\mathbf{C}^n(z_1, \dots, z_n)$. Fix $\varepsilon=(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i>0$. Then $\psi_\varepsilon(z)\to\psi(0)$ as $z\to 0$.*

Proof. By upper-semicontinuity of ψ, $\varlimsup_{z\to 0}\psi_\varepsilon(z)\leqq\psi(0)$. Suppose the lemma does not hold. Then $\psi(0)>-\infty$ and there exist $z^{(j)}\to 0$, $\delta>0$ so that

$$\psi_\varepsilon(z^{(j)})<\psi(0)-\delta. \tag{4}$$

Let $\bar\psi(z)=\psi_{(\pi,\dots,\pi)}(z)$, then $\bar\psi(z)=\bar\psi(|z_1|, \dots, |z_n|)$ is monotonically increasing in each variable and $\bar\psi(0)=\psi(0)$. But (4) and upper-semicontinuity give

$$\begin{aligned}\bar\psi(z^{(j)}) &= \frac{1}{(2\pi)^n}\int_0^{2\pi}\dots\int_0^{2\pi}\psi(z_1^{(j)}e^{i\theta_1}, \dots, z_n^{(j)}e^{i\theta_n})\,d\theta_1\dots d\theta_n\\ &< \frac{1}{(2\pi)^n}\Big(\int_{[-\varepsilon,\varepsilon]^n}\dots+\int_{[-\pi,\pi]^n\setminus[-\varepsilon,\varepsilon]^n}\Big)\dots\\ &\leqq \frac{(2\varepsilon)^n}{(2\pi)^n}(\psi(0)-\delta)+\frac{((2\pi)^n-(2\varepsilon)^n)(\psi(0)+1/2\,\delta(\frac{\varepsilon}{\pi})^n)}{(2\pi)^n}\\ &\leqq \psi(0)-\tfrac{1}{2}\delta\big(\tfrac{\varepsilon}{\pi}\big)^n,\end{aligned}$$

for all large enough j, a contradiction. □

Lemma 5. *Let Ω be a tube or a Reinhardt domain in $\mathbf{C}^n$ and let φ be a nonnegative plurisubharmonic function on Ω. Let K be a compact set in Ω. Then there exists a sequence $\{\varphi_{n,k}\}$ of continuous plurisubharmonic functions on Ω such that $\varphi_{n,k}\downarrow\varphi$ pointwise on K.*

Proof. If Ω is a tube domain we will use φ^ε to approximate φ, while if Ω is Reinhardt, we will use φ_ε. The proofs for both cases are similar, but the case of the Reinhardt domain is complicated because of the "degeneration" of φ_ε at the coordinate hyperplanes. We will restrict ourselves to the case of Reinhardt domains. Fix $\varepsilon=(\varepsilon_1, \dots, \varepsilon_n)$, $\varepsilon_i>0$. We show at first that φ_ε is continuous on Ω. Pick $z^0=(z_1^0, \dots, z_2^0)\in\Omega$. We can assume $z^0=(z_0', z_0'')$ where $z_0'=(z_1^0, \dots, z_k^0)$, $z_0''=(z_{k+1}^0, \dots, z_n^0)$ where $0\leqq k\leqq n$, $z_j^0\neq 0$ if $j\leqq k$, $z_j^0=0$ if $j>k$. Let $_jz=(_jz', _jz'')$ be a sequence tending to z^0, $z'\in\mathbf{C}^k$, $z''\in\mathbf{C}^{n-k}$, and let $_jw=(z_0', _jz'')$. We have

$$\varphi_\varepsilon(_jw)-\varphi_\varepsilon(_jz)\to 0 \quad \text{as} \quad j\to\infty,$$

by repeated application of the equicontinuity part of Proposition 1. Hence it suffices to show that

$$\varphi_\varepsilon(_jw)\to\varphi_\varepsilon(z^0) \quad \text{as} \quad j\to\infty.$$

By Lemma 4 we have for fixed $(\theta_1, \dots, \theta_k)$

$$\frac{1}{2^{n-k}\varepsilon_{k+1}\dots\varepsilon_n}\int_{-\varepsilon_{k+1}}^{\varepsilon_{k+1}}\dots\int_{-\varepsilon_n}^{\varepsilon_n}\varphi(z_1^0e^{i\theta_1}, \dots, z_k^0e^{i\theta_k}, {}_jz_{k+1}''e^{i\theta_{k+1}}, \dots, {}_jz_n''e^{i\theta_n})d\theta_n\dots d\theta_{k+1}$$
$$\to\varphi(z_1^0e^{i\theta_1}, \dots, z_k^0e^{i\theta_k}, 0, 0, \dots, 0).$$

So Lebesgue's dominated convergence theorem gives

$$\varphi_\varepsilon(_jw)$$
$$=\frac{1}{2^n\varepsilon_1\dots\varepsilon_n}\int_{-\varepsilon_1}^{\varepsilon_1}\dots\int_{-\varepsilon_k}^{\varepsilon_k}\Big(\int_{-\varepsilon_{k+1}}^{\varepsilon_{k+1}}\int_{-\varepsilon_n}^{\varepsilon_n}\varphi(z_1^0e^{i\theta_1}, \dots, {}_jz_n''e^{i\theta_n})\,d\theta_n\dots d\theta_{k+1}\Big)\,d\theta_k\dots d\theta_1$$
$$\to\frac{1}{2^n\varepsilon_1\dots\varepsilon_n}\int_{-\varepsilon_1}^{\varepsilon_1}\dots\int_{-\varepsilon_k}^{\varepsilon_k}\varphi(z_1^0e^{i\theta_1}, \dots, z_k^0e^{i\theta_k}, 0, \dots, 0)\,d\theta_k\dots d\theta_1=\varphi_\varepsilon(z^0),$$

as $j\to\infty$.

The mean value inequalities clearly hold for φ_ε, so we have shown that φ_ε is a continuous plurisubharmonic function. Next let $\eta>0$. By repeated application of Proposition 1, if $\varepsilon_1, \dots, \varepsilon_n\leqq\varepsilon(\eta)$ for small enough $\varepsilon(\eta)$, then

$$\varphi_\varepsilon(z)+\eta>\varphi(z) \quad \text{for each} \quad z\in K.$$

Also, by upper-semicontinuity of φ, $\overline{\lim}_{\varepsilon\to 0}\varphi_\varepsilon(z)\leqq\varphi(z)$. Now pick a sequence $\eta_m\searrow 0$ and corresponding $\varepsilon^m=\{\varepsilon_1^m, \dots, \varepsilon_n^m\}=\varepsilon(\eta_m)$ and let

$$\varphi_m(z)=\varphi_{\varepsilon^m}(z)+\eta_m.$$

Now a suitable subsequence φ_{m_k} will approximate φ monotonically from above on K.

Theorem 4. *Let Ω be a Reinhardt domain or a tube domain in $\mathbf{C}^n$ and let φ be a plurisubharmonic function on Ω. Then there exists a sequence of C^∞ strictly plurisubharmonic functions converging pointwise, monotonically, from above to φ.*

Proof. First assume φ is bounded from below. Let $\{K_m\}_{m=1}^\infty$ be a sequence of compact sets in Ω such that $K_m \subset \operatorname{int} K_{m+1}$, $\bigcup_1^\infty K_m = \Omega$. For each m there exists a sequence $\{\varphi_{n,m}\}_{n=1}^\infty$ of continuous plurisubharmonic functions on Ω with $\varphi_{n,m} \searrow$ on K_m, by Lemma 5. We now produce a new sequence: For each m, let $n(m) > m$ be so large that

$$\varphi_{n(m),m} < \varphi_{m,j} + \frac{1}{m} \quad \text{on} \quad K_j, \quad j = 1, \ldots, m.$$

This is possible because by Dini's theorem $\max(\varphi_{l,m}, \varphi_{m,j})$ converges uniformly to $\varphi_{m,j}$ on K_j as $l \to \infty$.

Let $\varphi_k = \sup_{m \geqq k} \varphi_{n(m),m}$. We claim that φ_k is continuous and plurisubharmonic on Ω and that $\varphi_k \searrow \varphi$. In fact we only need to show that φ_k is upper-semicontinuous. Pick $z \in \Omega$, $\delta > 0$, $l > k$ so that $z \in \operatorname{int} K_l$, $1/l < \delta$. Assume $m \geqq n(l)$ and $w \in K_l$. Then

$$\varphi_{n(m),m}(w) < \varphi_{m,l}(w) + \frac{1}{m} < \varphi_{n(l),l}(w) + \delta.$$

Hence

$$\overline{\lim_{w \to z}}\, \varphi_k(w) \leqq \big(\max_{k \leqq m \leqq n(l)} \varphi_{n(m),m}(z) \big) \vee \varphi_{n(l),l}(z) + \delta \leqq \varphi_k(z) + \delta.$$

For general φ let $\varphi_k^N \searrow \max(\varphi, -N)$ as $k \to \infty$. For each m let $n(m)$ be so large that

$$\varphi_{(n)m}^m < \varphi_m^j + \frac{1}{j} \quad \text{on} \quad K_j, \quad j = 1, \ldots, m.$$

Form $\varphi_k = \max_{m \geqq k} \varphi_{n(m)}^m$ and repeat the previous argument.

To obtain strictly plurisubharmonic functions we put

$$\tilde{\varphi}_k(z) = \varphi_k(z) + \frac{1}{k}|z|^2 + \frac{1}{k}.$$

By Richberg's theorem [4] there exist C^∞, strictly plurisubharmonic functions ψ_k such that

$$0 \leqq \psi_k(z) - \tilde{\varphi}_k(z) \leqq \frac{1}{k^2}.$$

Then $\{\psi_k(z)\}$ forms the required sequence. □

5. Approximation on Reinhardt and tube domains in $\mathbf{C}^2$

In this section we will prove that a result similar to Theorem 2 holds for tube domains in $\mathbf{C}^2$. Reinhardt domains can be treated similarly. The proof will be based on some lemmas. Let D be a bounded domain in $\mathbf{R}^2$ with C^2-boundary and let G be the tube with base D, i.e. $G = D \times i\mathbf{R}^2$. We will call a boundary point x

of G (strictly) convex if and only if its projection on the base is a (strictly) convex boundary point of D and similarly for (strictly) concave boundary points.

We split the boundary ∂D of D in three disjoint parts:

$$\partial D = Scx \cup Sce \cup Flt,$$

where Scx denotes the strictly convex part of ∂D, Sce the strictly concave part of ∂D and Flt the flat part of ∂D, i.e. $Flt=\partial D \setminus (Scx \cup Sce)$.

Lemma 6. *Let G be a tube domain and f a plurisubharmonic function on G. If a is a strictly concave boundary point of G, then f is bounded from above in a neighborhood of a.*

Proof. By an affine change of coordinates we can achieve that $a=(0,0)\in \mathbf{C}^2$ and that ∂D at $(0,0)\in \mathbf{R}^2$ is parametrized by $(t, \varphi(t))$, where $\varphi\colon [-2,2]\to \mathbf{R}$ is a C^2-function with

$$\varphi(0) = \varphi'(0) = 0, \quad \varphi''(t) < -1,$$

so that $D\cap[-2,2]\times[-10,10]$ is given by $x_2 > \varphi(x_1)$.

Consider the family of complex discs $\Delta_\varepsilon^a\colon \{|\lambda|<1\}\to \mathbf{C}^2$, associated to a

$$\Delta_\varepsilon^a(\lambda) = (\lambda, -\lambda^2/2+\varepsilon), \quad 0 \leqq \varepsilon < 1.$$

To $b\in\partial G$, $\|b\|<1$ we associate a similar family $\{\Delta_\varepsilon^b\}$ obtained by translation and rotation of $\{\Delta_\varepsilon^a\}$. The boundary of such a disc has distance to $\partial G \geqq 1/4$. It is then clear that $\cup \partial \Delta_\varepsilon^b$ is contained in a compact subset K of G, while $\cup \Delta_\varepsilon^b$ contains a neighborhood N of a in G. By the maximum principle for subharmonic functions applied to $f\circ \Delta_\varepsilon^b$ it follows that f is bounded on N by its maximum on K which is finite. □

Lemma 7. *Let G, f and a be as above and assume that f is continuous on G. Let L be a complex line tangent to a. Then $f|_L$ extends to a subharmonic function in a neighborhood of a relative to L.*

Proof. Without loss of generality we can take $a=(0,0)$, $L=\{z_2=0\}$. Observe that f is subharmonic on $L\setminus\{\operatorname{Re} z_1=0\}$ and by Lemma 6 bounded on a relative neighborhood U of $(0,0)$. There we define $f(iy, 0) =: \limsup_{z\to iy} f(z, 0)$. Clearly $f(z, 0)$ is upper semicontinuous on U. We have to check positivity of the Laplacian $\Delta_z f(z, 0)$ in distribution sense at $z=0$. Let φ be a nonnegative test function on U. Then using Lebesgue's dominated convergence theorem we have

$$\int \Delta\varphi(z) f(z,0) = \int \lim_{u\to 0} \Delta\varphi(z) f(z,u) = \lim_{u\to 0} \int \Delta\varphi(z) f(z,u) \geqq 0. \quad \square$$

The following lemma is very similar to Lemma 2 and Proposition 1. We will consider complex lines in G of the form

(3) $$L:\ \zeta \to (a\zeta+b, c\zeta+d), \quad \zeta\in R, \quad a, c\in\mathbf{R}, \quad b, d\in\mathbf{C}, \quad a^2+c^2=1,$$

where R is the square $|\mathrm{Re}\,\zeta|<C_1$, $|\mathrm{Im}\,\zeta|<C_2$. With f a plurisubharmonic function on some tube domain, recall that f_ε was defined as

$$\frac{1}{4\varepsilon_1\varepsilon_2}\int_{-\varepsilon_1}^{\varepsilon_1}\int_{-\varepsilon_2}^{\varepsilon_2} f(z_1+it, z_2+is)\,dt\,ds.$$

Lemma 8. *Let f, f_ε and R be as before and let L be a collection of lines as in* (3). *Suppose that f is bounded from below and that for $L\in\mathcal{L}$, $L(R)\subset(D\cup Sce)\times i\{\|\mathrm{Im}\,z\|<M\}$. Then for every compact subset K of R, $\{f_\varepsilon\circ L(\zeta)\}_{L\in\mathcal{L}}$ is an equicontinuous family on K.*

Proof. Fix $L\in\mathcal{L}$. By a Euclidean coordinate transform $L(\zeta)$ is transformed into $\zeta\to(\zeta, 0)$. In view of Lemmas 6 and 7 $f(\zeta+is, it)$ is a bounded subharmonic function on R, while the bound is independent of the choice of L and of $t, s<\max\{\varepsilon_1, \varepsilon_2\}$. We can assume $-1\leqq f\leqq 0$ on R and by enlarging R in the imaginary direction, $K+i[-2\|\varepsilon\|, 2\|\varepsilon\|]\subset\{\zeta:\ g(\zeta, 0)\geqq m>0\}$, where g is Green's function for R. As in the proof of Proposition 1 we introduce $f^\sim$ by

$$f^\sim(z_1, z_2) = \max\{f(z_1, z_2)-1,\ -2g(z_1, 0)/m\}.$$

It will be sufficient to prove the Lemma for $f^\sim$. In fact we only have to consider potentials of the form

$$F(\zeta+it, is) = c\int_R \Delta_\eta f^\sim(\eta+it, is)\log|\zeta-\eta|\,d\eta\wedge d\bar\eta.$$

Now we continue as in Lemma 2, keeping in mind that as a consequence of the coordinate transformation, the t, s-domain of integration becomes a parallelogram $P=P_L$ in the plane and is given by

$$P=\{(t, s):\ t\in I(s),\ s\in I\},$$

where I and $I(s)$ are intervals of length $\leqq 2\|\varepsilon\|$, depending on L. All P_L are contained in a fixed compact set determined by $\mathcal{L}$. Now by Fubini's theorem

$$4\varepsilon_1\varepsilon_2\big(F_\varepsilon(\zeta_1, 0)-F_\varepsilon(\zeta_2, 0)\big) = \int_P F(\zeta_1+it, is)-F(\zeta_2+it, is)\,dt\,ds$$

$$= \int_P c\int_R \Delta_\eta f^\sim(\eta+it, is)(\log|\zeta_1-\eta|-\log|\zeta_2-\eta|)\,d\eta\wedge d\bar\eta\,dt\,ds$$

$$= \int_{s\in I} c\int_R \Delta_\eta f^\sim(\eta, is)\int_{t\in I(s)}(\log|\zeta_1-\eta+it|-\log|\zeta_2-\eta+it|)\,dt\,d\eta\wedge d\bar\eta\,ds.$$

The inner integral can be computed explicitly, compare Lemma 1. It follows that for $|\zeta_1-\zeta_2|$ small enough the inner integral is uniformly small, independent of $\eta\in R$, $s\in I$. The proof now ends as that of Lemma 2. □

Lemma 9. *Let $G, f, f_\varepsilon(z)$ and a be as before and assume that f is bounded in a neighborhood of a, then for all small enough ε f_ε is continuous at a.*

Proof. We may assume $a=(0,0)$ and the same parametrization of ∂D as in Lemma 6. Let ${}_nz={}_nx+{}_ny\in G^-$ tend to a. Because f_ε is a convolution in the fiber direction, $f_\varepsilon({}_nz)-f_\varepsilon({}_nx)=O({}_nz-{}_nx)$ and we can assume ${}_ny=0$. We now consider two cases:

If ${}_nx_2\geqq 0$ then

$$|f({}_nx)-f(0)|\leqq|f({}_nx_1,{}_nx_2)-f(\delta,{}_nx_2)|+|f(\delta,{}_nx_2)-f(\delta,0)|+|f(\delta,0)-f(0)|,$$

which becomes arbitrarily small if δ is close to 0 and n is big enough, using Lemma 8 and the continuity of f_ε on G.

If ${}_nx_2\leqq 0$ then let ${}_nb$ be the point on ∂D which has minimal distance to ${}_nx$ and L^n the line through ${}_nx$ parallel to the tangent of ∂D at ${}_nb$. Let ${}_nc$ be the intersection of L^n and $\{x_2=0\}$. Then $|{}_nx-{}_nc|\leqq|{}_nx|$ and $|{}_nc|\leqq|{}_nx|$ by concavity, and application of Lemma 8 to the family $\{L^n\}$ gives the result. □

Lemma 10. *Let f be a continuous function on a convex open subset U of $\mathbf{R}^n$. Then there exists a convex function h on U with $h\geqq f$.*

Proof. We can assume $f>0$ on U. Let φ be a convex exhaustion function for U, i.e. φ is convex, $K_c:=\{\varphi\leqq c\}\subset\subset U$ for every $c\in\mathbf{R}$ and $U=\cup_c K_c$. Replace φ by $\max\{\varphi,0\}$ if necessary. For every m, $m=0,1,2,\ldots$ we define

$$\varphi_m(y):=\big(\max\{f(x)\colon x\in K_{m+2}\}\big)\big(\varphi(y)-m\big).$$

Now we take $h(x)=\sup\{\varphi_m(x)\colon m=0,1,2,\ldots\}$. On any of the K_j we only have to consider finitely many φ_m, hence h is well defined and convex. On $K_{m+2}\setminus K_{m+1}$ we have $\varphi_m\geqq f$, so $h\geqq f$. □

Lemma 11. *Let M be a compact set in the relative interior of Scx. Also, let f be a continuous plurisubharmonic function on G. If F is a compact subset of G, then there exists a plurisubharmonic function $f^\sim$ on G such that $f^\sim|_F=f$ and $f^\sim$ is continuous on $G\cup(M\times i\mathbf{R}^2)$. If $K\subset\partial D\setminus M$ and f is in addition continuous up to $K\times i\mathbf{R}^2$, then so is $f^\sim$.*

Proof. We can assume $f\geqq 0$ on G, replacing if necessary f by $\max\{f,m\}-m$, where $m=\min\{f(z)\colon z\in F\}$. If $Scx=\partial D$, then the technique of the proof of Theorem 2 readily gives the result. From now on we assume $Scx\neq\partial D$. It is convenient to consider Scx connected at first.

There exist an $\varepsilon>0$ and finitely many strictly convex subsets C_j of D contained in slightly larger strictly convex sets B_j, which are *not* contained in D, such that

$$M = \bigcup_j \text{interior}\,(M\cap\partial B_j), \quad \text{interior}\,(M\cap\partial B_j) \subset\subset \text{interior}\,(M\cap\partial C_j)$$

and

$$\{x\in D\setminus C_j\colon d(x, C_j) < \varepsilon\} \subset\subset B_j,$$

where the interior is relative to ∂D. There exist bounded convex exhaustion functions g_j for B_j. We multiply g_j with a positive constant, so that $g_j<-1$ on $\{x\in D\setminus C_j\colon d(x, C_j)<\varepsilon\}$. We take $h_j=\max\{g_j, -1\}$ on C_j, and extend it as -1 on $D^-\setminus C_j$. Now put $h^*=\max h_j$ and consider it as a function on G^-. Next we set $c=-1/2\max\{h^*(z)\colon z\in F\}$ and $h:=h^*+c$; h is a continuous convex, and hence plurisubharmonic, function on G^-, $h=c$ on a relative neighborhood of $M\times i\mathbf{R}^2$ in ∂G, $h\leqq -c$ on F.

We can find a curve Γ in contained in D except for the endpoints p_1, p_2 which are on $Scx\setminus M$, such that the arc determined by p_1, p_2 in Scx contains M. In addition we require that $\Gamma\subset\{h>c/2\}$ and that at the endpoints Γ consists of straight line segments $\Gamma_i:=(p_i; q_i]$, $i=1, 2$, which are orthogonal to Scx and so small that the line passing through q_i and perpendicular to Γ_i will meet Scx before it meets any other point of ∂D, if traversed from q_i in any direction. Let G_i be the component containing $G_i\times i\mathbf{R}^2$ of the subset of G which projects orthogonally onto $\Gamma_i\times i\mathbf{R}^2$; let P_i be the projection. In addition we can take the Γ_i so small that $G_1\cap G_2=\emptyset$. That all of this is possible is a simple consequence of the strict convexity. The set $G\setminus\Gamma\times i\mathbf{R}^2$ consists of two components, G^1 containing F and the other one, G^2.

Let Γ_i' be a line segment in D strictly containing Γ_i. We apply Lemma 10 to f restricted to $\Gamma_i'\times i\mathbf{R}^2$. This furnishes a convex function $\psi_i\geqq f$ on $\Gamma_i'\times i\mathbf{R}^2$. We extend ψ_i to G_i by $\psi_i(z):=\psi(P_i(z))$. The functions ψ_i will be continuous on $G_i^-\setminus(p_i\times i\mathbf{R}^2)$. Let $\Gamma_3=\Gamma\setminus(\Gamma_1\cup\Gamma_2)$ and let

$$\eta(y) = \max\{\psi_1(q_1+iy), \psi_2(q_2+iy), \max\{f(x+iy)\colon x\in\Gamma_3^-\}\}+1$$

on $\mathbf{R}^2$. Lemma 10 gives a convex function $\psi_3\geqq\eta$ on $\mathbf{R}^2$. We will view ψ_3 as a function on $\mathbf{R}^2\times i\mathbf{R}^2$ by $\psi_3(x+iy)=\psi_3(y)$. Next we define a function ψ as follows: on G_i put $\psi=\max\{\psi_i, \psi_3\}$, $i=1, 2$, while on $G\setminus(G_1\cup G_2)$ we set $\psi=\psi_3$. Observe that by definition of ψ_3, $\psi_3>\psi_i$ on $bG_i\cap G$. Hence ψ is a convex function on G and ψ is continuous on $G^-\setminus(\{p_1, p_2\}\times i\mathbf{R}^2)$. Obviously $\psi\geqq f$ on $G\times i\mathbf{R}^2$.

Now $f^\sim$ is constructed as follows: we can find a positive constant c such that $\psi+ch<f$ on F. Also $\psi+ch>f$ on $G\times i\mathbf{R}^2$. Hence the function $f^\sim$ defined by $f^\sim=\psi+ch$ on G^2 and by $f^\sim=\max\{f, \psi+ch\}$ on $G\setminus G^2$ satisfies the requirements of the lemma.

If Scx is not connected, then M might consist of at most finitely many components. For each of them there exists a function like $\psi+ch$. The maximum of these functions should be used instead of $\psi+ch$ in the preceding paragraph to obtain $f^{\sim}$ in the general case.

The second assertion is obvious since h is continuous and the functions ψ_i will be continuous up to $K\times i\mathbf{R}^2$ if f is. □

The next theorem is the main result of this section. In the proof the methods used in Theorem 1 and 2 are combined.

Theorem 4. *Let G be a tube domain in $\mathbf{C}^2$ with C^2-boundary as above. Every function in $PSH(G)\cap C(G)$ can be approximated uniformly on compact sets in G by functions in $PSH(G^-)\cap C(G^-)$.*

Proof. Let F be a compact set in G and $f\in PSH(G)\cap C(G)$. Replacing f by $\max\{f, \min\{f(z)\colon z\in F\}\}$ if necessary, we can assume that f is bounded from below. For ε small, f_ε will be uniformly close to f on F, hence it will be sufficient to approximate f_ε. By Lemma 6 and 9 f_ε is continuous up to the strictly concave boundary points of G.

Since D^- is compact and ∂D is smooth, there exists a finite open cover $\{B_0, \ldots, B_m\}$ of D^- with the same properties as the cover used in the proof of Theorem 1. We will use the same notation as in Theorem 1. We can arrange that the cover has the additional property:

$$\text{(4)} \qquad \text{For } \; j = 1, 2, \ldots, m, \;\; \partial B_j \cap bD \subset Scx \cup Sce.$$

Here we used that the normal of a C^2-curve in $\mathbf{R}^2$ remains constant along components of the flat points. For all small enough n, (4) remains valid for $B_{j,\nu}:=B_j+\nu n_j$ instead of B_j, more precisely, there exists $\nu_0>0$ such that

$$\{\partial B_{j,\nu}\cap\partial D,\; j = 1, 2, \ldots, m,\; 0 \leqq \nu < \nu_0\} \subset M_0 \subset \operatorname{int} M_1,$$

where M_0 and M_1 are compact subsets of $Scx\cup Sce$. There exist compact sets $K_{j,k}$, K_k and functions χ_k with the same properties as in Theorem 1. Let $R=\max\{\|y\|\colon x+iy\in F\}$. Let $\mathbf{B}=\{\|y\|<3R\}$, $\mathbf{B}_j=B_j\times i\mathbf{B}$, $\mathbf{B}_{j,\nu}=B_{j,\nu}\times i\mathbf{B}$, $\mathbf{K}_k= =K_k+i\mathbf{B}$, $\mathbf{K}_{j,k}=K_{j,k}\times i\mathbf{B}$ and $\chi_k(z)=\chi_k(x)$.

We use Lemma 11 to modify f_ε above $M:=M_1\cap Scx$. This yields a plurisubharmonic function h with $h=f_\varepsilon$ on F and h is continuous on G up to $M\times i\mathbf{R}^2\cup$ {strictly concave boundary points}. Now we can copy the proof of Theorem 1 verbatim for h, using $\mathbf{B}_j$, $\mathbf{B}_{j,\nu}$ etc., and observing that in Theorem 1 we really needed the uniform continuity of f only on a small neighborhood relative to Ω^- of the $K_{j,k}$'s. This then yields for arbitrary $\delta>0$ a continuous plurisubharmonic function $h^{\sim}$ defined on a neighborhood of $D^-\times i\mathbf{B}$, with $\|h-h^{\sim}\|_F<d$. However, similar

to (the proof of) Lemma 10 there exists a convex function $\varkappa(y)$, such that $\varkappa<h$ on F, but $\varkappa>h^{\sim}$ on $D^{-}\times i\{\|y\|=2R\}$. Hence the function $g=\max\{h^{\sim},\varkappa\}$ for $\|y\|<2R$, $g=h^{\sim}$ for $\|y\|>2R$ is continuous on a neighborhood of G^{-} and approximates f on F. One can now smoothen g by convolution as usual. □

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