A Titchmarsh-type convolution theorem on the group Z

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The classical Titchmarsh convolution theorem [1] states that for functions with compact support the convex hull of the support of their convolution is equal to the sum of the convex hulls of their supports. Recently Y. Domar [2, 3] extended this statement. Namely, he proved that

(1) $$\inf \operatorname{supp} f*g = \inf \operatorname{supp} f + \inf \operatorname{supp} g,$$

for functions rapidly decreasing on $\mathbf{R}_-$ and satistying some growth restrictions on $\mathbf{R}_+$. A similar result was obtained for functions on $\mathbf{Z}$ (i.e. for sequences of complex numbers). But the decrease and growth conditions in Domar's theorem were only sufficient for (1) to hold: sharp necessary and sufficient conditions had not been found. Note, however, that in the case of summable functions (or even for measures with finite variation) necessary and sufficient conditions were obtained by I. V. Ostrovskiĭ [4].

In this paper we obtain, under some regularity assumptions, a condition necessary and sufficient for the validity of (1) on the group $\mathbf{Z}$ of integers. To this aim we use the technique of the so-called Dyn'kin transform [5, 6]. An application to operator theory is also given.

Let us first discuss Domar's theorem. Let $\{v_n\}_{n\in\mathbf{Z}}$ be a sequence of positive numbers such that, for some $m\in\mathbf{N}$, $\{\log v_n\}_{n<-m}$ is convex, $\{\log v_n\}_{n>m}$ is concave and

$$\lim_{|n|\to\infty} \frac{\log v_n}{n} = -\infty.$$

Theorem ([2, 3]). *Let $\{a_n\}_{n\in\mathbf{Z}}$ and $\{b_n\}_{n\in\mathbf{Z}}$ be sequences such that*

(A) $$\sum_{n\in\mathbf{Z}} |a_n| v_n < \infty, \quad \sum_{n\in\mathbf{Z}} |b_n| v_{-n}^{-1} < \infty$$

(B) $$a*b = 0 \quad \text{on } \mathbf{Z}_-$$

and let $\{v_n\}$ *satisfy the condition*

$$\text{(C)} \qquad \begin{cases} \varliminf_{n\to-\infty} \left(\dfrac{\log v_n}{n} + \alpha \log|n|\right) < \infty \\ \varliminf_{n\to\infty} \left(\dfrac{\log v_n}{n} + \beta \log n\right) < \infty \end{cases}$$

where $\alpha > 0$, $\beta > 0$, $\alpha+\beta \geqq 2$ *and at least one of the limits is equal to* $-\infty$. *Then*

(D) *there exists* $k \in \mathbf{Z} \cup \{-\infty, +\infty\}$ *such that* $a_n = 0$, $n < k$ *and* $b_n = 0$, $n < -k$.

Domar pointed out in [3, 7] that if condition (C) does not hold then the theorem can fail. He presented an example showing that under the assumption

$$\varliminf_{|n|\to\infty} \left(\frac{\log v_n}{n} + \log|n|\right) > -\infty$$

the implication (A) & (B)$\Rightarrow$(D) does not hold. Namely, if we set

$$v_n = (n!)^{-1}, \quad a_n = 0, \quad b_n = 0, \quad n > 0,$$

$$v_n = 2^n |n|!, \quad a_n = (|n|!)^{-1}, \quad b_n = (-1)^n (|n|!)^{-1}, \quad n \leqq 0,$$

then (A) and (B) hold, but not (D).

This implies that $\max(\alpha, \beta) \geqq 1$ is necessary for the conclusion of Domar's theorem to be valid. There is a gap between this condition and the sufficient one $\alpha+\beta \geqq 2$.

Now, let us observe that condition (A) is rather weak: it guarantees the existence of the convolution $a*b$ only on $\mathbf{Z}_- \cup \{0\}$. So one can hope to move further when imposing stronger conditions on sequences $\{a_n\}$ and $\{b_n\}$:

$$(\mathrm{A}_k) \qquad \sum_{n\in\mathbf{Z}} |a_{n+k}| v_n < \infty, \quad \sum_{n\in\mathbf{Z}} |b_{n+k}| v_{-n}^{-1} < \infty,$$

$$(\mathrm{B}') \qquad a*b = 0 \text{ on } \mathbf{Z}_-, \quad (a*b)_0 = 1.$$

Now let us introduce additional regularity assumptions on $\{v_n\}$:

$$\text{(E)} \qquad \varliminf_{|n|\to\infty} \frac{|\log v_n|}{|n| \log|n|} > 0,$$

$$\text{(F)} \qquad \varliminf_{n\to\infty} (u_{\pm n} - u_{\pm 2^n}) > 0, \quad \text{where} \quad u_n = \log \frac{v_{n+1}}{v_n}.$$

Theorem 1. *Suppose that* (E) *and* (F) *are satisfied. Then for some k, that depends only on the value of the limit in* (E), *the implication* (A_k) & $(B')\Rightarrow(D)$ *holds if and only if*

$$\text{(G)} \qquad \varlimsup_{|n|\to\infty}\left(\frac{\log v_n}{n}+\log|n|\right)=-\infty.$$

Thus, under conditions (A_k), (E), (F), (G), either (1) is fulfilled or $a*b=0$, on $\mathbf{Z}\cup\{0\}$.

This theorem can be applied to the theory of invariant subspaces.

Define the spaces l_v^{ind} and l_v^{pro}:

$$l_v^{\text{ind}}=\bigcup_{k\in\mathbf{Z}} l^1(v_{n+k}),\quad l_v^{\text{pro}}=\bigcap_{k\in\mathbf{Z}} l^1(v_{n+k}),$$

where $l^1(w_n)\overset{\text{def}}{=}\left\{\{a_n\}_{n\in\mathbf{Z}}\colon \sum_{n\in\mathbf{Z}}|a_n|\,w_n<\infty\right\}$.

Right translation $\tau\colon \{a_n\}\to\{a_{n-1}\}$ is a bounded operator on l_v^{ind}, l_v^{pro}.

The subspaces

$$l_{v,m}^{\text{ind}}=\left\{\{a_n\}\in l_v^{\text{ind}}\colon a_n=0,\ n<m\right\},$$
$$l_{v,m}^{\text{pro}}=\left\{\{a_n\}\in l_v^{\text{pro}}\colon a_n=0,\ n<m\right\}$$

will be called standard. They are closed and invariant under τ. For a more complete discussion of this topic one can refer to [8].

Now an application of Theorem 1 gives the following result.

Theorem 2. *Under the assumptions* (E) *and* (F) *all closed right translation-invariant subspaces E of* l_v^{ind} *and* l_v^{pro} *with* $\tau E\subset E$, $\tau E\neq E$ *are standard if and only if the condition* (G) *is fulfilled.*

The question of existence of 2-invariant subspaces E (that is, subspaces with $\tau E=E$) remains open.

Proof of Theorem 1. The necessity of (G) was demonstrated in [7], the sufficiency follows immediately from the next two lemmas.

Lemma 1. *Under the assumption* (E), *there is a k such that if* (A_k) *and* (B') *do not imply* (D) *then there exists* $f\in A(\mathbf{C}), f\neq\text{const}$, *such that*

$$|\exp f(z)|=O\left(\sum_{n\geq 0}v_n|z|^n\right),$$
$$|\exp(-f(z))|=O\left(\sum_{n\geq 0}v_{-n}^{-1}|z|^n\right),\quad |z|\to\infty.$$

Lemma 2. *Under the assumption* (F), *if for some positive c*

$$\sum_{n\geq 0}v_n r^n>c\exp cr,\quad r\in\mathbf{R}_+,$$

then for some positive d,

$$v_n>d^n/n!,\ n>0.$$

Proof of Lemma 1. Let us introduce some auxiliary objects. To begin with, consider the sequence $\{v_n\}_{n\geq 0}$. Without loss of generality one can assume that $\{\log v_n\}_{n\geq 0}$ is concave. Put

$$p(r) = \sup_{n\geq 0}(n\log r+\log v_n)$$

and choose a C^1-function q such that $q(\exp r)$ is convex, $p(r)-1<q(r)<p(r)$. Further, let

$$q_n^* = \int_0^\infty r^n e^{-q(r)}\,dr.$$

Then $q_n^*>v_{n+1}^{-1}/(n+1)$. Indeed, the convexity of the sequence $\{-\log v_n\}$ implies that

$$-\log v_{n+1} = \sup_{r>0}\big((n+1)\log r-p(r)\big).$$

Denote by x the point where this supremum is attained. Then

$$q_n^* = \int_0^\infty r^n e^{-q(r)}\,dr > \int_0^\infty r^n e^{-p(r)}\,dr$$

$$\geq \int_0^x \frac{1}{r} e^{(n+1)\log(r/x)} e^{(n+1)(\log x)-p(x)}\,dr = v_{n+1}^{-1}\int_0^x \frac{r^n}{x^{n+1}}\,dr = v_{n+1}^{-1}/(n+1).$$

It should be mentioned that a similar estimate is contained in a recent book of P. Koosis "The logarithmic integral".

In the proof of Lemma 1 we need the following extension of the theorem of Dyn'kin [5] to the case of the plane.

Lemma 3. *In the notation introduced earlier, if the sequence* $\{c_n\}_{n\geq 1}$ *satisfies*

(2) $$\sum_{n\geq 1}\left|\frac{nc_n}{q_{n+2}^*}\right| <\infty$$

and if the function u is defined by

$$u(z) = \sum_{n\geq 1}\frac{c_n z^{-n+1}}{2q_{n+2}^*}<\infty,\quad z\in\mathbf{T},$$

then the function

(3) $$f(z) = \frac{1}{\pi}\iint_{\mathbf{C}}\frac{u(\zeta/|\zeta|)e^{-q(|\zeta|)}|\zeta|^2}{z-\zeta}\,dm_2(\zeta)$$

is well defined, $f\in C^1(\mathbf{C})$, $\lim_{z\to\infty}f(z)=0$,

(4) $$|\bar\partial f(z)| \leq \text{const}\cdot e^{-p(|z|)}|z|^2$$

and the limits

$$\hat{f}(n) \overset{\text{def}}{=} \lim_{r\to\infty} \frac{1}{2\pi i} \int_{r\mathbf{T}} f(z)\, z^{-n-1}\, dz$$

exist and satisfy

(5)
$$\hat{f}(n) = \begin{cases} 0, & n \geqq 0, \\ c_{-n}, & n < 0. \end{cases}$$

Proof of Lemma 3. The C^1-smoothness of f and the estimate (4) for $|\bar{\partial} f|$ follow from the C^1-smoothness of the right-hand side of (3). In fact, (2) implies that $u \in C^1(\mathbf{T})$. The aim of introducing the function q instead of p was to provide the required smoothness. At last, the multiplier $|\zeta|^2$ gives the required smoothness at zero.

We check now the equality (5):

(6)

$$\frac{1}{2\pi i} \int_{r\mathbf{T}} f(z)\, z^{-n-1}\, dz = \frac{1}{2\pi i} \int_{r\mathbf{T}} z^{-n-1}\, dz \cdot \frac{1}{\pi} \iint_{|\zeta|<r} \frac{u(\zeta/|\zeta|)\, e^{-q(|\zeta|)}\, |\zeta|^2}{z-\zeta}\, dm_2(\zeta)$$

$$= \frac{1}{2\pi i} \int_0^{2\pi} r^{-n} e^{-in\theta}\, d\theta \cdot \frac{1}{\pi} \int_0^r \int_0^{2\pi} \frac{u(e^{it})\, e^{-q(s)}\, s^2}{re^{i\theta} - se^{it}}\, s\, ds\, dt$$

$$= \lim_{R\to r} \sum\nolimits_{k\geqq 0} \frac{1}{2\pi i \cdot \pi} \int_0^R r^{-n-1} e^{-q(s)}\, s^3\, ds \int_0^{2\pi} \int_0^{2\pi} u(e^{it}) \frac{s^k}{r^k}\, e^{-i\theta + ikt - ik\theta - in\theta}\, dt\, d\theta$$

$$= \lim_{R\to r} \frac{2\pi}{2\pi i \cdot \pi} \int_0^R r^{-n-1} e^{-q(s)}\, s^3\, ds \int_0^{2\pi} u(e^{it}) \frac{r^{n+1}}{s^{n+1}}\, e^{-i(n+1)t}\, dt$$

$$= \hat{u}(n+1) \cdot 2 \int_0^r e^{-q(s)}\, s^{2-n}\, ds.$$

Hence for $n \geqq 0$ $\hat{f}(n) = 0$ and for $n < 0$

$$\hat{f}(n) = \lim_{r\to\infty} \frac{c_{-n}}{q^*_{-n+2}} \int_0^r e^{-q(s)}\, s^{2-n}\, ds = c_{-n}.$$

The relation $\lim_{z\to\infty} f(z) = 0$ is due to the fact that the numerator of the right-hand side of (3) is summable and vanishes at infinity, Lemma 3 is proved.

To prove Lemma 1, let us assume that $\{a_n\}_{n\geqq 1}$ and $\{b_n\}_{n\geqq 1}$ satisfy (B′). The sequence $\{c_n\}_{n\geqq 1} = \{a_{n+5}\}_{n\geqq 1}$ satisfies (2), for by (A_k) (for sufficiently large k, depending only on the value of the limit in (E)), we find:

$$\sum\nolimits_{n\geqq 1} \left| \frac{n a_{n+5}}{q^*_{n+2}} \right| < \sum\nolimits_{n\geqq 1} n(n+1) a_{n+5}\, e^{\log v_{n+3}}$$

$$< \sum\nolimits_{n\geqq 1} a_{n+5} v_{n+5-k} \exp\left(\log v_{n+3} - \log v_{n+5-k} + 2\log(n+1)\right) < \infty.$$

Now we apply Lemma 3 to obtain the function f mentioned in this lemma. Recall that it satisfies

$$|\bar\partial f(z)| \leqq \text{const}\cdot e^{-p(|z|)}|z|^2,$$

$$\hat f(n) = \begin{cases} 0, & n \geqq 0, \\ c_{-n}, & n < 0. \end{cases}$$

Let us introduce the function F, $F\in C^1(\mathbf{C})$, such that

$$F(z) = \sum_{n\geqq 0} a_{-n}z^n + \sum_{n=1}^{5} a_n z^{-n} + f(z)z^{-5}, \quad |z| \geqq 1.$$

Then $|\bar\partial F(z)| \leqq \text{const}\cdot e^{-p(|z|)}|z|^{-3}$,

$$\hat F(n) = a_{-n}, \quad n\in\mathbf{Z}.$$

The function F is called the Dyn'kin transform of the sequence $\{a_n\}$.

Starting with the sequences $\{v_{-n}^{-1}\}_{n\geqq 0}$, $\{b_n\}_{n\in\mathbf{Z}}$ and proceeding in the same way, we can construct functions p', q', a sequence $\{q'^*\}_{n\geqq 0}$ and a function F'.

Let $\Phi = F\cdot F'$. Then

(7) $$|\bar\partial\Phi(z)| \leqq \text{const}\cdot e^{-p(r)}r^{-3}\sum_{n\geqq 0}|b_{-n}|r^n + \text{const}\cdot e^{-p'(r)}r^{-3}\sum_{n\geqq 0}|a_{-n}|r^n$$

$$\leqq \text{const}\cdot\big(e^{-p(r)}r^{-3}\sup_{n\geqq 0} r^n v_n + e^{-p'(r)}r^{-3}\sup_{n\geqq 0} r^n v_{-n}^{-1}\big) \leqq \text{const}\cdot r^{-3}, \quad \text{where} \quad r = |z| \geqq 1.$$

Thus the function a_Φ,

(8) $$a_\Phi(z) = \Phi(z) - \frac{1}{\pi}\iint_{\mathbf{C}} \frac{\bar\partial\Phi(\zeta)}{z-\zeta}\,dm_2(\zeta),$$

and the moments

$$\hat\Phi(n) \overset{\text{def}}{=} \lim_{r\to\infty}\frac{1}{2\pi i}\int_{r\mathbf{T}}\Phi(z)\,z^{-n-1}\,dz, \quad n \geqq 0$$

are well defined, and a_Φ is an entire function.

Now we shall compute $\hat\Phi(n)$. Denote

$$F_r(z) = F(z) - \frac{1}{\pi}\iint_{|\zeta|>r}\frac{\bar\partial F(\zeta)}{z-\zeta}\,dm_2(\zeta),$$

$$F_r'(z) = F'(z) - \frac{1}{\pi}\iint_{|\zeta|>r}\frac{\bar\partial F'(\zeta)}{z-\zeta}\,dm_2(\zeta),$$

$$f_r(z) = \frac{1}{\pi}\iint_{|\zeta|<r}\frac{\bar\partial f(\zeta)}{z-\zeta}\,dm_2(\zeta),$$

$$\Phi_r = F_r\cdot F_r'.$$

Then

$$|\Phi-\Phi_r|\,|rT \leqq |F(F'-F_r')|\,|rT+|F'(F-F_r)|\,|rT+o(1)$$

$$\leqq \text{const}\cdot\Big(\sum_{n\geqq 0}|b_{-n}|r^n\int_r^\infty e^{-p(s)}s^{-2}\,ds+\sum_{n\geqq 0}|a_{-n}|r^n\int_r^\infty e^{-p'(s)}s^{-2}\,ds\Big)+o(1)$$

$$\leqq \text{const}\cdot\frac{1}{r}\big(e^{-p(r)}\sup_{n\geqq 0}r^n v_n+e^{-p'(r)}\sup_{n\geqq 0}r^n v_{-n}^{-1}\big)+o(1)=o(1),\quad r\to\infty.$$

Hence

$$\hat{\Phi}(n)=\lim_{r\to\infty}\frac{1}{2\pi i}\int_{rT}\Phi_r(z)z^{-n-1}\,dz=\lim_{r\to\infty}\sum_{m\in Z}(F_r|rT)\hat{}\,(n-m)\cdot(F_r'|rT)\hat{}\,(m),$$

(in the last expression ^ denotes the usual Fourier transform of a continuous function on the circle). Further, for $m\geqq 0$

$$(F_r|rT)\hat{}\,(m)=a_{-m},\quad (F_r'|rT)\hat{}\,(m)=b_{-m},$$

Besides,

$$\frac{1}{2\pi i}\int_{rT}z^{-n-1}\,dz\cdot\frac{1}{\pi}\iint_{r<|\zeta|<R}\frac{u(\zeta/|\zeta|)e^{-q(|\zeta|)}|\zeta|^2}{z-\zeta}\,dm_2(\zeta)$$

$$=2\hat{u}(n+1)\int_r^R e^{-q(s)}s^{2-n}\,ds,$$

as in (6). Hence for $m<0$

$$\big|(F_r|rT)\hat{}\,(m)-a_{-m}\big|=\lim_{R\to\infty}\big|((F-F_r)|RT)\hat{}\,(m)\big|$$

$$=\lim_{R\to\infty}\big|((f-f_r)|RT)\hat{}\,(m+5)\big|\leqq \text{const}\cdot\int_r^\infty e^{-p(s)}s^{-m-3}\,ds.$$

Similarly,

$$\big|(F_r'|rT)\hat{}\,(m)-b_{-m}\big|\leqq \text{const}\cdot\int_r^\infty e^{-p'(s)}s^{-m-3}\,ds.$$

Finally, we obtain

$$\big|\hat{\Phi}(n)-\sum_{m\in Z}a_{m-n}b_{-m}\big|\leqq \lim_{r\to\infty}\Big(\sum_{m\geqq n}|b_{-m}|\int_r^\infty e^{-p(s)}s^{-n+m-3}\,ds$$

$$+\sum_{m<0}|a_{m-n}|\int_r^\infty e^{-p'(s)}s^{-m-3}\,ds\Big)\leqq \lim_{r\to\infty}\int_r^\infty \text{const}\,s^{-n-3}\,ds$$

(compare with (7)).

This implies that for $n\geqq 0$

(9) $$\hat{\Phi}(n)=\sum_{m\in Z}a_{m-n}b_{-m}=\begin{cases}1, & n=0\\ 0, & n>0.\end{cases}$$

Thus we have shown that the Dyn'kin transform has a property similar to the standard property of Fourier transform: it maps convolutions to products.

It follows from (8) and (9) that

$$a_\Phi \equiv 1, \quad \lim_{z\to\infty} \Phi(z) = 1.$$

Let the disk K, $K=\{z\colon |z|<c\}$, be such that

$$|\Phi(z)| > 1/2 \quad \text{for} \quad z\notin K.$$

Taking into account that

$$|\bar\partial F\cdot F'|, \quad |\bar\partial F'\cdot F| \in L^\infty(\mathbf{C})\cap L^1(\mathbf{C})$$

(this follows from (7)), we can define functions

$$F^*(z) \overset{\text{def}}{=} F(z)\exp\left(-\frac{1}{\pi}\cdot\iint_{|z|>c}\left(\frac{\bar\partial F\cdot F'}{\Phi}\right)(\zeta)\frac{1}{z-\zeta}\,dm_2(\zeta)\right),$$

$$F'^*(z) \overset{\text{def}}{=} F'(z)\exp\left(-\frac{1}{\pi}\cdot\iint_{|z|>c}\left(\frac{\bar\partial F'\cdot F}{\Phi}\right)(\zeta)\frac{1}{z-\zeta}\,dm_2(\zeta)\right).$$

Then

$$F^*,\ F'^*\in A(\mathbf{C}\setminus\bar K),\ \lim_{z\to\infty}\frac{F^*(z)}{F(z)} = \lim_{z\to\infty}\frac{F'^*(z)}{F'(z)} = 1.$$

Thus $\lim_{z\to\infty} F^*(z)F'^*(z)=1$.

If the singularities of the functions F^* and F'^* at ∞ are at most poles, then the same is valid for F and F'. Hence the sequences $\{a_{-n}\}_{n\geqq 0}$, $\{b_{-n}\}_{n\geqq 0}$ are finite, and this implies (D).

If not, then both F^* and F'^* have essential singularities at infinity. Further, since F^* and F'^* have no roots in some neighbourhood of ∞, one can factorize them as follows:

$$F^* = g_1 g_2, \quad F'^* = g_1' g_2',$$

where $g_1, g_1'\in A(\mathbf{C})$, $g_2, g_2'\in A(\hat{\mathbf{C}}\setminus\bar K)$, $g_1(z)\neq 0$, $g_1'(z)\neq 0$, $z\in\mathbf{C}$, and there is an $m\geqq 0$ such that $(z^{-m}g_2g_2')(\infty)=1$.

Then $g_1g_1'\equiv 1$, $g_1\neq\text{const}$, and

$$|g_1(z)| \leqq \text{const}\,|F^*(z)| \leqq \text{const}\,|F(z)| \leqq \text{const}\textstyle\sum_{n\geqq 0} v_{-n}^{-1}|z|^n,$$

$$|g_1^{-1}(z)| \leqq \text{const}\textstyle\sum_{n\geqq 0} v_n|z|^n.$$

Thus Lemma 1 is proved (the function $f=\log g_1$ is the one we need).

Proof of Lemma 2. If for each $d>0$ there exists $\varepsilon>0$ such that the inequalities

$$v_m \leqq d^m/m!, \quad u_m - u_{2^m} > \varepsilon$$

hold for arbitrarily large m's, then for $r=\exp(-u_m)$

$$c\exp(c\exp(-u_m)) = c\exp cr < \sum\nolimits_{n\geqq 0}\exp(\log v_n+n\log r)$$
$$< 2^m\exp(\log v_m+m\log r)+\sum\nolimits_{n>2^m}\exp(\log v_n+n\log r)$$
$$< (2^m+\sum\nolimits_{n>2^m}\exp((n-2^m)(u_{2^m}-u_m)))\exp(\log v_m+m\log r).$$

For m sufficiently large,

$$c\exp(c\exp(-u_m)) < \exp(m+m\log d-m\log(m/e)-mu_m),$$
$$m\log m < m(3+\log d)+(-mu_m-c\exp(-u_m)).$$

The maximum value of the expression $mx-ce^x$ for $x\geqq 0$ is $m\log(m/ce)$, therefore

$$m\log m < m(3+\log d)+m\log(m/ce).$$

Then $3+\log d-\log ce>0$. This leads to a contradiction, if d is small. Lemma 2 is proved.

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