Interval estimates

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§ 1. Introduction

In this paper we prove that there exists an absolute constant $l>0$ such that, for every univalent H^1 function f in the open unit disk D and every $z_0 \in D$, there are $\vartheta \in \mathbf{R}$ and ε, $l(1-|z_0|) \leqq \varepsilon \leqq \pi$, such that

$$f(z_0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt.$$

Let f be a holomorphic function in the open unit disk D which belongs to the Hardy class H^1 ([4]). According to [1] and [2], every value $f(z_0)$, $|z_0|<1$, is of the form

$$f(z_0) = \frac{1}{|I|} \int_I f(e^{i\vartheta})\, d\vartheta,$$

where I is an interval on the unit circle with length $|I|$, $0<|I| \leqq 2\pi$. A sketch of the proof is given in Prop. 1, § 2 below. The proof does not provide information on the size or the location of the interval I. Extensions of the previous result in [5, 6] are related to BMO, measures and holomorphic mappings in several variables; still they do not contain quantitative information on the size of I. Some preliminary quantitative results concerning univalent functions can be found in [7] and [8]. Their proof makes use of the classical distortion theorems and especially of the 1/4-Koebe theorem.

The purpose of the present paper is to furnish a brief and complete presentation of the above quantitative results on univalent functions; the general H^1 case is, as far as I know, still open.

The main result, thus, states that if f is H^1 and univalent then $|I| \geqq 2l(1-|z_0|)$, where $l>0$ is an absolute constant independent of f and z_0. In the particular case where $f(z)=\log(1-z)$, the length $|I|$ is exactly of the order of $(1-|z_0|)$; however, I do not know the best value of the constant l.

In the special case of a function univalent in a larger disk $D_r=\{z\in\mathbf{C}\colon |z|<r\}$ with $r>1$ we have $|I|\geqq 2C_f(1-|z_0|)^{1/2}$ with $C_f>0$ a constant independent of $z_0\in D$. An easy calculation with the function $f(z)=z$ shows that $|I|$ is exactly of the order of $(1-|z|)^{1/2}$.

§ 2. Proofs

Let f be an H^1 function in the open unit disk D. For ε, $0<\varepsilon\leqq\pi$, and z, $|z|\leqq 1$, we denote

$$f_\varepsilon(z)=\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon} f(ze^{it})\,dt.$$

We also denote $f_0(z)=f(z)$ for all $z\in D$ and for almost all z in the unit circle $|z|=1$. We prove first the following version of Theorem 1 in [2] (see also Theorem 8 in [6]).

Proposition 1. *Let f be an H^1 function in the open unit disk D and let J be a Jordan curve in $\bar{D}$ the closed unit disk. For every point z in the interior of J and for every ε, $0\leqq\varepsilon\leqq\pi$, there are* $\tilde{z}\in J$ and $\tilde{\varepsilon}$, $\varepsilon\leqq\tilde{\varepsilon}\leqq\pi$, $0<\tilde{\varepsilon}$, *such that* $f_\varepsilon(z)=f_{\tilde{\varepsilon}}(\tilde{z})$.

Proof. We distinguish three cases.

i) $\varepsilon=0$ and $f_\varepsilon=f_0$ is constant; then $f_{\tilde{\varepsilon}}=f=$constant for all $\tilde{\varepsilon}$ and the result is obvious.

ii) $0<\varepsilon\leqq\pi$ and f_ε is constant; then the result holds with $\tilde{\varepsilon}=\varepsilon$ and $\tilde{z}$ any point of J.

iii) f_ε is non-constant in D. We argue by contradiction and thus we suppose that $f_{\tilde{\varepsilon}}(\tilde{z})\neq f_\varepsilon(z)$ for all $\tilde{\varepsilon}$, $\varepsilon<\tilde{\varepsilon}\leqq\pi$, and $\tilde{z}\in J$ (in this case $\varepsilon<\pi$, because f_ε is non-constant). The curves $f_{\tilde{\varepsilon}|J}$, $\varepsilon<\tilde{\varepsilon}\leqq\pi$, are homotopic in $\mathbf{C}-\{f_\varepsilon(z)\}$; therefore, $\mathrm{Ind}\,(f_{\tilde{\varepsilon}|J}, f_\varepsilon(z))=\mathrm{Ind}\,(f_{\pi|J}, f_\varepsilon(z))$ for all $\tilde{\varepsilon}$, $\varepsilon<\tilde{\varepsilon}\leqq\pi$, where Ind denotes the winding number. Since the function f_π is constant, we have $\mathrm{Ind}\,(f_{\tilde{\varepsilon}|J}, f_\varepsilon(z))=0$ for all ε, $\varepsilon<\tilde{\varepsilon}\leqq\pi$. We observe that each function $f_{\tilde{\varepsilon}}$ is continuous on $\bar{D}$ and holomorphic in D. The argument principle implies that $f_{\tilde{\varepsilon}}(w)\neq f_\varepsilon(z)$ for all $\tilde{\varepsilon}$, $\varepsilon<\tilde{\varepsilon}\leqq\pi$, and all w in the interior of J.

We also observe that $f_{\tilde{\varepsilon}}\to f_\varepsilon$ uniformly on compacta in D, as $\tilde{\varepsilon}\to\varepsilon$. Hurwitz's theorem states that either f_ε is constant or $f_\varepsilon(w)\neq f_\varepsilon(z)$ for all w in the interior of J. In our case f_ε is not constant; therefore, $f_\varepsilon(w)\neq f_\varepsilon(z)$ for all w in the interior of J. This contradicts the fact that z is in the interior of J and the proof is complete. Q.E.D.

Proposition 2. *Let $0<\lambda<1$. Then, there exists a constant $l_\lambda>0$, such that for every univalent function f in $|z|<1$ the following holds:*

If z_0, z and ε are such that $|z_0|=1-\delta$, $0<\delta\leqq 1$, $|z-z_0|=\lambda\delta$, $0<\varepsilon\leqq\pi$ and $f(z_0)=f_\varepsilon(z)$, then $\varepsilon\geqq l_\lambda\delta$.

Proof. Let μ be such that $\lambda<\mu<1$. We denote $D(z_0, \mu\delta)=\{w\in\mathbf{C}: |w-z_0|\leqq\mu\delta\}$ and $I_{z,\varepsilon}=\{ze^{it}: -\varepsilon\leqq t\leqq\varepsilon\}$.

The distortion theorems (Ch. 2 in [3] or Ch. 1 in [9]) imply that for $w\in D(z_0, \mu\delta)$ we have

$$|f'(w)| \leqq \frac{2}{(1-\mu)^3}|f'(z_0)| \quad \text{and} \quad |f''(w)| \leqq \frac{6}{(1-\mu)^4}\cdot\frac{1}{\delta}\cdot|f'(z_0)|.$$

The 1/4-Koebe theorem ([3], [9]) yields the following

$$\frac{1}{4}\lambda\delta|f'(z_0)| \leqq |f(z_0)-f(z)| = |f_\varepsilon(z)-f(z)| = \left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}[f(ze^{it})-f(z)]\,dt\right|.$$

We set $g(t)=f(ze^{it})$, which defines a C^∞ function: thus, we have the Taylor development $g(t)-g(0)=tg'(0)+t^2/2\cdot u(t)$, which implies

$$\frac{1}{4}\lambda\delta\,|f'(z_0)| \leqq \left|\frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}[g(t)-g(0)]\,dt\right| \leqq \frac{\varepsilon^2}{6}\sup_{|t|\leqq\varepsilon}|u(t)| \leqq \frac{\varepsilon^2}{6}\sup_{|t|\leqq\varepsilon}|g''(t)|.$$

Since $|z-z_0|=\lambda\delta$, one can easily verify that $I_{z,\varepsilon}\subset D(z_0, \mu\delta)$ or $\varepsilon/\delta\geqq\mu-\lambda$.

We consider the case $I_{z,\varepsilon}\subset D(z_0, \mu\delta)$. Since $g''(t)=-ze^{it}f'(ze^{it})-z^2e^{i2t}f''(ze^{it})$, using the above mentioned bounds for $|f'(w)|$ and $|f''(w)|$ in $D(z_0\mu\delta)$, we have the inequality

$$\frac{1}{4}\lambda\delta\,|f'(z_0)| \leqq \frac{\varepsilon^2}{6}\left[\frac{2}{(1-\mu)^3}+\frac{6}{(1-\mu)^4}\frac{1}{\delta}\right]|f'(z_0)|.$$

Since $0<\delta\leqq1$, $0<\mu<1$ and $f'(z_0)\neq0$ by the univalence of f, we obtain $\varepsilon/\delta\geqq C_{\lambda,\mu}>0$.

If $I_{z,\varepsilon}$ is not contained in $D(z_0, \mu\delta)$, then $\varepsilon/\delta\geqq\mu-\lambda$. Therefore, we always have $\varepsilon/\delta\geqq l_{\lambda,\mu}=\min(C_{\lambda,\mu}, \mu-\lambda)>0$. Now the result follows with $l_\lambda=\sup_{\mu\in(\lambda,1)}l_{\lambda,\mu}$ or $l_\lambda=l_{\lambda,\mu_\lambda}$ with $\mu_\lambda=\dfrac{1+\lambda}{2}$. Q.E.D.

Theorem 3. *There is an absolute constant $l>0$ such that the following holds: For every univalent H^1 function f in $D=\{z\in\mathbf{C}: |z|<1\}$ and for every $z_0\in D$, $|z_0|=1-\delta$, $0<\delta\leqq1$, there exist $\vartheta\in\mathbf{R}$ and ε, $l\delta<\varepsilon\leqq\pi$, such that*

$$f(z_0) = \frac{1}{2\varepsilon}\int_{-\varepsilon}^{\varepsilon}f(e^{i\vartheta}e^{it})\,dt.$$

Proof. Let J be the circle with center z_0 and radius $\delta/4$. Then, according to Prop. 1, there are $\tilde{z}\in J$ and $\tilde{\varepsilon}$, $0<\tilde{\varepsilon}\leqq\pi$, such that $f(z_0)=f_0(z_0)=f_{\tilde{\varepsilon}}(\tilde{z})$. Prop. 2 implies now that $\tilde{\varepsilon}\geqq l_{1/4}\cdot\delta$. We use Prop. 1 once more and we obtain $\vartheta\in\mathbf{R}$ and ε,

$\pi \geqq \varepsilon \geqq \tilde{\varepsilon} \geqq l_{1/4} \cdot \delta > 0$, such that

$$f(z_0) = f_{\tilde{\varepsilon}}(\tilde{z}) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt.$$

Therefore, we have the result with $l = l_{1/4}$. A slight modification in the proof gives the result with $l = \sup_{\lambda \in (0,1)} l_\lambda$. Q.E.D.

In the particular case of a function univalent in a larger disk we have:

Proposition 4. *Suppose that f is univalent function in a disk $D_r = \{z \in \mathbf{C}\colon |z| < r\}$ with $r > 1$. Then there is a constant $c_f > 0$ such that, for every z_0, $|z_0| = 1 - \delta$, $0 < \delta \leqq 1$, and for every $\vartheta \in \mathbf{R}$ and ε, $0 < \varepsilon \leqq \pi$, related by*

$$f(z_0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt,$$

we have $\varepsilon \geqq c_f \delta^{1/2}$.

Proof. We set $g(t) = f(e^{i\vartheta} e^{it})$, which defines a C^∞ function g. Since f is holomorphic in D_r with $r > 1$, it follows that $|g''(t)| \leqq M_f < +\infty$ for all $t \in \mathbf{R}$. The Taylor development of g gives $g(t) - g(0) = tg'(0) + t^2/2 \cdot u(t)$ with $|u(t)| \leqq M_f$.

This implies

$$\left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt - f(e^{i\vartheta}) \right| = \left| \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} \left[tg'(0) + \frac{t^2}{2}\, u(t) \right] dt \right| \leqq M_f \cdot \frac{\varepsilon^2}{6}.$$

On the other hand the 1/4-Koebe theorem yields

$$|f(z_0) - f(e^{i\vartheta})| \geqq \frac{1}{4} |f'(z_0)| \cdot (1 - |z_0|) = \frac{\delta}{4} |f'(z_0)|.$$

Since $f(z_0) = \frac{1}{2\varepsilon} \int_{-\varepsilon}^{\varepsilon} f(e^{i\vartheta} e^{it})\, dt$, we find $\frac{\delta}{4} |f'(z_0)| \leqq M_f \frac{\varepsilon^2}{6}$. As $\min_{|z_0| \leqq 1} |f'(z_0)| > 0$ we find $\varepsilon \geqq c_f \cdot \delta^{1/2}$, with $c_f > 0$.

Q.E.D.

§ 3. Examples

Let $f(z) = z$ and z_0, $|z_0| = 1 - \delta$, $0 < \delta \leqq 1$. If $f(z_0) = f_\varepsilon(e^{i\vartheta})$, then we easily obtain $1 - \delta = \frac{\sin \varepsilon}{\varepsilon}$; this implies that ε is exactly of the order of $\delta^{1/2}$, as $\delta \to 0$. We see, therefore, that the exponent 1/2 is best possible in Prop. 4.

Next let us consider the function $f(z) = \log(1 - z)$, which is univalent and $\boldsymbol{H}^1$ in D. Let $\vartheta \in \mathbf{R}$, $\varepsilon \in [0, \pi]$ and $z_0 \in D$, $z_0 = 1 - \delta$, $0 < \delta \leqq 1/2$, be such that $f(z_0) =$

$f_\varepsilon(e^{i\vartheta})$. It is easy to see, e.g. geometrically, that $e^{i\vartheta}=1$; it follows that

$$f(z_0) = \log\delta = \frac{1}{\varepsilon}\int_0^\varepsilon \log 2\sin\frac{t}{2}\,dt.$$

This implies that $-1+\log\frac{2\varepsilon}{\pi}\leqq\log\delta\leqq-1+\log\varepsilon$, which gives $e\delta\leqq\varepsilon\leqq\pi e/2\cdot\delta$. Therefore, the exponent 1 is the best possible in Theorem 3. Finally the exponent 1 is the best possible in Prop. 2; this can be seen by the examples $f(z)=\log(1-z)$ or $f(z)=(1-z)^{-2}$ as well.

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