An analytic algebra without analytic structure in the spectrum

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Introduction

Let A be a uniform algebra. Denote by $\hat{A}$ the algebra of all Gelfand transforms of A and by $M(A)$ the spectrum of A. A is called an analytic algebra provided any function $f \in \hat{A}$ which vanishes on a nonvoid open subset of $M(A)$ is identically zero. In [1] Wermer constructed a new "Stolzenberg example", i.e. he constructed a compact set $Y \subset \mathbf{C}^2$ such that X — the polynomially convex hull of Y — contains no analytic disc and $X \setminus Y \neq \emptyset$. In the present note we want to show that $P(X)$ (X the set in Wermer's example) is an analytic algebra, where $P(X)$ denotes the algebra of all continuous complex-valued functions on X which can be approximated uniformly on X by polynomials.

This seems to be interesting for two reasons. Firstly we get more information about Wermer's example.

Secondly, we learn that the identity theorem is a phenomenon in the theory of uniform algebras which occurs not only in connection with analytic structure. To be more precise we give two definitions.

We say that A has analytic structure in a point $x \in M(A)$ if there is a neighbourhood U of x in $M(A)$, an analytic set V in a polycylinder $P \subset \mathbf{C}^n$ and a homeomorphism $\Phi \colon V \to U$ such that $f \circ \Phi$ is an analytic function on V for all $f \in \hat{A}$.

A satisfies the weak identity theorem in $x \in M(A)$ if there is a (fixed) neighbourhood U of x in $M(A)$ such that $f_{|U} \equiv 0$ for each $f \in \hat{A}$ which vanishes in an arbitrary neighbourhood of x.

It follows from the theory of several complex variables that A satisfies the weak identity theorem in x if A has analytic structure in x. Since we can naturally identify $P(X)$ with $\widehat{P(X)}$ and X with $M(P(X))$, we have an example where the identity theorem does not depend on analytic structure in the spectrum.

The example

We shall show that $P(X)$ — X the polynomially convex compact set of Wermer's example (cf. [1]) — is an analytic algebra.

First we recall shortly the construction of X. We denote by $a_1, a_2, \dots$ the points in the disc $\{|z|<1/2\}$ with rational real and imaginary part. For each $j\in\mathbf{N}$ denote by B_j the algebraic function

$$B_j(z) = (z-a_1)(z-a_2)\dots(z-a_{j-1})\sqrt{(z-a_j)}$$

and set

$$g_j(z) = \sum_{k=1}^{j} c_k B_k(z)$$

where $c_1, \dots, c_j$ are positive constants. Denote by $\Sigma(c_1, \dots, c_n)$ the subset of the Riemann surface of g_n which lies in $\{|x|\leqq 1/2\}$, i.e.

$$\Sigma(c_1, \dots, c_n) = \{(z, w)\colon |z| \leqq 1/2,\ w = w_j^{(n)}(z),\ j = 1, \dots, 2^n\},$$

where $w_j^{(n)}(z)$, $j=1, \dots, 2^n$ are the values of g_n at z.

(1) For a sequence of positive constants (c_n) define $X_{(c_n)}$ to be the set of all points $(z, w)\in\mathbf{C}^2$ such that

i) $|z|\leqq 1/2$, and

ii) there is a sequence $(z, w_n)\in\Sigma(c_1, \dots, c_n)$ with $w_n\to w$ as $n\to\infty$.

(2) There exists a sequence of positive constants (c_n) with $c_1=1/10$ and $c_{j+1}\leqq 1/10c_j$ $(j=1, 2, \dots)$ such that $X=X_{(c_n)}$ is a polynomially convex compact set in $\mathbf{C}^2$ which contains no analytic disc. (cf. [1], Lemma 1, Lemma 2 and p. 132).

(3) X is the polynomially convex hull of the compact set

$$Y = X\cap\{|z| = 1/2\} \quad ([1], \text{Lemma } 3).$$

(4) In the following we set $\Sigma_n=\Sigma(c_1, \dots, c_n)$ and remark that g_{n+1} has at z the values

$$w_j^{(n)}(z)\pm c_{n+1}B_{n+1}(z), \quad j = 1, \dots, 2^n.$$

(5) As a direct consequence of (4) or (1) we see that for each $i\in\mathbf{N}$ the set $X\cap\{z=a_i\}$ contains exactly 2^{i-1} points which we denote by $(a_i, b_1), \dots, (a_i, b_{2^{i-1}})$.

It is proved in [1] (cf. Assertion 4. (12) resp. Assertion 2.) that for each $n\in\mathbf{N}$.

$$(6)\qquad |w_j^{(n)}(z)-w_l^{(n)}(z)| \geqq 3/2c_n|B_n(z)| \quad (|z| \leqq 1/2,\ j \neq l).$$

(Note that this inequality remains true if $z=a_l$ for $l\in\mathbf{N}$.) If $(z, w)\in X$ is an arbitrary point, there is $j\in\{1, \dots, 2^n\}$ such that

$$(7)\qquad |w_j^{(n)}(z)-w| \leqq 1/2c_n|B_n(z)|$$

(cf. proof of [1] Assertion 4. and note again that this inequality remains true if $z=a_l$).

Let $(z, w_j^{(n)}(z))$ be an arbitrary point of Σ_n. For $k>n$ set

$$w_k = w_j^{(n)}(z) + \sum_{l=n+1}^{k} c_l B_l(z),$$

where for each l $B_l(z)$ denotes one of the values of B_l at z, chosen arbitrarily. Then $(z, w_k)\in\Sigma_k$ and it follows that (z, w_k) converges to a point (z, w) — hence $(z, w)\in X$ by (1) — with

$$|w - w_j^{(n)}(z)| \leqq 1/2c_n|B_n(z)|.$$

To prove this use

$$|w_k - w_j^{(n)}(z)| \leqq \left(\sum_{l=n+1}^{k} c_l\right)|B_n(z)|$$

and $c_{l+1} \leqq (1/10)\ c_l$ by (2).

Thus, for each $(x, y)\in\Sigma_n$ there is $(x, w)\in X$ with

(8) $$|w-y| \leqq 1/2c_n|B_n(x)|.$$

Denote by Π the projection

$$\mathbf{C}^2 \to \mathbf{C}, \quad (z, w) \to z.$$

It follows from (8) that

(9) $$\Pi(X) = \{|z| \leqq 1/2\}.$$

(10) Finally we claim that the set

$$L = \bigcup_{i\in\mathbf{N}} \{(a_i, b_1), \dots, (a_i, b_{2^i-1})\}$$

is dense in X.

To prove this choose an arbitrary point $(x, y)\in X\setminus L$ and $\varepsilon>0$. By (1) we can choose $n\in\mathbf{N}$ such that there is a point $(x, w)\in\Sigma_n$ with $|w-y|<\varepsilon/4$ and $c_n<\varepsilon/4$. Choose a positive constant $\delta<\varepsilon$ such that $g_n(z)=\sum_{j=1}^{n} c_n B_n(z)$ has 2^n single-valued continuous branches defined on $\{|z-x|<\delta\}$. Denote by $w_1^{(n)}(z)$ the branch with $w_1^{(n)}(x)=w$. Choose $j>n$ such that $|a_j - x|<\delta$ and

$$|w_1^{(n)}(a_j) - w_1^{(n)}(x)| < \varepsilon/4.$$

By (8) there is a point $(a_j, b)\in X$ — this implies $(a_j, b)\in L$ by (5) — with

$$|w_1^{(n)}(a_j) - b| \leqq 1/2c_n|B_n(a_j)| < \varepsilon/4.$$

Hence $|b-y|<\varepsilon$.

(11) We recall the notion of a maximum modulus algebra (m.m.a.) from [3].

Let X be a locally compact Hausdorff space and A be an algebra of continuous complex-valued functions defined on X, let Ω be a region in $\mathbf{C}$ and f an element of A with $f(X)\subset\Omega$. We call (A, X) a m.m.a. over Ω with projection function f provided

i) A separates the points on X and contains the constants.

ii) For each compact set $K\subset\Omega$

$$f^{-1}(K) = \{x\in X\colon f(x)\in K\} \quad \text{is compact.}$$

iii) If N is a compact subset of X, $x \in N$ and $g \in A$, then

$$|g(x)| \leqq \max_{\partial N} |g|,$$

where ∂N denotes the boundary of N.

Claim 1. If $g \in P(X)$ vanishes in an open neighbourhood U of $(a_1, b_1) = (a_1, 0)$ then $g \equiv 0$.

We first note that there is an open neighbourhood W of a_1 in $\{|z| < 1/2\}$ such that $\Pi^{-1}(W) \cap X$ is contained in U. (Assume W does not exist. Then there is a sequence (d_n, y_n) in $X \setminus U$ with $d_n \to a_1$. By compactness a subsequence converges to a point $(a_1, y) \in X \setminus U$. This is a contradiction to

$$X \cap \{z = a_1\} = \{(a_1, b_1)\}.)$$

Since X is polynomially convex we can identify the spectrum of $P(X)$ with X. By (2) the Shilov boundary of $P(X)$ is contained in Y, hence $(P(X)|_{X \setminus Y}, X \setminus Y)$ is a m.m.a. with projection function $\Pi|_{X \setminus Y}$ over $\{|z| < 1/2\}$ by Rossi's maximum modulus principle ([2], Theorem 9.3) and (9). (Clearly $\Pi|_{X \setminus Y} \in P(X)|_{X \setminus Y}$ and $\Pi^{-1}|_{X \setminus Y}(N)$ is a compact subset of $X \setminus Y$ for each compact subset $N \subset \{|z| < 1/2\}$.)

By Wermer's subharmonicity theorem ([3], Lemma 1)

$$\log Z_g\colon \{|z| < 1/2\} \to \mathbf{R} \cup \{-\infty\}, \quad z \to \log \max_{\Pi^{-1}|_{X \setminus Y}(z)} |g|$$

is a subharmonic function. Since $\log Z_g \equiv -\infty$, on W, g vanishes on $X \setminus Y$. Hence — by the continuity of g and (10) — $g \equiv 0$.

Now we proceed by induction. We assume that $g \equiv 0$ if $g \in P(X)$ vanishes in an open neighbourhood of one of the points $(a_n, b_1), \ldots, (a_n, b_{2^n-1})$.

Let $g \in P(X)$ vanish in an open neighbourhood U of one of the points $(a_{n+1}, b_1), \ldots, (a_{n+1}, b_{2^n})$. For abbreviation denote this point by (a, b). In the following we have to make slight modifications in the case $n > 1$.

Let γ be a simple closed curve in $\{|z| < 1/2\}$ such that $a_1, \ldots, a_{n-1}$ lie outside and a, a_n lie inside γ (if $n > 1$). Then $g_{n-1}(z)$ has 2^{n-1} single-valued continuous branches on the domain G which contains a, a_n and is bounded by γ. By (4) there is one branch of $g_{n-1}(z)$ on G, which we denote by $w_1^{(n-1)}(z)$, such that

$$b = w_1^{(n-1)}(a) + c_n B_n(a),$$

or

$$b = w_1^{(n-1)}(a) - c_n B_n(a).$$

Claim 2. g vanishes in an open neighbourhood of $(a_n, w_1^{(n-1)}(a_n))$, hence $g \equiv 0$ by the induction hypothesis (here set $w_1^{(0)}(z) = 0$ for $|z| \leqq 1/2$).

(12) We first note that no point (z, w) of $X \cap \{|z| < 1/2\}$ is an isolated point of X, since otherwise the function

$$f\colon X \to \mathbf{C}, \quad (x, y) \to \begin{cases} 1 & \text{for} \quad (x, y) = (z, w) \\ 0 & \text{for} \quad (x, y) \neq (z, w) \end{cases}$$

would be an element of $P(X)$ by Shilov's idempotent theorem ([2], Theorem 8.6), contradicting the fact that (z, w) is a point of the polynomially convex hull of Y (cf. (3)).

If $n=1$ choose an arbitrary point $(x_0, y_0) \in X \setminus Y$ with $x_0 \neq a_1$. If $n>1$ set

$$W = \{(z, w) \in X\colon\ z \in G \ \text{ and } \ |w - w_1^{(n-1)}(z)| < 2/3 c_{n-1} |B_{n-1}(z)|\}.$$

W is an open neighbourhood of $(a_n, w_1^{(n-1)}(a_n))$.

Let $(x_0, y_0) \in W$, $x_0 \neq a_n$, be an arbitrary point. We shall show in both cases that $g(x_0, y_0) = 0$. This proves claim 2 by (12).

Now let S be a simple closed curve in $\{|z| < 1/2\}$ (if $n=1$), resp. in G (if $n>1$), such that a_n lies inside and x_0 and a_{n+1} lie on S. Set $S' = \{(z, w_1^{(n-1)}(z) \pm c_n B_n(z))\colon z \in S\}$. Here $\pm c_n B_n(z)$ denotes both values of $c_n B_n(z)$ at z, i.e. $(z, w_1^{(n-1)}(z) \pm c_n B_n(z))$ denotes two points. S' is a closed curve in Σ_n which contains (a, b). Set

$S'' = \{(z, w) \in S'$: There is an open neighbourhood V of (z, w) in Σ_n such that

g vanishes in all points $(x, y) \in X$ with

i) there is $w' \in \mathbf{C}$ with $(x, w') \in V$ and

ii) $|y - w'| < 3/4 c_n |B_n(x)|.\}$.

We want to show that $S'' = S'$.

i) S'' is an open subset of S' by definition.

ii) $S'' \neq \emptyset$.

Let (d_k, y_k) be a sequence of points in Σ_n converging to (a, b) and assume there is a sequence of points (d_k, y_k') in X such that $g(d_k, y_k') \neq 0$ and

$$|y_k' - y_k| < 4/5 c_n |B_n(d_k)|.$$

By compactness and since g vanishes on U by induction hypothesis a subsequence of (d_k, y_k') converges to a point $(a, y) \in X \setminus U$ with

$$|y - b| \leqq 4/5 c_n |B_n(a)|.$$

But this implies that $y = b$ by (5) and (6), a contradiction to $(a, y) \in X \setminus U$.

iii) S'' is a closed subset of S'.

Let (x_1, y_1) be a boundary point of S'' in S'. Choose $\delta > 0$ (in particular $\delta < 1/2(1/2 - |x_1|)$) such that $g_n(z)$ has 2^n single-valued branches on $\{|z - x_1| < \delta\}$. Denote by $w_1^{(n)}(z)$ the branch with $w_1^{(n)}(x_1) = y_1$.

Set

$$W' = \{(z, w) \in X\colon\ |z-x_1| < \delta \ \text{ and } \ |w-w_1^{(n)}(z)| < 2/3c_n|B_n(z)|\}.$$

W' is an open set in X and

$$\{(z, w_1^{(n)}(z))\colon\ |z-x_1| < \delta\}$$

is a neighbourhood of (x_1, y_1) in Σ_n.

It follows from (8) that

$$\Pi(W') = \{|z-x_1| < \delta\}$$

and from (6) and (7) that

$$\partial W' \subset \{|z-x_1| = \delta\}$$

hence $\Pi^{-1}|_{W'}(L)$ is a compact subset of W' for each compact subset $L \subset \{|z-x_1| < \delta\}$. Let $(x_2, w_1^{(n)}(x_2))$ be a point of S'' with $|x_2-x_1| < \delta$. Choose a neighbourhood V of $(x_2, w_1^{(n)}(x_2))$ in $\{(z, w_1^{(n)}(z))\colon\ |z-x_1| < \delta\}$ according to the definition of S''. $\Pi(V)$ is an open subset of $\{|z-x_1| < \delta\}$ and $g(z, w)=0$ for all points $(z, w) \in W'$ with $\Pi(z) \in \Pi(V)$ by the definition of S'' and W'.

Now we can apply Wermer's subharmonicity theorem as in the proof of claim 1 to $(P(X)|_{W'}, W')$, $\{|z-x_1| < \delta\}$, $\Pi|_{W'}$ and g to show that $g|_{W'} \equiv 0$, hence $(x_1, y_1) \in S''$ by (6) and (7).

If $n=1$ we have

$$|y_0 - w_i^{(1)}(x_0)| \leqq 1/2c_1|B_1(x_0)| \quad \text{for} \quad i=1 \quad \text{or} \quad i=2$$

by (7), hence $g(x_0, y_0)=0$ by the definition of S'' (recall that $x_0 \in S$).

If $n>1$

$$|y_0 - w_j^{(n-1)}(x_0)| > 3/4c_{n-1}|B_{n-1}(x_0)| \quad \text{for} \quad j>1$$

by (6), (7) and the definition of W. Hence

$$|y_0 - w_j^{(n-1)}(x_0) \pm c_n B_n(x_0)| > 3/4c_{n-1}|B_{n-1}(x_0)| - c_n|B_n(x_0)| > 6c_n|B_n(x_0)| \quad \text{by} \quad (2).$$

It follows again from (7) and (4) that

$$|y_0 - w_1^{(n-1)}(x_0) + c_n B_n(x_0)| \leqq 1/2c_n|B_n(x_0)|$$

or

$$|y_0 - w_1^{(n-1)}(x_0) - c_n B_n(x_0)| \leqq 1/2c_n|B_n(x_0)|$$

and $g(x_0, y_0)=0$ by the definition of S''.

If $g \in P(X)$ vanishes in an arbitrary open set $U \subset X$ then g must vanish in a neighbourhood of a point (a_n, b_i) for suitable $n, i \in \mathrm{N}$ by (10), hence $g \equiv 0$.

Remark 1. Originally I used in the proof a lemma of Glicksberg (cf. [2] Lemma 10.4) instead of Wermer's subharmonicity theorem. The use of this theorem was pointed out to me by Tom Ransford and Bernard Aupetit.

Remark 2. Let $X \subset \mathbf{C}^n$ be a compact set and denote by $\hat{X}$ the polynomially convex hull of X. There is no general principle which implies that $P(\hat{X})$ satisfies the local identity theorem in a point $x \in \hat{X} \setminus X$. Take for instance

$$X = [0, 1] \times \partial E \subset \mathbf{C}^2$$

then

$$\hat{X} = [0, 1] \times \bar{E}.$$

($\bar{E}$ denotes the closed unit disc, $[0, 1]$ the closed unit interval.) Let $(x, y) \in [0, 1] \times E$. Choose for each $n \in \mathbf{N}$ a continuous function f_n on $[0, 1]$ such that $f_n(x') = 0$ if $|x' - x| \leqq 1/n$ and $f(x') \neq 0$ if $|x' - x| > 1/n$. Then

$$g_n \colon \hat{X} \to \mathbf{C}, \quad (z, w) \to f_n(z)$$

defines an element of $P(\hat{X})$ by the Stone—Weierstraß theorem for each $n \in \mathbf{N}$. Hence $P(\hat{X})$ does not satisfy the local identity theorem in (x, y).

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