Generalized analyticity in UMD spaces

Earl Berkson, T. A. Gillespie and Paul S. Muhly

1. Introduction

Let G be a compact abelian group with dual group $\hat{G}$ and let $u \to R_u$ be a strongly continuous representation of G on a Banach space X. Associated with each $\gamma \in \hat{G}$, define $E_\gamma : X \to X$ by

$$E_\gamma x = \int_G \gamma(u) R_{-u} x \, du \quad (x \in X),$$

where du denotes Haar measure on G and the integral exists as a Bochner integral. Then E_γ is a bounded projection which maps X onto the corresponding eigenspace

(1.1) $$X_\gamma = \{x \in X : R_u x = \gamma(u) x \quad \text{for all} \quad u \in G\}.$$

Furthermore,

(1.2) $$\|E_\gamma\| \leqq c = \sup\{\|R_u\| : u \in G\} < \infty \quad (\gamma \in \hat{G})$$

and

(1.3) $$E_\gamma E_\nu = 0 \quad (\gamma, \nu \in \hat{G}, \ \gamma \neq \nu).$$

Also, the injectivity of the Fourier transform implies that

(1.4) $$\text{clm}\{X_\gamma : \gamma \in \hat{G}\} = X \quad \text{and} \quad \cap\{\ker E_\gamma : \gamma \in \hat{G}\} = \{0\},$$

where "clm" denotes "closed linear span". We shall refer to E_γ as the spectral projection associated with $\gamma \in \hat{G}$.

Suppose now that G is connected. Then $\hat{G}$ can be given (in a non-canonical and in general non-unique way) a total ordering with respect to which it becomes an ordered group. Let $\geqq$ be any such ordering and define the subspaces X^+, X^- of X by

(1.5) $$X^+ = \text{clm}\{X_\gamma : \gamma \in \hat{G}, \ \gamma \geqq 0\}, \quad X^- = \text{clm}\{X_\gamma : \gamma \in \hat{G}, \ \gamma < 0\}.$$

Then (1.3) and (1.4) imply that $X^+ \cap X^- = \{0\}$ and that the algebraic direct sum $X^+ \oplus X^-$ is dense in X. The main result of this paper (Theorem (4.1)) is that, under a suitable geometric condition on X,

$$X = X^{+} \oplus X^{-}.$$

In addition, the norm of the corresponding projection E^{+} of X onto X^{+} satisfies $\|E^{+}\| \leqq c^2 K_X$, where c is the constant in (1.2) and K_X is a constant dependent on X but not on G or $\geqq$. The precise condition on X is that it should belong to the class of Banach spaces having the unconditionality property for martingale differences, the so-called UMD spaces.

A special case of this result is Bochner's generalization of the M. Riesz harmonic conjugacy theorem. In the present context, the M. Riesz theorem is most conveniently formulated as asserting the boundedness (with norm denoted by C_p) of the natural projection of $L^p(\mathbf{T})$ onto $H^p(\mathbf{T})$ for the circle group $\mathbf{T}$ $(1<p<\infty)$. Bochner's theorem [6, Theorem 16] states that, for G compact, connected and abelian, and $\geqq$ as above, the 'analytic' projection of a trigonometric polynomial $\sum_{\gamma \in \hat{G}} c_\gamma \gamma$ on G onto $\sum_{\gamma \geqq 0} c_\gamma \gamma$ is bounded relative to the L^p norm and so extends to all of $L^p(G)$, provided $1<p<\infty$. In addition, the norm of this analytic projection is dominated by a constant which depends on p but not on G or $\geqq$. To obtain Bochner's result from Theorem (4.1), take X to be the UMD space $L^p(G)$ and R_u to be the translation operator on $L^p(G)$ associated with $u \in G$. In this case, the constant K_X mentioned above is majorized by C_p. (See Remark (a) after the proof of Theorem (4.1).) Thus C_p is seen to be the best possible constant for Bochner's theorem, a fact shown earlier in [1].

Two techniques combine to give our extension of Bochner's theorem, namely the use of a generalization of the method of transference of Coifman and Weiss [10] and the boundedness of the vector-valued Hilbert transform for UMD spaces. We discuss these ideas in §§ 2, 3 before proving the main result in § 4. (In § 2, we also indicate how to obtain a version of the homomorphism theorem for multipliers using transference.) In the final section, we discuss briefly how the extension of Bochner's theorem can be refined in the case when X is a subspace of an L^p-space $(1<p<\infty)$ and the operators R_u $(u \in G)$ are invertible isometries. For such X, the analytic projection of $L^p(\mathbf{T})$ onto $H^p(\mathbf{T})$ is extremal in the sense that it gives rise to the largest possible norm for the 'analytic' projection E^{+} of X onto X^{+}. Indeed, such a result is valid for linear combinations of E^{+}, the identity operator I and the spectral projection E_0 associated with the zero element of $\hat{G}$ (Theorem (5.1)). A more restricted version of this extremal result (in the context of Bochner's theorem) was obtained earlier in [1, Theorem (3.8) and Corollary 3.13)]. Other aspects of spectral theory in UMD spaces have been studied by the authors [3, 4], where the notion of transference has also played an important role.

As usual, $\mathbf{R}$, $\mathbf{C}$, $\mathbf{Z}$ and $\mathbf{N}$ denote the real numbers, the complex numbers, the integers and the positive integers respectively. All Banach spaces are taken to have complex scalars and all integrals of vector-valued functions are to be interpreted as appropriate Bochner integrals. Given sets A and B, we denote their set-theoretic difference by $A \setminus B$. If A and B are also subsets of an abelian group, their algebraic difference $\{a-b\colon a \in A,\ b \in B\}$ will be written $A-B$.

2. Generalized transference

We begin by recalling the general transference result of Coifman and Weiss [10, Theorem 2.4]. This is concerned with the representation of a locally compact amenable group $\mathcal{G}$ on a subspace Y of an L^p-space, and the transfer of norm estimates for certain convolution operators on $L^p(\mathcal{G})$ to obtain similar estimates for associated operators on Y. For simplicity, we confine ourselves here to the case when $\mathcal{G}$ is locally compact abelian (and hence automatically amenable), since this suffices for our applications.

To fix notation, let $\mathcal{G}$ be a locally compact abelian group, let $(\mathcal{M}, \mu)$ be an arbitrary measure space, and let Y be a closed subspace of $L^p(\mathcal{M}, \mu)$, where $1 \leqq p < \infty$. Let $u \to R_u$ be a strongly continuous representation of $\mathcal{G}$ on Y with

$$c = \sup \{\|R_u\|\colon u \in \mathcal{G}\} < \infty.$$

Let $k \in L^1(\mathcal{G})$ and put

$$T_k y = \int_{\mathcal{G}} k(u) R_{-u} y \, du \quad (y \in Y). \tag{2.1}$$

Integration (in Bochner's sense) is with respect to Haar measure du on $\mathcal{G}$ and (2.1) defines T_k as a bounded linear operator on Y, with

$$\|T_k\| \leqq c \|k\|_1. \tag{2.2}$$

The aim of transference is to improve the order of magnitude of the majorant in (2.2) as follows.

(2.3) **Theorem** ([10, Theorem 2.4]). *With the above hypotheses and notation,*

$$\|T_k\| \leqq c^2 N_p(k),$$

where $N_p(k)$ denotes the norm of the convolution operator $f \to k * f$ on $L^p(\mathcal{G})$.

A few comments about (2.3) and its proof are in order. Firstly, Coifman and Weiss consider the case $Y = L^p(\mathcal{M}, \mu)$, but their proof applies when Y is only a subspace of $L^p(\mathcal{M}, \mu)$. Secondly, they take k to have compact support; however, (2.2) shows that $\|T_k\|$ is continuous as a function of k relative to the L^1-norm and hence the extension to an arbitrary integrable kernel k is immediate. (Such an extension is indicated in [10, p. 11].) Of more significance, the proof in [10] relies in part on an application of Fubini's theorem, and so μ is assumed to be σ-finite therein. Furthermore, Coifman and Weiss interpret (2.1) in a pointwise sense as

$$(2.4) \qquad (T_k y)(\omega) = \int_{\mathscr{G}} k(u)(R_{-u}y)(\omega)\,du \quad (\omega \in \mathscr{M}, \mu - \text{a.e.})$$

and, in order to apply Fubini's theorem, implicitly require an appropriate joint measurability condition as a function of (u, ω) of expressions such as the integrand $k(u)(R_{-u}y)(\omega)$ in (2.4). Whilst we do not wish to dwell on these technicalities, we record here the following lemma. With the aid of this, the proof of [10, Theorem 2.4] may be adapted to give (2.3) in the generality stated here.

(2.5) **Lemma.** *Let $u \to R_u$ be a strongly continuous representation of the locally compact abelian group $\mathscr{G}$ on a closed subspace Y of $L^p(\mathscr{M}, \mu)$ with*

$$c = \sup\{\|R_u\| : u \in \mathscr{G}\} < \infty,$$

where $(\mathscr{M}, \mu)$ is an arbitrary measure space and $1 \leqq p < \infty$. Let $k \in L^1(\mathscr{G})$ have compact support K, let T_k be as in (2.1), and let V be a relatively compact open subset of $\mathscr{G}$. Then, given $y \in Y$, there exist a σ-finite measurable subset $\mathscr{M}_0$ of $\mathscr{M}$ and a jointly measurable function $F\colon \mathscr{G} \times \mathscr{M} \to \mathbf{C}$ such that

(i) *F vanishes off $(V-K) \times \mathscr{M}_0$;*

(ii) *for almost all $u \in V-K$, $F(u, \cdot)$ is a representing function for the equivalence class (modulo equality μ-a.e.) $R_u y$;*

(iii) *for all $v \in V$,*

$$\int_{\mathscr{G}} k(u) F(v-u, \cdot)\,du$$

is a representing function for the equivalence class $R_v T_k y$.

Outline of proof. Fix $y \in Y$. The uniform continuity of $u \to R_u y$ on $V-K$ gives a sequence $\{f_n\}$ of simple functions, say

$$f_n(u) = \sum_m \chi_{nm}(u) y_{nm},$$

with each χ_{nm} the characteristic function of a measurable subset of $V-K$ and each $y_{nm} \in Y$, such that $f_n(u) \to R_u y$ in $L^p(\mathscr{M}, \mu)$ norm uniform on $V-K$. Since each y_{nm} vanishes almost everywhere off some σ-finite subset of $\mathscr{M}$, there is a σ-finite measurable subset $\mathscr{M}_0$ of $\mathscr{M}$ such that y_{nm} vanishes off $\mathscr{M}_0$ for all n, m. With

$$F_n(u, \omega) = \sum_m \chi_{nm}(u) y_{nm}(\omega),$$

it is easy to check that $\{F_n\}$ is Cauchy in $L^p((V-K) \times \mathscr{M}_0)$ and so converges in $L^p((V-K) \times \mathscr{M}_0)$ to some function F. We may consider F as a measurable function $\mathscr{G} \times \mathscr{M} \to \mathbf{C}$ satisfying (i).

We have

$$\int_{V-K}\left\{\int_{\mathcal{M}}|F(u,\omega)-(R_u y)(\omega)|^p\,d\mu(\omega)\right\}du \leqq 2^p\|F-F_n\|^p_{L^p((V-K)\times\mathcal{M}_0)}$$
$$+2^p\int_{V-K}\left\{\int_{\mathcal{M}}|F_n(u,\omega)-(R_u y)(\omega)|^p\,d\mu(\omega)\right\}du\to 0$$

as $n\to\infty$. Property (ii) now follows.

To obtain (iii), fix $v\in V$. Thus $R_v T_k y$ is given by the Bochner integral $\int_{\mathcal{G}} k(u)R_{v-u}y\,du$. For any $g\in L^q(\mathcal{M},\mu)$ (where $p^{-1}+q^{-1}=1$), we have, with the aid of Fubini's theorem,

$$\langle R_v T_k y, g\rangle=\int_K k(u)\langle R_{v-u}y, g\rangle\,du=\int_{\mathcal{M}_0} g(\omega)\left\{\int_K k(u)F(v-u,\omega)\,du\right\}d\mu(\omega)$$
$$=\int_{\mathcal{M}} g(\omega)\left\{\int_K k(u)F(v-u,\omega)\,du\right\}d\mu(\omega).$$

The proof of the Lemma is easily completed by letting g run through the characteristic functions of the subsets of $\mathcal{M}$ having finite measure.

Suppose now that $\mathcal{G}_0$ is a σ-compact, locally compact abelian group, and that $u\to S_u$ is a representation of $\mathcal{G}_0$ by measure-preserving transformations of the points in an arbitrary measure space $(\mathcal{M},\mu)$. For $1\leqq p<\infty$, let $u\to R_u^{(p)}$ be the representation (by isometries) of $\mathcal{G}_0$ in $L^p(\mathcal{M},\mu)$ defined as follows:

$$(R_u^{(p)})f(\omega)=f(S_{-u}\omega),\quad\text{for}\quad f\in L^p(\mathcal{M},\mu),\quad u\in\mathcal{G}_0,\quad \omega\in\mathcal{M}.$$

It is easily seen that, for any given value of p, strong continuity of the representation $R^{(p)}$ is equivalent to the following condition (which is independent of p): for each measurable subset E of $\mathcal{M}$ such that $\mu(E)<\infty$, $\mu(E\Delta(S_u E))\to 0$ as $u\to 0$ in $\mathcal{G}_0$, where Δ denotes symmetric difference. We shall also assume this latter condition of strong continuity. In [9, Theorem 3.7], Coifman and Weiss apply their original version of the above Theorem (2.3) to the present setting in order to establish a method for transferring "normalized" $L^p(\mathcal{G}_0)$-multipliers with their bounds to $L^p(\mathcal{M},\mu)$. The proof in [9] explicitly requires μ to be σ-finite, and implicitly requires the representation S to provide joint measurability in (u,ω) of $f(S_{-u}\omega)$ for appropriate measurable functions f. Use of Theorem (2.3) above in the proof of [9, Theorem 3.7] removes both of these requirements. Specifically, the Transference Result for Multipliers takes the following form.

(2.6) **Proposition.** *Let $\mathcal{G}_0$ be a σ-compact, locally compact abelian group, let $(\mathcal{M},\mu)$ be an arbitrary measure space, and suppose that $u\to S_u$ is a representation of $\mathcal{G}_0$ by measure-preserving transformations of $\mathcal{M}$. Assume further that, for each measurable subset E of $\mathcal{M}$ of finite measure, $\mu(E\Delta(S_u E))\to 0$ as $u\to 0$ in $\mathcal{G}_0$. Denote by $\varepsilon(\cdot)$ the (regular Borel) spectral measure of the unitary representation $u\to R_u^{(2)}$ described above. If $1\leqq p<\infty$ and ψ is an $L^p(\mathcal{G}_0)$-multiplier, normalized in the sense of*

[9, § 3], *then the operator* $\int_{\hat{\mathscr{G}}_0} \psi(\gamma)\, d\varepsilon(\gamma)$ *can be extended from* $L^2(\mathscr{M}, \mu) \cap L^p(\mathscr{M}, \mu)$ *to a bounded linear mapping of* $L^p(\mathscr{M}, \mu)$ *into* $L^p(\mathscr{M}, \mu)$ *whose norm does not exceed the* $L^p(\mathscr{G}_0)$*-multiplier norm of* ψ.

This Proposition allows us to deduce by transference a version of the Homomorphism Theorem for Multipliers [11, Appendix B]. For a locally compact abelian group $\mathscr{G}$, we shall denote the space of $L^p(\mathscr{G})$-multipliers by $M_p(\hat{\mathscr{G}})$, and the corresponding multiplier norm by $\|\cdot\|_{M_p(\hat{\mathscr{G}})}$.

(2.7) **Theorem.** (Homomorphism Theorem for Multipliers). *Let* Γ_1 *and* Γ_2 *be locally compact abelian groups. Suppose that* $\hat{\Gamma}_2$ *is* σ*-compact, and* ϱ *is a continuous homomorphism of* Γ_1 *into* Γ_2. *If* $1 \leqq p < \infty$, *and* $\psi \in M_p(\Gamma_2)$ *is normalized in the sense of* [9, § 3], *then the composition* $\psi \circ \varrho \in M_p(\Gamma_1)$, *and*

$$\|\psi \circ \varrho\|_{M_p(\Gamma_1)} \leqq \|\psi\|_{M_p(\Gamma_2)}.$$

Outline of Proof. Since ψ is normalized, we can also assume without loss of generality that ψ is continuous. The argument with standard tools in [11, Lemma B.1.1] now allows us to assume further that ψ is continuous with compact support. Let $\hat{G}_j = \Gamma_j$, $j = 1, 2$, and let $\hat{\varrho}\colon G_2 \to G_1$ be the dual homomorphism of ϱ. For $u \in G_2$, let S_u be translation in G_1 by $\hat{\varrho}(u)$. Thus $u \to S_u$ represents G_2 by (Haar) measure-preserving transformations of G_1. It is straightforward to check that, in the notation of the preceding Proposition, the spectral measure $\varepsilon(\cdot)$ for the representation $u \to R_u^{(2)}$ of G_2 in $L^2(G_1)$ satisfies the following whenever B is a Baire subset of Γ_2:

$\varepsilon(B)$ is the $L^2(G_1)$-multiplier transformation corresponding to the characteristic function of $\varrho^{-1}(-B)$.

Use of this fact, together with the preceding Proposition, completes the proof.

We now return to the context of (2.3) and obtain a generalization of that result which applies to representations of a locally compact abelian group $\mathscr{G}$ on an arbitrary Banach space X. To this end, let $L^p(\mathscr{G}, X)$ denote the usual Lebesgue—Bochner space of p-integrable X-valued functions on $\mathscr{G}$, where $1 \leqq p < \infty$. Given $k \in L^1(\mathscr{G})$, let $N_{p,X}(k)$ denote the norm of convolution by k on $L^p(\mathscr{G}, X)$.

(2.8) **Theorem.** *Let* $u \to R_u$ *be a strongly continuous representation of* $\mathscr{G}$ *on* X *such that*

$$c = \sup\{\|R_u\| : u \in \mathscr{G}\} < \infty,$$

let $k \in L^1(\mathscr{G})$, *and let* $T_k\colon X \to X$ *be defined by*

$$T_k x = \int_{\mathscr{G}} k(u) R_{-u} x\, du \quad (x \in X).$$

Then $\|T_k\| \leqq c^2 N_{p,X}(k)$ *for* $1 \leqq p < \infty$.

Proof. This is a mild adaptation of the proof of the Coifman—Weiss result. (Indeed, it is somewhat simpler since there are no measure-theoretic technicalities.) Firstly, the inequality (2.2) is still valid, so we may restrict to the case when k has compact support K. Fix $\varepsilon>0$ and let V be a relatively compact open subset of $\mathcal{G}$ such that

$$m(V-K)/m(V) \leqq 1+\varepsilon,$$

where m temporarily denotes Haar measure [14, Lemma (31.36)]. Let χ denote the characteristic function of $V-K$. Fix $x\in X$ and let $f\in L^p(\mathcal{G}, X)$ be defined by $f(u)=\chi(u)R_u x$. Averaging the inequality $\|T_k x\|^p\leqq c^p\|R_v T_k x\|^p=c^p\|T_k R_v x\|^p$ over $v\in V$, we have

$$\|T_k x\|^p \leqq c^p\{m(V)\}^{-1}\int_V\Big\|\int_K k(u)R_{v-u}x\,du\Big\|^p dv$$

$$= c^p\{m(V)\}^{-1}\int_V\Big\|\int_{\mathcal{G}} k(u)\chi(v-u)R_{v-u}x\,du\Big\|^p dv$$

$$= c^p\{m(V)\}^{-1}\int_V\|(k*f)(v)\|^p dv \leqq c^p\{m(V)\}^{-1}\{N_{p,X}(k)\|f\|_p\}^p$$

$$\leqq c^{2p}(1+\varepsilon)\{N_{p,X}(k)\}^p\|x\|^p.$$

The required estimate for $\|T_k\|$ now follows, completing the proof of (2.8).

3. UMD spaces

A Banach space X has the unconditionality property for martingale differences (written as $X\in\text{UMD}$) if, for $1<p<\infty$, there are constants $C(p, X)$ such that

$$\Big\|\sum_{j=1}^n \varepsilon_j d_j\Big\|_p \leqq C(p,X)\Big\|\sum_{j=1}^n d_j\Big\|_p \quad (n\in\mathbf{N})$$

for every martingale difference sequence $\{d_j\colon j\in\mathbf{N}\}$ in $L^p([0,1],X)$ and all choices of numbers $\varepsilon_1, \varepsilon_2, \dots,$ with $\varepsilon_j=\pm1$. A more extended discussion of this definition, together with further references, may be found in [8]. The class UMD contains a number of classical Banach spaces such as the L^p-spaces of an arbitrary measure $(1<p<\infty)$ and their non-commutative analogues, including the von Neumann—Schatten p-classes $(1<p<\infty)$. (See [4, 5] for more background and further details.)

In the present note, we shall be concerned with a characterization of the UMD property in terms of the Hilbert transform, due to Burkholder and Bourgain. To state the result precisely, we denote by H_X the periodic Hilbert transform

$$H_X f(s) = \lim_{\varepsilon\to 0+}(2\pi)^{-1}\int_{\pi\geqq|t|\geqq\varepsilon}\cot(t/2)f(s-t)\,dt,$$

of a strongly measurable function $f\colon \mathbf{T}\to X$, provided this exists almost everywhere on $\mathbf{T}$.

(3.1) **Theorem** ([7, 8]). *For a Banach space X, the following statements are equivalent.*

(i) $X\in$ UMD.

(ii) *H_X is a bounded linear operator $L^p(\mathbf{T}, X)\to L^p(\mathbf{T}, X)$ for every p in the range $1<p<\infty$.*

(iii) *H_X is a bounded linear operator $L^p(\mathbf{T}, X)\to L^p(\mathbf{T}, X)$ for some p in the range $1<p<\infty$.*

It should perhaps be remarked that the UMD property may analogously be characterized in terms of the Hilbert kernels associated with $\mathbf{R}$ and $\mathbf{Z}$ (see [4] for the case of $\mathbf{Z}$). However, in the present context, it is the periodic case which is most appropriate.

Given a UMD space X, define the constant K_X as

(3.2) $$K_X = \inf_{1<p<\infty} \|2^{-1}(I+P_X+iH_X)\|_p,$$

where $P_X f$ denotes the constant term in the Fourier series of an integrable function $f\colon \mathbf{T}\to X$ and $\|\cdot\|_p$ denotes the operator norm on $L^p(\mathbf{T}, X)$. We can now state a preliminary version (for the circle group) of our main result.

(3.3) **Theorem.** *Let $e^{it}\to R_{e^{it}}$ be a strongly continuous representation of* $\mathbf{T}$ *on the* UMD *space X and, for $n\in\mathbf{Z}$, let E_n be the associated spectral projection*

$$E_n x = (2\pi)^{-1}\int_{-\pi}^{\pi} e^{int} R_{e^{-it}} x\, dt.$$

Then there is a unique bounded linear operator T on X such that

(3.4) $$Tx = \begin{cases} x & (x\in E_n X,\ n\geqq 0)\\ 0 & (x\in E_n X,\ n<0).\end{cases}$$

Furthermore,

(3.5) $$\|T\| \leqq c^2 K_X,$$

where K_X is given by (3.2) *and* $c=\sup\{\|R_{e^{it}}\|\colon e^{it}\in\mathbf{T}\}$.

Proof. Let k_n denote the nth Fejér kernel for $\mathbf{T}$ and let $h_n\colon \mathbf{T}\to\mathbf{C}$ be defined by $h_n(e^{it})=\sum_{j\geqq 0}\hat{k}_n(j)e^{ijt}$, where $\wedge$ denotes the Fourier transform. Let T_n be the transferred operator on X defined by

$$T_n x = (2\pi)^{-1}\int_{-\pi}^{\pi} h_n(e^{it}) R_{e^{-it}} x\, dt \quad (x\in X).$$

By Theorem (2.8),

(3.6) $$\|T_n\| \leqq c^2 N_{p,X}(h_n) \quad (n\in\mathbf{N},\ 1<p<\infty).$$

Since $h_n * f = k_n * (2^{-1}(I+P_X+iH_X)f)$ for $f \in L^p(\mathbf{T}, X)$ and $\|k_n\|_1 = 1$, we have

$$N_{p,X}(h_n) \leqq N_{p,X}(k_n)\|2^{-1}(I+P_X+iH_X)\|_p \leqq \|2^{-1}(I+P_X+iH_X)\|_p$$

for $n \in \mathbf{N}$ and $1<p<\infty$. Hence, by (3.2) and (3.6),

$$\|T_n\| \leqq c^2 K_X \quad (n \in \mathbf{N}). \tag{3.7}$$

For $j \in \mathbf{Z}$ and $x \in E_j X$, $R_{e^{it}} x = e^{ijt} x$; hence, for all n,

$$T_n x = \hat{h}_n(j) x \to \begin{cases} x & (x \in E_j X,\ j \geqq 0) \\ 0 & (x \in E_j X,\ j < 0). \end{cases}$$

It follows from (3.7) and the density in X of the linear span of the eigenspaces $E_j X$ $(j \in \mathbf{Z})$ that $\{T_n\}$ converges strongly to an operator T on X satisfying (3.4) and (3.5). It is easy to see from (3.4) that $X = X^+ \oplus X^-$, and T is the projection onto X^+ along X^-. This settles the uniqueness assertion and completes the proof of Theorem (3.3).

Remark. The existence of the operator T satisfying (3.4) was also shown in [2, Theorem (4.2)]. However, the methods used there do not lead to the estimate (3.5), which will be needed for the proof of our main result (Theorem (4.1)).

4. Analytic projections

Using the notation of §§ 1, 3, we can now state precisely our main result.

(4.1) **Theorem.** *Let G be a compact, connected, abelian group and let $\geqq$ be a linear ordering of its dual group $\hat{G}$. Let $u \to R_u$ be a strongly continuous representation of G on a* UMD *space X. Then*

$$X = X^+ \oplus X^- \tag{4.2}$$

and the corresponding projection E^+ of X onto X^+ satisfies

$$\|E^+\| \leqq c^2 K_X, \tag{4.3}$$

where X^+, X^- and K_X are given by (1.5) *and* (3.2) *respectively and* $c = \sup\{\|R_u\| : u \in G\}$.

The proof of (4.1) proceeds by considering first the case $G = \mathbf{T}^N$ and then deducing the general case.

Proof of (4.1) *when* $G = \mathbf{T}^N$. Assume that $u \to R_u$ is a strongly continuous representation of $\mathbf{T}^N$ on the UMD space X and that $\geqq$ is some linear ordering on $\hat{G} = \mathbf{Z}^N$ which respects the additive group structure. Fix $x = \sum_{\gamma \in \hat{G}} x_\gamma$, where each x_γ belongs to the eigenspace X_γ given by (1.1) and $x_\gamma \neq 0$ for only finitely many

$\gamma \in \hat{G}$. We must show that

$$\left\|\sum_{\gamma \geqq 0} x_\gamma\right\| \leqq c^2 K_X \|x\| \tag{4.4}$$

in order to established (4.2) and (4.3).

By Lemma 2.5 of [1], there exists $a \in \mathbf{R}^N$ such that

$$a \cdot \gamma > 0 \quad \text{if} \quad x_\gamma \neq 0 \quad \text{and} \quad \gamma > 0 \tag{4.5}$$

and

$$a \cdot \gamma < 0 \quad \text{if} \quad x_\gamma \neq 0 \quad \text{and} \quad \gamma < 0. \tag{4.6}$$

(See the discussion on p. 283 of [1]; the dot here denotes the standard inner product on $\mathbf{R}^N$.) We may further assume that the coordinates of a are rational and then, by multiplying by a suitable positive integer, take $a = (k_1, \ldots, k_N) \in \mathbf{Z}^N$.

Now consider the homomorphism θ of $\mathbf{T}$ into $\mathbf{T}^N$ defined by $\theta(e^{it}) = (e^{ik_1 t}, \ldots, e^{ik_N t})$ and the associated representation

$$\tilde{R}_{e^{it}} = R_{\theta(e^{it})} \tag{4.7}$$

of $\mathbf{T}$ on X. By Theorem (3.3), there is a bounded linear operator $T\colon X \to X$ such that, if $y \in X$ belongs to the eigenspace

$$X_n = \{z\colon \tilde{R}_{e^{it}} z = e^{\mathrm{int}} z \quad \text{for all} \quad e^{it} \in \mathbf{T}\}$$

for some $n \in \mathbf{Z}$, then

$$Ty = \begin{cases} y & (n \geqq 0) \\ 0 & (n < 0). \end{cases}$$

Furthermore,

$$\|T\| \leqq \sup\{\|\tilde{R}_{e^{it}}\|^2\colon e^{it} \in \mathbf{T}\} K_X \leqq c^2 K_X. \tag{4.8}$$

Noting that

$$\tilde{R}_{e^{it}} x_\gamma = R_{\theta(e^{it})} x_\gamma = \gamma(\theta(e^{it})) x_\gamma = e^{i(a\cdot\gamma)t} x_\gamma$$

for each $\gamma \in \hat{G}$, it follows from (4.5) and (4.6) that

$$Tx = \sum_{\gamma \geqq 0} x_\gamma;$$

(4.8) now gives the desired inequality (4.4) and completes the proof in this special case.

Proof of (4.1) *in the general case.* Now let G be an arbitrary compact, connected, abelian group and $u \to R_u$ a representation of G on X as in the statement of (4.1). As before, we must establish (4.4) for an arbitrary element $x = \sum_\gamma x_\gamma$ in the linear span of the eigenspaces X_γ defined by (1.1).

Fix such an element $x = \sum_\gamma x_\gamma$ and put $\Gamma = \{\gamma \in \hat{G}\colon x_\gamma \neq 0\}$. Assume that Γ is non-empty (otherwise $x = 0$ and there is nothing to prove), let Λ denote the subgroup of $\hat{G}$ generated by Γ, and let K denote the annihilator of Λ in G. Since Γ is finite and $\hat{G}$ is torsion-free, Λ may be identified with $\mathbf{Z}^N$ and its dual G/K with $\mathbf{T}^N$ for some positive integer N. Also, $\Lambda=\mathbf{Z}^N$ inherits a linear ordering from $\hat{G}$. Let

$$Y=\left(\sum_{\gamma\in\Gamma}E_\gamma\right)X=\oplus\{X_\gamma:\ \gamma\in\Gamma\}.$$

Consider the induced strongly continuous representation $\tilde{R}$ of G/K on Y defined by

$$\tilde{R}_{u+K}=R_u|Y.$$

It is easily verified with the aid of (1.1) that the eigenspaces Y_γ $(\gamma\in\Lambda)$ for $\tilde{R}$ satisfy

$$Y_\gamma=X_\gamma\quad(\gamma\in\Gamma),\quad Y_\gamma=\{0\}\quad(\gamma\in\Lambda\setminus\Gamma).$$

Applying the result of (4.1) in the special case of $\mathbf{T}^N$, noting that Y is a UMD space with $K_Y\leqq K_X$, we conclude that

$$\left\|\sum_{\gamma\geqq 0}x_\gamma\right\|\leqq K_Y\sup\{\|\tilde{R}_{u+K}\|^2:\ u+K\in G/K\}\|x\|\leqq K_Xc^2\|x\|.$$

This gives (4.4) and completes the proof of Theorem (4.1) in the general case.

Remarks. (a) Let $(\mathcal{M},\mu)$ be an arbitrary measure space and let X be a closed subspace of $L^p(\mathcal{M},\mu)$, where $1<p<\infty$. Given a trigonometric polynomial Q: $\mathbf{T}\to X$, say

$$Q(e^{it})=\sum_{n\in\mathbf{Z}}e^{int}x_n,$$

with each $x_n\in X$, we have

$$\left(2^{-1}(I+P_X+iH_X)Q\right)(e^{it})=\sum_{n\geqq 0}e^{int}x_n.$$

A simple application of Fubini's theorem (justified by replacing $\mathcal{M}$ by a σ-finite subset which carries each x_n) shows that

$$\|2^{-1}(I+P_X+iH_X)Q\|_p\leqq C_p\|Q\|_p,$$

where C_p is the norm of the classical Riesz projection of $L^p(\mathbf{T})$ onto $H^p(\mathbf{T})$. Hence, for X a subspace of $L^p(\mathcal{M},\mu)$, the constant K_X given by (3.2) satisfies $K_X\leqq C_p$. This shows that, for such a space X, the norm of E^+ in Theorem (4.1) is dominated by a constant which depends only on p and c but not on the particular measure space $(\mathcal{M},\mu)$. Thus, as observed in § 1, Theorem (4.1) is indeed a full extension of Bochner's generalization of the M. Riesz conjugacy theorem.

(b) Using the Cotlar bootstrap method, it was shown in [13, IV. 4] that the Hilbert transform is bounded on $L^p(\mathbf{R},\mathcal{C}_p)$ for $1<p<\infty$, where $\mathcal{C}_p$ denotes the von Neumann—Schatten p-class. Hence Theorem (4.1) applies to $X=\mathcal{C}_p$ for $1<p<\infty$. Let $\mathcal{C}_p$ be realized as an operator ideal acting on the space $H^2(\mathbf{T})$ and let the group $\mathbf{T}$ be represented on $\mathcal{C}_p$ by the isometries R_u of $\mathcal{C}_p$ given by

$$R_ux=S_{u^{-1}}xS_u\quad(x\in\mathcal{C}_p,\ u\in\mathbf{T}),$$

where S_u: $H^2(\mathbf{T})\to H^2(\mathbf{T})$ corresponds to rotation by u. A straightforward calculation shows that, when $\hat{\mathbf{T}}=\mathbf{Z}$ has its natural order, the projection E^+ in Theorem (4.1) is the upper triangular projection which maps the matrix (relative to the standard basis in $H^2(\mathbf{T})$) of $x\in\mathscr{C}_p$ to its upper triangular truncation (obtained by replacing with 0 all entries strictly below the main diagonal). That this is bounded on $\mathscr{C}_p$ for $1<p<\infty$ is a celebrated result of Macaev [15]. Whilst this proof of Macaev's result is somewhat indirect (for a more direct approach using the Cotlar bootstrap method, see [12, III. 6]), it is of some interest to note that Theorem (4.1) does provide an extension of Macaev's theorem along the lines of Bochner's extension of the M. Riesz conjugacy theorem. Another extension of Macaev's theorem, again involving the UMD property in its proof, appears in [5].

5. Representations on subspaces of L^p-spaces

In this final section, we give a refinement of the inequality (4.3) when X is a subspace of an L^p-space $(1<p<\infty)$ and the operators R_u are invertible isometries. To state the result precisely, let P^+ denote the classical Riesz projection of $L^p(\mathbf{T})$ onto $H^p(\mathbf{T})$ and let P_0 denote the rank one projection $f\to\hat{f}(0)$ on $L^p(\mathbf{T})$. Denote the norm of an operator on $L^p(\mathbf{T})$ by $\|\cdot\|_p$.

(5.1) **Theorem.** *Let $(\mathscr{M},\mu)$ be an arbitrary measure space and let X be a closed subspace of $L^p(\mathscr{M},\mu)$, where $1<p<\infty$. Let G be a compact, connected, abelian group and let $u\to R_u$ be a strongly continuous representation of G by invertible isometries on X. Let E^+ be the analytic projection on X corresponding to some linear ordering on $\hat{G}$ and let E_0 be the spectral projection associated with $0\in\hat{G}$. Then*

$$\|\xi I+\eta E_0+\zeta E^+\|\leqq\|\xi I+\eta P_0+\zeta P^+\|_p$$

for $\xi,\eta,\zeta\in\mathbf{C}$.

Remark. With $X=L^p(\mathbf{T})$, $G=\mathbf{T}$, and R_u given by rotation by $u\in\mathbf{T}$, the operators $\xi I+\eta E_0+\zeta E^+$ and $\xi I+\eta P_0+\zeta P^+$ are equal. Thus, Theorem (5.1) shows that the regular representation of $\mathbf{T}$ on $L^p(\mathbf{T})$ is extremal (with respect to the size of the norm of a linear combination of the identity operator and the projections E_0, E^+) amongst isometric representations of compact, connected, abelian groups on subspace of L^p-spaces. A restricted version of this result, applicable to certain multipliers, was obtained in [1, Theorem (3.8) and Corollary (3.13)].

Proof of (5.1). This is a mild adaptation of the proof of Theorem (4.1) and so we indicate the necessary modifications without giving complete details.

As in the proof of Theorem (4.1), it suffices to consider the case when $G=\mathbf{T}^N$. Assume then that $G=\mathbf{T}^N$ and that $\geqq$ is some linear ordering on $\hat{G}=\mathbf{Z}^N$. Fix

$x=\sum x_\gamma$ in the linear span of the eigenspaces X_γ ($\gamma \in \mathbf{Z}^N$) and let $a \in \mathbf{Z}^N$ satisfy (4.5) and (4.6). We may then represent $\mathbf{T}$ on X as in (4.7). The proof is completed by applying the following result (in the case $c=1$).

(5.2) **Lemma.** *Let $e^{it} \to R_{e^{it}}$ be a strongly continuous representation of* $\mathbf{T}$ *on X, where X is as in* (5.1). *Let E^+ and E_0 be the associated projections on X corresponding to the natural order on* $\mathbf{Z}$, *and let $\xi, \eta, \zeta \in \mathbf{C}$. Then*

$$\|\xi I+\eta E_0+\zeta E^+\| \leqq c^2 \|\xi I+\eta P_0+\zeta P^+\|_p,$$

where $c=\sup \{\|R_{e^{it}}\|: e^{it} \in \mathbf{T}\}$.

Proof of Lemma. This is along the same lines as the proof of Theorem (3.3), but uses the original transference result (2.3) rather than the Banach space version (2.8). In this case, we transfer the convolution kernel

$$f_n = \xi k_n+\eta+\zeta h_n,$$

where k_n is the nth Fejér kernel and h_n is as in the proof of Theorem (3.3), to X by the representation R of $\mathbf{T}$ and let $n \to \infty$. We omit the details.

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References

1. Berkson, E. and Gillespie, T. A., The generalized M. Riesz Theorem and transference, *Pacific J. Math.* **120** (1985), 279—288.
2. Berkson, E. and Gillespie, T. A., Fourier series criteria for operator decomposability, *Integral Equations and Operator Theory* **9** (1986), 767—789.
3. Berkson, E., Gillespie, T. A. and Muhly, P. S., Théorie spectrale dans les espaces UMD, *C.R. Acad. Sc. Paris* **302**, Série I (1986), 155—158.
4. Berkson, E., Gillespie, T. A. and Muhly, P. S., Abstract spectral decompositions guaranteed by the Hilbert transform, *Proc. London Math. Soc.* **53** (1986), 489—517.
5. Berkson, E., Gillespie, T. A. and Muhly, P. S., A generalization of Macaev's theorem to non-commutative L^p-spaces, *Integral Equations and Operator Theory* **10** (1987), 164—186.
6. Bochner, S., Additive set functions on groups, *Annals of Math.* **40** (1939), 769—799.
7. Bourgain, J., Some remarks on Banach spaces in which martingale difference sequences are unconditional, *Ark. Mat.* **21** (1983), 163—168.
8. Burkholder, D. L., A geometric condition that implies the existence of certain singular integrals of Banach-space-valued functions, in: Proc. of Conference on Harmonic Analysis in Honor of Antoni Zygmund (Chicago, 1981), ed. by W. Beckner *et al.*, Wadsworth Publishers, Belmont, Calif., 1983.

9. Coifman, R. R. and Weiss, G., Operators associated with representations of amenable groups, singular integrals induced by ergodic flows, the rotation method and multipliers, *Studia Math.* **47** (1973), 285—303.
10. Coifman, R. R. and Weiss, G., *Transference methods in analysis*, Regional Conf. Series in Math., No. 31, Amer. Math. Soc., Providence, 1977.
11. Edwards, R. E. and Gaudry, G. I., *Littlewood—Paley and multiplier theory*, Springer-Verlag, Berlin, 1977.
12. Gohberg, I. C. and Krein, M. G., *Theory and applications of Volterra operators in Hilbert space*, Transl. Math. Monographs, vol. 24, Amer. Math. Soc., Providence, 1970.
13. Gutiérrez, J. A., *On the boundedness of the Banach space-valued Hilbert transform*, thesis, Univ. of Texas (Austin), 1982.
14. Hewitt, E. and Ross, K. A., *Abstract harmonic analysis* II, Springer-Verlag, Berlin, 1970.
15. Macaev, V. I., Volterra operators obtained from self-adjoint operators by perturbation, *Dokl. Akad. Nauk SSSR* **139** (1961), 810—813 [*Soviet Math. Dokl.*, **2** (1961), 1013—1016].

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Earl Berkson
Department of Mathematics
University of Illinois
1409 West Green Street
Urbana, Illinois 61801
U.S.A.

T. A. Gillespie
Department of Mathematics
James Clerk Maxwell Building,
The King's Buildings
Edinburgh EH9 3JZ
Scotland

Paul S. Muhly
Department of Mathematics
The University of Iowa
Iowa City, Iowa 52242
U.S.A.