On uniqueness of hyperfunction solutions of holonomic systems*

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Dedicated to Professor Takeyuki Hida on his sixtieth birthday

Introduction

Let M be a real analytic manifold of dimension n and let X be its complexification. Consider a system of linear differential equations $\mathfrak{M}$: $P_i(x, \partial/\partial x)u(x)=0$, $(i=1, \ldots, m)$, where $P_i(x, \partial/\partial x)$'s are differential operators with holomorphic coefficients on X. Suppose that $\mathfrak{M}$ is an elliptic system on M, — i.e., the characteristic variety of $\mathfrak{M}$ is contained in the zero-section $M \times \{0\}$ in the cotangent bundle T^*M — and let N be a real analytic subset of M of real codimension $\geqq 1$. If two hyperfunction solutions coincide with each other on $M-N$, then they coincide on the whole domain M. This is a direct consequence of the fact that there is no solution whose support is contained in N since any hyperfunction solution of $\mathfrak{M}$ is real analytic. This means that a solution is determined by its data on $M-N$. However, in general, there may exist a solution supported in N when $\mathfrak{M}$ is not elliptic, especially when N is contained in the projection of $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}-M$ to M, where $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$ is the real locus of the characteristic variety of $\mathfrak{M}$.

In a general situation, a system of linear differential equations can be interpreted as a left coherent $\mathscr{D}_X$-module on X where $\mathscr{D}_X$ is the sheaf of linear differential operators on X. In particular, there is an important class of left coherent $\mathscr{D}_X$-modules called holonomic systems. The purpose of this paper is to give a natural and practical extension of the above fact when $\mathfrak{M}$ is a *holonomic system* on X, i.e., a left coherent $\mathscr{D}_X$-module whose characteristic variety is of dimension n. Namely we have the following theorem.

Let $\mathfrak{M}$ be a holonomic system on X. We denote by $\mathrm{ch}(\mathfrak{M})$ the characteristic variety of $\mathfrak{M}$. Let Z be an analytic subset in $\mathrm{ch}(\mathfrak{M})$. We shall con-

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sider the microfunction solution of $\mathfrak{M}$. As we have seen in [Sm—Kw—Ka], p. 273, we may identify T_M^*X with T^*M. In this paper, we regard T^*M as a subset T_M^*X in T^*X and deal with the sheaf $\mathscr{C}_M$ as a sheaf on T^*M. The real loci of $\mathrm{ch}(\mathfrak{M})$ and Z are denoted by $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}:=\mathrm{ch}(\mathfrak{M})\cap T^*M$ and $Z_{\mathbf{R}}:=Z\cap T^*M$, respectively. The spectral map from the sheaf of hyperfunctions $\mathscr{B}_M$ to the direct image $\pi_*(\mathscr{C}_M)$ by $\pi;\ T^*M\to M$ of the sheaf of microfunctions $\mathscr{C}_M$ is denoted by sp. Our main theorem (Theorem 1.1) is the following.

Theorem. *We suppose that Z is a conic analytic subset of complex codimension $\geqq 1$ in $\mathrm{ch}(\mathfrak{M})$. For a hyperfunction solution $u(x)$ of the holonomic system $\mathfrak{M}$, if $sp(u(x))|_{\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}-Z_{\mathbf{R}}}=0$, then $u(x)=0$ as a hyperfunction on M.*

When $\mathfrak{M}$ is a non-trivial elliptic system, $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$ is the zero section $M\times\{0\}$ and $Z_{\mathbf{R}}$ is a real analytic subset of real codimension $\geqq 1$. Thus the assumption of Theorem means that $u(x)$ is zero on $M-Z_{\mathbf{R}}$. The above theorem may be seemed as an extension of the fact that any solution of an elliptic system is uniquely extended to M from the data on $M-N$. In particular, we may put $Z=\mathrm{ch}(\mathfrak{M})_{\mathrm{sing}}:=$ the singular locus of the characteristic variety. Theorem means that a solution of $\mathfrak{M}$ is completely determined by the value on the regular locus $\mathrm{ch}(\mathfrak{M})_{\mathrm{reg}}=\mathrm{ch}(\mathfrak{M})-\mathrm{ch}(\mathfrak{M})_{\mathrm{sing}}$ and we need not pay attention to the value on the singular locus of $\mathrm{ch}(\mathfrak{M})$.

The proof of this theorem is not difficult. We can prove it by some arguments on characteristic varieties of $\mathfrak{M}$ and Holmgren's uniqueness theorem. Readers may feel that this theorem is nothing but one easy application of Holmgren's uniqueness theorem. However this theorem is important from the practical point of view. For example, we may take Z to be the set of singular points in $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$. When $\mathfrak{M}$ is a holonomic system with regular singularity, the structure of $\mathfrak{M}$ is very simple at least microlocally near any point of $\mathrm{ch}(\mathfrak{M})-Z$ (see [Ka—Kw]). In particular, as stated in [Ka—Kw], the microfunction solution is expressed explicitly by making use of the theory of principal symbols. It is often possible to calculate the microfunction solution near a point of $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}-Z_{\mathbf{R}}$. Thus the above theorem guarantees that we need not consider the structure of the solution on $Z_{\mathbf{R}}$ and hence it is helpful for the proof of uniqueness of hyperfunction solutions to a holonomic system. As far as we are dealing with the uniqueness problem of hyperfunction solutions of a holonomic system, it is desirable to reformulate the Holmgren's uniqueness theorem as presented in the above theorem from a practical position. Indeed, a weak form of the above theorem was used in [Mr 1] and [Mr 2] in order to determine relatively invariant hyperfunctions.

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1. Formulation of the problem and the key lemma

Let M be a real analytic manifold of dimension n and let X be its complexification. We denote by $\mathscr{D}_X$ be the sheaf of linear differential operators on X with holomorphic coefficients. Following Kashiwara [Ka 2], we interpret a system of linear differential equation $\mathfrak{M}$ as a left coherent $\mathscr{D}_X$-module. Then the solution space of $\mathfrak{M}$ may be expressed using a Hom-functor. Namely, let $\mathscr{F}_M$ be a sheaf of $\mathscr{D}_X$-module on M like, for example, the sheaf of C^∞-functions or hyperfunctions on M. We denote by $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M}, \mathscr{F}_M)$ the sheaf of $\mathscr{D}_M$-homomorphisms from $\mathfrak{M}$ to $\mathscr{F}_M$ and call a section of it a *$\mathscr{F}_M$-solution of* $\mathfrak{M}$. For details, see [Ka 2], Chapter 2, § 1.

Throughout this paper, we always consider hyperfunction solutions, so a solution of $\mathfrak{M}$ always means a hyperfunction solution or a microfunction solution as its spectral map image. We denote by $\mathscr{B}_M$ the sheaf of *hyperfunctions* on M and by $\mathscr{C}_M$ the sheaf of *microfunctions* on the cotangent bundle T^*M. There is a natural sheaf isomorphism $\mathscr{sp}$ called the *spectral map* which gives an isomorphism from $\mathscr{B}_M$ to $\pi_*(\mathscr{C}_M)$: $\mathscr{sp}\colon \mathscr{B}_M \cong \pi_*(\mathscr{C}_M)$.

Remark. In [Ka—Kw—Ki] the sheaf of microfunctions $\mathscr{C}_M$ is defined as a sheaf on the pure imaginary cospherical bundle $\sqrt{-1}S^*M$, which is naturally extended to the sheaf on T^*M-M constant to the direction by the action of $\mathbf{R}_+^\times$. The sheaf $\mathscr{C}_M$ in [Ka—Kw—Ki] corresponds to the restriction sheaf of $\mathscr{C}_M$ defined above to T^*M-M, i.e., $\mathscr{C}_M|_{T^*M-M}$. The sheaf $\mathscr{C}_M$ defined above is denoted by $\hat{\mathscr{C}}_M$ in [Ka—Kw—Ki]. See [Ka—Kw—Ki] Chapter III § 8, Definition (3.8.1).

Let $\bar{u}$ be a section of $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M}, \mathscr{B}_M)$: a *hyperfunction solution* of $\mathfrak{M}$. Then, through the spectral map $\mathscr{sp}$, $\mathscr{sp}\circ\bar{u}$ is regarded as a section of $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M}, \pi_*(\mathscr{C}_M))$ naturally, and defines a section of microfunction solution of $\mathfrak{M}$. We call the support of $\mathscr{sp}\circ\bar{u}$ in T^*M the *support of* $\bar{u}$ *as a microfunction solution.* In particular, we put $\mathrm{S.S.}(\bar{u}) := \mathrm{supp}(\mathscr{sp}\circ\bar{u}) - M$, and call it the *singular spectrum* of $\bar{u}$. The support $\mathrm{supp}(\bar{u})$ (resp. $\mathrm{supp}(\mathscr{sp}\circ\bar{u})$) is by definition $\bigcup_f \mathrm{supp}(\bar{u}(f))$ (resp. $\bigcup_f \mathrm{supp}(\mathscr{sp}\circ\bar{u}(f))$) where f runs through the set of sections of $\mathfrak{M}$.

Let $\mathrm{ch}(\mathfrak{M})$ be the characteristic variety of $\mathfrak{M}$. (*Remark.* $\mathrm{ch}(\mathfrak{M})$ is denoted by $\mathrm{S.S.}(\mathfrak{M})$ in [Ka 2].) The characteristic variety $\mathrm{ch}(\mathfrak{M})$ is an analytic subset in T^*X. In particular, $\mathfrak{M}$ is called a *holonomic system* if the complex dimension of $\mathrm{ch}(\mathfrak{M})$ coincides with the dimension of X. The real locus of $\mathrm{ch}(\mathfrak{M})$ is denoted by $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$, i.e., $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}} := \mathrm{ch}(\mathfrak{M}) \cap T^*M$. Then the support of any solution of $\mathfrak{M}$ is contained in $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$. The main theorem of this paper is the following.

Theorem 1.1. *Let $\mathfrak{M}$ be a holonomic system on X. Let Z be an analytic subset of complex codimension $\geqq 1$ in $\mathrm{ch}(\mathfrak{M})$. The real locus of Z is denoted by $Z_{\mathbf{R}} := Z \cap T^*M$. For a local section $\bar{u}$ of $\mathscr{H}om_{\mathscr{D}_X}(\mathfrak{M}, \mathscr{B}_M)$, if $\mathscr{sp}\circ\bar{u}|_{\mathrm{ch}(\mathfrak{M})_{\mathbf{R}} - Z_{\mathbf{R}}} = 0$, then $\bar{u} = 0$.*

Corollary 1.2. *Let* $\mathfrak{M}$ *be a holonomic system on X. For a local section* $\bar{u}$ *of* $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_M)$, *if* $sp\circ\bar{u}|_{ch(\mathfrak{M})_{reg\,R}}=0$, *then* $\bar{u}=0$.

Corollary 1.2 follows from Theorem 1.1 since $ch(\mathfrak{M})_{sing}=ch(\mathfrak{M})-ch(\mathfrak{M})_{reg}$ is an analytic subset of codimension $\geqq 1$ in $ch(\mathfrak{M})$.

Lastly we give a key lemma for the proof of Theorem 1.1.

Lemma 1.3. *Let M be a real analytic manifold and let* $v(x)$ *be a hyperfunction on M defined near a point* $x_0\in M$. *Let* $p(x)$ *be a real valued real analytic function defined near* x_0 *such that* $p(x_0)=0$ *and* $dp(c_0)\neq 0$. *If* $\mathrm{supp}\,(v(x))$ *is contained in the set* $\{x\in M; p(x)\geqq 0\}$ *and, if* $(x_0, dp(x_0))\notin \mathrm{S.S.}(v(x))$ *or* $(x_0, -dp(x_0))\notin \mathrm{S.S.}(v(x))$, *then* $v(x)=0$ *near* x_0.

Lemma 1.3 is the so-called the "Holmgren's uniqueness theorem" from the point of view of microfunction theory. The proof of this lemma is found in [Ka—Kw—Ki], Proposition 3.5.2 in Chapter III.

Corollary 1.4. *Under the same situation as Lemma* 1.3, *let N be a non-singular real analytic subvariety in M defined near* $x_0\in M$. *We suppose that;* 1) *the support* $\mathrm{supp}\,(v(x))$ *is contained in N, and;* 2) $((T_N^*M-M)\cap\pi^{-1}(x_0))-\mathrm{S.S.}\,(v(x))$ *is not an empty set where* π *is the projection map* $T^*M\to M$. *Then* $v(x)=0$ *near* x_0.

Proof. Since N is non-singular, there exists a local coordinate $(x_1, \dots, x_n)$ near x_0 whose origin is x_0 and such that N is defined as $\{x\in M; x_1=\dots=x_l=0\}$. From the assumption 2), there exists a point $(x_0, \xi_0)\in T_N^*M$ with $\xi_0\neq 0$ and $(x_0, \xi_0)\notin \mathrm{S.S.}\,(v(x))$. We may put $\xi_0=\sum_{i=1}^l c_i\cdot dx_i(x_0)$ with some $c_i\in\mathbf{R}$ $(i=1, \dots, l)$. Then by putting $p(x)=\sum_{i=1}^l c_i\cdot x_i$, we have $p(x)|_N=0$, and hence $\mathrm{supp}\,(v(x))\subset\{x\in M; p(x)\geqq 0\}$. Furthermore since $(x_0, \xi_0)=(x_0, dp(x_0))\notin \mathrm{S.S.}\,(v(x))$, we have $v(x)=0$ near x_0 by Lemma 1.3. (Q.e.d.)

The next section is devoted to the proof of Theorem 1.1.

2. Proof of the main theorem

2.1. Preliminaries of the proof. Let X be a complex manifold. A subset A of X is called *analytic* at a point $x\in X$ if there exist a neighbourhood U of x in X and finitely many holomorphic functions $f_1(x), \dots, f_k(x)$ on U such that $A\cap U=\{x\in U; f_1(x)=\dots=f_k(x)=0\}$. The set A is called a *locally analytic subset* of X if A is analytic in X at every point of A. We say that a locally analytic subset A is *non-singular* at $x_0\in A$ if there exist a neighbourhood U of x_0 in X and holomorphic functions $f_1(x), \dots, f_k(x)$ on U such that $A\cap U=\{x\in U; f_1(x)=\dots=f_k(x)=0\}$ and $df_1\wedge\dots\wedge df_k(x_0)\neq 0$. We denote by A_{reg} the set of non-singular points of A. We denote by A_{sing} the set $A-A_{\text{reg}}$ and call it *the singular locus* of A. A point in A_{sing} is called a *singular point of* A. For a locally analytic subset A, its singular locus is a locally analytic subset in X and A_{reg} is an open dense subset in A.

We want to consider a stratification of a complex manifold X. We say that $\{A_\alpha\}_{\alpha\in I}$ is a *stratification* of X if:

(2.1) 1) $X=\coprod_{\alpha\in I}A_\alpha$, (a disjoint union).
2) Each A_α is a locally analytic subset *in* X.
3) $\coprod_{\alpha\in I}A_\alpha$ is a locally finite covering of X.

We call each A_α a *stratum* of the stratification $\{A_\alpha\}_{\alpha\in I}$. In particular, when the index set I is a finite set, we say that $\{A_\alpha\}_{\alpha\in I}$ is a finite stratification. Let $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\beta\}_{\beta\in J}$ be two stratifications of X. We say that $\{A_\alpha\}_{\alpha\in I}$ is a *refinement* of $\{B_\beta\}_{\beta\in J}$ if any stratum in $\{A_\alpha\}_{\alpha\in I}$ is contained in one of the strata in $\{B_\beta\}_{\beta\in J}$.

Let A be a locally analytic subset in a complex manifold X. Then we get a finite stratification $\{A_\alpha\}_{\alpha\in I}$ of X satisfying the conditions (2.1) in a canonical way. Namely we put:

$$A_{(0)}:=A \quad\text{and}\quad A_0:=X-A,$$

$$A_{(1)}:=A_{(0)\text{sing}} \quad\text{and}\quad A_1:=A_{(0)\text{reg}}=A_{(0)}-A_{(1)},$$

and by induction on i we put:

$$A_{(i)}:=A_{(i-1)\text{sing}} \quad\text{and}\quad A_i:=A_{(i-1)\text{reg}}=A_{(i-1)}-A_{(i)},$$

for $i\geqq 1$. Since $\dim A_{(i)}<\dim A_{(i-1)}$, $A_{(i)}$ is an empty set if i is sufficiently large. We put l the number such that $A_{(l)}\neq\emptyset$ and $A_{(l+1)}=\emptyset$. Let $\coprod_{j\in J_i}A_{i,j}$ be the connected component decomposition of A_i, which is a disjoint union of open subsets in A_i. Then we have a partition of A:

$$A=\coprod_{1\leqq i\leqq l}\left(\coprod_{j\in J_i}A_{i,j}\right)=:\coprod_{\alpha\in I}A_\alpha, \tag{2.2}$$

with $I:=\{\alpha=(i,j);\ 1\leqq i\leqq l \text{ and } j\in J_i\}$. The following lemma is easily checked.

Lemma 2.1. *The above partition* (2.2) *of the set A is a stratification of A. Each stratum is a connected non-singular locally analytic subset in X. In particular, if A is a closed analytic subvariety of X, then* (2.2) *is a finite stratification.*

Definition. (Canonical stratification.) We call the stratification $\{A_\alpha\}_{\alpha\in I}$ defined in (2.2) the *canonical stratification of X by A*.

Next we shall consider the conormal bundle of a stratum of a stratification $\{A_\alpha\}_{\alpha\in I}$ whose strata are all connected non-singular locally analytic subset. Let $\{A_\alpha\}_{\alpha\in I}$ be a stratification of a complex manifold X consisting of non-singular connected strata. The conormal bundle of A_α in X is denoted by $T^*_{A_\alpha}X$.

Here $(T^*X)_x$ is the cotangent vector space of X at x and $(TA_\alpha)_x$ is the tangent space of A_α at the point $x \in X$. Then we see that:

Lemma 2.2. *Let $\{A_\alpha\}_{\alpha\in I}$ and $\{B_\beta\}_{\beta\in J}$ be two stratifications of X. We suppose that $\{B_\beta\}_{\beta\in J}$ is a refinement of $\{A_\alpha\}_{\alpha\in I}$ and that each strata A_α and B_β are connected and non-singular. Then we have:*

$$\bigcup_{\alpha\in I} T^*_{A_\alpha} X \subset \bigcup_{\beta\in J} T^*_{B_\beta} X.$$

This is easily checked.

We shall construct the characteristic variety of a holonomic system $\mathfrak{M}$ by a stratification of X. Let $\bigcup_{i=1}^{l} \Lambda_i = \mathrm{ch}(\mathfrak{M})$ be the decomposition into irreducible components of $\mathrm{ch}(\mathfrak{M})$. As stated in [Ka 2], each Λ_i is an irreducible conic Lagrangian analytic subset in T^*X. We put $X_i := \pi(\Lambda_i)$ where π is the projection map from T^*X to X. Then X_i is an irreducible analytic subset in X. We put:

$$Y_i := (X_i)_{\mathrm{reg}} \quad \text{and} \quad \Lambda'_i := \overline{T^*_{Y_i} X} \quad \text{for} \quad i = 1, \dots, l.$$

Then Y_i is a non-singular locally analytic subset in X and Λ_i is a closed conic Lagrangian analytic subset in T^*X which is contained in Λ_i. It is easily checked that there is a finite stratification consisting of non-singular strata of X which is a refinement of all the canonical stratification of X by X_i $(i=1, \dots, l)$. Let $\{C_\beta\}_{\beta\in L}$ be one of such finite stratifications of X. By Lemma 2.2, we have

$$\mathrm{ch}(\mathfrak{M}) = \bigcup_{i=1}^{l} \Lambda_i \subset \bigcup_{\beta\in L} T^*_{C_\beta} X. \tag{2.3}$$

For the proof of Theorem 1.1, we need a slightly finer stratification than $\{C_\beta\}_{\beta\in L}$.

Definition. (Full-fiberness) Let A be a connected non-singular locally analytic subset in X and let Z be an analytic subset in $T^*_A X$. We say that Z is *full-fiber* with respect to A at $x_0 \in A$ if $\pi^{-1}(x_0) \cap T^*_A X = \pi^{-1}(x_0) \cap Z$.

Lemma 2.3. *Let Z be an analytic subset of codimension $\geqq 1$ in $\mathrm{ch}(\mathfrak{M})$ and let $\{C_\beta\}_{\beta\in L}$ be a finite stratification of X. Suppose that $\{C_\beta\}_{\beta\in L}$ is a refinement of the canonical stratification of X by X_i for all $i=1, \dots, l$. Let C'_β be the subset of C_β consisting of the points at which $Z \cap T^*_{C_\beta} X$ is full-fiber with respect to C_β. Then C'_β is a strictly less dimensional analytic subset in C_β.*

Proof. We shall show that C'_β is a closed locally analytic subset in C_β. Let $x_0 \in C_\beta - C'_\beta$. From the assumption that Z is of codimension $\geqq 1$ in $\mathrm{ch}(\mathfrak{M})$, there exists a point $(x_0, \xi_0) \in T^*_{C_\beta} X$ such that $(x_0, \xi_0) \notin Z$. Since $Z \cap T^*_{C_\beta} X$ is a closed set in $T^*_{C_\beta} X$, there exists an open neighbourhood U of (x_0, ξ_0) in T^*X such that $U \cap Z = \varphi$. The projection image $\pi(U)$ is a neighbourhood of x_0 in C_β and $\pi(U)$ is contained in $C_\beta - C'_\beta$. Hence $C_\beta - C'_\beta$ is an open set in C_β. Thus C'_β is a closed set in C_β. Next we show that C'_β is a locally analytic subset in C_β. Let $x_0 \in C'_\beta$ and let $(x_1, \dots, x_n)$ be a local coordinate of X defined in a neighbourhood U of x_0 such that C_β is written as $\{x_1 = \dots = x_p = 0\}$ in U. Then

$$T^*_{C_\beta}X = \{(x, \xi) \in T^*X;\ x_1 = \dots = x_p = 0,\ \xi_{p+1} = \dots = \xi_n = 0\}$$

where $\xi = (\xi_1, \dots, \xi_n)$ is the dual coordinate of $(x_1, \dots, x_n)$. We put $\xi' := (\xi_1, \dots, \xi_p)$ and $A_{\xi'} := \pi(Z \cap \{(x, \xi) \in T^*_{C_\beta}X;\ \xi = (\xi', 0)\})$. Then $A_{\xi'}$ is an analytic subset in U. Since $C'_\beta \cap U = \bigcap_{\xi' \in C^p} A_{\xi'}$ and since any intersection of analytic subsets is an analytic subset (by the Nullstellensatz), C'_β is an analytic subset in U, which means that C'_β is a locally analytic subset in C_β. Thus C'_β is an analytic subset in C_β. Since Z is of codimension $\geqq 1$ in $T^*_{C_\beta}X$, there exists at least one point in $T^*_{C_\beta}X - Z$, hence $C_\beta - C'_\beta$ is not empty. Then C'_β is strictly less dimensional than C_β. (Q.e.d.)

By Lemma 2.3, C'_β is a strictly less dimensional analytic subset in C_β. We take the canonical stratification of C_β by C'_β and denote it by $\{D_{\beta,\gamma}\}_{\gamma \in \Gamma_\beta}$. Then $\{D_{\beta,\gamma}\}_{\gamma \in \Gamma_\beta}$ is a finite stratification of C_β and each stratum $D_{\beta,\gamma}$ is a connected non-singular locally analytic subset in X. Then

(2.4) $$\{D_\delta\}_{\delta \in \Delta} \quad \text{with} \quad \Delta := \{\delta = (\beta, \gamma);\ \beta \in L,\ \gamma \in \Gamma_\beta\},$$

is a finite stratification of X and it is a refinement of $\{C_\beta\}_{\beta \in L}$. After all we have the following proposition.

Proposition 2.4. *Let $\mathfrak{M}$ be a holonomic system on X and let Z be a conic analytic subset of codimension $\geqq 1$ in* ch $(\mathfrak{M})$. *Then there exists a finite stratification $\{D_\delta\}_{\delta \in \Delta}$ of X satisfying the following conditions:*

(2.5) 1) *Each D_δ is a connected non-singular locally analytic subset in X.*
2) ch $(\mathfrak{M}) \subset \bigcup_{\delta \in \Delta} T^*_{D_\delta}X$.
3) *For any stratum D_δ, Z is not full-fiber with respect to D_δ at any point $x \in D_\delta$.*

Proof. We may take the stratification $\{D_\delta\}_{\delta \in \Delta}$ given by (2.4). We see easily that the conditions (2.5) 1) and 2) are satisfied from the definition of $\{D_\delta\}_{\delta \in \Delta}$ and Lemma 2.2. We shall prove 3). The stratification $\{D_\delta\}_{\delta \in \Delta}$ is a refinement of the stratification $\{C_\beta\}_{\beta \in L}$. The subset C'_β in C_β is the set of points where Z is full-fiber with respect to C_γ. If $D_{\beta,\gamma}$ is a stratum in $\{D_{\beta,\gamma}\}_{\gamma \in \Gamma_\beta}$ is contained in $C_\beta - C'_\beta$, then Z is not full-fiber with respect to $D_{\beta,\gamma}$ at any point of $D_{\beta,\gamma}$. This is because $D_{\beta,\gamma}$ is an open set in C_β. Otherwise $D_{\beta,\gamma}$ is contained in C'_β and it is strictly less dimensional than C_β. Then we have:

$$\pi^{-1}(x_0) \cap T_{D_{\gamma,\beta}}X \supsetneqq \pi^{-1}(x_0) \cap T_{C_\beta}X = \pi^{-1}(x_0) \cap Z,$$

for any $x_0 \in D_{\beta,\gamma}$. This is because Z is full-fiber with respect to C_β at x_0. Thus Z is not full-fiber with respect to $D_{\beta,\gamma}$ at $x_0 \in D_{\beta,\gamma}$. Hence $\{D_\delta\}_{\delta \in \Delta}$ satisfies the condition 3) of (2.5). (Q.e.d.)

2.2. Proof of Theorem 1.1. Let $\bar{u}$ be a local section of $\mathcal{H}om_{\mathcal{D}_X}(\mathfrak{M}, \mathcal{B}_M)$ satisfying $\mathcal{sp}\circ\bar{u}|_{\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}-Z_{\mathbf{R}}}=0$. We shall show that the hyperfunction $\bar{u}(f)=0$ for any section f of $\mathfrak{M}$. From Proposition 2.4, we have a stratification $\{D_\delta\}_{\delta\in\Delta}$ of X satisfying the condition (2.5). Then we have:

$$\text{(2.6)} \qquad \operatorname{supp}(\mathcal{sp}\circ\bar{u}(f)) \subset \left(\bigcup_{\delta\in\Delta} T^*_{D_\delta}X\cap T^*M\right) \subset \bigcup_{\delta\in\Delta} T^*_{D_{\delta\mathbf{R}}}M$$

where $D_{\delta\mathbf{R}}:=D_\delta\cap M$. Since $D_{\delta\mathbf{R}}$ is a real locus of D_δ, we have $T^*_{D_\delta}X\cap T^*M\subset T^*_{D_{\delta\mathbf{R}}}M$ because the real dimension of $D_{\delta\mathbf{R}}$ is not greater than the complex dimension of D_δ. Here $T^*_{D_{\delta\mathbf{R}}}M$ means the real conormal bundle of $D_{\delta\mathbf{R}}$ in M.

The support of $\mathcal{sp}\circ\bar{u}$ is contained in $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}$ and $\mathcal{sp}\circ\bar{u}=0$ on $\mathrm{ch}(\mathfrak{M})_{\mathbf{R}}-Z_{\mathbf{R}}$ from the assumption. That is to say, $\mathcal{sp}\circ\bar{u}=0$ on $T^*M-Z_{\mathbf{R}}$. In particular, $\mathcal{sp}\circ\bar{u}=0$ on $\bigcup_{\delta\in\Delta}T^*_{D_{\delta\mathbf{R}}}M-Z_{\mathbf{R}}$. We shall show in the following that $\bar{u}=0$ from the assumption that $\mathcal{sp}\circ\bar{u}=0$ on $\bigcup_{\delta\in\Delta}T^*_{D_{\delta\mathbf{R}}}M-Z_{\mathbf{R}}$.

Since $\{D_\delta\}_{\delta\in\Delta}$ is a finite stratification, there exists a partition of Δ:

$$\text{(2.7)} \qquad \Delta_{(0)}\cup\Delta_{(1)}\cup\ldots\cup\Delta_{(m)} = \Delta,$$

satisfying the following condition:

(2.8) We put $\Delta^k:=\bigcup_{i\geqq k}\Delta_{(i)}$ and $D^k:=\coprod_{\delta\in\Delta^k}D_\delta$ with $k=0, 1, 2, \ldots$. For any $\delta\in\Delta_{(k)}$ and $x_0\in D_\delta$, there exists a neighbourhood U of x_0 satisfying $U\cap D^k\subset D_\delta$.

We shall show that $\bar{u}=0$ near any point of $D_{\delta\mathbf{R}}$ for any $\delta\in\Delta_{(i)}$ by induction on i. First suppose that $\delta\in\Delta_{(0)}$. Then, from the condition (2.8), D_δ is an open set in X. Hence the real locus $D_{\delta\mathbf{R}}$ is an open set in M. Since $\operatorname{supp}(\mathcal{sp}\circ\bar{u}(f))\subset T^*_{D_{\delta\mathbf{R}}}M=D_{\delta D}\times\{0\}$, $\bar{u}(f)$ is a real analytic function on $D_{\delta\mathbf{R}}$ for any section f of $\mathfrak{M}$ by Proposition 1.1 2), and $\bar{u}(f)=0$ on $D_{\delta\mathbf{R}}-Z_{\mathbf{R}}$ from the assumption. The subset $Z_{\mathbf{R}}$ is of real codimension $\geqq 1$ in $D_{\delta\mathbf{R}}$ because Z is of complex codimension $\geqq 1$ in D_δ. Thus $\bar{u}(f)=0$ on the whole $D_{\delta\mathbf{R}}$ for an arbitrary section f of $\mathfrak{M}$ and hence $\bar{u}=0$ on $D_{\delta\mathbf{R}}$ for any $\delta\in\Delta_{(0)}$.

Next we suppose that we have proved the induction hypothesis for k:

(2.9) For any $\delta\in\Delta_{(0)}\cup\Delta_{(1)}\cup\ldots\cup\Delta_{(k-1)}=\Delta-\Delta^k$, $\bar{u}$ is zero near any point in $D_{\delta\mathbf{R}}$.

Then we have:

$$\text{(2.10)} \qquad \begin{aligned} &1)\quad \operatorname{supp}(\bar{u}(f))\subset D^k\cap M=\bigcup_{\delta\in\Delta^k}D_{\delta\mathbf{R}},\\ &2)\quad \operatorname{supp}(\mathcal{sp}\circ\bar{u}(f))\subset\bigcup_{\delta\in\Delta^k}T^*_{D_{\delta\mathbf{R}}}M,\end{aligned}$$

from (2.9) and (2.6), (2.8). We shall show that $\bar{u}=0$ near any point in $D_{\delta\mathbf{R}}$ for $\delta\in\Delta_{(k)}$. Take $\delta\in\Delta_{(k)}$ and $x_0\in D_{\delta\mathbf{R}}$. From the condition (2.8), we can take U a neighbourhood of x_0 in X satisfying $U\cap D^k\subset D_\delta$. Let f be a local section of $\mathfrak{M}$ and consider the hyperfunction $\bar{u}(f)$. Then since $(U\cap(D^k\cap M))\subset D_{\delta\mathbf{R}}$, we have:

$$\text{(2.11)}\qquad \begin{aligned} &1)\quad \operatorname{supp}(\bar{u}(f))\cap U\subset D_{\delta\mathbf{R}},\\ &2)\quad \operatorname{supp}(\mathit{sp}\circ\bar{u}(f)|_U)\subset T^*_{D_{\delta\mathbf{R}}}M,\end{aligned}$$

by (2.10).

Lemma 2.5. *For any fixed* $x_0\in D_{\delta\mathbf{R}}$, *there exists a point* $(x_0,\xi_0)\in T^*_{D_{\delta\mathbf{R}}}M$ *such that* $\xi_0\neq 0$ *and* $(x_0,\xi_0)\notin Z_{\mathbf{R}}$.

Proof. Let A be a real conic subset in T^*M, i.e., if $(x,\xi)\in A$, then $(x,c\cdot\xi)\in A$ for all $c\in\mathbf{R}^{\times}_{+}$. We define the *fiber complexification* $A\otimes\mathbf{C}$ of A by:

$$\text{(2.12)}\qquad A\otimes\mathbf{C}:=\{(x,y)\in T^*X;\ \text{there exists a point } (x,\xi)\in A \text{ such that } y=c\cdot\xi \text{ with some } c\in\mathbf{C}^{\times}\}.$$

Since Z is a conic subset in X, $Z_{\mathbf{R}}:=Z\cap T^*M$ is a real conic subset in T^*M. For a point $x_0\in\pi(Z_{\mathbf{R}})$, $(\pi^{-1}(x_0)\cap Z_{\mathbf{R}})\otimes\mathbf{C}$ is contained in $\pi^{-1}(x_0)\cap Z$. On the other hand, we have $\pi^{-1}(x_0)\cap T^*_{D_\delta}X=\pi^{-1}(x_0)\cap(T^*_{D_\delta}X\cap T^*M)\otimes\mathbf{C}\subset(\pi^{-1}(x_0)\cap T^*_{D_{\delta\mathbf{R}}}M)\otimes\mathbf{C}$ for any $x_0\in D_{\delta\mathbf{R}}$.

For a fixed point $x_0\in D_{\delta\mathbf{R}}$, if there is no point $(x_0,\xi_0)\in\pi^{-1}(x_0)\cap T^*_{D_{\delta\mathbf{R}}}M$ such that $\xi_0\neq0$ and $(x_0,\xi_0)\notin Z_{\mathbf{R}}$, then we have:

$$\text{(2.13)}\qquad (\pi^{-1}(x_0)\cap T^*_{D_{\delta\mathbf{R}}}M)=\pi^{-1}(x_0)\cap Z_{\mathbf{R}},$$

since $Z_{\mathbf{R}}$ is a real conic subset. Thus $(x_0,0)$ is automatically contained in $\pi^{-1}(x_0)\cap Z_{\mathbf{R}}$. Taking the fiber complexifications of both sides of (2.13), we have:

$$\text{(2.14)}\qquad \begin{aligned}\pi^{-1}(x_0)\cap T^*_{D_\delta}X&\subset(\pi^{-1}(x_0)\cap T^*_{D_{\delta\mathbf{R}}}M)\otimes\mathbf{C}\\ &=(\pi^{-1}(x_0)\cap Z_{\mathbf{R}})\otimes\mathbf{C}\subset\pi^{-1}(x_0)\cap Z.\end{aligned}$$

This contradicts the condition that Z is not full-fiber with respect to D_δ at $x_0\in D_\delta$: the condition 3) of (2.5). Thus we have the result. (Q.e.d.)

We shall apply Corollary 1.4 to our case: $N=D_{\delta\mathbf{R}}$ with $\delta\in\Delta_{(k)}$ and $v(x)=\bar{u}(f)$. From the condition (2.11) 1), the assumption 1) in Corollary 1.2 is satisfied. On the other hand, we have $\operatorname{supp}(\mathit{sp}\circ\bar{u}(f)|_U)\subset T^*_{D_{\delta\mathbf{R}}}M\cap Z_{\mathbf{R}}$. By Lemma 2.5, there exists a point $(x_0,\xi_0)\in T^*_{D_{\delta\mathbf{R}}}M$ such that $\xi_0\neq0$ and $(x_0,\xi_0)\notin\operatorname{supp}(\mathit{sp}\circ\bar{u}(f)|_U)$. This means that the assumption 2) in Corollary 1.4 is satisfied. Thus we have $\bar{u}(f)=0$ near x_0. That is to say, we have $\bar{u}(f)=0$ for any section f of $\mathfrak{M}$ near any point x_0 in $D_{\delta\mathbf{R}}$ for all $\delta\in\Delta_{(k)}$. This means that $\bar{u}=0$ near any point x_0 in $D_{\delta\mathbf{R}}$. We have:

$$\text{(2.15)}\qquad \text{For any } \delta\in\Delta_{(0)}\cup\ldots\cup\Delta_{(k)},\ \bar{u} \text{ is zero near any point in } D_{\delta\mathbf{R}}$$

in addition to (2.9). Thus by induction on k, we have $\bar{u}=0$ on M. We complete the proof of Theorem 1.1.

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