Whitney's extension theorem for ultradifferentiable functions of Beurling type

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Classes of non-quasianalytic functions on $\mathbf{R}^n$ are usually defined by imposing conditions on the derivatives of the functions. For example, if $(M_p)_{p\in\mathbf{N}_0}$ is an appropriate sequence of positive numbers, one defines

$$\mathscr{E}^{(M_p)}(\mathbf{R}^n):=\{f\in C^\infty(\mathbf{R}^n)|\ \text{for each compact set } K \text{ in } \mathbf{R}^n \text{ and each } h>0$$
$$\sup_{\alpha\in\mathbf{N}_0^n}\sup_{x\in K}|f^{(\alpha)}(x)|(h^{|\alpha|}M_{|\alpha|})^{-1}<\infty\};$$

$\mathscr{E}^{\{M_p\}}(\mathbf{R}^n)$ is defined similarly (replacing the all quantifier over h by an existence quantifier). Continuing the classical work of E. Borel [5] many authors (see Bronshtein [7], Bruna [8], Carleson [9], Dzanasija [10], Ehrenpreis [11], Komatsu [15], Mityagin [22], Petzsche [24] and Wahde [30]) have investigated conditions on $(M_p)_{p\in\mathbf{N}_0}$ and on sequences $(a_\alpha)_{\alpha\in\mathbf{N}_0^n}$ implying the existence of $f\in\mathscr{E}^{(M_p)}(\mathbf{R}^n)$ (resp. $\mathscr{E}^{\{M_p\}}(\mathbf{R}^n)$) with

$$f^{(\alpha)}(0)=a_\alpha \quad \text{for all} \quad \alpha\in\mathbf{N}_0^n.$$

In the present note we study this question and a version of Whitney's extension theorem for the non-quasianalytic classes $\mathscr{E}_\omega(\mathbf{R}^n)$ which have been introduced by Beurling [2] and Björck [3] using the Fourier transform. Most familiar function classes, like the Gevrey classes, can be obtained by both methods ($M_p=(p!)^s$ or $\omega(x)=|x|^{1/s}$, $s>1$). However, in general, the two definitions lead to different classes.

To define $\mathscr{E}_\omega(\mathbf{R}^n)$ we vary Beurling's approach a bit. We assume that $\omega\colon \mathbf{R}\to[0,\infty[$ is a continuous function having the following properties:

(i) $$\limsup_{t\to\infty}\frac{\omega(2t)}{\omega(t)}<\infty \quad\text{and}\quad \lim_{t\to\infty}\frac{\log t}{\omega(t)}=0;$$

(ii) $$\int_{-\infty}^{+\infty}\frac{\omega(t)}{1+t^2}\,dt<\infty;$$

(iii) $\varphi\colon t\mapsto\omega(e^t)$ is a convex function on $\mathbf{R}$.

By φ^* we denote the Young conjugate of $\varphi|[0, \infty[$, and we put

$$\mathscr{E}_\omega(\mathbf{R}^n) := \left\{ f \in C^\infty(\mathbf{R}^n) \mid \sup_{\alpha \in \mathbf{N}_0^n} \sup_{|x| \leqq m} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \infty \quad \text{for all} \quad m \in \mathbf{N} \right\}.$$

Then the main result of the present paper states that the following assertions are equivalent:

(1) *For each compact convex set K in $\mathbf{R}^n$ with $\mathring{K} \neq \emptyset$ a Whitney field $f = (f_\alpha)_{\alpha \in \mathbf{N}_0^n} \in C(K)^{\mathbf{N}_0^n}$ is of the form $f = (g^{(\alpha)}|K)_{\alpha \in \mathbf{N}_0^n}$ for some $g \in \mathscr{E}_\omega(\mathbf{R}^n)$ iff for each $m \in \mathbf{N}$*

$$\sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in K} |f_\alpha(x)| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \infty.$$

(2) *The characterization in* (1) *holds for each set $K = \bar{\Omega} \subset \mathbf{R}^n$, where Ω is a bounded open set with real-analytic boundary.*

(3) *A sequence $(a_\alpha)_{\alpha \in \mathbf{N}_0^n} \in \mathbf{C}^{\mathbf{N}_0^n}$ is of the form $B_n(g) := (g^{(\alpha)}(0))_{\alpha \in \mathbf{N}_0^n}$ for some $g \in \mathscr{E}_\omega(\mathbf{R}^n)$ iff for each $m \in \mathbf{N}$*

$$\sup_{\alpha \in \mathbf{N}_0^n} |a_\alpha| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \infty.$$

(4) *There exist $K > 1$ and $t_0 > 0$ with $\omega(Kt) \leqq \frac{K}{2}\omega(t)$ for all $t \geqq t_0$.*

(5) *There exists $C > 0$ with $t \int_t^\infty \frac{\omega(s)}{s^2} ds \leqq C\omega(t) + C$ for all $t > 0$.*

Moreover, we show that for every continuous increasing function $\omega\colon [0, \infty[\to [0, \infty[$ with $\omega(0) = 0$ and $\lim_{t \to \infty} \omega(t) = \infty$ which satisfies (5), the function $\varkappa\colon t \mapsto t \int_t^\infty \frac{\omega(s)}{s^2} ds$ is an increasing concave function with $\varkappa(0) = 0$ which satisfies (5) and

$$\omega(t) \leqq \varkappa(t) \leqq C\omega(t) + C \quad \text{for all} \quad t > 0.$$

This implies that the class $\mathscr{E}^{(M_p)}$ coincides with a class of type $\mathscr{E}_\omega$ as soon as the sequence $(M_p)_{p \in \mathbf{N}_0}$ satisfies the conditions (M1), (M2) and (M3) of Komatsu [14]. As a consequence, our main theorem extends previous results of Ehrenpreis [11] and Komatsu [15].

The proof of our main result is based on the methods which were introduced by Carleson [9] and Ehrenpreis [11]. To sketch the idea, let $\mathscr{E}_\omega(K)$ denote the space of Whitney fields on K defined by the estimates in (1). Then we use an argument of Taylor [26] and Whitney's extension theorem to show that $\mathscr{E}_\omega(K)_b'$, the dual space of $\mathscr{E}_\omega(K)$ equipped with the topology of uniform convergence on the bounded subsets of $\mathscr{E}_\omega(K)$, is isomorphic to a weighted space $A_{\mathbf{P}_K}(\mathbf{C}^n)$ of entire functions by the Fourier—Laplace transform. Since $\mathscr{E}_\omega(\mathbf{R}^n)'_b$ is also isomorphic to a weighted space $A_p(\mathbf{C}^n)$ of entire functions, a theorem of Dieudonné and Schwartz implies that the restriction map ϱ_K: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(K)$ is surjective if and only if the inclusion map J: $A_{\mathrm{P}_K}(\mathbf{C}^n)\to A_p(\mathbf{C}^n)$ is an injective topological homomorphism. Because of this characterization one can use the Phragmén—Lindelöf principle and estimates of the harmonic extension P_ω of ω to show that ϱ_K is surjective if $P_\omega=O(\omega)$, which is equivalent to (4) and (5). If this condition does not hold then one can use Hörmander's L^2-estimates for the solutions of $\bar\partial$-problems to show that J is not a topological homomorphism.

To indicate a further consequence of our main theorem, assume that ω satisfies (i)—(iii) and (4) and let $\mathscr{E}_\omega(\{0\},\mathbf{R}^n)$ denote the Fréchet space of all sequences $(a_\alpha)_{\alpha\in\mathbf{N}_0^n}$ in $\mathbf{C}^{\mathbf{N}_0^n}$ which are defined by the estimates in (3). Then one would like to know whether the Borel map B_n: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(\{0\},\mathbf{R}^n)$ admits a continuous linear right inverse E_n. Using a result of [19] we show that the condition

for each $C>1$ there exist $\delta>0$ and $R_0>0$ with

$$(*)\qquad \omega^{-1}(CR)\,\omega^{-1}(\delta R)\leqq(\omega^{-1}(R))^2 \quad\text{for all}\quad R\geqq R_0$$

is necessary for the existence of E_n. In [21] we show that $(*)$ is also sufficient.

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1. Weight functions

In this section we fix some notation and introduce the weight functions ω which will be used subsequently. Without further reference we shall use the standard notation from complex analysis (see Hörmander [12]) and from functional analysis (see e.g. Schaefer [25]).

1.1. Weight functions ω. Let ω: $\mathbf{R}\to[0,\infty[$ be a continuous even function which is increasing on $[0,\infty[$ and satisfies $\omega(0)=0$ and $\lim_{t\to\infty}\omega(t)=\infty$. We consider the following conditions on ω:

$(\alpha)'$ $0=\omega(0)\leqq\omega(s+t)\leqq\omega(s)+\omega(t)$ *for all* $s,t\in\mathbf{R}$;

(α) $\omega(2t)=0(\omega(t))$ *as* t *tends to* ∞;

(β) $\int_{-\infty}^{+\infty}\frac{\omega(t)}{1+t^2}\,dt<\infty$;

(γ) $\log(1+|t|)=0(\omega(t))$ *as* t *tends to* ∞;

(γ)′ $\lim_{t\to\infty}\frac{\log t}{\omega(t)}=0$;

(δ) $\varphi\colon t\mapsto\omega(e^t)$ *is convex on* $\mathbf{R}$;

(ε) *there exists* $C>0$ *with* $\int_1^\infty\frac{\omega(yt)}{t^2}\,dt\leqq C\omega(y)+C$ *for all* $y\geqq 0$.

1.2. Remark. (a) The conditions 1.1(α)′, (β) and (γ) are basically those which are used in Björck [3] (see also Beurling [2]) to develop a theory of ultradifferentiable functions and ultradistributions.

(b) Note that a function ω which satisfies the general conditions in 1.1 and 1.1(β) has the property $\lim_{t\to\infty}\frac{\omega(t)}{t}=0$, since for $t\geqq 1$

$$\frac{\omega(t)}{t}=\int_t^\infty\frac{\omega(t)}{s^2}\,ds\leqq\int_t^\infty\frac{\omega(s)}{s^2}\,ds.$$

1.3. Proposition. *Let* $\omega\colon[0,\infty[\to[0,\infty[$ *be a continuous increasing function with* $\omega(0)=0$ *and* $\lim_{t\to\infty}\omega(t)=\infty$. *Then the following conditions are equivalent:*

(1) $\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{\varepsilon\omega(t)}{\omega(\varepsilon t)}=0$;

(2) *there exists* $K>1$ *with* $\limsup_{t\to\infty}\frac{\omega(Kt)}{\omega(t)}<K$;

(3) *there exists* $C>0$ *with* $\int_1^\infty\frac{\omega(yt)}{t^2}\,dt\leqq C\omega(y)+C$ *for all* $y>0$;

(4) *there exists an increasing concave function* $\varkappa\colon[0,\infty[\to[0,\infty[$ *with* $\varkappa(0)=0$ *and*

(i) $\omega(y)\leqq\varkappa(y)\leqq C\omega(y)+C$;

(ii) $\int_1^\infty\frac{\varkappa(yt)}{t^2}\,dt\leqq C\varkappa(y)+C$.

Proof. (1)⇒(2): By (1) there is $0<\varepsilon<1$ with $\limsup_{t\to\infty}\frac{\varepsilon\omega(t)}{\omega(\varepsilon t)}<1$. Hence (2) holds with $K=\frac{1}{\varepsilon}$.

(2)⇒(3): From (2) we get the existence of $T>0$ and $0<\varepsilon<1$ with

$$\omega(Kt)\leqq(K-\varepsilon)\,\omega(t)\quad\text{for all } t\geqq T.$$

This implies that for each $y \geqq T$ we have

$$\int_1^\infty \frac{\omega(yt)}{t^2}\,dt = \sum_{j=0}^\infty \int_{K^j}^{K^{j+1}} \frac{\omega(yt)}{t^2}\,dt \leqq \sum_{j=0}^\infty \frac{\omega(K^{j+1}y)}{K^{2j}}(K^{j+1}-K^j)$$

$$\leqq K(K-1)\,\omega(y)\sum_{j=0}^\infty \left(\frac{K-\varepsilon}{K}\right)^{j+1} = \frac{1}{\varepsilon}K(K-1)(K-\varepsilon)\,\omega(y).$$

Hence (3) holds.

(3)⇒(4): We define $\varkappa$ by

$$\varkappa(y) := \int_1^\infty \frac{\omega(yt)}{t^2}\,dt = y\int_y^\infty \frac{\omega(s)}{s^2}\,ds.$$

Then (i) and (ii) follow easily from (3) and the fact that ω is increasing. It is also obvious that $\varkappa$ is increasing and satisfies $\varkappa(0)=0$.

To show that $\varkappa$ is concave, note that $\varkappa$ is differentiable on $]0, \infty[$ with $\varkappa'(y)= \int_y^\infty \frac{\omega(s)}{s^2}\,ds-\frac{\omega(y)}{y}=\int_y^\infty \frac{d\omega(s)}{s}$. Hence $\varkappa'$ is decreasing, so $\varkappa$ is concave.

(4)⇒(1): Because of (4) (i) it suffices to show $\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{\varepsilon\varkappa(t)}{\varkappa(\varepsilon t)}=0$. To prove this, note that $\varkappa$ satisfies $\frac{\varkappa(2y)}{2}\leqq\varkappa(y)$ and hence

$$\frac{\varkappa(2^n y)}{2^n} \leqq \frac{\varkappa(2^{n-1}y)}{2^{n-1}}$$

for each $n\in\mathrm{N}$ and each $y\geqq 0$. Since $\varkappa$ is increasing, this implies by (4) (ii)

$$n\frac{\varkappa(2^n y)}{2^n} \leqq \sum_{j=1}^n \frac{\varkappa(2^j y)}{2^j} \leqq 4\sum_{j=1}^n \int_{2^j}^{2^{j+1}} \frac{\varkappa(ty)}{t^2}\,dt$$

$$\leqq 4\int_1^\infty \frac{\varkappa(yt)}{t^2}\,dt \leqq 4C(\varkappa(y)+1) \leqq 8C\varkappa(y)$$

for $y\geqq y_0$. Next let $0<\varepsilon\leqq\frac{1}{2}$ be given and choose $n\in\mathrm{N}$ with $\frac{1}{2^{n+1}}\leqq\varepsilon<\frac{1}{2^n}$. Then we have for all large $t>0$ and $t=2^{n+1}y$

$$\frac{\varepsilon\varkappa(t)}{\varkappa(\varepsilon t)} \leqq \frac{\frac{1}{2^n}\varkappa(t)}{\varkappa\left(\frac{1}{2^{n+1}}t\right)} = \frac{2\varkappa(2^{n+1}y)}{2^{n+1}\varkappa(y)} \leqq \frac{16C}{n},$$

which proves $\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{\varepsilon\varkappa(t)}{\varkappa(\varepsilon t)}=0.$

Remark. If in Proposition 1.3 the function ω has the additional property that $t\to\omega(e^t)$ is convex on $\mathbf{R}$ then the function $\varkappa$ in 1.3 (4) has this property, too.

1.4. Corollary. *If ω satisfies one of the equivalent conditions in* 1.3, *then there exists $0<\alpha<1$ with $\omega(t)=O(t^\alpha)$.*

Proof. By 1.3 (1) there exist $K>1$ and $t_0>0$ with

$$\frac{\omega(Kt)}{\omega(t)}<\frac{K}{2}\quad\text{for all}\quad t\geqq t_0.$$

Hence there exist $0<\alpha<1$ and $m\geqq 0$ with

$$\frac{\omega(Kt)}{\omega(t)}\leqq K^\alpha\quad\text{for all}\quad t\geqq K^m.$$

This implies for all $n\in\mathbf{N}_0$

$$\omega(K^{m+n})\leqq K^{\alpha n}\omega(K^m)=K^{\alpha(m+n)}\frac{\omega(K^m)}{K^{\alpha m}}.$$

Since ω is increasing, we get from this for all $t\in[K^{m+n}, K^{m+n+1}]$

$$\omega(t)\leqq\omega(K^{m+n+1})\leqq K^{\alpha(m+n+1)}\frac{\omega(K^m)}{K^{\alpha m}}=K^{(m+n)\alpha}\frac{\omega(K^m)}{K^{\alpha(m-1)}}\leqq t^\alpha\frac{\omega(K^m)}{K^{\alpha(m-1)}}.$$

1.5. Definition. Assume that ω satisfies the general conditions in 1.1 as well as 1.1(β).

(a) The harmonic extension $P_\omega\colon \mathbf{C}\to[0,\infty[$ of ω is defined by

$$P_\omega(x+iy):=\begin{cases}\dfrac{|y|}{\pi}\displaystyle\int_{-\infty}^{+\infty}\frac{\omega(t)}{(t-x)^2+y^2}\,dt & \text{for}\quad |y|>0;\\ \omega(x) & \text{for}\quad y=0.\end{cases}$$

(b) The radial extension $\tilde\omega\colon \mathbf{C}^n\to[0,\infty[$ of ω is defined by $\tilde\omega(z)=\omega(|z|)$.

1.6. Remark. (a) It is well-known that P_ω is continuous on $\mathbf{C}$ and harmonic in the (open) upper and lower half plane.

(b) If ω satisfies the general conditions in 1.1 as well as 1.1(δ) then $\tilde\omega$ is a continuous plurisubharmonic function on $\mathbf{C}^n$. In this case we have $\tilde\omega\leqq P_\omega$ for $n=1$, provided that ω satisfies 1.1(β), too.

1.7. Proposition. *Assume that ω satisfies the general conditions of* 1.1 *as well as* 1.1(β). *Then the conditions* (1)—(4) *in* 1.3 *are equivalent to*

$$(*)\qquad P_\omega(z)=O(\tilde\omega(z))\quad\textit{as}\quad |z|\quad\textit{tends to}\quad\infty.$$

Proof. If $(*)$ holds then there exists $C>0$ with

$$P_\omega(z) \leqq C\tilde{\omega}(z)+C \quad \text{for all} \quad z\in\mathbf{C}.$$

Since ω is increasing on $[0, \infty[$, this implies for $y>0$

$$C+C\omega(y) = C+C\tilde{\omega}(iy) \geqq P_\omega(iy) \geqq \frac{y}{\pi}\int_{|t|\geqq y} \frac{\omega(t)}{t^2+y^2}\,dt$$

$$= \frac{2}{\pi}\int_1^\infty \frac{\omega(yt)}{t^2+1}\,dt \geqq \frac{1}{\pi}\int_1^\infty \frac{\omega(yt)}{t^2}\,dt.$$

Hence ω satisfies 1.3(3).

Next assume that ω satisfies one of the equivalent conditions (1)—(4) of Proposition 1.3. Then it follows easily from 1.3(2) that ω satisfies condition 1.1(α). This and 1.3(3) implies the existence of positive numbers C_1 and C_2 such that for $z=x+iy$ with $x\geqq 0, y\geqq 0$ and $x+y\geqq 1$ we have

$$P_\omega(z) = \frac{y}{\pi}\int_{|t|\leqq x+y} \frac{\omega(t)}{(t-x)^2+y^2}\,dt+\frac{y}{\pi}\int_{|t|>x+y} \frac{\omega(t)}{(t-x)^2+y^2}\,dt$$

$$\leqq \omega(x+y)+\frac{2}{\pi}\int_1^\infty \frac{\omega(x+sy)}{1+s^2}\,ds \leqq \omega(x+y)+\frac{2}{\pi}\int_1^\infty \frac{\omega\big(s(x+y)\big)}{1+s^2}\,ds$$

$$\leqq \omega(x+y)+C_1\big(1+\omega(x+y)\big) \leqq C_2\big(1+\tilde{\omega}(z)\big).$$

By the symmetry properties of ω and the Poisson kernel, this implies $(*)$.

1.8. Examples. (a) It is easy to check that the following functions ω satisfy the general conditions in 1.1 and 1.1(α), (γ), (δ) and (ε) (after a suitable change on $[-A, A]$ for some $A>0$).

By Proposition 1.3 they are equivalent (in the sense of 1.3(4) (i)) to an increasing concave function having the same properties

(1) $\omega(t) = \big(\log(1+|t|)\big)^s, \quad s \geqq 1;$

(2) $\omega(t) = \big(\log(1+|t|)\big)^q \exp\big((\log(1+|t|))^p\big), \quad 0<p<1, \quad 0\leqq q<\infty;$

(3) $\omega(t) = |t|^p\big(\log(1+|t|)\big)^q, \quad 0<p<1, \quad 0\leqq q<\infty.$

(b) For $a>1$ there exists ω satisfying the general conditions of 1.1 as well as 1.1(α)′, (β), (γ)′ and (δ) such that for all large $|t|$ we have $\omega(t)=|t|(\log|t|)^{-a}$. By Corollary 1.4 this function does not satisfy condition 1.2(ε).

In Section 2 we shall use the following lemma:

1.9. Lemma. *For ω as in* 1.1 *assume that* 1.1(α) *and* (β) *are satisfied. Then there exists $A>0$ such that for all $z\in\mathbf{C}$ we have*

$$P_\omega(z+w) \leqq AP_\omega(z)+A \quad \textit{for all} \quad w\in\mathbf{C} \quad \textit{with} \quad |w|\leqq 1.$$

Proof. It is easy to check that 1.1(α) implies the existence of $K \geqq 1$ with

(1) $$\omega(t+x) \leqq K(1+\omega(t)+\omega(x)) \quad \text{for all} \quad t, x \in \mathbf{R}$$

(see Braun, Meise and Taylor [6], 1.2). Hence there exists $K_1 \geqq 1$ with $\omega(t+x) \leqq K_1(1+\omega(x))$ for all $x \in \mathbf{R}$ and $t \in \mathbf{R}$ with $|t| \leqq 1$. By the properties of the Poisson kernel, this implies

(2) $$P_\omega(z+t) \leqq K_1 P_\omega(z)+K_1 \quad \text{for all} \quad z \in \mathbf{C} \quad \text{and all} \quad t \in \mathbf{R}, \quad |t| \leqq 1.$$

Furthermore, (1) implies for all $y \neq 0$

$$P_\omega(x+iy) = \frac{|y|}{\pi} \int_{-\infty}^{+\infty} \frac{\omega(t+x)}{t^2+y^2} \, dt \leqq K(1+\omega(x))+KP_\omega(iy).$$

Then there exists $K_2 \geqq 1$ with

(3) $$P_\omega(z+iy) \leqq K_2 P_\omega(z)+K_2 \quad \text{for all} \quad z \in \mathbf{C} \quad \text{and all} \quad y \in \mathbf{R}, \quad |y| \leqq 1.$$

Now the result is an obvious consequence of (2) and (3).

2. Spaces of entire functions

In this section we prove the main results of the present article, formulated in terms of entire functions. To state them in an appropriate way, we first introduce the (DF)-spaces which we are going to use. In doing this, we denote by $A(\mathbf{C}^n)$ the algebra of all entire functions on $\mathbf{C}^n$.

2.1. Definition. Let $\mathbf{P}=(p_j)_{j\in\mathbf{N}}$ be an increasing sequence of continuous functions on $\mathbf{C}^n$ with $\lim_{|z|\to\infty}(p_{j+1}(z)-p_j(z))=\infty$ for all $j \in \mathbf{N}$. Then we put

$$A_{\mathbf{P}}(\mathbf{C}^n) := \{f \in A(\mathbf{C}^n) \mid \text{ there exists } j \in \mathbf{N} \text{ with } \|f\|_j := \sup_{z\in\mathbf{C}^n} |f(z)| \exp(-p_j(z)) < \infty\},$$

and we endow $A_{\mathbf{P}}(\mathbf{C}^n)$ with its natural inductive limit topology. If $\mathbf{P}=(jp)_{j\in\mathbf{N}}$ for some function p on $\mathbf{C}^n$, then we write $A_p(\mathbf{C}^n)$ instead of $A_{\mathbf{P}}(\mathbf{C}^n)$.

Remark. From Montel's theorem it follows easily that $A_{\mathbf{P}}(\mathbf{C}^n)$ is a (DFS)-space, i.e. the strong dual of a Fréchet—Schwartz space.

2.2. Proposition. *For ω as in* 1.1 *assume that* 1.1(α) *and* (β) *are satisfied. Let $g, h\colon \mathbf{R}^n \to [0, \infty[$ be continuous positive homogeneous functions with $h \leqq Mg$ for some $M > 0$ and define p and $\mathbf{P}=(p_j)_{j\in\mathbf{N}}$ by*

$$p(z) := g(\operatorname{Im} z)+\tilde{\omega}(z), \quad p_j(z) := h(\operatorname{Im} z)+j\tilde{\omega}(z), \quad j \in \mathbf{N}, \quad z \in \mathbf{C}^n.$$

If $P_\omega(z)=O(\tilde\omega(z))$ *then the inclusion map*

$$J\colon A_{\mathrm{P}}(\mathbf{C}^n)\to A_p(\mathbf{C}^n)$$

is a topological homomorphism, i.e. J is a linear topological isomorphism between $A_{\mathrm{P}}(\mathbf{C}^n)$ *and* $\operatorname{im}(J)$.

Proof. Since $h\leqq Mg$, it follows easily that J is a continuous linear map. Since $A_{\mathrm{P}}(\mathbf{C}^n)$ and $A_p(\mathbf{C}^n)$ are (DFS)-spaces, the lemma of Baernstein [1], p. 29, implies that J is a topological homomorphism iff for each bounded set B in $A_p(\mathbf{C}^n)$ the set $J^{-1}(B)$ is bounded in $A_{\mathrm{P}}(\mathbf{C}^n)$. To show that this holds, let B be fixed. Without loss of generality we can assume that for suitable $A, D>0$ we have

$$B=\{f\in A_p(\mathbf{C}^n)\,|\,|f(z)|\leqq A\exp\big(Dg(\operatorname{Im}z)+D\tilde\omega(z)\big)\quad\text{for all}\quad z\in\mathbf{C}^n\}. \tag{1}$$

$J^{-1}(B)$ will be bounded in $A_{\mathrm{P}}(\mathbf{C}^n)$ if we show that there are positive numbers A' and D' with

$$J^{-1}(B)\subset\{f\in A_{\mathrm{P}}(\mathbf{C}^n)\,|\,|f(z)|\leqq A'\exp\big(h(\operatorname{Im}z)+D'\tilde\omega(z)\big)\quad\text{for all}\quad z\in\mathbf{C}\}. \tag{2}$$

To prove (2) let $f\in J^{-1}(B)\subset A_{\mathrm{P}}(\mathbf{C}^n)$ be given. Then there exist positive numbers A_f and D_f with

$$|f(z)|\leqq A_f\exp\big(h(\operatorname{Im}z)+D_f\tilde\omega(z)\big)\quad\text{for all}\quad z\in\mathbf{C}^n. \tag{3}$$

Next fix $z=(z_1,\dots,z_n)=(x_1+iy_1,\dots,x_n+iy_n)\in\mathbf{C}^n$ with $\operatorname{Im}z=(y_1,\dots,y_n)\neq0$. Then choose an orthonormal basis $\{e_1,\dots,e_n\}$ of $\mathbf{R}^n$ with $(y_1,\dots,y_n)=\eta e_n$ for some $\eta>0$ and note that $(x_1,\dots,x_n)=\sum_{j=1}^{n-1}c_je_j+\xi e_n$. Now put $a:=\sum_{j=1}^{n-1}c_je_j$, $b:=e_n$, $\zeta:=\xi+i\eta$ and note that

$$F\colon\mathbf{C}\to\mathbf{C},\quad F(w):=f(a+wb)$$

is an entire function of exponential type and that

$$F(\zeta)=f(a+\zeta b)=f(z).$$

The definition of a and b implies

$$|a+wb|^2=|a|^2+|w|^2=\sum_{j=1}^{n-1}c_j^2+|w|^2:=c^2+|w|^2.$$

Since ω satisfies 1.1(α), we can assume w.l.o.g. that $\omega(2t)\leqq K\omega(t)$ for all $t>0$. Then the definition of $\tilde\omega$ implies

$$\tilde\omega(a+wb)=\omega\big((c^2+|w|^2)^{1/2}\big)\leqq\omega(c+|w|)\leqq K\big(\omega(c)+\tilde\omega(w)\big).$$

Hence the choice of a and b and the properties of h and g together with (1) and (3) imply that F satisfies the following estimates for all $w\in\mathbf{C}$ with $\operatorname{Im}w>0$:

$$|F(w)|\leqq A\exp[D(\operatorname{Im}w)g(b)+DK\omega(c)+DK\tilde\omega(w)]. \tag{4}$$

$$|F(w)|\leqq A_f\exp[(\operatorname{Im}w)h(b)+D_fK\omega(c)+D_fK\tilde\omega(w)]. \tag{5}$$

Since ω satisfies 1.1(β) and 1.2(b), the Phragmén—Lindelöf principle (see Boas [4], 6.5.4) implies

$$\log|F(u+iv)| \leqq \frac{v}{\pi}\int_{-\infty}^{+\infty}\frac{\log|F(t)|}{(u-t)^2+v^2}\,dt+vd \tag{6}$$

$$\leqq \frac{v}{\pi}\int_{-\infty}^{+\infty}\frac{DK\omega(t)+DK\omega(c)+\log A}{(u-t)^2+v^2}\,dt+vd$$

$$= DKP_\omega(u+iv)+DK\omega(c)+\log A+vd,$$

where

$$d = \lim_{r\to\infty}\frac{2}{\pi}\frac{1}{r}\int_0^\pi \log|F(re^{i\Theta})|\sin\Theta\, d\Theta \tag{7}$$

$$\leqq \lim_{r\to\infty}\frac{2}{\pi}\frac{1}{r}\int_0^\pi (\log A_f+h(b)r\sin\Theta+D_fK(\omega(c)+\omega(r)))\sin\Theta\, d\Theta$$

$$= \lim_{r\to\infty}\left(\frac{2}{\pi}h(b)\int_0^\pi \sin^2\Theta\, d\Theta+\frac{2}{\pi}\frac{D_fK\omega(r)}{r}\int_0^\pi \sin\Theta\, d\Theta\right) = h(b),$$

because of 1.2(b).

Now note that by hypothesis there exists $E\geqq 1$ with

$$P_\omega(w) \leqq E\tilde{\omega}(w)+E \quad \text{for all} \quad w\in \mathbf{C}. \tag{8}$$

Hence we get from (6), (7) and (8)

$$|F(w)| \leqq A\exp\big((\operatorname{Im} w)h(b)+DKE\tilde{\omega}(w)+DKE+DK\omega(c)\big)$$

$$\leqq Ae^{DKE}\exp\big(h(\operatorname{Im}(a+wb))+DKE(\tilde{\omega}(w)+\omega(c))\big)$$

$$\leqq Ae^{DKE}\exp\big(h(\operatorname{Im}(a+wb))+2DEK\tilde{\omega}(a+wb)\big)$$

and consequently

$$|f(z)| \leqq A'\exp\big(h(\operatorname{Im} z)+B'\tilde{\omega}(z)\big),$$

where $A':=Ae^{DKE}$ and $D':=2DEK$. This proves that (2) holds and completes the proof.

2.3. Proposition. *For ω as in 1.1 assume that 1.1(α), (β), (γ), (δ) are satisfied. Let $h\colon \mathbf{R}^n\to[0,\infty[$ be a continuous positive homogeneous function and define p and $\mathbf{P}=(p_j)_{j\in\mathbf{N}}$ by*

$$p(z):=|\operatorname{Im} z|+\tilde{\omega}(z),\quad p_j(z):=h(\operatorname{Im} z)+j\tilde{\omega}(z),\quad j\in\mathbf{N},\quad z\in\mathbf{C}^n.$$

If $\sup\left\{\frac{P_\omega(z)}{\tilde{\omega}(z)}\,\Big|\, z\in\mathbf{C},\ |z|\geqq 1\right\}=\infty$ *then the continuous inclusion map*

$$J\colon A_{\mathbf{P}}(\mathbf{C}^n)\to A_p(\mathbf{C}^n)$$

is not a topological homomorphism.

Proof. Obviously, the result follows from:

(1) There exists an unbounded sequence $(g_j)_{j\in\mathbf{N}}$ in $A_{\mathbf{P}}(\mathbf{C}^n)$ for which $(J(g_j))_{j\in\mathbf{N}}$ is bounded in $A_p(\mathbf{C}^n)$.

To prove this, we use the hypothesis to find a sequence $(a_j)_{j\in\mathbf{N}}$ in $\mathbf{C}$ with $\lim_{j\to\infty}\frac{P_\omega(a_j)}{\tilde{\omega}(a_j)}=\infty$. Without loss of generality we can assume that $\operatorname{Im} a_j>0$ for all $j\in\mathbf{N}$. We claim that the following holds:

Claim. There exist a sequence $(f_j)_{j\in\mathbf{N}}$ in $\mathbf{C}[z]$ and $C, D>0$ with

$$\sup_{j\in\mathbf{N}}\sup_{z\in\mathbf{C}}|f_j(z)|\exp\big(-DP_\omega(z)\big)=C<\infty. \tag{2}$$

$$f_j(a_j)=\exp\big(P_\omega(a_j)\big) \quad \text{for each} \quad j\in\mathbf{N}. \tag{3}$$

To show that our claim implies (1), we assume without loss of generality that $h(1, 0, \dots, 0)=\min\{h(y)\,|\,y\in\mathbf{R}^n,\ |y|=1\}$. Then, for $j\in\mathbf{N}$, we define $g_j\in A(\mathbf{C}^n)$ by

$$g_j(z_1, \dots, z_n):=f_j(z_1)\exp\big(-h(1, 0, \dots, 0)iz_1\big).$$

By Braun, Meise and Taylor [6], 2.2, there exists $A\geqq 1$ with

$$P_\omega(z)\leqq|\operatorname{Im} z|+A\tilde{\omega}(z) \quad \text{for all} \quad z\in\mathbf{C}.$$

Hence (2) implies for each $j\in\mathbf{N}$ and all $z\in\mathbf{C}^n$

$$\begin{aligned}|g_j(z)| &\leqq C\exp\big((D+1+h(1, 0, \dots, 0))\,|\operatorname{Im} z|+AD\tilde{\omega}(z)\big)\\ &\leqq C\exp\big(AD'p(z)\big),\end{aligned}$$

which proves that $\big(J(g_j)_{j\in\mathbf{N}}\big)$ is bounded in $A_p(\mathbf{C}^n)$.

To see that g_j is in $A_{\mathbf{P}}(\mathbf{C}^n)$ for all $j\in\mathbf{N}$, note that f_j is a polynomial and that our choice of the direction $(1, 0, \dots, 0)$ implies

$$|\exp\big(-h(1, 0, \dots, 0)iz_1\big)|\leqq\exp\big(h(\operatorname{Im} z_1, \dots, \operatorname{Im} z_n)\big)$$

for all $z=(z_1, \dots, z_n)\in\mathbf{C}^n$. Next note that (3) and our choice of the sequence $(a_j)_{j\in\mathbf{N}}$ imply that for each $m\in\mathbf{N}$ and all sufficiently large $j\in\mathbf{N}$ we have

$$\begin{aligned}&|g_j(a_j, 0, \dots, 0)|\exp\big(-p_m(a_j, 0, \dots, 0)\big)\\ &=|f_j(a_j)|\exp\big(-m\tilde{\omega}(a_j)\big)=\exp P_\omega(a_j)-m\tilde{\omega}(a_j)\big)\\ &\geqq\exp\left(P_\omega(a_j)\left(1-\frac{m\tilde{\omega}(a_j)}{P_\omega(a_j)}\right)\right)\geqq\exp\left(\frac{1}{2}P_\omega(a_j)\right).\end{aligned}$$

This shows that $(g_j)_{j\in\mathbf{N}}$ is unbounded in $A_{\mathbf{P}}(\mathbf{C}^n)$, since $\lim_{j\to\infty}|a_j|=\infty$.

Proof of the claim. Since the function $t\to\omega(e^t)$ is convex and since ω satisfies 1.1(β) by hypothesis, we can choose an increasing sequence $(R_j)_{j\in\mathbf{N}}$ in $]0, \infty[$

as well as sequences $(A_j)_{j\in\mathbf{N}}$ in $]0, \infty[$ and $(B_j)_{j\in\mathbf{N}}$ in $\mathbf{R}$ such that the following conditions (4) and (5) are satisfied.

(4) For each $j\in\mathbf{N}$ the function $t\to\omega_j(e^t)$ is continuous, strictly increasing and convex and satisfies $\omega_j\leqq\omega$, where $\omega_j\colon \mathbf{R}\to[0, \infty[$ is defined by

$$\omega_j(t) := \begin{cases} \omega(t) & \text{if} \quad |t| \leqq R_j \\ A_j \log|t| + B_j & \text{if} \quad |t| > R_j. \end{cases}$$

(5) For each $j\in\mathbf{N}$ we have

$$\sup_{|z-a_j|\leqq 1} |P_\omega(z) - P_{\omega_j}(z)| \leqq \frac{1}{j}.$$

Then we choose $\varphi\in\mathcal{D}(\mathbf{C})$ with $\operatorname{supp}\varphi\subset\{z\in\mathbf{C}\,\big|\,|z|\leqq 1\}$, $0\leqq\varphi\leqq 1$ and $\varphi(z)=1$ for $|z|\leqq\frac{1}{2}$ and we define $u_j\in\mathcal{D}(\mathbf{C})$ by

$$u_j(z) := \left(1-\frac{z}{a_j}\right)^{-1} \exp\left(P_\omega(a_j)\right) \bar\partial\varphi(z-a_j). \tag{6}$$

Since ω satisfies 1.1(γ) we get from (5) and Lemma 1.9:
There exist $L\geqq 1$ and $M>0$ such that for all $j\in\mathbf{N}$

$$\int_{|z-a_j|\leqq 1} |u_j(z)|^2 \exp\left(-2LP_{\omega_j}(z)\right) d\lambda(z) \leqq M. \tag{7}$$

Now note that $w_j\colon \mathbf{C}\to[0, \infty[$, defined by

$$w_j(z) := \begin{cases} P_{\omega_j}(z) & \text{if} \quad \operatorname{Im} z > 0; \\ \tilde\omega_j(z) & \text{if} \quad \operatorname{Im} z \leqq 0; \end{cases} \tag{8}$$

is a continuous subharmonic function because of (4). From Lemma 1.9 it follows that we can assume w.l.o.g. that $\operatorname{Im} a_j\geqq 1$ for all $j\in\mathbf{N}$. Hence (6) and (8) imply by (7)

$$\int_{\mathbf{C}} |u_j(z)|^2 \exp\left(-2Lw_j(z)\right) d\lambda(z) \leqq M \quad \text{for all} \quad j\in\mathbf{N}. \tag{9}$$

Since $\bar\partial u_j=0$, it follows from Hörmander [12], 4.4.2, that there exist $v_j\in C^\infty(\mathbf{C})$ with $\bar\partial v_j=u_j$ satisfying the estimate

$$\int_{\mathbf{C}} |v_j(z)|^2 \exp\left(-2Lw_j(z) - 2\log(1+|z|^2)\right) d\lambda(z) \leqq \frac{M}{2} \quad \text{for all} \quad j\in\mathbf{N}. \tag{10}$$

Now note that the function $f_j\colon \mathbf{C}\to\mathbf{C}$, defined by

$$f_j(z) := \varphi(z-a_j)\exp\left(P_\omega(a_j)\right) - \left(1-\frac{z}{a_j}\right) v_j(z)$$

is holomorphic and that (10) implies:

There exists $D' \geqq 1$ and $M' > 0$ such that for all $j \in \mathbf{N}$

$$\int_{\mathbf{C}} |f_j(z)|^2 \exp\left(-2D' w_j(z) - 5\log(1+|z|^2)\right) d\lambda(z) \leqq M'. \tag{11}$$

Since $w_j = O(\log(1+|z|^2))$ it follows by standard arguments that f_j is a polynomial. By (4) and 1.6(b) we have $w_j \leqq P_\omega$ for each $j \in \mathbf{N}$. Hence it follows from (11) and Lemma 1.9 that $(f_j)_{j\in\mathbf{N}}$ satisfies condition (2). Since $f_j(a_j) = \exp(P_\omega(a_j))$, condition (3) also holds.

3. Extension of ultradifferentiable functions

In this section we introduce the classes $\mathscr{E}_\omega(\mathbf{R}^n)$ (resp. $\mathscr{E}_\omega(K)$) of ω-ultradifferentiable functions of Beurling type on $\mathbf{R}^n$ (resp. on a compact subset of $\mathbf{R}^n$). Then we use the results of the preceding section to derive necessary and sufficient conditions on ω for the restriction map $\varrho_K\colon \mathscr{E}_\omega(\mathbf{R}^n) \to \mathscr{E}_\omega(K)$ to be surjective.

3.1. Definition. (a) A function $\omega\colon \mathbf{R} \to [0, \infty[$ will be called a weight function if it satisfies the general conditions in 1.1 as well as 1.1(α), (β), (γ)′ and (δ).

(b) For a weight function ω let φ denote the function defined by 1.1(δ). We define its Young conjugate $\varphi^*\colon [0, \infty[\to [0, \infty[$ by

$$\varphi^*(x) := \sup\{xy - \varphi(y) \mid y \geqq 0\}.$$

3.2. Definition. Let ω be a weight function and let $\Omega \subset \mathbf{R}^n$ be open. Then we define

$\mathscr{E}_\omega(\Omega) := \{f \in C^\infty(\Omega) \mid$ for each $m \in \mathbf{N}$ and each compact set $K \subset \Omega$:

$$p_{K,m}(f) := \sup_{\alpha\in\mathbf{N}_0^n} \sup_{x\in K} |f^{(\alpha)}(x)| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) < \infty\},$$

where $f^{(\alpha)}$ denotes the α-th derivative of f. We endow $\mathscr{E}_\omega(\Omega)$ with the l.c. topology which is given by the system of seminorms $\{p_{K,m} \mid K \subset\subset \Omega,\ m \in \mathbf{N}\}$.

Furthermore we define

$$\mathscr{D}_\omega(\Omega) := \underset{K\subset\subset\Omega}{\underrightarrow{\operatorname{ind}}}\ \mathscr{D}_\omega(K),$$

where for a compact set K in $\mathbf{R}^n$ we put

$$\mathscr{D}_\omega(K) := \{f \in \mathscr{E}_\omega(\mathbf{R}^n) \mid \operatorname{supp}(f) \subset K\},$$

endowed with the induced topology.

3.3. Remark. Note that in Braun, Meise and Taylor [6], the following was shown:

(1) $\mathscr{E}_\omega(\Omega)$ and $\mathscr{D}_\omega(K)$ are nuclear Fréchet spaces and $\mathscr{D}_\omega(\Omega)$ is non-trivial.

(2) For $p\colon \mathbf{C}^n \to [0, \infty[$, $p(z) := |\operatorname{Im} z| + \tilde{\omega}(z)$, the Fourier—Laplace transform $\mathscr{F}\colon \mathscr{E}_\omega(\mathbf{R}^n)'_b \to A_p(\mathbf{C}^n)$, $\mathscr{F}(\mu) = \hat{\mu}\colon z \to \langle \mu_x, e^{-i\langle x|z\rangle}\rangle$, is a linear topological isomorphism.

(3) If the weight function ω satisfies also 1.1$(\alpha)'$ then the spaces $\mathscr{D}_\omega(\Omega)$ and $\mathscr{E}_\omega(\Omega)$ in 3.2 coincide with those which were introduced by Beurling [2] and Björck [3].

(4) Without restriction we can assume that for all $\alpha \in \mathbf{N}_0^n$ and all $m \in \mathbf{N}$ we have $nm\varphi^*\left(\frac{|\alpha|}{nm}\right) \leqq m \sum_{j=1}^n \varphi^*\left(\frac{\alpha_j}{m}\right) \leqq m\varphi^*\left(\frac{|\alpha|}{m}\right)$.

3.4. Definition. Let ω be a weight function and $K \subset \mathbf{R}^n$ a compact set.

(a) If K is the closure of its interior, then we define

$$\mathscr{E}_\omega(K) := \{f = (f_\alpha)_{\alpha \in \mathbf{N}_0^n} \in C(K)^{\mathbf{N}_0^n} \,|\, f_{(0)}|\mathring{K} \in C^\infty(\mathring{K}),$$

$$(f_{(0)}|\mathring{K})^{(\alpha)} = f_\alpha|\mathring{K} \text{ for each } \alpha \in \mathbf{N}_0^n \text{ and for each } m \in \mathbf{N}\colon$$

$$|f|_m := \sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in K} |f_\alpha(x)| \exp(-m\varphi^*(|\alpha|/m)) < \infty\}.$$

(b) If $K = \{a\}$ then we define

$$\mathscr{E}_\omega(\{a\}) := \Big\{f = (f_\alpha)_{\alpha \in \mathbf{N}_0^n} \in \mathbf{C}^{\mathbf{N}_0^n} \,\Big|\, |f|_m := \sup_{\alpha \in \mathbf{N}_0^n} |f_\alpha| \exp(-m\varphi^*(|\alpha|/m)) < \infty$$

$$\text{for each } m \in \mathbf{N}\Big\}.$$

Both spaces are endowed with the l.c. topology which is induced by the norm-system $(|\ |_m)_{m \in \mathbf{N}}$. From 3.3(4) it follows easily, that $(|\ |_m)_{m \in \mathbf{N}}$ is equivalent to $(\|\ \|_m)_{m \in \mathbf{N}}$, where

$$\|f\|_m := \sup_{\alpha \in \mathbf{N}_0^n} \sup_{x \in K} |f_\alpha(x)| \exp\Big(-m \sum_{j=1}^n \varphi^*(\alpha_j/m)\Big).$$

(c) For K as in (a) or (b) we define the restriction map

$$\varrho_K := \mathscr{E}_\omega(\mathbf{R}^n) \to \mathscr{E}_\omega(K) \quad \text{by} \quad \varrho_K(f) := (f^{(\alpha)}|K)_{\alpha \in \mathbf{N}_0^n}.$$

Obviously, ϱ_K is continuous and linear.

3.5. Lemma. *Let ω be a weight function and let K be a compact convex subset of $\mathbf{R}^n$ with $\mathring{K} \neq \emptyset$. Then $\varrho_K\colon \mathscr{E}_\omega(\mathbf{R}^n) \to \mathscr{E}_\omega(K)$ has dense range.*

Proof. Without loss of generality we can assume $0 \in K$. Then for each $0 \neq f \in \mathscr{E}_\omega(K)$ and $0 < t < 1$ we define $f_t\colon x \mapsto f_{(0)}(tx)$ for $x \in \frac{1}{t}\mathring{K}$. By the definition of

$\mathcal{E}_\omega(K)$ we have for each $\alpha \in \mathbf{N}^n$

(1) $$f_t^{(\alpha)}(x) = t^{|\alpha|} f_{(0)}^{(\alpha)}(tx) = t^{|\alpha|} f_\alpha(tx), \quad x \in \frac{1}{t} \mathring{K},$$

which implies $f_t \in \mathcal{E}_\omega\left(\frac{1}{t}\mathring{K}\right)$. By Braun, Meise and Taylor [6], there exists $\chi_t \in \mathcal{D}_\omega\left(\frac{1}{t}\mathring{K}\right)$ with $\chi_t|K \equiv 1$ so that $\chi_t f_t \in \mathcal{E}_\omega(\mathbf{R}^n)$. This implies that $(f_t^{(\alpha)}|K)_\alpha$ is in im ϱ_K for each $0<t<1$. Hence it suffices to show $\mathcal{E}_\omega(K)-\lim_{t\uparrow 1} f_t|K = f$.

To do this, first note that 1.1(α) implies the existence of $A \in \mathbf{N}$ with $\omega(x) \leqq A\omega\left(\frac{x}{e}\right)$ for all large $x>0$. Hence we have $\varphi(t) \leqq A\varphi(t-1)$ for large $t>0$ and consequently

$$\varphi^*(y) \geqq y + A\varphi^*\left(\frac{y}{A}\right) \quad \text{for all large} \quad y > 0.$$

This implies:

For each $m \in \mathbf{N}$ there exist $l \in \mathbf{N}$ and $C>0$ such that for all $j \in \mathbf{N}_0$:

(2) $$\exp\left(-m\varphi^*\left(\frac{j}{m}\right)\right) \leqq Ce^{-j} \exp\left(-l\varphi^*\left(\frac{j}{l}\right)\right).$$

Since K is convex and contains the origin it follows easily from (1) that f_t converges to f on K with all its derivatives as t tends to 1. Hence (2) implies that for each $m \in \mathbf{N}$ and $j \in \mathbf{N}$ we have

$$\begin{aligned}\varlimsup_{t\uparrow 1} |f-f_t|_m &\leqq \varlimsup_{t\uparrow 1} \sup_{|\alpha| \geqq j} \sup_{x\in K} |f_\alpha(x) - f_t^{(\alpha)}(x)| \exp\left(-m\varphi^*\left(\frac{|\alpha|}{m}\right)\right) \\ &\leqq Ce^{-j} |f-f_t|_l \leqq 2Ce^{-j}|f|_l,\end{aligned}$$

which completes the proof.

3.6. Proposition. *Let ω be a weight function and let K be a compact convex subset of $\mathbf{R}^n$ with $0 \in \mathring{K}$. Let H_k: $\mathbf{R}^n \to [0, \infty[$*

$$H_K(x) = \sup\{\langle x|\xi\rangle \,|\, \xi \in K\}$$

denote the support functional of K and put $\mathbf{P} := (p_m)_{m\in\mathbf{N}}$, where p_m: $\mathbf{C}^n \to [0, \infty[$ is defined by $p_m(z) := H_K(\operatorname{Im} z) + m\tilde{\omega}(z)$. Then the map $\mathcal{G}$: $\mathcal{E}_\omega(K)'_b \to A_{\mathbf{P}}(\mathbf{C}^n)$ defined by

$$\mathcal{G}(\mu): z \mapsto \langle \mu_x, \varrho_K(e^{-i\langle x|z\rangle})\rangle,$$

is a linear topological isomorphism.

Proof. It is easy to check that $\mathcal{G}(\mu)$ is in $A_{\mathbf{P}}(\mathbf{C}^n)$ for each $\mu \in \mathcal{E}_\omega(K)'$ (see Braun, Meise and Taylor [6]) and that $\mathcal{G}$ is a continuous linear map. Lemma 3.5 implies the injectivity of $\mathcal{G}$. Hence the result follows from the open mapping theorem provided that we prove the surjectivity of $\mathcal{G}$. To do this, we use the idea of proof of Taylor [26], 2.8. If $g\in A_P(\mathbf{C}^n)$ is given then there exist $m\in\mathbf{N}$ with

$$(1) \qquad |g(z)| \leqq \exp\left(H_K(\operatorname{Im} z)+m\tilde{\omega}(z)\right) \quad \text{for all} \quad z\in\mathbf{C}^n.$$

Since ω satisfies 1.1(α) and (γ)′ it is easy to check that a slight variation of the proof of Hörmander [12], 4.4.3, can be used to prove the existence of $A>0$, $l, \nu\in\mathbf{N}$ and of a function $G\in A(\mathbf{C}^n\times\mathbf{C}^n)$ which has the following properties:

$$(2) \qquad G(z, z) = g(z) \quad \text{for all} \quad z\in\mathbf{C}^n;$$

$$(3) \qquad |G(z, w)| \leqq A(1+|z|^2)^\nu \exp\left(H_K(\operatorname{Im} z)+l\sum_{j=1}^n \tilde{\omega}(w_j)\right)$$

$$\text{for all} \quad (z, w)\in\mathbf{C}^n\times\mathbf{C}^n.$$

If we fix $z\in\mathbf{C}^n$ then $G(z, \cdot)$ is in $A(\mathbf{C}^n)$. Hence we have

$$(4) \qquad G(z, w) = \sum_{\alpha\in\mathbf{N}_0^n} Q_\alpha(z) w^\alpha,$$

where

$$(5) \qquad Q_\alpha(z) = \left(\frac{1}{2\pi i}\right)^n \int_{|\zeta_1|=r_1} \cdots \int_{|\zeta_n|=r_n} \frac{G(z,\zeta)}{\zeta_1^{\alpha_1+1}\cdots\zeta_n^{\alpha_n+1}}\, d\zeta_1 \ldots d\zeta_n,\ \alpha\in\mathbf{N}_0^n$$

for $r=(r_1, \ldots, r_n)\in\mathbf{R}_+^n$. Estimating (5) by (3) we get for each $\alpha\in\mathbf{N}_0^n$

$$(6) \qquad |Q_\alpha(z)| \leqq A(1+|z|^2)^\nu \exp\left(H_K(\operatorname{Im} z)\right) \inf_r \exp\left[l\sum_{j=1}^n\left(\tilde{\omega}(r_j)-\frac{\alpha_j}{l}\log r_j\right)\right]$$

$$\leqq A(1+|z|^2)^\nu \exp\left(H_K(\operatorname{Im} z)\right) \exp\left(-l\sum_{j=1}^n \varphi^*(\alpha_j/1)\right).$$

By the theorem of Paley—Wiener—Schwartz, (6) implies that for each $\alpha\in\mathbf{N}_0^n$ there exists $T_\alpha\in C^\infty(\mathbf{R}^n)'$ with $\operatorname{supp}(T_\alpha)\subset K$ such that the Fourier—Laplace transform $\hat{T}_\alpha$ of T_α satisfies $(-i)^{|\alpha|}\hat{T}_\alpha(z)=Q_\alpha(z)\exp\left(l\sum_{j=1}^n \varphi^*(\alpha_j/l)\right)$. Moreover, (6) implies the existence of $t\in\mathbf{N}$ and $D>0$ such that:

For each $\alpha\in\mathbf{N}_0^n$ and each $f\in C^\infty(\mathbf{R}^n)$:

$$(7) \qquad |\langle T_\alpha, f\rangle| \leqq D \sup_{|\beta|\leqq t-1}\ \sup_{x\in 2K} |f^{(\beta)}(x)|.$$

Next note that by Whitney's extension theorem (see Malgrange [16], I.3) there exists a continuous linear extension map $R\colon C^t(K)\to C^t(\mathbf{R}^n)$. Hence (7) and the proof of Hörmander [13], 1.5.4, imply:

There exists $E>0$ such that for each $\alpha\in\mathbf{N}_0^n$ there exists

$$(8) \qquad \mu_\alpha\in C^t(K)' \quad \text{with} \quad |\langle\mu_\alpha, f\rangle| \leqq E \sup_{|\beta|\leqq t}\ \sup_{x\in K} |f_\beta(x)| \quad \text{for each}$$

$$f\in C^t(K) \quad \text{and} \quad \langle\mu_\alpha, \varrho_K(f)\rangle = \langle T_\alpha, f\rangle \quad \text{for all} \quad f\in C^\infty(\mathbf{R}^n).$$

Next we use 3.5(2) to choose $k>1$ and $M>0$ such that

$$\sum_{j=0}^{\infty} \exp\left(-l\varphi^*\left(\frac{j}{l}\right)+k\varphi^*\left(\frac{j}{k}\right)\right) = M < \infty. \tag{9}$$

Then we note that the multiplication operator $M_z\colon A_{\tilde{\omega}}(\mathbf{C})\to A_{\tilde{\omega}}(\mathbf{C})$, $M_z(f)\colon z\mapsto zf(z)$ is continuous. Hence its adjoint is also continuous. By Meise [17], 2.4, and Meise and Taylor [18], 1.10, $A_{\tilde{\omega}}(\mathbf{C})'_b$ can be identified with the sequence space $\mathcal{E}_\omega(\{0\})$ $(0\in\mathbf{R})$. With this identification M_z^t equals the backward shift operator. It is easy to check that the continuity of this operator implies:

There exist $p\in\mathbf{N}$ and $L>0$ such that for each $j\in\mathbf{N}_0$ and

$$1\leqq\tau\leqq t \quad\text{we have}\quad \exp\left(-k\varphi^*\left(\frac{j}{k}\right)\right)\leqq L\exp\left(-p\varphi^*\left(\frac{j+\tau}{p}\right)\right). \tag{10}$$

Hence we get from (8), (9) and (10) and each $f=(f_\alpha)_{\alpha\in\mathbf{N}_0^n}\in\mathcal{E}_\omega(K)$:

$$\sum_{\alpha\in\mathbf{N}_0^n}\left|\left\langle\mu_\alpha\exp\left(-l\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{l}\right)\right), f_\alpha\right\rangle\right|$$

$$\leqq \sum_{\alpha\in\mathbf{N}_0^n} E\sup_{|\beta|\leqq t}\sup_{x\in K}|f_{\alpha+\beta}(x)|\exp\left(-l\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{l}\right)\right)$$

$$\leqq \sum_{\alpha\in\mathbf{N}_0^n} E\exp\left(-l\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{l}\right)\right.$$

$$\left.+k\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{k}\right)\right)L^n\sup_{|\beta|\leqq t}\sup_{x\in K}|f_{\alpha+\beta}(x)|\exp\left(-p\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j+\beta_j}{p}\right)\right)\leqq EL^nM^n\|f\|_p.$$

This proves that $\mu\colon f\mapsto\sum_{\alpha\in\mathbf{N}_0^n}\langle\mu_\alpha,f_\alpha\rangle\exp\left(-1\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{1}\right)\right)$ is a continuous linear form on $\mathcal{E}_\omega(K)$. By the definition of $\hat{T}_\alpha$ we get from (8), (4) and (2)

$$\mathcal{G}(\mu)[z] = \sum_{\alpha\in\mathbf{N}_0^n}\langle\mu_{\alpha,x},\varrho_K(e^{-i\langle x|z\rangle})\rangle\exp\left(-l\sum_{j=1}^n\varphi^*\left(\frac{\alpha_j}{l}\right)\right)(-iz)^\alpha$$

$$= \sum_{\alpha\in\mathbf{N}_0^n}\hat{T}_\alpha(z)\exp\left(-l\sum_{j=1}^n\varphi^*(a_j/l)\right)(-iz)^\alpha = \sum_{\alpha\in\mathbf{N}_0^n}Q_\alpha(z)z^\alpha = G(z,z) = g(z).$$

This shows that $\mathcal{G}$ is surjective and completes the proof.

Remark. From the proof of Proposition 3.6 one can easily derive the following result on the local structure of ultradistributions in $\mathcal{D}_\omega(\mathbf{R}^n)'$, which corresponds to Komatsu [14], 8.1: For every $T\in\mathcal{D}_\omega(\mathbf{R}^n)'$ and each compact convex subset $K\subset\mathbf{R}^n$ there exist $m\in\mathbf{N}$ and a family $(\mu_\alpha)_{\alpha\in\mathbf{N}_0^n}$ in $C(K)'$ with

$$\sup_{\alpha\in\mathbf{N}_0^n}\|\mu_\alpha\|_{C(K)'}\exp\left(-m\sum_{j=1}^n\varphi^*(\alpha_j/m)\right)<\infty$$

such that for each $f\in\mathcal{D}_\omega(K)$ we have

$$\langle T, f\rangle = \sum_{\alpha\in\mathbf{N}_0^n}\langle\mu_\alpha, f^{(\alpha)}\rangle.$$

3.7. Proposition. *Let ω be a weight function which satisfies* 1.1(ε). *Then the restriction map $\varrho_K\colon \mathcal{E}_\omega(\mathbf{R}^n)\to\mathcal{E}_\omega(K)$ is a surjective topological homomorphism for each compact convex set K in $\mathbf{R}^n$ which is either a singleton or has non-empty interior.*

Proof. It suffices to prove this for $0\in\mathring{K}$ (resp. $K=\{0\}$). Assuming this, we define $p\colon \mathbf{C}^n\to[0,\infty[$ by $p(z)=|\operatorname{Im} z|+\tilde{\omega}(z)$ and $\mathbf{P}=(p_m)$, where $p_m(z)=H_K(\operatorname{Im} z)+m\tilde{\omega}(z)$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{E}_\omega(K)'_b & \xrightarrow{\varrho_K^t} & \mathcal{E}_\omega(\mathbf{R}^n)'_b \\ \mathcal{G}\big\downarrow & & \big\downarrow\mathcal{F} \\ A_{\mathbf{P}}(\mathbf{C}^n) & \xrightarrow{J} & A_p(\mathbf{C}^n) \end{array} \tag{1}$$

where $\mathcal{F}$ denotes the Fourier—Laplace transform, J denotes the inclusion and where $\mathcal{G}$ is defined in 3.6 if $\mathring{K}\neq\emptyset$. If $K=\{0\}$, then we identify $\mathcal{E}_\omega(\{0\})'_b$ in the usual way with the sequence space $\lambda(A, \mathbf{N}_0^n)'_b$, where

$$\lambda(A,\mathbf{N}_0^n) := \Big\{(y_\alpha)_{\alpha\in\mathbf{N}_0^n}\in\mathbf{C}^{\mathbf{N}_0^n}\,\Big|\, \|y\|_m := \sum_{\alpha\in\mathbf{N}_0^n}|y_\alpha|\exp\Big(-m\sum_{j=1}^n\varphi^*(\alpha_j/m)\Big)<\infty \text{ for each } m\in\mathbf{N}\Big\}$$

and we define $\mathcal{G}\colon \lambda(A,\mathbf{N}_0^n)'_b\to A_{\mathbf{P}}(\mathbf{C}^n)=A_{\tilde{\omega}}(\mathbf{C}^n)$ by

$$\mathcal{G}\big((y_\alpha)_{\alpha\in\mathbf{N}}\big)[z] := \sum_{\alpha\in\mathbf{N}_0^n} y_\alpha(-iz)^\alpha.$$

It is easy to check that $\mathcal{G}$ is a linear topological isomorphism and that the diagram (1) is also commutative in this case. Since $\mathcal{G}$ and $\mathcal{F}$ are linear topological isomorphisms it is obvious that ϱ_K^t is an injective topological homomorphism if and only if J has this property. Since ω satisfies condition 1.1(ε), Proposition 1.7 implies that the hypotheses of Proposition 2.2 are satisfied, which proves that ϱ_K^t is an injective topological homomorphism. This implies that ϱ_K is a surjective topological homomorphism (see e.g. Schaefer [25], IV, 7.8).

3.8. Corollary. *Let ω be a weight function which satisfies the conditions* 1.1(ε) *and let $\Omega\subset\mathbf{R}^n$ be an open bounded set with a real analytic boundary. Then the restriction map*

$$\varrho_\Omega\colon \mathcal{E}_\omega(\mathbf{R}^n)\to\mathcal{E}_\omega(\bar{\Omega})$$

is a surjective topological homomorphism.

Proof. First we treat a special case: Put $Q:=(]-1,1[)^n$ and assume that U is an open set in $\mathbf{R}^n$ for which there exists a real-analytic diffeomorphism $\varphi\colon Q\to U$

with

$$(*)\qquad \begin{cases} U\cap\Omega = \varphi(\{(\xi_1, \dots, \xi_n)\in Q \mid \xi_n > 0\}) = \varphi(\Omega_+) \\ U\cap\partial\Omega = \varphi(\{(\xi_1, \dots, \xi_n)\in Q \mid \xi_n = 0\}). \end{cases}$$

Furthermore, assume that for $g\in\mathscr{E}_\omega(\bar\Omega)$ there exists a compact set $L\subset U$ such that $g_\alpha(x)=0$ for all $\alpha\in\mathbf{N}_0^n$ and all $x\in L\cap\bar\Omega$. Since φ is real-analytic, there exists $f\in\mathscr{E}_\omega(\bar\Omega_+)$ with $f_{(0)}=g_{(0)}\circ\varphi$ by Braun, Meise and Taylor [6]. Since $\bar Q_+$ is a compact convex set with non-empty interior, Proposition 3.7 implies the existence of $F\in\mathscr{E}_\omega(\mathbf{R}^n)$ with $\varrho_{\bar Q_+}(F)=f$. Of course we can assume that $\operatorname{supp}(F)$ is contained in a compact subset of Q. Then the function $G\colon \mathbf{R}^n\to\mathbf{C}$, defined by

$$G(x):=\begin{cases}0 & \text{if } x\notin U;\\ F(\varphi^{-1}(x)) & \text{if } x\in U;\end{cases}$$

is in $\mathscr{E}_\omega(\mathbf{R}^n)$ and satisfies $\varrho_{\bar\Omega}(G)=g$.

For the general case let $g=(g_\alpha)\in\mathscr{E}_\omega(\bar\Omega)$ be given. Then the compactness of $\partial\Omega$ implies the existence of $N\in\mathbf{N}$ so that for $1\leqq j\leqq N$ we can find open sets U_j in $\mathbf{R}^n$, maps $\varphi_j\colon Q\to U_j$ and functions $\varphi_j\in\mathscr{D}_\omega(U_j)$ which have the following properties

(1) $\partial\Omega\subset\bigcup_{j=1}^N U_j$;

(2) $\sum_{j=1}^N \varphi_j(x)=1$ for all x in some open neighbourhood V of $\partial\Omega$;

(3) φ_j is a real-analytic diffeomorphism which satisfies $(*)$ if U in $(*)$ is replaced by U_j.

Then it is easy to check that $\varphi_j g:=\left(\sum_{\beta\leqq\alpha}\binom{\alpha}{\beta}\varphi_j^{(\alpha-\beta)}g_\beta\right)_{\alpha\in\mathbf{N}_0^n}$ is in $\mathscr{E}_\omega(\bar\Omega)$ and has the properties which we required in the special case. Hence there exists $G_j\in\mathscr{E}_\omega(\mathbf{R}^n)$ with $\varrho_{\bar\Omega}(G_j)=\varphi_j g$ for $1\leqq j\leqq N$. Because of (2), the function $G_0:=\left(1-\sum_{j=1}^N\varphi_j\right)g_0$ has compact support in Ω. Hence it can be considered as a function in $\mathscr{D}_\omega(\Omega)\subset\mathscr{E}_\omega(\mathbf{R}\ )$. Then $G:=\sum_{j=0}^N G_j$ is in $\mathscr{E}_\omega(\mathbf{R}^n)$ and satisfies

$$\varrho_{\bar\Omega}(G)[x] = \sum_{j=0}^N \varrho_{\bar\Omega}(G_j)[x] = \left(1-\sum_{j=1}^N\varphi_j(x)\right)g_0(x)+\sum_{j=1}^N\varphi_j(x)g_0(x) = g_0(x)$$

for each $x\in\Omega$, which implies $\varrho_{\bar\Omega}(G)=g$.

Remark. The proof of Corollary 3.8 shows that the following more general version of 3.8 holds, too: Let Ω be an open set in $\mathbf{R}^n$ with a real-analytic boundary, which means that for each $x\in\partial\Omega$ there exists an open neighbourhood U of x and a real-analytic diffeomorphism $\varphi\colon Q\to U$ which satisfies 3.8$(*)$. Then the restriction map $\varrho_{\bar\Omega}\colon\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(\bar Q)$ is surjective, where we extend Definition 3.3 in an obvious way to the present situation.

3.9. Proposition. *Let ω be a weight function and assume that the restriction map ϱ_K: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(K)$ is surjective for some compact convex set K in $\mathbf{R}^N$ with $\mathring{K}\neq 0$ or for some singleton K. Then ω satisfies* 1.1(ε).

Proof. Since $\mathscr{E}_\omega(\mathbf{R}^n)$ and $\mathscr{E}_\omega(K)$ are Fréchet spaces, the surjectivity of ϱ_K implies that ϱ_K is a surjective topological homomorphism. Hence ϱ_K^t is an injective topological homomorphism. By the commutative diagram 3.7(1) this implies that J: $A_{\mathbf{P}}(\mathbf{C}^n)\to A_p(\mathbf{C}^n)$ is an injective topological homomorphism. Hence the result follows from Proposition 2.3 and Proposition 1.7.

3.10. Theorem. *Let ω be a weight function. Then the following assertions are equivalent:*

(1) *For each $n\in\mathbf{N}$ and each compact convex set $K\subset\mathbf{R}^n$ with $\mathring{K}\neq\emptyset$ the restriction map ϱ_K: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(K)$ is surjective.*

(2) *For each $n\in\mathbf{N}$ and each bounded open set $\Omega\subset\mathbf{R}^n$ with real-analytic boundary the map $\varrho_{\bar{\Omega}}$: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(\bar{\Omega})$ is surjective.*

(3) *There exists $n\in\mathbf{N}$ and a compact convex set $K\subset\mathbf{R}^n$ with $\mathring{K}\neq\emptyset$ such that ϱ_K is surjective.*

(4) *For each $n\in\mathbf{N}$ the Borel map B_n: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega\{0\}$, $B_n(f):=(f^{(\alpha)}(0))_{\alpha\in\mathbf{N}_0^n}$, is surjective.*

(5) *There exists $n\in\mathbf{N}$ such that B_n is surjective.*

(6) *There exists $C>0$ with* $\int_1^\infty \frac{\omega(yt)}{t^2}\,dt \leqq C\omega(y)+C$ *for all* $y>0$.

(7) $\lim_{\varepsilon\downarrow 0}\limsup_{t\to\infty}\frac{\varepsilon\omega(t)}{\omega(\varepsilon t)}=0$.

(8) *There exists $K>1$ with* $\limsup_{t\to\infty}\frac{\omega(Kt)}{\omega(t)}<K$.

(9) *There exists a weight function $\varkappa$ which satisfies* 1.1(ε), *which is concave on* $[0,\infty[$ *and which is equivalent to ω in the sense of* 1.3(4) (i).

Proof. The implications (1)⇒(3), (2)⇒(3) and (4)⇒(5) hold trivially. By Proposition 3.9 we have the implications (3)⇒(6) and (5)⇒(6). The implications (6)⇒(1) and (6)⇒(4) hold by Proposition 3.7, while (6)⇒(2) holds by Corollary 3.8. The equivalence of (6), (7), (8) and (9) was proved in 1.3 and the remark following 1.3.

3.11. Remark. Let $(M_j)_{j\in\mathbf{N}_0}$ be a sequence of positive numbers which has the following properties:

(M1) $M_j^2\leqq M_{j-1}M_{j+1}$ for all $j\in\mathbf{N}$;

(M2) there exist $A, H>1$ with $M_n\leqq AH^n\min_{0\leqq j\leqq n}M_jM_{n-j}$ for all $n\in\mathbf{N}$;

(M3) there exists $A>0$ with $\sum_{q=j+1}^\infty\frac{M_{q-1}}{M_q}\leqq Aj\frac{M_j}{M_{j+1}}$ for all $j\in\mathbf{N}$; and define

$\omega_M\colon \mathbf{R}\to[0,\infty[$ by

$$\omega_M(t)=\begin{cases}\sup\limits_{j\in\mathbf{N}_0}\log\dfrac{|t|^j M_0}{M_j} & \text{for} \quad |t|>0\\ 0 & \text{for} \quad t=0.\end{cases}$$

Then ω_M is a continuous even function with $\omega_M(0)=0$ and $\lim_{t\to\infty}\omega_M(t)=\infty$, which satisfies 1.1$(\gamma)'$ and 1.1(δ). By Komatsu [14], 4.4, (M3) implies that ω_M satisfies 1.1(ε). Hence Proposition 1.3 implies the existence of a weight function $\varkappa$ which satisfies 1.1(ε), which is concave on $[0,\infty[$ and which satisfies

(1) $\qquad \omega_M(t)\leqq\varkappa(t)\leqq C\omega_M(t)+C$ for some $C>0$ and all $t>0$.

Since $\varkappa$ is subadditive, this implies

(2) $\qquad \omega_M(2t)\leqq\varkappa(2t)\leqq 2\varkappa(t)\leqq 2C\omega_M(t)+2C$ for all $t>0$.

By Komatsu [14], 3.6, (M2) implies the existence of $D>0$ with

(3) $\qquad 2\omega_M(t)\leqq\omega_M(Dt)+D$ for all $t>0$.

From (1)—(3) it follows that for each open set Ω (resp. compact set K) in $\mathbf{R}^n$ we have

$$\mathscr{E}_\varkappa(\Omega)=\mathscr{E}^{(M_j)}(\Omega)=\Big\{f\in C^\infty(\Omega)\Big|\ \sup_{\alpha\in\mathbf{N}_0^n}\sup_{x\in K}\frac{|f^{(\alpha)}(x)|}{h^{|\alpha|}M_{|\alpha|}}<\infty$$
$$\text{for each } h>0 \text{ and each } K\subset\Omega \text{ compact}\Big\}$$

and $\mathscr{E}_\varkappa(K)=\mathscr{E}^{(M_j)}(K)$, where $\mathscr{E}^{(M_j)}(K)$ is defined similarly as in 3.4.

Because of this, Theorem 3.10 extends the results of Ehrenpreis [11], p. 447, and Komatsu [15], 4.5.

Remark. It is an interesting question to know if Whitney's extension theorem holds for the classes $\mathscr{E}_\omega$ and arbitrary compact subsets K of $\mathbf{R}^n$ when ω is a weight function satisfying 3.10(6). For the classes $\mathscr{E}^{\{M_p\}}$ Bruna [8] has shown this to be the case, provided that the sequence $(M_p)_{p\in\mathbf{N}_0}$ satisfies some conditions which are stronger than (M1), (M2) and (M3). We suspect it is also the case here. However, our use of Fourier transform methods makes it difficult to treat the case of arbitrary compact sets.

By Theorem 3.10 we know that the maps ϱ_K and B_n are surjective in many cases. In these cases one would like to know whether they also admit a continuous linear right inverse, i.e. whether one can do the extension with a continuous linear operator. For the ordinary C^∞-functions on $\mathbf{R}^n$ it is known that B_n does not admit a continuous linear right inverse (see Mityagin [23]) and that a continuous linear extension operator exists for compact convex sets $K\subset\mathbf{R}^n$ with $\mathring{K}\neq\emptyset$ (see Tidten [27], 4.6). For the present classes of functions we shall treat this question in [21] (see the announcement [20]). Here we use a result of our paper [19] to derive a necessary condition for the existence of a continuous linear right inverse of the Borel map B_n. In [21] we prove that this condition is also sufficient.

3.12. Corollary. *Let ω be a weight function. If the Borel map B_n: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(\{0\})$ admits a continuous linear right inverse then ω has the following property:*

(*) *For each $C>1$ there exist $\delta>0$ and $R_0>0$ with*

$$\omega^{-1}(CR)\omega^{-1}(\delta R)\leqq(\omega^{-1}(R))^2 \quad \textit{for all} \quad R\geqq R_0.$$

Proof. Let $\mathscr{E}^{2\pi}_\omega(\mathbf{R}^n)$ denote the closed linear subspace of $\mathscr{E}_\omega(\mathbf{R}^n)$ consisting of all functions which are periodic with respect to hypercube $[-\pi,\pi]^n$. Since $\mathscr{E}^{2\pi}_\omega(\mathbf{R}^n)$ is the kernel of a system of difference equations it follows from Meise [17], 3.7, that $\mathscr{E}^{2\pi}_\omega(\mathbf{R}^n)$ is isomorphic to a power series space of infinite type (for subadditive weight functions ω see Vogt [29], 7.7). If B_n admits a continuous linear right inverse, then $\tilde{B}_n:=B|\mathscr{E}^{2\pi}_\omega(\mathbf{R}^n)$ has this property too, and hence $\mathscr{E}_\omega(\{0\})$ is isomorphic to a linear topological subspace of $\mathscr{E}^{2\pi}_\omega(\mathbf{R}^n)$. This implies that $\mathscr{E}_\omega(\{0\})$ has the property (DN) of Vogt [23]. In the proof of Proposition 3.7 we have already remarked that $\mathscr{E}_\omega(\{0\})'_b\cong A_{\bar{\omega}}(\mathbf{C}^n)$. Hence ω satisfies (*) by [19], 3.1.

3.13. Example. For $1<s<\infty$ define ω_s: $t\mapsto(\max(0,\log|t|))^s$. Then ω is a weight function which satisfies 1.1(ε). It is easy to check that ω does not satisfy 3.12(*). Hence the Borel map B_n: $\mathscr{E}_\omega(\mathbf{R}^n)\to\mathscr{E}_\omega(\{0\})$ is surjective but does not admit a continuous linear right inverse.

References

1. Baernstein II, A., Representation of holomorphic functions by boundary integrals, *Trans. Amer. Math. Soc.* **160** (1971), 27—37.
2. Beurling, A., *Quasi-analyticity and general distributions,* Lectures 4 and 5, AMS Summer Institute, Stanford, 1961.
3. Björck, G., Linear partial differential operators and generalized distributions, *Ark. Mat.* **6** (1965), 351—407.
4. Boas, R. P., *Entire functions,* Academic Press, 1954.
5. Borel, E., Sur quelques points de la théorie des fonctions, *Ann. École Norm. Sup.* (3) **12** (1895), 9—55.
6. Braun, R. W., Meise, R. and Taylor, B. A., Ultradifferentiable functions and Fourier analysis, manuscript.
7. Bronshtein, M. D., Continuation of functions in Carleman's non-quasianalytical classes, Soviet Math. (Iz. VUZ) **30**, No 12 (1986), 11—14.
8. Bruna, J., An extension theorem of Whitney type for non-quasianalytic classes of functions, *J. London Math. Soc.* (2), **22** (1980), 495—505.
9. Carleson, L., On universal moment problems, *Math. Scand.* **9** (1961), 197—206.
10. Dzanasija, G. A., Carleman's problem for functions of the Gevrey class, *Soviet Math. Dokl.* **3** (1962), 969—972, translated from *Dokl. Akad. Nauk SSSR* **145** (1962), 259—262.

11. EHRENPREIS, L., *Fourier analysis in several complex variables,* Wiley-Interscience Publ. 1970.
12. HÖRMANDER, L., *An introduction to complex analysis in several variables,* Van Nostrand, 1967.
13. HÖRMANDER, L., *Linear partial differential operators,* Springer Verlag, 1969.
14. KOMATSU, H., Ultradistributions I, Structure theorems and a characterization, *J. Fac. Sci. Tokyo, Sect. IA Math.* **20** (1973), 25—105.
15. KOMATSU, H., Ultradistributions II, The kernel theorem and ultradistributions with support in a submanifold, *J. Fac. Sci. Tokyo, Sect. IA Math.* **24** (1977), 607—628.
16. MALGRANGE, B., *Ideals of differentiable functions,* Oxford University Press, 1966.
17. MEISE, R., Sequence space representations for (DFN)-algebras of entire functions modulo closed ideals, *J. reine angew. Math.* **363** (1985), 59—95.
18. MEISE, R. and TAYLOR, B. A., Sequence space representations for (FN)-algebras of entire functions modulo closed ideals, *Studia Math.* **85** (1987), 203—227.
19. MEISE, R. and TAYLOR, B. A., Splitting of closed ideals in (DFN)-algebras of entire functions and the property (DN), *Trans. Amer. Math. Soc.* **302** (1987), 321—370.
20. MEISE R. and TAYLOR, B. A., Opérateurs linéaires continus d'extension pour les fonctions ultradifférentiables sur des intervalles compacts, *C.R. Acad. Sci. Paris Sér. I Math.* **302** (1986), 219—222.
21. MEISE, R. and TAYLOR, B. A., Linear extension operators for ultradifferentiable functions on compact sets, to appear in *Amer. J. Math.* (1988).
22. MITYAGIN, B. S., An indefinitely differentiable function with the values of its derivatives given at a point, *Soviet Math. Dokl.* **2** (1961), 594—597, translated from *Dokl. Akad. Nauk SSSR* **138** (1961), 289—292.
23. MITYAGIN, B. S., Approximate dimension and bases in nuclear spaces, *Russian Math. Surveys* **16** (1961), 59—127, translated from *Uspekhi Mat. Nauk* **16** (1961), 63—132.
24. PETZSCHE, H.-J., On E. Borel's theorem, preprint, (1987).
25. SCHAEFER, H. H., *Topological vector spaces,* Springer, 1971.
26. TAYLOR, B. A., Analytically uniform spaces of infinitely differentiable functions, *Comm. Pure. Appl. Math.* **24** (1971), 39—51.
27. TIDTEN, M., Fortsetzung von C^∞-Funktionen, welche auf einer abgeschlossenen Menge in $\mathbf{R}^n$ definiert sind, *Manuscripta Math.* **27** (1979), 291—312.
28. VOGT, D., Charakterisierung der Unterräume von s, *Math. Z.* **155** (1977), 109—117.
29. VOGT, D., Sequence space representations of spaces of test functions and distributions, pp. 405—433, *Functional analysis, holomorphy and approximation theory,* G. Zapata (Ed.), Marcel Dekker, Lect. Notes in Pure and Appl. Math. **83** (1983).
30. WAHDE, G., Interpolation in non-quasi-analytic classes of infinitely differentiable function *Math. Scand.* **20** (1967), 19—31.

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