The geometry of complete linear maps

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Introduction

During the previous century an amazing amount of knowledge was accumulated about complete quadrics, collineations and correlations and a profound insight into their geometric and enumerative properties was acquired. A considerable effort has been made in our century to construct parameter spaces for these complete objects and to explain the results of the previous century in terms of the geometry and intersection theory of these parameter spaces. Severi [13], [14], van der Waerden [19] and Semple [10] studied such spaces for complete conics and Semple [11], [12] and Alguneid [1] extended the results to quadrics in dimension 3 and 4 respectively. Moreover, Semple [11] studied parameter spaces for complete collineations in dimension 2 and 3.

The fundamental ideas for the construction of parameter spaces in arbitrary dimension were suggested by Semple [11] and performed by Tyrrell [15]. Their work is important, not only because they show how to construct spaces whose points are complete quadrics, collineations or correlations, but also because it suggests several different approaches to the construction of such spaces. During the last decade several of these suggestions have been followed by various authors. De Concini and Procesi [2], Demazure [3], Kleiman and Thorup [7] and Vainsencher [17], [18] all have different approaches to the subject and Finat [4] and Uzava [16] give variations on the same methods. The first part of the following work represents an addition to this literature. We do, however, take a different point of view from the above authors. While these start out by constructing a space which is a more or less likely candidate for a parameter space and then (at best) prove that the points of the space can be interpreted as complete objects in the classical sense, we start at the opposite end by generalizing the classical concepts of complete collineations and correlations to families of complete linear maps. We then define maps between such objects and show that there exists a unique complete linear map which is an attractor for such maps. This complete linear map is then a natural candidate for a parameter space in the functorial sense.

The definition of (families of) complete linear maps is one of the main contributions of the first part of the present work. A second main contribution is the construction of the characteristic maps associated to a complete linear map. These characteristic maps are the main tool in defining maps between complete objects. Our definition of families of complete linear maps is designed to extend the methods of Semple and Tyrrell from fields to arbitrary commutative rings (with unity). As a result we obtain a treatment that is based upon (multi-) linear algebra over commutative rings and which extends to arbitrary base schemes.

Two previous versions of the first part of this work has circulated during the last three years. In both we start with the definition of a category of complete linear maps and then solve the problem of constructing a final object in this category in an as coordinate free way as possible. This approach is in many ways more elegant and is sketched in the last section below. We have, however, chosen a more "concrete" presentation here because this is technically simpler.

As has long been realized, the methods that can be used to study complete collineations and correlations can also be used to study complete quadrics. The methods of this work confirms that observation.

For the history of complete quadrics, collineations and correlations of the previous century we refer to Zeuthens article [20]. An exposition and history of complete conics can be found in Kleiman [5] and Kleiman's article [6] contains many historical remarks and references to works on completed objects. In [8] we gave a sketch of the development of complete correlations and collineations and announced the results of the present work and in [9] we described the works mentioned above on completed objects in this century and gave some historical comments.

§ 1. Definitions, notations, examples

In this section we shall give the definition of complete linear maps over an arbitrary base scheme S and give their diagonal representation. Such a representation was obtained by J. A. Tyrrell [15] in the case of quadrics over a field and our definition of complete maps is motivated by the desire to have a similar representation over arbitrary rings.

Let E and F be vector bundles over a scheme S, of ranks $r+1$ respectively $s+1$ with $r \leqq s$. Moreover, let T be an S-scheme and

$$\alpha\colon E_T \to F_T \otimes L$$

a T-linear map, where L is a line bundle on T and G_T denotes the pull-back to T

of a bundle G on S. For each integer $i=0, 1, \ldots, r$ we denote by $I(i, \alpha)$ the determinant ideal which is the image of the map

$$(1.1) \qquad \bigwedge^{i+1} E_T \otimes \bigwedge^{i+1} (F_T \otimes L)^* \to \mathcal{O}_T$$

obtained from the $(i+1)'$st exterior power of α. Here, as in the following, G^* denotes the dual of a bundle G.

It is convenient to fix an open subset S_0 over which E and F are free and to fix bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$ of $E|S_0$ respectively $F|S_0$. We denote by $E(i)$ and $F(i)$ the subbundle of $E|S_0$ generated by $e_0, e_1, \ldots, e_i$ respectively the canonical quotient bundle of $F|S_0$ generated by $f_0, f_1, \ldots, f_i$. Moreover, we write

$$e(i_0, i_1, \ldots, i_j) = e_{i_0} \wedge e_{i_1} \wedge \ldots \wedge e_{i_j}$$

and

$$f(i_0, i_1, \ldots, i_j) = f_{i_0} \wedge f_{i_1} \wedge \ldots \wedge f_{i_j}.$$

We shall choose as bases for $\bigwedge^{k+1} E|S_0$ and $\bigwedge^{k+1} F|S_0$ the elements $e(i_0, i_1, \ldots, i_k)$ respectively $f(i_0, i_1, \ldots, i_k)$ with $0 \leqq i_0 < i_1 < \ldots < i_k \leqq r$ and we shall consider these bases as ordered in the lexicographical ordering.

Throughout we shall consider the elements of $E|S_0$ and $F|S_0$ as row vectors and consequently write the image of a vector e by a map corresponding to a matrix A with respect to some base as eA.

Lemma 1. *Let t be an integer such that $0 \leqq t \leqq r$ and assume that $I(0, \alpha) = \mathcal{O}_T$ and the ideals in the sequence*

$$(1.2) \qquad I(r-t, \alpha) \subseteqq \ldots \subseteqq I(1, \alpha) \subseteqq I(0, \alpha) = \mathcal{O}_T$$

are invertible.

(i) *Let V be an open subset of T which maps to S_0 and is such that the maps*

$$(1.3) \qquad \bigwedge^{i+1} E(i)_V \otimes \bigwedge^{i+1} F(i)_V^* \to I(i, \alpha) \otimes L^{\otimes(i+1)}$$

induced by the maps (1.1) *are surjective for $i=0, 1, \ldots, r-t$. Then choosing the trivialization*

$$\tau\colon \mathcal{O}_V \cdot (e_0 \otimes f_0^*) \to L$$

of L, given by (1.3) *for $i=0$, we have that the matrix $M(\alpha)$ which represents*

$$E_V \xrightarrow{\alpha_V} (F \otimes L)_V^* \xrightarrow{id \cdot \tau^*} F_V$$

with respect to the given bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$ can be written as a *product* $A\cdot D\cdot B$ *of matrices of the following form*

$$(M_A)\qquad A(a_{i,j})=\begin{bmatrix}1 & 0 & & & \cdots & 0\\ a_{1,0} & 1 & & & & \\ & & \ddots & & & \\ & & & 1 & & \vdots\\ \vdots & & & a_{r-t+1,r-t} & 1 & \\ & & & & 0 & \ddots & \\ & & & \vdots & \vdots & & 0\\ a_{r,0} & & & a_{r,r-t} & 0 & \cdots 0 & 1\end{bmatrix},$$

$$(M_B)\qquad B(b_{i,j})=\begin{bmatrix}1 & b_{0,1} & & \cdots & & b_{0,s}\\ 0 & 1 & & & & \\ & & \ddots & & & \vdots\\ \vdots & & 1 & b_{r-t,r-t+1} & \cdots & b_{r-t,s}\\ & & & 1 & 0 \;\cdots & 0\\ & & & & \ddots & \vdots\\ & & & & & 0\\ 0 & & \cdots & & 0 & 1\end{bmatrix},$$

$$(M_D)\qquad D(d_{i,j})=\begin{bmatrix}1 & 0 & & & \cdots & & 0\\ \vdots & d_1 & & & & & \\ & & d_1d_2 & & & & \vdots\\ & & & \ddots & & & \\ \vdots & & & & d_1d_2\dots d_{r-t} & 0 & \cdots 0\\ & & & & 0 & d'_{1,1} & \cdots d'_{1,s-r+t}\\ & & & & \vdots & \vdots & \vdots\\ 0 & \cdots & & & 0 & d'_{t,1} & \cdots d'_{t,s-r+t}\end{bmatrix}$$

with $a_{i,j}$ *and* $b_{i,j}$ *in* $\Gamma(V,\mathcal{O}_V)$, *where* d_i*'s are non-zero divisors in* $\Gamma(V,\mathcal{O}_V)$ *and* $d'_{i,j}=d_1\cdot d_2\dots d_{r-t}\cdot d_{i,j}$ *with* $d_{i,j}\in\Gamma(V,\mathcal{O}_V)$.

Moreover, written in the above form we have that $I(r-t+1,\alpha)_V$ *is generated by the determinants* $d_1^{r-t}\cdot d_2^{r-t-1}\dots d_{r-t}\cdot d'_{i,j}$ *of the* $(r-t+2)\times(r-t+2)$*-submatrices of* $M(\alpha)$ *containing the* $(r-t+1)\times(r-t+1)$*-submatrix in the upper left corner.*

(ii) *Let* x *be a point of* T *that maps to* S_0. *Then we can find a neighbourhood* V *of* x *in* T *which maps into* S_0 *and such that, after possibly renumbering the elements of the bases* $e_0, e_1, \dots, e_r$ *and* $f_0, f_1, \dots, f_s$, *the maps* (1.3) *are all surjections.*

(iii) *If we have that* $I(r-t+1,\alpha)=0$, *then* $(\ker\alpha)_V$ *is the subbundle of rank* t *generated by* $e_{r-t+1}A^{-1}, e_{r-t+2}A^{-1}, \dots, e_rA^{-1}$ *and* $(\operatorname{im}\alpha)_V$ *is contained in the bundle*

$$(I(r-t,\alpha)\colon \operatorname{im}(\alpha))_V$$

of rank $r-t+1$ *generated by* $f_0B, f_1B, \dots, f_{r-t}B$.

Proof. We shall prove assertions (i) and (ii) simultaneously for $0 \leqq t \leqq r+1$, starting with the trivial case $t=r+1$, and proceeding by descending induction on t. Assume that assertions (i) and (ii) holds for some t with $t>0$. Then by (ii) we can, after possibly renumbering the elements of the bases, obtain that the maps (1.3) are surjective for $i=0, 1, \ldots, r-t$ over some neighbourhood V of x. Denote by $M(\alpha_V)$ the matrix representing α restricted to V in the given bases and with the given trivialization of L. By the second part of (i) we have that $I(r-t+1, \alpha)_V$ is generated by the determinants $d_1^{r-t} d_2^{r-t-1} \ldots d_{r-t} d'_{i,j}$ of the $(r-t+2)\times(r-t+2)$-submatrices of $M(\alpha)$ containing the $(r-t+1)\times(r-t+1)$-submatrix in the upper left corner.

If $I(r-t+1, \alpha)$ is invertible, then after possibly shrinking V to a neighbourhood V' of x, we have that it is generated by one of these determinants and after possibly renumbering $e_{r-t+1}, e_{r-t+2}, \ldots, e_r$ and $f_{r-t+1}, f_{r-t+2}, \ldots, f_s$ we may assume that the generator is the determinant $d_1^{r-t} d_2^{r-t-1} \ldots d_{r-t} d'_{1,1}$ of the $(r-t+2)\times(r-t+2)$-matrix in the upper left corner of $M(\alpha)_{V'}$, expressed with respect to the renumbered bases. We see that with respect to these bases the map (1.3) is then surjective over V' for $i=0, 1, \ldots, r-t+1$ and we have proved assertion (ii).

Moreover, when the assumptions of (i) of the Lemma holds we see that all the coordinates of $M(\alpha)$, that are both in the last t rows and the last $s-r+t$ columns, are divisible by the $(r-t+2, r-t+2)$-coordinate $d_1 d_2 \ldots d_{r-t+1}$ where we write $d_{r-t+1} = d_{1,1}$. Hence, subtracting a multiple of row $(r-t+2)$ of $M(\alpha)$ from the last $(t-1)$-rows we obtain zeroes in column $(r-t+2)$ except in the $(r-t+2)'$nd coordinate. These subtractions correspond to the multiplication of D to the left by a matrix of the form M_A with non-zero coordinates only in the $(r-t+2)'$nd column. Similarly, we can multiply D to the right by a matrix of the form M_B with non-zero coordinates only in the $(r-t+2)'$nd column and obtain zeroes in row $(r-t+2)$ except in the $(r-t+2)'$nd coordinate. Hence, we have the first assertion of part (i) of the Lemma.

To obtain the second assertion we notice that all we have done is to add multiples of row and column $(r-t+2)$ to the last $t-1$ rows respectively columns. Hence, the determinants of the $(r-t+3)\times(r-t+3)$-matrices containing the $(r-t+2)\times(r-t+2)$-matrix in the upper left corner are the same before and after the subtractions, and clearly, after the subtraction, these determinants generate $I(r-t+2, \alpha)_{V'}$.

Assertion (iii) follows from assertion (i). Indeed, with respect to the bases $e_0 \cdot A^{-1}, e_1 \cdot A^{-1}, \ldots, e_r \cdot A^{-1}$ and $f_0 \cdot B, f_1 \cdot B, \ldots, f_s \cdot B$ the map α_V is represented by the matrix D. Hence, $(\ker \alpha)_V$ is generated by $e_{r-t+1} \cdot A^{-1}, e_{r-t+2} \cdot A^{-1}, \ldots, e_r \cdot A^{-1}$ and $(\operatorname{im} \alpha)_V$ is contained in the direct summand

$$\{f \in F_V \mid d_1^{r-t} d_2^{r-t-1} \ldots d_{r-t} f \in (\operatorname{im} \alpha)_V\} = \big(I(r-t, \alpha) \colon (\operatorname{im} \alpha)_V\big)$$

generated by the elements $f_0 \cdot B, f_1 \cdot B, \ldots, f_{r-t} \cdot B$.

We are now ready for the main definition of this work.

Definition. Let t be an integer such that $0 \leqq t \leqq r$ and let $\varrho = (r_1, r_2, \ldots, r_k)$ be a k-tuple of integers satisfying the inequalities

$$r = r_1 > r_2 > \ldots > r_k \geqq t.$$

We shall, as a convention, put $r_{k+1} = t-1$.

A *t-completed T-linear map* α_ϱ between E and F, consists of T-linear maps

$$\alpha_j \colon E_j \to F_j \otimes L_j \quad \text{for} \quad j = 1, 2, \ldots, k,$$

where the L_j are line bundles on T, such that the following three conditions hold:

(i) For $j = 1, 2, \ldots, k$ the ideals in the sequence

$$I(r_j - r_{j+1} - 1, \alpha_j) \subseteqq \ldots \subseteqq I(1, \alpha_j) \subseteqq I(0, \alpha_j) = \mathcal{O}_T$$

are invertible.

(ii) For $j = 1, 2, \ldots, k-1$ we have that

$$I(r_j - r_{j+1}, \alpha_j) = 0.$$

(iii) We have that $E_1 = E_T$ and $F_1 = F_T$ and that

$$E_{j+1} = \ker \alpha_j$$

and

$$F_{j+1} \otimes L_j = F_j \otimes L_j / \big(I(r_j - r_{j+1} - 1, \alpha_j) \colon \operatorname{im} \alpha_j\big) \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

A 0-completed map we call a *complete T-linear map.*

From Lemma 1 (ii) it follows that E_j and F_j are bundles of ranks $r_j + 1$ respectively $s - r + r_j + 1$. We shall call ϱ *the rank* of α_ϱ.

From the definition it follows that E_j is a subbundle of E and that there is a natural map $F \to F_j$ making F_j a quotient of F.

We shall say that two t-completed T-linear maps of rank ϱ given by

$$\alpha_j \colon E_j \to F_j \otimes L_j$$

and

$$\alpha_j' \colon E_j' \to F_j' \otimes L_j' \quad \text{for} \quad j = 1, 2, \ldots, k$$

are (projectively) *equivalent* if there are isomorphisms

$$\gamma_j \colon E_j \to E_j'$$

$$\pi_j \colon F_j \to F_j'$$

$$\delta_j \colon L_j \to L_j'$$

such that the diagrams

$$\begin{array}{ccc} E_j & \xrightarrow{\alpha_j} & F_j\otimes L_j \\ {\scriptstyle \gamma_j}\downarrow & & \downarrow{\scriptstyle \pi_j\otimes\delta_j} \\ E_j' & \xrightarrow{\alpha_j'} & F_j'\otimes L_j' \end{array}$$

commute for $j=1, 2, \ldots, k$.

We note that from the above definition it follows that E_j and E_j' are equal as subbundles of E and that F_j and F_j' are *equivalent quotients* of F, that is, the diagrams

$$\begin{array}{ccc} F & \to & F_j \\ \| & & \downarrow{\scriptstyle \pi_j} \\ F & \to & F_j' \end{array},$$

where the horizontal maps are the natural quotient maps, commute for $j=1, 2, \ldots, k$.

Example 2. (i) *The diagonal form*

Denote by $E'(j)$ and $F'(j)$ the free S_0-bundles generated by $e_{r-j}, e_{r-j+1}, \ldots, e_r$ respectively $f_{r-j}, f_{r-j+1}, \ldots, f_s$.

Let $\varrho=(r_1, r_2, \ldots, r_k)$ be a k-tuple of integers such that $r=r_1>r_2>\ldots>r_k>t$ and let T be an S_0-scheme. Moreover, we let

$$\delta_j\colon E'(r_j)_T \to F'(r_j)_T$$

be the map given with respect to the bases $e_{r-r_j}, e_{r-r_j+1}, \ldots, e_r$ and $f_{r-r_j}, f_{r-r_j+1}, \ldots, f_s$ by the $(r_j+1)\times(r_j+1)$-matrix

$$(M_{j,D}) \qquad D_j(d_i) = \begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & d_{r-r_j+1} & & & & & \vdots \\ & & d_{r-r_j+1}d_{r-r_j+2} & & & & \\ \vdots & & & \ddots & & & 0 \\ & & & & d_{r-r_j+1}d_{r-r_j+2}\cdots d_{r-r_{j+1}-1} & & \\ & & & & & 0 & \\ & & & & & & \ddots \\ 0 & & \cdots & & & & 0 \end{bmatrix}$$

for $j=1, 2, \ldots, k-1$ and for $j=k$ by the $(r_k+1)\times(r_k+1)$-matrix

$$D_k(d_{i,j}) =$$

$$(M_{k,D}) \qquad \begin{bmatrix} 1 & 0 & \cdots & & & & 0 \\ 0 & d_{r-r_k+1} & & & & & \\ \vdots & & d_{r-r_k+1}d_{r-r_k+2} & & & & \vdots \\ & & & \ddots & & & \\ & & & & d_{r-r_k+1}d_{r-r_k+2}\cdots d_{r-t} & & 0 \\ & & & & 0 & d_{1,1}' \cdots & d_{1,s-r+t}' \\ & & & & \vdots & \vdots & \vdots \\ 0 & & \cdots & & 0 & d_{t,1} \cdots & d_{t,s-r+t}' \end{bmatrix}$$

Here $d_1, d_2, \ldots, d_{r-t}$ are non-zero divisors in $\Gamma(T, \mathcal{O}_T)$ and

$$d'_{i,k} = d_{r-r_k+1} d_{r-r_k+2} \ldots d_{r-t} d_{i,k}$$

with $d_{i,j} \in \Gamma(T, \mathcal{O}_T)$.

The maps δ_j clearly give a t-completed T-linear map δ_ϱ of rank ϱ that we say is in *diagonal form* with respect to the given bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$.

(ii) *The diagonal representation*

Let A and B be matrices of the form M_A and M_B of Lemma 1(i). With the notation and assumptions of Example 2(i) we let $E_j = E'(r_j)_T \cdot A^{-1}$ and $F_j = F'(r_j)_T \cdot B$. The maps

$$\alpha_j \colon E_j \to F_j \quad \text{for} \quad j = 1, 2, \ldots, k$$

that are given with respect to the bases $e_{r-r_j} \cdot A^{-1}, e_{r-r_j+1} \cdot A^{-1}, \ldots, e_r \cdot A^{-1}$ and $f_{r-r_j} \cdot B, f_{r-r_j+1} \cdot B, \ldots, f_s \cdot B$ by matrix $D_j(D_{i,j})$ above clearly give a t-completed T-linear map α_ϱ of rank ϱ. We say that the matrices A, B and D_j give a *diagonal representation of* α_ϱ with respect to the given bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$.

(iii) *The "generic" diagonal representation*

A particular case of the diagonal representation is sufficiently important to deserve its own terminology.

Assume that S_0 is affine and equal to $\operatorname{Spec} A$. Let

$$\begin{array}{lll}
x_{i,j} & \text{for} & 0 \leqq j \leqq r-t, \quad 1 \leqq i \leqq r \quad \text{and} \quad j < i, \\
y_{i,j} & \text{for} & 0 \leqq i \leqq r-t, \quad 1 \leqq j \leqq s \quad \text{and} \quad i < j, \\
z_i & \text{for} & i \in \{1, 2, \ldots, r-t\} \setminus \{r-r_2, r-r_3, \ldots, r-r_k\} \\
z_{i,j} & \text{for} & 1 \leqq i \leqq t \quad \text{and} \quad 1 \leqq j \leqq s-r+t
\end{array}$$

be $(r+1)(s+1)-k-1$ independent variables over A. We denote by $B_t(\varrho) = A[x, y, z]$ the polynomial ring over A in these variables and let $CL_t^0(\varrho) = \operatorname{Spec} B_t(\varrho)$. The t-completed $CL_t^0(\varrho)$-linear map of rank ϱ which has the diagonal representation given by $A(x_{i,j})$, $B(y_{i,j})$ and $D_j(z_{i,l})$ we denote by $v_t^0(\varrho)$ and we let

$$v_j^0 \colon E_j \to F_j \quad \text{for} \quad j = 1, 2, \ldots, k$$

be the individual maps that define $v_t^0(\varrho)$.

(iv) *Restriction to open sets*

Let α_ϱ be a t-completed T-linear map of rank ϱ given by maps

$$\alpha_j \colon E_j \to F_j \otimes L_j.$$

Then for each open subset V of T the restricted maps

$$(\alpha_j)_V\colon (E_j)_V \to (F_j \otimes L_j)_V$$

give a t-completed V-linear map that we denote by $(\alpha_\varrho)_V$.

2. The caracteristic maps

Our next task is to define maps between completed linear maps. The main ingredient in our definition is the characteristic maps associated to a completed map. In the particular case that the completed map is of rank (r) the characteristic maps consist simply of multiples of the adjugates of the map α_1 defining the completed map. These adjugates, when α_1 has coefficients in a field, played a central role in the presentation of Tyrrell [15]. It is crucial for our presentation that we are able to construct the characteristic maps over an arbitrary base and for completed maps of any rank.

Construction of the characteristic maps 3. Let α_ϱ be a t-completed T-linear map of rank $\varrho=(r_1, r_2, \ldots, r_k)$ given by maps

$$\alpha_j\colon E_j \to F_j \otimes L_j \quad \text{for} \quad j = 1, 2, \ldots, k.$$

Put

$$I_j(\alpha_\varrho) = I(r-r_2-1, \alpha)\cdot I(r_2-r_3-1, \alpha_2)\ldots I(r_{j-1}-r_j-1, \alpha_{j-1})$$

and

$$L_j(\alpha_\varrho) = L^{r-r_2} \otimes L_2^{r_2-r_3} \otimes \ldots \otimes L_{j-1}^{r_{j-1}-r_j} \quad \text{for} \quad j = 1, 2, \ldots, k-1.$$

We shall construct canonical surjections, called the *characteristic* maps of α_ϱ,

$$\alpha_\varrho(r-i)\colon \bigwedge^{r-i+1} E_T \otimes \bigwedge^{r-i+1} F_T^* \to L(r-i+t) \quad \text{for} \quad i = t, t+1, \ldots, r,$$

where

$$L(r-i+1) = I_j(\alpha) \otimes I(r_j-i, \alpha_j) \otimes L_j(\alpha_\varrho) \otimes L_j^{r_j-i+1},$$

when j is determined by the inequalities $r_{j+1} < i \leqq r_j$.

The construction takes place in four steps. We first recall that whenever we have an exact sequence

$$0 \to P \to Q \to R \to 0$$

of bundles on T with P and Q of rank $p+1$ and $q+1$ respectively, then for each integer i such that $0 \leqq i \leqq p+1$ there is a canonical surjection

$$\bigwedge^{q-i+1} Q \to \bigwedge^{q-p} R \otimes \bigwedge^{p-i+1} P.$$

Similarly for each integer i such that $0\leqq i\leqq q-p$ there is a canonical injection

$$\bigwedge^{p+1} P\otimes \bigwedge^{q-p-i} R \to \bigwedge^{q-i+1} Q.$$

Step 1. Denote by G_j the cokernel of the inclusion $E_{j+1}\subset E_j$. Then if $0\leqq i\leqq r_{j+1}+1$ we obtain a canonical surjection

$$\bigwedge^{r_j-i+1} E_j \to \bigwedge^{r_j-r_{j+1}} G_j\otimes \bigwedge^{r_{j+1}-i+1} E_{j+1}. \tag{2.1}$$

Similarly, if we denote by H_j the kernel of the map $F_j\otimes L_j\to F_{j+1}\otimes L_j$ and if $0\leqq i\leqq r_{j+1}+1$, we obtain a canonical injection

$$\bigwedge^{r_j-r_{j+1}} H_j\otimes \bigwedge^{r_{j+1}-i+1} (F_{j+1}\otimes L_j) \to \bigwedge^{r_j-i+1} (F_j\otimes L_j). \tag{2.2}$$

Step 2. The map

$$\alpha_j\colon E_j\to F_j\otimes L_j$$

factors through a unique map $G_j\to H_j$. Consequently, the resulting map

$$\bigwedge^{r_j-r_{j+1}} E_j \to I(r_j-r_{j+1}-1, \alpha_j)\otimes \bigwedge^{r_j-r_{j+1}} (F_j\otimes L_j)$$

factors through a canonical isomorphism

$$\bigwedge^{r_j-r_{j+1}} G_j \to I(r_j-r_{j+1}-1, \alpha_j)\otimes \bigwedge^{r_j-r_{j+1}} H_j. \tag{2.3}$$

Step 3. Given a positive integer $l\leqq k$. Let

$$G(l) = \bigwedge^{r-r_2} G_1\otimes \bigwedge^{r_2-r_3} G_2\otimes\ldots\otimes \bigwedge^{r_{l-1}-r_l} G_{l-1}$$

and

$$H(l) = \bigwedge^{r-r_2} H_1\otimes \bigwedge^{r_2-r_3} H_2\otimes\ldots\otimes \bigwedge^{r_{l-1}r_l} H_{l-1}.$$

Then, taking the composites of the maps (2.1) and (2.2) for $j=1, 2, \ldots, l-1<k$ and tensoring by the appropriate bundles, we obtain maps

$$\bigwedge^{r-i+1} E_T \to G(l)\otimes \bigwedge^{r_l-i+1} E_l \tag{2.4}$$

respectively

$$H(l)\otimes \bigwedge^{r_l-i+1} F_l \to \bigwedge^{r-i+1} F_T\otimes L_l(\alpha_\varrho). \tag{2.5}$$

Moreover, the tensor product of the maps (2.3) gives a map

$$\alpha_1(l)\colon G(l)\to I_l(\alpha_\varrho)\otimes H(l). \tag{2.6}$$

Step 4. The map

$$\alpha_2(r_l-i)\colon \bigwedge^{r_l-i+1} E_l \to I(r_l-i, \alpha_l)\otimes \bigwedge^{r_l-i+1} (F_l\otimes L_l) \tag{2.7}$$

obtained from the $(r_l-i+1)'$st exterior power of α_l when $r_{l+1}<i\leqq r_l$ gives, together with the maps (2.4), (2.5) and (2.6), a map

$$\alpha_0(r-i)\colon \bigwedge^{r-i+1} E_T \to \bigwedge^{r-i+1} F_T\otimes L(r-i+1)$$

which makes the diagram

$$(2.8)\qquad \begin{array}{ccc} \bigwedge^{r-i+1} E_T & \xrightarrow{\alpha_0(r-i)} & \bigwedge^{r-i+1} F_T\otimes L(r-i+1) \\ \Big\downarrow & & \| \\ & & I_l(\alpha_\varrho)\otimes I(r_l-i,\alpha_l)\otimes \bigwedge^{r-i+1} F_T\otimes L_l(\alpha_\varrho)\otimes L_l^{r_l-i+1} \\ & & \uparrow \\ G(l)\otimes \bigwedge^{r_l-i+1} E_l & \xrightarrow{\alpha_l(l)\otimes\alpha_\varrho(r_l-i)} & I_l(\alpha_\varrho)\otimes H(l)\otimes I(r_l-i,\alpha_l)\otimes \bigwedge^{r_l-i+1} F_l\otimes L_l^{r_l-i+1} \end{array}$$

commutative.

The map $\alpha_\varrho(r-i)$ of Construction 3 is the map associated to $\alpha_0(r-i)$ when $r_{l+1}<i\leqq r_l$. We see that, since the left and right vertical maps are surjective respectively (split) injective, the image of $\alpha_0(r-i)$ is the subbundle $I_l(\alpha_\varrho)\otimes H(l)\otimes I(r_l-i,\alpha_l)\otimes \bigwedge^{r_l-i+1} F_l\oplus L_l^{r_l-i+1}$. In particular, the associated map $\alpha_\varrho(r-i)$ is surjective.

Example 4. (i) The diagonal form. Let δ_ϱ be the t-completed T-linear map given in Example 2(i). We see from (2.6) and (2.7) of the construction that for $r_{j+1}<i\leqq r_j$ the map

$$\bigwedge^{r-i+1} E|S_0 \to \bigwedge^{r-i+1} F^*|S^0$$

corresponding to $\sigma(r-i)$ is given, with respect to the given well ordered bases $e(i_0, i_1, \ldots, i_{r-i})$ and $f(i_0, i_1, \ldots, i_{r-i})$, by the matrix

(2.9)

$$[\bigwedge_0^{r_j-i+1} D_j] = \begin{bmatrix} 1 & 0 & & \cdots & & & 0 \\ 0 & d_{r-i+1} & & & & & \\ \vdots & & d_{r-i+1}d_{r-i+2} & & & & \vdots \\ & & & \ddots & & & \\ & & & & d_{r-i+1}d_{r-i+2}\cdots d_{r-r_{j+1}-1} & & \\ & & & & & \ddots & \\ & & & & & & 0\ldots 0 \\ & & & & & & \vdots \quad \vdots \\ 0 & & & \cdots & & & 0\ldots 0 \end{bmatrix}$$

where $\bigwedge_0^{r_j-i+1} D_j$ is the $(r_j-i+1)'$st exterior power of the matrix D_j of Example 2(i) divided by the common factor $d_{r-r_j+1}^{r_j-i} d_{r-r_j+2}^{r_j-i-1}\ldots d r_{r-i}$ and $[\bigwedge^{r_j-i+1} D_j]$ is the $\binom{r+1}{i}\times\binom{s+1}{s-r+i}$-matrix with $\bigwedge_0^{r_j-i+1} D_j$ in the upper left corner and zeroes elsewhere.

(ii) *The diagonal representation*

Let α_ϱ be the t-completed T-linear map having the diagonal representation of Example 2 (ii). With the notation of Example 4 (i), it is clear from Construction 3, that we have

$$\alpha_\varrho(r-i) = (\overset{r-i+1}{\wedge} A)\cdot\sigma(r-i)(\overset{r-i+1}{\wedge} B) \tag{2.10}$$

for $i=t, t+1, \ldots, r$. Hence, for $r_{j+1}<i\leqq r_j$ the map

$$\overset{r-i+1}{\wedge} E|S_0 \to \overset{r-i+1}{\wedge} F|S_0$$

corresponding to $\alpha_\varrho(r-i)$ is given, with respect to the well ordered bases $e(i_0, i_1, \ldots, i_{r-i})$ and $f(i_0, i_1, \ldots, i_{r-i})$, by the matrix $\overset{r-i+1}{\wedge} A\cdot[\overset{r_j-i+1}{\wedge_0} D_j]\cdot\overset{r-i+1}{\wedge} B$. A short calculation shows that the latter matrix takes the form

$$\begin{bmatrix} 1 & b_{r-i,r-i+1} & b_{r-i,r-i+2} \cdots b_{r-i,s} & \times & \times & \times \\ a_{r-i+1,r-i}, & d_{r-i+1}+a_{r-i+1,r-i}b_{r-i,r-i+1} & \times & \times & & \times \\ a_{r-i+2,r-i} & \times & & & & \\ \vdots & & & & & \\ a_{r,r-i} & \times & & & & \\ \times & & & & & \\ \times & & & & & \\ \times & \times & \times & \times & & \times \end{bmatrix} \tag{2.11}$$

where the crosses indicate elements in $\Gamma(T, \mathcal{O}_T)$.

(iii) *The "generic" diagonal representation*

We keep the notation and assumptions of Example 2 (ii) and (iii) and Example 4 (ii). Let

$$f\colon T \to CL_t^0$$

be the map given on coordinate rings by the A-algebra homomorphism

$$\varphi\colon B_t^0(\varrho) \to \Gamma(T, \mathcal{O}_T)$$

defined by $\varphi(x_{i,j})=a_{i,j}$, $\varphi(y_{i,j})=b_{i,j}$ and $\varphi(z_{i,j})=d_{i,j}$ for all the indices i and j appearing in the definition of Example 2 (iii) and

$$\varphi(z_i) = d_i \quad \text{for} \quad i\in\{1, 2, \ldots, r-t\}\setminus\{r-r_2, r-r_3, \ldots, r-r_t\}$$

$$\varphi(z_i) = 0 \quad \text{for} \quad i\in\{r-r_2, r-r_3, \ldots, r-r_t\}.$$

Then we have that

$$f^* v^0_{(r)}(r-i) = \alpha_\varrho(r-i) \quad \text{for} \quad i = t, t+1, \ldots, r \tag{2.12}$$

and f is the *unique* map $T\to CL_t^0$ that satisfies (2.12). Indeed, from the form (2.10) for $v^0_{(r)}(r-i)$ and $\alpha_\varrho(r-i)$ we see that to prove (2.12) it suffices to show that

$$f^*[\overset{r-i+1}{\wedge_0} D] = [\overset{r_j-i+1}{\wedge} D_0] \quad \text{for} \quad i = t, t+1, \dots, r \quad \text{and} \quad r_{j+1} < i \leqq r_j.$$

However, the latter equations are immediate consequences of the following obvious observation:

If $r_{j+1} < i \leqq r_{j+1}$ then the $\binom{r_j+1}{r_j-i+1} \times \binom{r_j+1}{r_j-i+1}$-matrix in the upper left corner of $\overset{r-i+1}{\wedge_0} D(z_{i,j})$ is the matrix $\overset{r_j-i+1}{\wedge} D_j(z_{i,j})$ and all the other coordinates are divisible by z_{r-r_j}.

The uniqueness of f follows immediately from the form (2.11) of the matrices representing $v^0_{(r)}(r-i)$ and $\alpha_\varrho(r-i)$ and from the above observation. Indeed, it follows from (2.12) that f^* must send the coordinates of $v_{(r)}(r-i)$ written in the form (2.11) to the corresponding coordinates of $\alpha_\varrho(r-i)$ written in the same form. Hence f^* sends the $x_{i,j}$ to $a_{i,j}$ and the $y_{i,j}$ to $b_{i,j}$ for all choices of indexes. Moreover, it sends $z_j + x_{j,j-1}y_{j-1,j}$ to $d_j + a_{j,j-1}b_{j-1,j}$ and hence z_j to d_j for $j \in \{1, 2, \dots, r-t\} \setminus \{r-r_2, r-r_3, \dots, r-r_t\}$. Finally, it follows from the above observations that f^* sends z_j to zero for $i \in \{r-r_2, r-r_3, \dots, r-r_k\}$.

The above map f in the case when T is the space $CL^0_t(\varrho)$ gives a *unique* map

$$i_\varrho\colon CL^0_t(\varrho) \to CL^0_t(r)$$

such that

$$i^*_\varrho v^0_{(r)}(r-i) = v_\varrho(r-i) \quad \text{for} \quad i = t, t+1, \dots, r$$

and we see that i_ϱ is the map that identifies $CL^0_t(\varrho)$ with the affine subbundle of codimension $k-1$ of CL^0_t over $\operatorname{Spec} A$ which is defined by the equations

$$z_i = 0 \quad \text{for} \; i \in \{r-r_2, r-r_3, \dots, r-r_k\}.$$

Moreover, we see that f factors via i_ϱ and a *unique* map

$$g\colon T \to CL^0_t(\varrho)$$

such that

$$g^* v_\varrho(r-i) = \alpha_\varrho(r-i) \quad \text{for} \quad i = t, t+1, \dots, r.$$

We have that

$$g^* v^0_j = \alpha_j \quad \text{for} \quad j = 1, 2, \dots, k.$$

Finally, we see that, since $\varphi(z_i)$ is a non-zero element of $\Gamma(T, \mathcal{O}_T)$ for

$$i \in \{1, 2, \dots, r-t\} \setminus \{r-r_2, r-r_3, \dots, r-r_t\},$$

the map f factors via $CL^0_t(\sigma)$ with $\sigma = (r = s_1, s_2, \dots, s_l)$ if and only if

$$\{s_1, s_2, \dots, s_l\} \subseteqq \{r_1, r_2, \dots, r_k\}.$$

(iv) *Restriction to an open subset*

Let V be an open subset of T and $(\alpha_\varrho)_V$ the restriction of a t-completed T-linear map α_ϱ as in Example 2(iv). Then it follows from Construction 3 that we have

$$\alpha_\varrho(r-i)_V = (\alpha_\varrho)_V(r-i)$$

for $i=t, t+1, \ldots, r$.

(v) *Equivalent maps*

Let α_ϱ and α'_ϱ be two t-completed T-linear maps of rank ϱ that are equivalent. Then it follows from Construction 3 that their characteristic maps $\alpha_\varrho(r-i)$ and $\alpha'_\varrho(r-i)$ are equivalent for $i=t, t+1, \ldots, r$ as surjective maps (that is they have the same kernels).

The converse assertion of that in Example 4 (v) also holds and explains the term characteristic maps. However, the proof of the converse is considerably more difficult and will follow from our next result.

Proposition 5. *Given a sequence $\varrho=(r, r_2, \ldots, r_k)$ of integers such that $r>r_2>\ldots>r_k\geqq t$ and an integer l such that $1\leqq l\leqq k$.*

For $i=r_{l+1}+1, r_{l+1}+2, \ldots, r$ we let

$$\pi(r-i)\colon \Big(\bigwedge^{r-i+1} E\otimes \bigwedge^{r-i+1} F^*\Big)_T \to M(r-i+1)$$

be a surjection onto an invertible bundle $M(r-i+1)$ over an S-scheme T.

Moreover, let $\{U_\gamma\}_{\gamma\in I}$ be a covering of T by open subsets and let $\{\alpha_{\gamma,\varrho}\}_{\gamma\in I}$ be a collection of t-completed U_γ-linear maps of rank ϱ between $E|U_\gamma$ and $F|U_\gamma$.

Assume that for all γ, the restriction of $\pi(r-i)$ to U_γ is equivalent to the characteristic map $\alpha_\gamma(r-i)$ of $\alpha_{\gamma,\varrho}$ for $i=r_{l+1}+1, r_{l+1}+2, \ldots, r$.

Then there is an $(r_{l+1}+1)$-completed T-linear map $\alpha_{\varrho(l)}$ of rank $\varrho(l)=(r, r_2, \ldots, r_l)$ which, when restricted to U_γ in the sense of Example 4 (iv), is equivalent to $\alpha_{\gamma,\varrho(l)}$ and which satisfies the following property:

(*) *The characteristic map $\alpha_{\varrho(l)}(r\pm i)$ of $\alpha_{\varrho(l)}$ is equivalent to the surjection $\pi(r-i)$ for $i=r_{l+1}+1, r_{l+1}+2, \ldots, r$.*

Moreover, if $\alpha'_{\varrho(l)}$ is another $(r_{l+1}+1)$-completed T-linear map that satisfies property (*), then $\alpha'_{\varrho(l)}$ and $\alpha_{\varrho(l)}$ are equivalent as completed linear maps.

Proof. We shall prove the Proposition by induction on l starting by the case $l=1$. For $l=1$ we have that $\pi(0)$ defines a map

$$E\to F\otimes M(1)$$

which, by the assumptions of the Proposition, becomes equivalent to $\alpha_{\gamma,1}$ when restricted to U_γ. Since the properties (i), (ii) and (iii) of the Definition of completed maps hold for α_1 restricted to U_γ for all γ, we have that α_1 is an (r_2+1)-completed T-linear map. It follows from Example 4 (iii) and (iv) that the characteristic map $\alpha_{\varrho(1)}(r-i)$ restricted to U_γ is equivalent to $\alpha_\gamma(r-i)$ for $i=r_2+1, r_2+2, \dots, r$. The same is true for $\pi(r-i)$. Hence the kernels of $\alpha_{\varrho(1)}(r-i)$ and $\pi(r-i)$ are equal and thus these maps are equivalent.

Moreover, let

$$\alpha_1'\colon E' \to F' \otimes L'$$

be another map whose characteristic maps $\alpha'_{\varrho(1)}(r-i)$ are equivalent to $\pi(r-i)$ for $i=r_2+1, r_2+2, \dots, r$. Then $\alpha'(0)$ and $\alpha(0)$ are equivalent and their associated maps α_1' respectively α_1 are then equivalent as completed maps.

Assume that the Proposition holds for $l-1 \geqq 1$. Then we have bundles E_j, F_j and L_j and maps

$$\alpha_j\colon E_j \to F_j \otimes L_j$$

for $j=1, 2, \dots, l-1$ that define an (r_l+1)-completed T-linear map which satisfies property $(*)$ of the Proposition for $i=r_l+1, r_l+2, \dots, r$ and whose restriction to U_γ is equivalent to the completed map defined by $\alpha_{\gamma,1}, \alpha_{\gamma,2}, \dots, \alpha_{\gamma,l-1}$. Moreover, if α_ϱ' is a completed map as in the last part of the Proposition we have that the completed maps given by $\alpha_1, \alpha_2, \dots, \alpha_{l-1}$ and $\alpha_1', \alpha_2', \dots, \alpha_{l-1}'$ are equivalent.

We now define the subbundle E_l of E and the quotient bundle F_l of F by

$$E_l = \ker \alpha_{l-1}$$

and

$$F_l \otimes L_{l-1} = F_{l-1} \otimes L_{l-1} / \big(I(r_{l-1}-r_l+1, \alpha_{l-1})\colon \operatorname{im} \alpha_{l-1}\big).$$

Then E_l and F_l satisfy property (iii) of the Definition of completed maps.

In Construction 3 we have that the bundles $G(l)$, $H(l)$, $L_l(\alpha_\varrho)$ and $I_l(\alpha_\varrho)$ are all determined by $\alpha_1, \alpha_2, \dots, \alpha_{l-1}$ and the same is true for the map $\alpha_1(l)$. Moreover, the equivalence between the completed maps given by $\alpha_1, \alpha_2, \dots, \alpha_{l-1}$ and $\alpha_1', \alpha_2', \dots, \alpha_{l-1}'$ gives canonical isomorphism from the above bundles to the corresponding bundles $G'(l)$, $H'(l)$, $L_l'(\alpha_\varrho')$ and $I_l(\alpha_\varrho')$ constructed from $\alpha_1', \alpha_2', \dots, \alpha_{l-1}'$ and a commutative diagram

$$\begin{tikzcd} G(l) \arrow[r, "\alpha_1(l)"] \arrow[d, "\varphi"] & I_l(\alpha_\varrho') \otimes H(l) \arrow[d, "\psi \otimes v"] \\ G'(l) \arrow[r, "\alpha_1'(l)"] & I_l(\alpha_\varrho') \otimes H'(l) \end{tikzcd} \tag{2.13}$$

where the maps φ, ψ and v are the natural isomorphisms mentioned above.

From these induction assumptions, the above definition of E_l and F_l and property (iii) in the definition of α_ϱ' it immediately follows that there are isomorphisms $\gamma_l\colon E_l \to E_l'$ and $\pi_l\colon F_l \to F_l'$ identifying E_l and E_l' as subbundles of E and making F_l and F_l' equivalent as quotient bundles of F.

We define the invertible sheaf L_l by the equality

$$M(r-r_l+1) = I_l(\alpha_\varrho)\otimes L_l(\alpha_\varrho)\otimes L_l \tag{2.14}$$

and obtain a diagram

$$\begin{tikzcd}
\bigwedge^{r-r_l+1} E_T \arrow[r, "\pi_0(r-r_l)"] \arrow[dd] & \bigwedge^{r-r_l+1} F_T\otimes M(r-r_l+1) \arrow[d, equal] \\
 & \bigwedge^{r-r_l+1} F_T\otimes L_l(\alpha_\varrho)\otimes I_l(\alpha_\varrho)\otimes L_l \\
G(l)\otimes E_l & F_l\otimes L_l\otimes I_l(\alpha_\varrho)\otimes H_l \arrow[u]
\end{tikzcd} \tag{2.15}$$

where the horizontal map is obtained from $\pi(r-r_l)$ and the left and right vertical maps are those obtained from the maps (2.4) respectively (2.5) of the construction. Since $\pi(r-i)$ restricted to U_γ is equivalent to $\alpha_\gamma(r-i)$ by the assumption of the Proposition and since the diagram (2.15) restricted to U_γ is "equivalent" to the top part of diagram (2.8) of Construction 3 for $\alpha_{\gamma,\varrho}$, we have that there is a map

$$\varepsilon_\gamma\colon\ (G(l)\otimes E_l)|U_\gamma \to (F_l\otimes L_l\otimes I_l(\alpha_\varrho)\otimes H_l)|\,U_\gamma$$

such that when (2.14) is restricted to U_γ and completed by ε_γ it is commutative. Moreover, since the left and right horizontal maps of diagram (2.15) are surjective respectively injective we have that the maps ε_γ are unique and glue together into a map

$$\varepsilon\colon\ G(l)\otimes E_l \to F_l\otimes L_l\otimes I_l(\alpha_\varrho)\otimes H_l$$

such that when diagram (2.15) is completed by this map it becomes commutative. The map ε together with the isomorphism $\alpha_1(l)$ define uniquely a map

$$\alpha_l\colon\ E_l \to F_l\otimes L_l$$

such that

$$\varepsilon = \alpha_1(l)\otimes\alpha_l.$$

Moreover, we have that there is a commutative diagram

$$\begin{tikzcd}
E_l\,|\,U_\gamma \arrow[r, "\alpha_l|U_\gamma"] \arrow[d] & (F_l\otimes L_l)\,|\,U_l \arrow[d] \\
E_{\gamma,l} \arrow[r, "\alpha_{\gamma,l}"] & (F_{\gamma,l}\otimes L_{\gamma,l})\,|\,U_\gamma
\end{tikzcd}$$

such that the properties (i) and (ii) of the Definition of completed maps hold for α_l. Hence $\alpha_1, \alpha_2, \ldots, \alpha_l$ define an $(r_{l+1}+1)$-completed T-linear map which, when restricted to U_γ, is equivalent to the completed map defined by the maps $\alpha_{\gamma,1}, \alpha_{\gamma,2}, \ldots, \alpha_{\gamma,l}$.

From the latter property we obtain from Example 4 (iii) and (iv) that the restriction of the associated characteristic map $\alpha_{\varrho(l)}(r-i)$ to U_γ is equivalent to $\alpha_\gamma(r-i)$

and hence to $\pi(r-i)|U_\gamma$ for $i=r_{l+1}+1, r_{l+1}+2, \ldots, r$. Consequently, the maps $\alpha_{\varrho(l)}(r-i)$ and $\pi(r-i)$ have the same kernel over U_γ and hence over T and thus property $(*)$ of the Proposition holds.

Finally, it remains to see that the completed map defined by $\alpha_1, \alpha_2, \ldots, \alpha_l$ is equivalent to that defined by $\alpha_1', \alpha_2', \ldots, \alpha_l'$. To this end we first remark that by Construction 3 we have that

$$L'(r-r_l+1) = I_l(\alpha_\varrho')\otimes L_l'(\alpha_\varrho')\otimes L_l'$$

and that, by assumption, we have that $L'(r-r_l+1)$ and $M(r-r_l+1)$ are isomorphic. Together with the isomorphism mentioned above between $I_l(\alpha_\varrho)$ and $I_l(\alpha_\varrho')$ and between $L_l(\alpha_\varrho)$ and $L_l'(\alpha_\varrho')$, we obtain from the definition (2.14) of L_l an isomorphism $\delta_l\colon L_l\to L_l'$.

We have now defined all the maps in the diagram

$$\begin{tikzcd} E_l \arrow[r, "\alpha_l"] \arrow[d, "\gamma_l"'] & F_l\otimes L_l \arrow[d, "\pi_l\otimes\delta_l"] \\ E_l' \arrow[r, "\alpha_l'"] & F_l'\otimes L_l' \end{tikzcd} \tag{2.16}$$

and it only remains to prove that this diagram commutes. However, from the commutativity of the diagram (2.15) completed with ε and from the commutativity of the diagram

$$\begin{tikzcd} \bigwedge^{r-r_l+1} E_T \arrow[r, "\alpha_0'(r-r_l)"] \arrow[d] & \bigwedge^{r-r_l+1} F_T\otimes L(r-r_l+1) \\ G'(l)\otimes E_l' \arrow[r, "\alpha_1(l)\otimes\alpha_l'"] & I_l(\alpha_\varrho)\otimes H'(l)\otimes F_l'\otimes L_l' \arrow[u] \end{tikzcd}$$

obtained from diagram (2.8) of Construction 3 for α_ϱ', it follows from the way all the above maps were defined that there is a commutative diagram

$$\begin{tikzcd} G(l)\otimes E_l \arrow[r, "\varepsilon"] \arrow[d, "\varphi\otimes\gamma_l"'] & I_l(\alpha_\varrho)\otimes H(l)\otimes F_l\otimes L_l \arrow[d, "\psi\otimes\nu\otimes\pi_l\otimes\delta_l"] \\ G'(l)\otimes E_l' \arrow[r, "\alpha_1'(l)\otimes\alpha_l'"] & I_l(\alpha_\varrho')\otimes H'(l)\otimes F_l'\otimes L_l' \end{tikzcd} \tag{2.17}$$

where φ, ψ and ν are the isomorphisms of diagram (2.13). The commutativity of (2.16) follows from the commutativity of diagrams (2.13) and (2.17) together with the equality $\varepsilon=\alpha_1(l)\otimes\alpha_l$.

Corollary 6. *Given two t-completed T-linear maps α_ϱ and α_ϱ' such that the characteristic maps $\alpha_\varrho(r-i)$ and $\alpha_\varrho'(r-i)$ are equivalent as surjective maps for $i=t, t+1, \ldots, r$. Then α_ϱ and α_ϱ' are equivalent as t-completed T-linear maps.*

Proof. Let $\{U_\gamma\}_{\gamma\in I}$ consist of the single element T and let $\alpha_{\gamma,\varrho}=\alpha_\varrho$. Then α'_ϱ satisfies the assumption of the second part of the Proposition and is consequently equivalent to α_ϱ.

Lemma 7. *Let T be an S-scheme and α_ϱ a t-completed T-linear map of rank $\varrho=(r_1, r_2, \ldots, r_k)$ given by the maps*

$$\alpha_j\colon E_j \to F_j\otimes L_j.$$

(i) *Let V be an open subset of T which maps to S_0 and is such that the maps $\alpha_\varrho(r-i)$ restricted to*

$$(2.18)\qquad \Big(\bigwedge^{r-i+1} E(r-i)\otimes \bigwedge^{r-i+1} F(r-i)^*\Big)_V$$

are surjective for $i=t, t+1, \ldots, r$. We denote by

$$\mu_{r-i+1}\colon \mathcal{O}_V \to L(r-i+1)$$

the resulting trivialization of $L(r-i+1)$ given by the basis $e(0, 1, \ldots, r-i)\otimes f(0, 1, \ldots, r-i)^*$ of the bundle (2.18).

Then from the maps μ_{r-i+1} we obtain trivializations

$$\tau_i\colon \mathcal{O}_V \to L^2$$

such that the maps

$$\beta_j\colon (E_j)_V \to (F_j\otimes L_j)_V \xrightarrow{id\otimes l_i^{-1}} (F_j)_V$$

define a t-completed T-linear map of rank ϱ which has a diagonal representation

$$A(a_{i,j})\cdot D_j(d_{i,j})\cdot B(b_{ij})$$

by matrices of the form M_A, M_B and $M_{j,D}$ of Lemma 1 (i) *and Example* 2 (ii) *with respect to the bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$ as in Example* 2 (ii).

(ii) *Let x be a point of T that maps to S_0. Then we can find a neighbourhood V of x in T which maps to S_0 and such that, after possibly renumbering the elements of the bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$, the maps $\alpha_\varrho(r-i)_V$ restricted to the modules* (2.18) *are surjections for $i=t, t+1, \ldots, r$.*

Proof. We shall prove the Proposition by induction on k starting with $k=0$.

Assume that we have, under the assumptions of (i), chosen trivializations $\tau_j\colon \mathcal{O}_V\to L_j$ of L_j for $j=1, 2, \ldots, l-1<k$ and found matrices of the form A' and B' of the form M_A and M_B of Lemma 1 (i) with $t=r_l-1$ such that $E_j=E'(r_j)A^{-1}$ and $F_j=F'(r_j)B$ for $j=1, 2, \ldots, l$ and such that α_j with respect to the bases $e_{r-r_j}A^{-1}, e_{r-r_j+1}A^{-1}, \ldots, e_rA^{-1}$ and $f_{r-r_j}B, f_{r-r_j+1}B, \ldots, f_sB$ is represented by a matrix D_j as in Example 2 (i). We have trivializations

$$\mu_{r-r_l}\colon \mathcal{O}_V \to L(r-r_l) = I_l(\alpha_\varrho)\otimes L_l(\alpha_\varrho)$$

and

$$\mu_{r-r_l+1}\colon \mathcal{O}_V \to L(r-r_l+1) = L(r-r_l)\otimes L_l.$$

Hence we obtain a trivialization

$$\tau_l\colon \mathcal{O}_V \to L_l.$$

From Construction 3, diagram (2.8) it follows that with the above trivializations the map

$$\bigwedge^{r-i+1} E_V\otimes \bigwedge^{r-i+1} F_V \to L(r-i+1) \xrightarrow{\mu_{r-r_l}^{-1}\otimes id} I(r_l-i,\alpha_l)\otimes L_l^{r_l-i+1}$$

factors via the map

$$\bigwedge^{r_l-i+1} (E_l)_V\otimes \bigwedge^{r_l-i+1} (F_l)_V \to I(r_l-i,\alpha_l)\otimes L_l^{r_l-i+1} \tag{2.19}$$

obtained from α_l. Moreover, for $i=r_{l+1}+1, r_{l+1}+2, \dots, r_l$, the condition that $\alpha(r-i+1)$ restricted to the module (2.18) is surjective is the same as the condition that the map (2.19) restricted to

$$\mathcal{O}_V\big(e(r-r_l, r-r_l+1, \dots, r-i)\otimes f(r-r_l, r-r_l+1, \dots, r-i)^*\big) \tag{2.20}$$

is surjective. We can now use Lemma 1 (i) with $\tau=\tau_l$ to the map α_l to obtain a diagonal representation of this map with respect to the bases $e_{r-r_l}, e_{r-r_l+1}, \dots, e_r$ and $f_{r-r_l}, f_{r-r_l+1}, \dots, f_s$. However, we consider the $(r_l+1)\times(r_l+1)$ and $(s-r+r_l+1)\times(s-r+r_l+1)$-matrices A respectively B of Lemma 1 as $(r+1)\times(r+1)$ and $(s+1)\times(s+1)$-matrices A'' respectively B'' with A respectively B in the lower right corner, $1'$s on the diagonal and the remaining coordinates zero.

It is then clear that the matrices $A'\cdot A''$ and $B''\cdot B'$ give a diagonal representation of the map α_l and part (i) of the Lemma follow by induction.

For part (ii) we can assume that we have found a neighbourhood V such that the $\alpha(r-i)_V$ restricted to (2.18) are surjections for $i=r_l+1, r_l+2, \dots, r_{l-1}$. Then as in part (i) we can find matrices A' and B' that give a diagonal representation for $\alpha_1, \alpha_2, \dots, \alpha_{l-1}$. We can now use Lemma 1 (ii) to α_l so that, after possibly shrinking V and renumbering the elements of the bases $e_{r-r_l}, e_{r-r_l+1}, \dots, e_r$ and $f_{r-r_l}, f_{r-r_l+1}, \dots, f_s$, we have that for $i=r_{l+1}+1, r_{l+1}+2, \dots, r_l$ the map (2.19) restricted to the line bundle (2.20) is surjective. Consequently, as we saw in the proof of part (i) of the Lemma the same is true for $\alpha_\varrho(r-i+1)$ restricted to the line bundle (2.18). Hence, part (ii) follows by induction.

Theorem 8. *Assume that S_0 is affine and equal to* Spec A. *Let T be an S_0-scheme and α_ϱ a t-completed T-linear map of rank $\varrho=(r_1, r_2, \dots, r_k)$ given by the maps*

$$\alpha_j\colon E_j \to F_j\otimes L_j \quad \textit{for} \quad j=1,2,\dots,k.$$

Assume that the maps $\alpha_\varrho(r-i)$ *restricted to*

$$\bigwedge^{r-i+1} E(r-i)\otimes \bigwedge^{r-i+1} F(r-i)^*$$

are surjective for $i=t, t+1, \ldots, r$. *Then there is a unique morphism*

$$f\colon T\to CL_t^0(r)$$

such that

$$f^* v_{(r)}^0(r-i) = \alpha_\varrho(r-i) \quad \textit{for} \quad i = t, t+1, \ldots, r.$$

The morphism f *satisfies the following two properties:*

(i) *If factors, via* i_ϱ *of Example* 4 (iii), *through a map*

$$g\colon T\to CL_t^0(\sigma)$$

with $\sigma=(r=s_1, s_2, \ldots, s_l)$ *if and only if*

$$\{s_1, s_2, \ldots, s_l\} \subseteqq \{r_1, r_2, \ldots, r_k\}.$$

(ii) *With* g *as in* (i) *we have that*

$$g^* v_j^0 = \alpha_j \quad \textit{for} \quad j\in\{i|r_i\in\{s_1, s_2, \ldots, s_l\}\}.$$

Proof. It follows from part (i) of Lemma 7 that after trivializing the bundles $L(r-i+1)$ appropriately we can find a diagonal representation of α_ϱ of the form given in Example 2 (ii). As we saw in Example 4 (iii) there is then a unique map

$$f\colon T\to CL_t^0$$

satisfying all the assertions of the Theorem.

If

$$h\colon T\to CL_t^0$$

is another map such that

$$h^* v_{(r)}^0(r-i) = \alpha_\varrho(r-i) \quad \text{for} \quad i = t, t+1, \ldots, r$$

then it coincides with f with respect to the above trivialization and hence $h=f$.

3. The parameter space for completed linear maps

The schemes $CL_t^0(\sigma)$ of Theorem 9 can be considered as the parameter spaces of complete linear maps that can be trivialized in a particular way with respect to the bases $e_0, e_1, \ldots, e_r$ and $f_0, f_1, \ldots, f_s$. In this section we shall glue together these parameter spaces for different rearrangements of the bases and obtain a parameter space for all complete linear maps. The main tool in the glueing process is the characteristic maps

$$\alpha_\varrho(r-i)\colon (\bigwedge^{r-i+1} E\otimes \bigwedge^{r-i+1} F^*)_T \to L(r-i+1) \quad \text{for} \quad i = t, t+1, \ldots, r$$

that are associated to a t-completed T-linear map α_ϱ. These maps define maps

$$h_i\colon T \to \mathbf{P}(\bigwedge^{r-i+1} E\otimes \bigwedge^{r-i+1} F^*) = P(r-i+1) \quad \text{for} \quad i = t, t+1, \ldots, r$$

such that $\alpha_\varrho(r-i)$ is equivalent to the pull back by h_i of the universal quotient

$$\pi_{r-i}\colon (\bigwedge^{r-i+1} E\otimes \bigwedge^{r-i+1} F^*)_{P(r-i+1)} \to L_{P(r-i+1)}$$

on $P(r-i+1)$. Consequently we obtain a map

$$h\colon T \to P = \prod_{i=t}^{r} P(i)$$

and it is the latter map we shall use to define the parameter space of completed linear maps as a subscheme of P.

It follows from Exercise 2 (v) that two equivalent completed maps α_ϱ and β_ϱ give rise to the same map h. Conversely, if two completed maps α_ϱ and β_ϱ give rise to the same map h then for $i=t, t+1, \ldots, r$ the maps $\alpha_\varrho(r-i)$ and $\beta_\varrho(r-i)$ are equivalent as surjections. Hence it follows from Corollary 6 that α_ϱ and β_ϱ are equivalent as completed maps.

We collect the main results of this article in the following Theorem.

Theorem 9. *Let $\varrho=(r_1, r_2, \ldots, r_k)$ be a k-tuple of integers satisfying the inequalities*

$$r = r_1 > r_2 > \ldots > r_k \geqq t \geqq 0.$$

Then there exists an S-scheme $CL_t(\varrho)$ and a t-completed $CL_t(\varrho)$-linear map v_ϱ given by maps

$$v_j\colon E_j(\varrho) \to F_j(\varrho)\otimes L_j(\varrho) \quad \textit{for} \quad j = 1, 2, \ldots, k,$$

which satisfies the following properties:

(i) *Over an affine open subset $S_0=\operatorname{Spec} A$ of S the scheme $CL_t(\varrho)$ can be covered by open affine subsets of the form $CL_t^0(\varrho)$ of Example* 2 (iii) *and Theorem* 8.

In particular $CL_t(\varrho)$ is an affine bundle over S of relative dimension $(r+1)(s+1)-k$.

(ii) *Let α_ϱ be a t-linear T-complete map of rank ϱ given by the maps*

$$\alpha_j\colon E_j \to E_j\otimes L_j \quad \textit{for} \quad j = 1, 2, \ldots, k.$$

Then there exists a unique map

$$f(\alpha_\varrho)\colon T \to CL_t(r)$$

such that $\alpha_\varrho(r-i)$ and $f(\alpha_\varrho)^* v_\varrho(r-i)$ are equivalent as surjections for $i=t, t+1, \ldots, r$.

(iii) *The map*

$$f(v_\varrho)\colon\ CL_t(\varrho) \to CL_t(r)$$

defined in part (ii), *is a closed immersion which in the affine covering described in part* (i) *coincides with the map i_ϱ of Example* 4 (iii).

In particular we have that $f(v_\varrho)$ is a complete intersection defined by the ideal

$$\prod_{i=2}^{k} I(r-r_i, v_\varrho)\cdot I(r-r_i-1, v_\varrho)^{-2}\cdot I(r-r_i-2),$$

where we let $I(-1, v_\varrho)$ be the structure sheaf.

(iv) *Let $\sigma=(r=s_1, s_2, \ldots, s_l)$. Then $f(\alpha_\varrho)$ factors via $f(v_\sigma)$ if and only if*

$$\{s_1, s_2, \ldots, s_l\} \subseteqq \{r_1, r_2, \ldots, r_k\}. \tag{3.1}$$

(v) *If* (3.1) *holds and*

$$g(\alpha_\varrho)\colon\ T \to CL_t(\delta)$$

is the factorization of (iv), *then for $j\in\{i \mid s_i\in\{r_1, r_2, \ldots, r_k\}\}$ we have that there are isomorphisms γ_j, π_j and δ_j making the diagram*

$$\begin{tikzcd} g(\alpha_\varrho)^*E_j(\delta) \arrow[r, "g(\alpha_\varrho)^*v_j"] \arrow[d, "\gamma_j"'] & g(\alpha_\varrho)^*F_j(\varrho)\otimes g(\alpha_\varrho)^*L_j(\delta) \arrow[d, "\pi_j\otimes\delta_j"] \\ E_j \arrow[r, "\alpha_j"] & F_j\otimes L_j \end{tikzcd}$$

commutative.

Proof. We assume first that S is the affine scheme $S_0=\operatorname{Spec} A$.

Let π, τ be permutations of $(0, 1, \ldots, r)$ respectively $(0, 1, \ldots, s)$. We denote by $CL_t^{\pi,\tau}(\varrho)$ the space that has the "generic" diagonal presentation of Example 2 (iii) with respect to the rearrangement π of $e_0, e_1, \ldots, e_r$ and τ of $f_0, f_1, \ldots, f_r$. In particular $CL_t^{id,id}(\varrho)=CL_t^0(\varrho)$.

Let α_ϱ be a t-complete T-linear map and assume that T is an S_0-scheme. We denote by $T_i(\pi, \tau)$ the open subset of T where the map $\alpha_\varrho(r-i)$ restricted to the subspace of $(\bigwedge^{r-i+1} E\otimes \bigwedge^{r-i+1} F^*)_T$ generated by

$$\big(e(\pi(0), \pi(1), \ldots, \pi(r-i))\big)\otimes f\big(\tau(0), \tau(1), \ldots, \tau(r-i)\big)^* \tag{3.2}$$

is surjective. Then $T_i(\pi, \tau)$ is mapped by h_i into the subset $U_i(\pi, \tau)$ of $P(r-i+1)$ where the universal quotient map π_{r-i} restricted to (3.2) is surjective.

We introduce the following notation

$$T(\pi, \tau) = \bigcap_{i=t}^{r} T_i(\pi, \tau)$$

and

$$U(\pi, \tau) = \prod_{t=1}^{r} U_i(\pi, \tau).$$

Then $U(\pi, \tau)$ is an open subset of P and the maps h_i for $i=t, t+1, \ldots, r$ gives a map

$$h(\pi, \tau)\colon T(\pi, \tau) \to U(\pi, \tau) \subseteqq P.$$

In particular we obtain a map

$$j(\pi, \tau)\colon CL_t^{(\pi, \tau)}(\sigma) \to U(\pi, \tau)$$

for each $\sigma=(r=s_1, s_2, \ldots, s_l)$. If we assume that (3.1) holds we see that since the two latter maps are defined by the quotients $\alpha_\varrho(r-i)$ and $v_\sigma(r-i)$ respectively, we have that

$$h(\pi, \tau) = g(\pi, \tau) \cdot j(\pi, \tau)$$

where

$$g(\pi, \tau)\colon T(\pi, \tau) \to CL_t^{(\pi, \tau)}(\sigma)$$

is the map g of Theorem 9 defined with respect to the arrangement π and τ of the bases $e_0, e_1, \ldots, e_r$ respectively $f_0, f_1, \ldots, f_s$.

From the expressions (2.11) for the maps $v_\sigma(r-i)$ and the fact that $v_\sigma(r-i)$ is the pull-back of π_{r-i} by $j(\pi, \tau)$ followed by the projection of $U(\pi, \tau)$ onto the i'th factor $P(r-i+1)$ it follows that $j(\pi, \tau)$ is a closed immersion.

Let π_1 and τ_1 be two other permutations of $\{0, 1, \ldots, r\}$ respectively $\{0, 1, \ldots, s\}$. From the way in which the maps $h(\pi, \tau)$, $j(\pi, \tau)$ and $g(\pi, \tau)$ above were defined and the uniqueness of the maps into $CL_t^{(\pi, \tau)}(\sigma)$ asserted by Theorem 8 it follows that the following diagram is commutative:

$$\begin{tikzcd}
T(\pi, \tau) \arrow[r, "g(\pi,\tau)"] \arrow[dd, phantom, "\cup\|"] & CL_t^{(\pi, \tau)}(\sigma) \arrow[r, "j(\pi,\tau)"] \arrow[d, phantom, "\cup\|"] & U(\pi, \tau) \arrow[dd, phantom, "\cup\|"] \\
 & C \arrow[dr, "j(\pi,\tau)|C"] \arrow[dd, shift left, "h'"] & \\
T(\eta, \tau) \cap T(\pi_1, \tau_1) \arrow[ur, "g(\pi,\tau)|T'"] \arrow[dr, "g(\pi_1,\tau_1)|T'"'] \arrow[dd, phantom, "\cap\|"] & & U(\pi, \tau) \cap U(\pi_1, \tau_1) \arrow[dd, phantom, "\cap\|"] \\
 & C' \arrow[ur, "j(\pi_1,\tau_1)|C'"'] \arrow[uu, shift left, "h"] \arrow[d, phantom, "\cap\|"] & \\
T(\pi_1, \tau_1) \arrow[r, "g(\pi_1,\tau_1)"'] & CL_t^{(\pi_1, \tau_1)}(\sigma) \arrow[r, "j(\pi_1,\tau_1)"'] & U(\pi_1, \tau_1),
\end{tikzcd}$$

where C and C' are the open subspaces of $CL_t^{(\pi, \tau)}(\sigma)$ respectively $CL_t^{(\pi_1, \tau_1)}(\sigma)$ where the characteristic maps restricted to the spaces generated by the vector $e(\pi(0), \pi(1), \ldots, \pi(r-i)) \otimes f(\pi(0), \pi(1), \ldots, \pi(r-i))^*$ respectively

$$e(\pi_1(0), \pi_1(1), \ldots, \pi_1(r-i)) \otimes f(\tau_1(0), \tau_1(1), \ldots, \tau_1(r-i))^*,$$

are surjective for $i=t, t+1, \ldots, r$, and where h and h' are the unique (inverse) maps whose existence are asserted by Theorem 8.

It follows that the spaces $CL_t^{(\pi, \tau)}(\sigma)$ for all different rearrangements π and τ glue together to a closed subscheme $CL_t(\sigma)$ of the open subset $U = \cup\, U(\pi, \tau)$ of

P, where the union is over all rearrangements π and τ. Hence we have proved the existence of the scheme $CL_t(\sigma)$ that satisfies assertion (i) of the Theorem.

It also follows that the maps $g(\pi, \tau)$ glue together to a map

$$g(\alpha_\varrho)\colon \bigcup_{\pi,\tau} T(\pi,\tau) \to CL_t(\sigma)$$

and from the way maps into the projective space $P=\prod_{i=t}^{r} P(i)$ are defined it follows that this map is the unique map such that there are isomorphisms

$$\sigma(r-i+1)\colon g(\alpha_\varrho)^*(L_{P(r-i+1)}|CL_t(\sigma)) \to L(r-i+1) \quad \text{for} \quad i=t, t+1, \dots, r$$

such that the diagrams

$$\begin{tikzcd}[column sep=huge] (\overset{r-i+1}{\bigwedge} E \otimes \overset{r-i+1}{\bigwedge} F^*)_{\cup T(\pi,\tau)} \arrow[r, "g(\alpha_\varrho)^*(\pi_{r-i+1}|CL_t(\sigma))"] \arrow[dr, "\alpha_\varrho(r-i)"'] & g(\alpha_\varrho)^*(L_{P(r-i+1)}|CL_t(\sigma)) \arrow[d, "\sigma(r-i+1)"] \\ & L(r-i+1) \end{tikzcd}$$

are commutative. However, it follows from Lemma 7 (ii) that $\cup T(\pi,\tau)=T$. Consequently, $g(\alpha_\varrho)$ is defined on T and for $\sigma=(r)$ we get a map

$$f(\alpha_\varrho)\colon T \to CL_t(r),$$

such that $\alpha_\varrho(r-i)$ is equivalent to $f(\alpha_\varrho)^* L_{P(r-i+1)}$.

We see that, to prove assertion (iii) it remains to prove the existence of a universal map v_1 on $CL_t(r)$ such that $L_{P(r-i+1)}|CL_t(\sigma)$ is equivalent to $v_{(r)}(r-i+1)$ for $i=t, t+1, \dots, r$.

We therefore turn to the existence of the maps v_j asserted in the Theorem.

On each local piece $CL_t^{(\pi,\tau)}(\sigma)$ there are completed maps $v_\sigma^{(\pi,\tau)}$ represented by

$$v_j^{(\pi,\tau)}\colon E_j^{(\pi,\tau)} \to F_j^{(\pi,\tau)}$$

as in Example 2 (ii) and by the way the embedding of $CL_t^{(\pi,\tau)}(\sigma)$ in $U(\pi,\tau)$ was defined we have that the characteristic map of $v_\sigma^{(\pi,\tau)}$ is equal to the universal quotient π_{r-i} on $P(r-i+1)$ pulled back to $CL_t^{(\pi,\tau)}(\sigma)$. It follows from the first part of Proposition 5 that there is a t-complete $CL_t(\sigma)$ linear map v_σ of rank σ whose restriction to $CL_t^{(\pi,\tau)}(\sigma)$ is equivalent to the map $v_\sigma^{(\pi,\tau)}$ and whose characteristic map $v_\sigma(r-i)$ is equivalent to the pull-back of π_{r-i} to $CL_t(\sigma)$. Consequently, we have proved the existence of the map v_σ of the Theorem and we see that when $\sigma=(r)$ we have also finished the proof of assertion (ii). Moreover, since we have that the pull-back of π_{r-i} to $CL_t(\sigma)$ and $CL_t(r)$ are equivalent to the characteristic maps $v_\sigma(r-i)$ respectively $v_{(r)}(r-i)$, we see that the map $f(v_\sigma)\colon CL_t(\sigma)\to CL_t(r)$, defined by v_σ, is the inclusion $CL_t(\sigma)\subseteqq CL_t(r)\subseteqq U$ defined by glueing together the maps $j(\pi,\tau)$ above. Hence we have proved the first assertion of part (iii). The second assertion reduces to the easy verification made in Lemma 1 (i) that on the piece $CL_t^0(r)$ the ideal of part (iii) is generated by z_i for $i \in \{r-r_2, r-r_3, \ldots, r-r_k\}$.

Part (iv) follows immediately from the first part of Theorem 9 and the way in which we constructed $f(\alpha_\varrho)$ and $f(v_\sigma)$ from the maps $g(\pi, \tau)$.

From part (iv) it follows that, in order to prove part (v), we may assume that $\varrho=\sigma$. As we have seen above, we then have that $g(\pi, \tau)^* v_\varrho$ and $\alpha_\varrho | T(\pi, \tau)$ are equivalent completed maps. Moreover, we saw above that $g(\alpha_\varrho)^* v_\varrho(r-i)$ and $\alpha_\varrho(r-i)$ are equivalent surjections for $i=t, t+1, \ldots, r$. Hence it follows from Proposition 5 that $g(\alpha_\varrho)^* v_\varrho$ and α_ϱ are equivalent and we have proved part (v) of the Theorem.

Finally we notice that we have only proved our result over an affine subset $S_0=\operatorname{Spec} A$ of S. However, it is clear that all objects considered glue together over S exactly like those connected to P do and consequently that all results hold over any base S.

The parameter spaces that we are really interested in are the schemes $CL_0(r)$. We have throughout retained the additional complication of considering the t-completed maps because it makes it easy to display the connection between our approach and that of Vainsencher [17], [18]. This connection is explained by the following result from which we also, as an additional benefit, obtain that the schemes $CL_0(r)$ are proper over S. The properness of $CL_0(r)$ can however also be proved directly, with slightly less work, using the valuative criterion.

Proposition 10. *Assume that* $t>0$. *With the notation of Example* 2 (iii) *and Theorem* 10 *the following two assertions hold:*

(i) *Let V be the open subset of $CL_{t-1}(r)$ which maps to the open subset $S_0=\operatorname{Spec} A$ of S and where the restriction of $\alpha_\varrho(r-i)$ to*

$$\bigwedge^{r-i+1} E(r-i) \otimes \bigwedge^{r-i+1} F(r-i)^*$$

is surjective for $i=t, t+1, \ldots, r$. Then the canonical map

$$f(v_{t-1,(r)})\colon V \to CL_t^0(r)$$

makes V the monoidal transformation of $CL_t^0(r)$ with center on the subscheme defined by the ideal

$$I(r-t+1, v_{(r)}^0) \cdot I(r-t, v_{(r)}^0)^{-2} \cdot I(r-t-1, v_{(r)}^0) = (z_{i,j}) = (z_{i,j})_{1 \leq i, j \leq t}.$$

The exceptional locus is the subscheme $CL_{t-1}((r, t)) \cap V$ of V and the scheme $CL_{t-1}((r, r_2)) \cap V$ is the strict transform of $CL_t((r, r_2))$ for $r_2=t+1, t+2, \ldots, r-1$.

(ii) *We have that $CL_{t-1}(r)$ is the monoidal transformation of $CL_t(r)$ with center* *on the zeroes of the ideal*

$$I(r-t+1, v_t)\cdot I(r-t, v_t)^{-2}\cdot I(r-t-1, v_t).$$

The exceptional locus is the subscheme $CL_{t-1}(r, t)$ *and the strict transform of* $CL_t((r, r_2))$ *for* $r_2=t+1, t+2, \ldots, r-1$ *is* $CL_{t-1}((r, r_2))$.

Proof. It follows from Lemma 7 (ii) that V is covered by the open affine subsets $V(i, j)$ of V, where the map $v^0_{(r)}(r-t+1)_V$ restricted to the $\mathcal{O}_V$-module generated by the element

$$e_0\wedge e_1\wedge\ldots\wedge e_{r-t}\wedge e_k\otimes f_0\wedge f_1\wedge\ldots\wedge f_{r-t}\wedge f_l$$

is surjective where $r-t+1\leqq i\leqq r$ and $r-t+1\leqq j\leqq s$. We first consider the subset $V(r-t+1, r-t+1)=CL^0_{t-1}$. Both v^0_t and v^0_{t-1} have diagonal representations described in Example 2(iii). The matrix $A_{t-1}(x_{i,j})$ in the representation of v^0_{t-1} can be written as

$$\begin{bmatrix} 1 & 0 & \cdots & & & & 0\\ x_{10} & 1 & \ddots & & & & \\ & & 1 & & & & \vdots\\ \vdots & & x_{r-t+1,r-t} & 1 & \ddots & & \\ & & & 0 & \ddots & & 0\\ & & \vdots & \vdots & & \ddots & \\ x_{r0} & & x_{r,r-t} & 0 & & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & & \cdots & & & 0\\ 0 & \ddots & & & & & \vdots\\ & & 1 & & & & \\ \vdots & & 0 & 1 & & & \\ & & \vdots & x_{r-t+2,r-t+1} & 1 & \ddots & \\ & & & & 0 & \ddots & 0\\ & & & & \vdots & \ddots & \vdots\\ 0 & \cdots & 0 & x_{r,r-t+1} & 0 & \cdots\ 0 & 1 \end{bmatrix}.$$

We denote the left matrix by $A_t(x_{i,j})$ and the right by $A'(x_{i,j})$. Similarly $B(y_{i,j})$ in the representation of v^0_{t-1} can be written $B'(y_{i,j})\cdot B_t(y_{i,j})$. Hence the matrix representing v^0_{t-1} can be written

$$A_t(x_{i,j})A'(x_{i,j})D_{t-1}(z_{i,j})B'(y_{i,j})B_t(y_{i,j}).$$

The $t\times(s-r+t)$-matrix in the upper left corner of $A'(x_{i,j})D(z_{i,j})B'(y_{i,j})$ is

$$\begin{bmatrix} z'_{r-t+1} & y_{r-t+1,r-t+2}z'_{r-t+1}, \ldots, y_{r-t+1,s}z'_{r-t+1}\\ x_{r-t+2,r-t+1}z'_{r-i+1}, & (z'_{i+1,j+1}+x_{i,r-t+1}y_{r-t+1,j})z'_{r-t+1}, \ldots\\ \vdots & \vdots\\ x_{r,r-t+1}z'_{r-t+1} & \end{bmatrix}$$

where $z'_{r-t+1}=z_1\cdot z_2\ldots z_{r-t+1}$.

Comparing with the diagonal representation $A_t(x_{i,j})$, $B_t(x_{i,j})$ and $D(z_{i,j})$ of v^0_t we see that the map of rings

$$\varphi\colon B_t(r) = A[x, y, z] \to B_{t-1}(r)$$

that corresponds to the morphism $CL^0_{t-1}\to CL^0_t$ sends all the $x_{i,j}$, $y_{i,j}$, $z_{i,j}$, z_l

to themselves except the following

$$\varphi(z'_{11}) = z'_{r-t+1}$$
$$\varphi(z'_{i,1}) = x_{r-t+i,r-t+1}z'_{r-t+1} \quad \text{for} \quad i = 2, 3, \dots, t$$
$$\varphi(z'_{1,j}) = y_{r-t+1,r-t+j}z'_{r-t+1} \quad \text{for} \quad j = 2, 3, \dots, s-r+t$$
$$\varphi(z'_{i,j}) = (z'_{i-1,j-1}+x_{r-t+i,r-t+1}y_{r-t+1,r-t+j})z'_{r-t+j} \quad \text{for} \quad i = 2, 3, \dots, t \quad \text{and} \quad j = 2, 3, \dots, s-r+t.$$

We see that

$$\varphi(z_{i,1}/z_{11}) = \varphi(z'_{i,1}/z'_{11}) = x_{r-t+i,r-t+1} \quad \text{for} \quad i = 2, 3, \dots, t$$
$$\varphi(z_{1,j}/z_{11}) = \varphi(z'_{1,j}/z'_{11}) = y_{r-t+1,r-t+j} \quad \text{for} \quad j = 2, 3, \dots, s-r+t$$
$$\varphi(z_{i,j}/z_{11}) = \varphi(z'_{i,j}/z_{11}) = z_{i-1,j-1}+x_{r-t+i,r-t+1}y_{r-t+1,r-t+j} \quad \text{for} \quad i = 2, 3, \dots, t \quad \text{and} \quad j = 2, 3, \dots s-r+t.$$

Consequently we have that

$$B_{t-1}(r) = B_t(r)[z_{i,j}/z_{1,1}] \quad \text{where} \quad i = 1, 2, \dots, t \quad \text{and} \quad j = 1, 2, \dots, s-r+t.$$

Reordering the elements of the bases $e_{r-t+1}, e_{r-t+2}, \dots, e_r$ and $f_{r-t+1}, f_{r-t+2}, \dots, f_s$ we see that the coordinate ring of $V(k, l)$ is

$$B_t(r)[z_{i,j}/z_{k+1,l+1}].$$

Hence we see that the $V(k, l)$ have exactly the coordinate rings of the local pieces of the monoidal transformation of $CL^0_t(r)$ with center on the ideal $(z_{i,j})$, and it is clear that they also fit together on the intersections as the monoidal transformation does. We have proved the first part of assertion (i) of the Proposition.

For the second part of assertion (i) we have only to notice that the exceptional locus in $B_t(r)[z_{i,j}/z_{1,1}]$ is defined by the element $\varphi(z_{1,1})=\varphi(z'_{1,1}/z_1z_2\dots z_{r-t})= z'_{r-t+1}/z_1z_2\dots z_{r-t}=z_{r-t+1}$. Hence, by Example 4 (iii) it is equal to $CL^0_{t-1}(r, t)$. Moreover, we have that $CL^0_{t-1}(r, r_2)$ and $CL^0_t(r, r_2)$ are both irreducible codimension one subschemes of CL^0_{t-1} respectively CL^0_t defined by the equation z_{r-r_2}. Hence we have proved the last part of assertion (i).

Part (ii) of the Proposition follows immediately from part (i).

The purpose of the next result is to give the connection between our approach and that of Tyrrell [15]. He maps the open subset V of $CL_r(r)=\mathbf{P}(\mathrm{Hom}(E, F)^*)= \mathbf{P}(E\otimes F^*)$, consisting of maps α: $E\to F$ of maximal rank, into the product

$$P = \prod^r_{i=0} \mathbf{P}(\bigwedge^{i+1} E\otimes \bigwedge^{i+1} F^*)$$

by the map

$$(\alpha, \bigwedge^2 \alpha, \dots, \bigwedge^{r+1} \alpha).$$

Then he defines $CL_0(r)$ as the closure of the image by this map, and finds a characterization of the points of $CL_0(r)$ in terms of his characteristic maps.

Proposition 11. *The morphism*

$$CL_t(r) \to \prod_{i=0}^{t} \mathbf{P}(\overset{i+1}{\wedge} E \otimes \overset{i+1}{\wedge} F^*)$$

given by the characteristic map

$$(v_t(0), v_t(1), \dots, v_t(r-t))$$

of v_t is a closed immersion.

Proof. Over the open subset $CL_t^0(r)$ of $CL_t(r)$ the map v_t has the diagonal form $A(x_{i,j}) \cdot D(z_{i,j}) \cdot B(y_{i,j})$ described in Example 2 (ii) and (iii). The corresponding matrices representing the maps $v_t^0(r-i)$ take the form (2.11) of Example 4(ii) of § 1 with $j=1$ and the variables as coefficients. From the form of the latter matrices for $i=t, t+1, \dots, r$ we see that the characteristic map of v_t^0 make CL_t^0 a closed subscheme of the open affine subset of $\prod_{i=0}^{t} \mathbf{P}(\overset{i+1}{\wedge} E \otimes \overset{i+1}{\wedge} F^*)$ consisting of matrices with a 1 in the upper left corner. Hence, the morphism of the Proposition is an embedding. However, from Proposition 10 it follows that CL_t is proper over S. Consequently, the immersion is closed.

4. The category of completed linear maps

The following section contains a different point of view from that of the previous section. We shall consider 0-completed maps that we call simply *complete* and we denote by $CL(\varrho)$ and CL the spaces $CL_0(\varrho)$ respectively $CL_0(r)$.

We shall now define a category that we shall call the *category of complete maps*. The *objects* of this category are S-schemes T with an equivalence class of complete T-linear maps. By definition each member of such an equivalence class has the same rank which we call the *rank of the object.*

Let $\alpha=(T, [\alpha_\varrho])$ and $\beta=(U, [\beta_\sigma])$ be objects of the category of ranks $\varrho=(r, r_2, \dots, r_k)$ respectively $\sigma=(r=s, s_2, \dots, s_l)$, where $[\alpha_\varrho]$ and $[\beta_\sigma]$ denotes the classes containing the complete maps α_ϱ respectively β_σ. By a *morphism* from α to β we mean a morphism

$$f\colon T \to U$$

such that the maps $f^*\beta_\sigma(r-i)$ and $\alpha_\varrho(r-i)$ are equivalent as quotients maps from

$$(\overset{r-i+1}{\wedge} E \otimes \overset{r-i+1}{\wedge} F^*)_T$$

for $i=0, 1, \ldots, r$. It follows from Example 4 (v) that the definition of a morphism is independent of the choice of representants from the classes $[\alpha_\varrho]$ and $[\beta_\sigma]$. Moreover, it follows from Theorem 9 (iv) that if such a morphism exists, then we must have that

$$\{s, s_2, \ldots, s_l\} \subseteqq \{r, r_1, \ldots, r_k\}. \tag{4.1}$$

We shall write $\varrho \leqq \sigma$, and say that ϱ is at most equal to σ if the inequality (4.1) holds. With the above terminology the main contents of Theorem 9 is that, in the full subcategory of the category of complete maps whose objects are those of rank at most σ, there is a final object $(CL(\sigma), [v_\sigma])$. By a *final object* of a category we mean an object into which there is a *unique* map from each object in the category.

A more elegant approach to this subject might be to start with the definition of the category of complete maps and then pose and solve the problem of finding a final object in an as coordinate free way as possible. This approach was taken in two previous versions of this article. We have, however, chosen the approach of this version because it is more concrete and hopefully more understandable.

5. An auxiliary geometric construction

We shall in this section prove a result that illustrates the geometry of the space of complete linear maps and that provides the crucial inductive step in our forthcoming treatment of the enumerative theory of such maps.

Let p be an integer such that $0<p\leqq r$. We shall in the following denote by X and Y the grassmannians $G_p(E^*)$ respectively $G_p(F)$ and by Z the product $X\times Y$. The universal sequences on X and Y we shall denote by

$$0 \to Q^* \to E_X^* \to G^* \to 0$$

respectively

$$0 \to R \to F_Y \to H \to 0$$

Moreover, we let $CL=CL(E, F)$ and denote by $C(p)$ the degeneration subscheme $CL(r, r-p)$ of CL. On $C(p)$ there is a canonical inclusion

$$E_2 \to E_{C(p)} \tag{5.1}$$

and a canonical surjection

$$F_{C(p)} \to F_2. \tag{5.2}$$

These define a natural map

$$t\colon C(p) \to G_p(E^*)\times G_p(F)$$

such that the canonical maps $G \to E_X$ and $F_Y \to H$ pull back to (5.1) respectively (5.2). The restriction of the universal map

$$v\colon E_{CL} \to F_{CL} \otimes L$$

on CL restricts to a map

$$E_{C(p)} \to G_1 \to H_1 \otimes L \to F_{C(p)} \otimes L \tag{5.3}$$

on $C(p)$, where $G_1 = E_{C(p)}|E_2$ and $H_1 = \ker(F_{C(p)} \otimes L \to F_2 \otimes L)$. Hence we obtain a map

$$g\colon C(p) \to CL(Q_Z, R_Z)$$

into the complete linear maps between Q_Z and R_Z. We shall denote by $CL(p)$ the space $CL(Q_Z, R_Z)$ and by h the structure map

$$h\colon CL(p) \to Z.$$

The characteristic map $v(p=1)$ on CL defines a map

$$c\colon CL \to P$$

where $P = \mathbf{P}(\overset{p}{\wedge} E \otimes \overset{p}{\wedge} F^*)$ and there is a Segre map

$$s(p)\colon Z \to \mathbf{P},$$

defined by the natural quotient map

$$(\overset{p}{\wedge} E \otimes \overset{p}{\wedge} F^*)_Z \to (\overset{p}{\wedge} Q \otimes \overset{p}{\wedge} R^*)_Z.$$

Together with the inclusion $j\colon C(p) \to CL$ we obtain a diagram

$$\begin{tikzcd}
CL \arrow[r, "c"] & P \\
C(p) \arrow[u, "j"] \arrow[r, "t"] \arrow[dr, "g"'] & Z \arrow[u, "s(p)"'] \\
& CL(p) \arrow[u, "h"']
\end{tikzcd}$$

Proposition 12.

(i) *The above diagram is commutative.*

(ii) *Let* $L_p(i+1)$ *for* $i=0, 1, \ldots, p-1$ *be the universal quotients by the characteristic maps*

$$(\overset{i+1}{\wedge} Q \otimes \overset{i+1}{\wedge} R^*)_{CL(p)} \to L_p(i+1)$$

on $CL(p)$. *Then*

$$g^* L_p(i+1) = L(i+1)|C(p) \quad \textit{for} \quad i = 0, 1, \ldots, p-1.$$

(iii) *Let* $D=CL(G_{CL(p)}, H_{CL(p)}\otimes L_p(p)^{-1})$ *be the complete linear maps between* $G_{CL(p)}$ *and* $H_{CL(p)}\otimes L_p(p)^{-1}$ *over* $CL(p)$. *Then there is a canonical isomorphism*

$$C(p)\to D$$

such that, under this isomorphism, the morphism g *corresponds to the stucture map* g' *of* D *over* $CL(p)$.

(iv) *Under the isomorphism in* (iii) *the universal quotient* M *on* D *by the first characteristic map*

$$(G\otimes H^*\otimes \mathbf{L}_p(p))_D\to M$$

is pulled back to $L(p+1)|C(p)$.

Proof. (i) The map cj is defined by the restriction of the characteristic map

$$\nu(p-1)\colon (\overset{p}{\wedge} E\otimes \overset{p}{\wedge} F^*)_{CL}\to L(p)$$

to $C(p)$, that is by the characteristic map on $C(p)$. On the other hand the map $s(p)t$ is defined by the natural map

$$(\overset{p}{\wedge} E\otimes \overset{p}{\wedge} F^*)_{C(p)}\to \overset{p}{\wedge} G_1\otimes \overset{p}{\wedge}(H_1^*\otimes L). \tag{5.4}$$

By the Construction of the characteristic map on $C(p)$ these two maps are the same. Hence the top square of the diagram commutes. However, the bottom triangle is obviously commutative so we have proved part (i).

(ii) From the factorization (5.3) it follows that, for $i=0,1,\dots,p-1$ we have a commutative diagram

$$\begin{tikzcd}
(\overset{i+1}{\wedge} E\otimes \overset{i+1}{\wedge} F^*)_{C(p)} \arrow[r] \arrow[d] & L(i+1)|C(p) \\
\overset{i+1}{\wedge} G_1\otimes \overset{i+1}{\wedge}(H_1^*\otimes L) \arrow[ur] &
\end{tikzcd}$$

where the slanted map is obtained by exterior product from the map.

$$G_1\to H_1 \tag{5.5}$$

in the same way as $\nu(i)$ is obtained from ν. However, by the definition of g the map (5.5) is the pull-back of the universal map

$$\varepsilon\colon Q_{CL(p)}\to R_{CL(p)}\otimes L_p(1)$$

on $CL(p)$. We therefore have proved assertion (ii).

(iii) The universal map

$$\nu_2\colon E_2\to F_2\otimes L_2$$

on $C(p)$, defines a unique canonical map

$$C(p) \to D$$

of $CL(p)$ schemes such that the universal map

$$\alpha\colon G_D \to (H\otimes L_p(p)^{-1})_D\otimes M$$

pulls back to v_2. Conversely, on D, the pull back by g' of the map

$$E_D \to Q_D \xrightarrow{\varepsilon_D} (R\otimes L_p(1))_D \to (F\otimes L_p(1))_D$$

together with the map α, define a unique canonical map

$$D \to C(p)$$

such that the above two maps are pull-backs of the canonical maps v and v_2 on $C(p)$. By the uniqueness the maps are inverses of each other.

(iv) The pull-back of α by the isomorphism in (iii) is v_2. Hence $L_p(p)_D^{-1}\otimes M$ pulls back to L_2. However, by Construction 3 on $C(p)$, the characteristic map $v(p)'$ on $C(p)$ is given by maps

$$\bigwedge^{p+1} E_{C(p)}\otimes \bigwedge^{p+1} F^*_{C(p)} \to \bigwedge^{p} G_1\otimes\big(\bigwedge^{p} H_1^*\otimes L_1\big)\otimes E_2\otimes F_2^* \to \bigwedge^{p} G_1\otimes \bigwedge^{p} (H_1^*\otimes L)\otimes L_2.$$

Here the right hand side is $L(p+1)|C(p)$ and as we saw in the proof of part (i) we have that

$$\bigwedge^{p} G_1\otimes \bigwedge^{p} (H_1^*\otimes L) \cong L(p)|C(p).$$

Consequently, we have that $L_2\cong(L(p+1)\otimes L(p))^{-1}$. We have proved that M pulls back to

$$L_p(p)_D\otimes L_2 = L_p(p)_D\otimes(L(p)|C(p))\otimes L(p+1)|C(p).$$

However, by part (ii) and (iii) above $L_p(p)_D=(g')^*L_p(p)$ pulls back to $g^*L_p(p)= L(p)|C(p)$ and we have proved part (iv).

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