On a lattice-point problem in hyperbolic space and related questions in spectral theory

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1. Introduction

Let $\mathbf{H}^{N+1}$ denote the $(N+1)$-dimensional hyperbolic space and let $G=\mathrm{Con}\,(N)$ denote the group of isometries of $\mathbf{H}^{N+1}$. We shall take $B^{N+1}=\{\mathbf{x}\in\mathbf{R}^{N+1}\colon \|\mathbf{x}\|<1\}$, where $\|...\|$ denotes the euclidean norm, as a model for $\mathbf{H}^{N+1}$; then G is the group of diffeomorphisms $g\colon B^{N+1}\to B^{N+1}$ so that for all $\mathbf{x},\mathbf{x}'\in B^{N+1}$

$$L(g(\mathbf{x}), g(\mathbf{x}')) = L(\mathbf{x}, \mathbf{x}')$$

where

$$L(\mathbf{x}, \mathbf{x}') = 1+(\|\mathbf{x}-\mathbf{x}'\|^2/(1-\|\mathbf{x}\|^2)(1-\|\mathbf{x}'\|^2)).$$

Let $\Gamma\subset G$ be a discrete subgroup with limit set

$$L(\Gamma)\subset S^N = \{\mathbf{x}\in\mathbf{R}^{N+1}\colon \|\mathbf{x}\| = 1\}.$$

Let $\delta(\Gamma)$ denote the exponent of convergence of Γ. For such groups the basic lattice-point problem consists in determining the asymptotic behaviour of

$$N(X; \mathbf{x}, \mathbf{x}') = \mathrm{Card}\,\{\gamma\in\Gamma\colon L(\mathbf{x}, \gamma\mathbf{x}') \leqq X\}$$

as $X\to\infty$. This function counts the number of "lattice-points" $\gamma(\mathbf{x}')$ (the orbit of $\mathbf{x}'$ under Γ) which lie in a large sphere with centre $\mathbf{x}$, the word "sphere" being used in the sense of hyperbolic geometry. There are further analogous problems where the family of spheres is replaced by other increasing sequences of domains but we shall not be concerned with these here.

There have been several investigations into such questions. The most comprehensive and precise results are those obtained by Lax and Phillips [2] under the assumptions that Γ is geometrically finite and $\delta(\Gamma)>N/2$. To explain their result we need some further concepts. Let for $\mathbf{w}\in B^{N+1}$, $\zeta\in S^N$

$$P(\mathbf{w}, \zeta) = (1-\|\mathbf{w}\|^2)/\|\mathbf{w}-\zeta\|^2$$

be the Poisson kernel. Then, under the assumption that Γ is geometrically finite, there exists a unique probability measure μ supported on $L(\Gamma)$ so that

$$F(\mathbf{w}) = \int P(\mathbf{w}, \zeta)^{\delta(\Gamma)}\, d\mu(\zeta)$$

satisfies

$$F(\gamma(\mathbf{w})) = F(\mathbf{w}) \quad (\gamma\in\Gamma);$$

see [5], [7], [10]. (Such a measure exists even without the assumption that Γ be geometrically finite, but it will not be unique in general.) The function F is an eigenfunction of the Laplace operator on $\mathbf{H}^{N+1}$ with eigenvalue $-\delta(\Gamma)(N-\delta(\Gamma))$. Then, under the assumptions made above, Lax and Phillips show that there exist $c, \eta>0$ so that

$$N(X;\ \mathbf{x}, \mathbf{x}') = c\cdot F(\mathbf{x})F(\mathbf{x}')\cdot X^{\delta(\Gamma)}+O(X^{\delta(\Gamma)-\eta})$$

as $X\to\infty$; in fact they prove an even sharper result.

If we relax the condition that Γ be geometrically finite but continue to assume that $\delta(\Gamma)>N/2$ then one can still prove a positive result. There exists, in fact, a measure τ supported on $]N/2, \delta(\Gamma)]$, and a function $p_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ defined for $s\in]N/2, \delta(\Gamma)]$ and $\mathbf{x}, \mathbf{x}'\in B^{N+1}$ so that for some $\eta>0$

$$N(X;\ \mathbf{x}, \mathbf{x}') = \int p_\Gamma(\mathbf{x}, \mathbf{x}';\ s)\cdot X^s\cdot d\tau(s)+O(X^{\delta(\Gamma)-\eta})$$

as $X\to\infty$. In this the principal term dominates X^a for all $a<\delta(\Gamma)$. The function $s\mapsto p_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ is τ-measurable and is τ-almost everywhere smooth as a function of $\mathbf{x}$ and $\mathbf{x}'$; in either of these two variables it is an eigenfunction of the Laplace operator with eigenvalue $-s(N-s)$. One also has that

$$p_\Gamma(\mathbf{x}, \mathbf{x}'; s)^2 \leqq p_\Gamma(\mathbf{x}, \mathbf{x}; s)p_\Gamma(\mathbf{x}', \mathbf{x}'; s)$$

and that $p_\Gamma(\mathbf{x}, \mathbf{x}; s)>0$ τ-almost everywhere. See [6] for a discussion of the case $N=1$; the general case is analogous. The result of Lax and Phillips discussed above is equivalent to τ having, when Γ is geometrically finite, an atom at $\delta(\Gamma)$ which is isolated and $p_\Gamma(\mathbf{x}, \mathbf{x}'; \delta(\Gamma))=F(\mathbf{x})F(\mathbf{x}')$.

All of these results are based on the spectral theory of the Laplace operator and for this reason use heavily the assumption that $\delta(\Gamma)>N/2$. The objective of this paper is to obtain analogous results when $\delta(\Gamma)\leqq N/2$. We shall prove

Theorem 1. *Suppose Γ is convex cocompact; then there exists $c>0$, so that*

$$N(X; \mathbf{x}, \mathbf{x}') \sim c\cdot F(\mathbf{x})F(\mathbf{x}')\cdot X^{\delta(\Gamma)}$$

as $X\to\infty$.

For the notion of "convex cocompact" see [7] or [10]; it is a geometrical condition stronger than "geometrically finite", and excludes the possibility that Γ has parabolic elements. The assertion of Theorem 1 is almost certainly also valid under the assumption that Γ be geometrically finite but our proof will be based on some results about spectral theory which are not yet available in this generality. Note that in the theorem we have no information about the "error term".

Theorem 1 will be deduced from another theorem which we shall now describe. The Laplace operator Δ possesses a resolvent operator $(-\Delta-s(N-s))^{-1}$ on $L^2(\mathbf{H}^{N+1}, \sigma)$, where σ is the measure derived from the hyperbolic metric, with kernel

$$r(\mathbf{x}, \mathbf{x}'; s) =$$

$$2^{-1-2s}\frac{\pi^{-N/2}\Gamma(s)}{\Gamma(s-N/2+1)} L(\mathbf{x}, \mathbf{x}')^{-s}{}_2F_1(s, s-(N-1)/2; 2s-(N-1); L(\mathbf{x}, \mathbf{x}')^{-1}),$$

where $\mathrm{Re}\,(s)>N/2$ (but we shall use $r(\mathbf{x}, \mathbf{x}'; s)$ without this restriction on s). Since

$$r(\mathbf{x}, \mathbf{x}'; s) \sim 2^{-1-2s}\frac{\pi^{-N/2}\Gamma(s)}{\Gamma(s-N/2+1)} L(\mathbf{x}, \mathbf{x}')^{-s}$$

as $L(\mathbf{x}, \mathbf{x}')\to\infty$ the series

$$r_\Gamma(\mathbf{x}, \mathbf{x}'; s) = \sum\nolimits_{\gamma\in\Gamma} r(\mathbf{x}, \gamma\mathbf{x}'; s)$$

converges absolutely when $\mathrm{Re}\,(s)>\delta(\Gamma)$. It has been proved recently by Mazzeo and Melrose [3] that under the assumption that Γ be convex cocompact $r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ has an analytic continuation as a meromorphic function to the entire plane (see [1] for a simpler treatment of the case $N=1$). Moreover, by a theorem of Sullivan ([10], see [7] Lecture 3, Theorem 1 for a formulation in the language we are using here) $\Gamma(s-N/2+1)r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ has at most a simple pole at $\delta(\Gamma)$. We shall now prove:

Theorem 2. *Suppose that Γ is such that $r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ has an analytic continuation to a neighbourhood of $\{s\in\mathbf{C}\colon \mathrm{Re}\,(s)\geqq\delta(\Gamma)\}$ as a meromorphic function, and that Γ is infinite. Then $\Gamma(s-N/2+1)r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ has a simple pole at $\delta(\Gamma)$ and no further poles on $\{s\in\mathbf{C}\colon \mathrm{Re}\,(s)=\delta(\Gamma)\}$. There exists a probability measure μ supported on $L(\Gamma)$ and a constant $c_1(\Gamma)>0$, so that the residue of $\Gamma(s-N/2+1)\cdot r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ is $c_1(\Gamma)F(\mathbf{x})F(\mathbf{x}')$, where $F(\mathbf{x})=\int P(\mathbf{x}, \zeta)^{\delta(\Gamma)}\,d\mu(\zeta)$.*

Note that μ is, in Sullivan's terminology [10], a conformal density of dimension $\delta(\Gamma)$. The assumption on Γ is, as we noted above, satisfied when Γ is geometrically finite. It seems very likely that it is only satisfied for geometrically finite groups.

We shall now deduce Theorem 1 from Theorem 2. If Γ is geometrically finite it follows from the remarks above and Theorem 2 that $\sum_{\gamma\in\Gamma} L(\mathbf{x}, \gamma\mathbf{x}')^{-s}$ has an analytic continuation to a neighbourhood of $\{s\in\mathbf{C}\colon \mathrm{Re}\,(s)\geqq\delta(\Gamma)\}$ as a meromorphic function with a simple pole at $\delta(\Gamma)$. The residue at this pole is of the form $c_2(\Gamma)F(\mathbf{x})F(\mathbf{x}')$ for a certain constant $c_2(\Gamma)>0$. Theorem 1 follows from this and the Wiener—Ikehara Tauberian theorem.

The proof of Theorem 2 is based on ideas from ergodic theory although these might not be too evident. It was suggested by work of Nicholls [4] on the hyperbolic lattice-point problem, and by that of Ruelle [9] and of Parry and Pollicot [8] on the Ruelle zeta-function and its geometric applications. The question as to whether poles could exist on the abscissa of convergence was posed in [1] and [7].

2. Proof of Theorem 2

As we have already remarked $\Gamma(s-N/2+1)r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ can have at most a simple pole at $\delta(\Gamma)$. As $\delta(\Gamma)$ is the abscissa of convergence of a Dirichlet series with positive terms, it follows from Landau's theorem that $\delta(\Gamma)$ must be a singularity of $\Gamma(s-N/2+1)r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$, and hence $\delta(\Gamma)$ is a pole of order 1.

In particular it follows that Γ is of divergence type. By a theorem of Aaronson and Sullivan (see [11], Theorem 4) it follows that, if μ is a conformal density of dimension $\delta(\Gamma)$, then both μ and the measure on $\{(\zeta, \zeta'): \zeta, \zeta' \in S^N, \zeta \neq \zeta'\}$ given by $\|\zeta-\zeta'\|^{-2\delta(\Gamma)}\, d\mu(\zeta) \cdot d\mu(\zeta')$ are ergodic. The construction of μ given in [5], [7] and [10] now shows that the residue of $\Gamma(s-N/2+1)r_\Gamma(\mathbf{x}, \mathbf{x}'; s)$ has the form stated.

We now have to show that there are no further poles on $(\delta(\Gamma))=\{s \in \mathbf{C}: \operatorname{Re}(s)=\delta(\Gamma)\}$. Suppose that s_0 were such a pole, $\operatorname{Re}(s_0)=\delta(\Gamma)$, $s_0 \neq \delta(\Gamma)$. Since $\delta(\Gamma)$ is a simple pole it follows that s_0 cannot be of order greater than 1, and is therefore again a simple pole. Now consider for a sequence (s_j), $s_j \to s_0$, $\operatorname{Re}(s_j) > \delta(\Gamma)$ the sequence of measures

$$(s_j-s_0) \sum_{\gamma \in \Gamma} L(\mathbf{x}, \gamma\mathbf{x}')^{-s_j}\, \delta_{\gamma(\mathbf{x}')}$$

where δ_u is the Dirac measure supported at u. The total mass of elements of this sequence is bounded above since $\delta(\Gamma)$ is a simple pole. Thus by Helly's theorem there exists a subsequence on which this converges weakly to a measure μ_0. We can apply this measure to the constant function 1 and we see that, because s_0 is a simple pole, μ_0 is non-zero, at least for almost all $\mathbf{x}$, $\mathbf{x}'$. We take, for convenience, $\mathbf{x}'=\mathbf{0}$; the argument of [5], [7], Lecture 2 now shows that

$$\int P(w, \zeta)^{s_0}\, d\mu_0(\zeta)$$

is a Γ-invariant eigenfunction of the Laplace operator with eigenvalue $-s_0(N-s_0)$. In particular, μ_0 is a conformal density of dimension s_0.

The measure μ referred to above was constructed by the same process with s_0 replaced by $\delta(\Gamma)$. Since the elements in the sequence used to construct μ_0 can be dominated by those used to construct μ, we see that μ_0 is absolutely continuous with respect to μ. Thus $\|\zeta-\zeta'\|^{-2s_0}\, d\mu_0(\zeta) \cdot d\mu_0(\zeta')$, which is Γ-invariant, is absolutely continuous with respect to $\|\zeta-\zeta'\|^{-2\delta(\Gamma)}\, d\mu(\zeta) \cdot d\mu(\zeta')$. But this latter measure is ergodic and so, as $\mu_0 \neq 0$ it follows that there is a $c \neq 0$ so that

$$\|\zeta-\zeta'\|^{-2s_0} d\mu_0(\zeta)\, d\mu_0(\zeta') = c\,.\, \|\zeta-\zeta'\|^{-2\delta(\Gamma)} d\mu(\zeta)\, d\mu(\zeta').$$

Let now $\varphi = d\mu_0/d\mu$. Then we have for μ-almost all ζ, ζ'

$$(*) \qquad \varphi(\zeta)\,\varphi(\zeta') = c\,.\, \|\zeta-\zeta'\|^{2(s_0-\delta(\Gamma))}.$$

Note that the right-hand side is continuous on $L(\Gamma)\times L(\Gamma)$-diagonal; it follows that φ can be taken to be continuous and the equation $(*)$ is valid on $L(\Gamma)\times L(\Gamma)$-diagonal. This follows as every non-trivial open subset of $L(\Gamma)\times L(\Gamma)$ has positive measure, this itself being a consequence of the ergodicity of the invariant measure.

We shall derive a contradiction from $(*)$. To do this it is convenient to pass to the upper half-space model of hyperbolic space. We do this so that $O\in \mathbf{R}^N$ and ∞ are the fixed points of a loxodromic element γ of Γ. Let $s_0-\delta(\Gamma)=it$. From $(*)$ and ergodicity it follows that $|\varphi|$ is constant; in particular we may assume that $\varphi(O)\neq 0$. We note also that the form that $(*)$ takes in the upper half-space model is exactly the same as $(*)$ and we shall therefore introduce no new notations. Set $\zeta'=O$; we see that there exists a constant c_1 so that for $\zeta\in L(\Gamma)-\{0\}$, we have

$$\varphi(\zeta) = c_1\cdot \|\zeta\|^{2it}.$$

We substitute this into $(*)$ and obtain

$$(\|\zeta-\zeta'\|^2/\|\zeta\|^2\|\zeta'\|^2)^{it} = c_1^2/c.$$

Let $\gamma(x)=\lambda A(x)$ where $A\in O(N)$ and $\lambda\in \mathbf{R}_+^\times$. Then

$$\frac{\|\gamma(\zeta)-\gamma(\zeta')\|^2}{\|\gamma\zeta\|^2\|\gamma\zeta'\|^2} = \lambda^{-2}\frac{\|\zeta-\zeta'\|^2}{\|\zeta\|^2\|\zeta'\|^2}$$

and it follows that $\lambda^{2it}=1$. Next one has

$$(**) \qquad \frac{\|\gamma^k\zeta-\zeta'\|^2}{\|\gamma^k\zeta\|^2\|\zeta'\|^2} = \lambda^{-2k}(\|\zeta'\|^{-2}-2(\zeta', A^k\zeta)\|\zeta\|^{-2}\|\zeta'\|^{-2}\lambda^k+\|\zeta\|^{-2}\lambda^{2k}).$$

Suppose that $\lambda>1$ and that $k\to-\infty$. Then the only way that $(**)$ can be satisfied for sufficiently negative k is that for such k one has

$$2\cdot(\zeta', A^k\zeta) = \lambda^k\|\zeta'\|^2.$$

Since A^k returns infinitely often to a neighbourhood of the identity in $O(N)$ it follows that $(\zeta',\zeta)=0$. Thus ζ and ζ' are orthogonal. The only assumption that we made on ζ and ζ' was that $\zeta,\zeta'\in L(\Gamma)$, $\zeta\neq\zeta'$, $\zeta,\zeta'\neq 0$, ∞. It would now follow that Card $(L(\Gamma)-\{0\})\leqq N$. This is a contradiction if Γ is non-elementary. This completes the proof of the theorem.

References

1. Colin de Verdiere, Y., Théorie spectrale des surfaces de Riemann d'aire infinie. *Astérisque* **132** (1985), 259—275.
2. Lax, P. and Phillips, R. S., The asymptotic distribution of lattice points in Euclidean and non-Euclidean spaces. *J. Funct. Anal.* **46** (1982), 280—350.
3. Mazzeo, R. R. and Melrose, R. B., Meromorphic extension of the resolvent on complete spaces with asymptotically constant negative curvature. *Preprint,* 1986.
4. Nicholls, P. J., A lattice point problem in hyperbolic space. *Michigan Math. J.* **30** (1982), 273—287.
5. Patterson, S. J., The limit set of a Fuchsian group. *Acta Math.* **136** (1976), 241—273.
6. Patterson, S. J., Spectral theory and Fuchsian groups. *Math. Proc. Cambridge Philos. Soc.* **81** (1977), 59—75.
7. Patterson, S. J., Lectures on measures on limit sets of Kleinian groups. *Analytic and geometric aspects of hyperbolic space, Warwick and Durham, 1984,* Ed. Epstein, D. B. A., Cambridge Univ. Press, 1986.
8. Parry, W. and Pollicot, M., An analogue of the prime number theorem for closed orbits of Axiom A flows. *Ann. of Math.* **118** (1983), 573—591.
9. Ruelle, D., Repellers for real analytic maps. *Ergodic Theory and Dynamical Systems* **2** (1982), 99—107.
10. Sullivan, D., The density at infinity of a discrete group of hyperbolic motions. *Inst. Hautes Études Sci. Publ. Math.* **50** (1979), 419—450.
11. Sullivan, D., Discrete conformal groups and measurable dynamics. *Bull. Amer. Math. Soc.* **6** (1982), 57—73.

Received July 28, 1986

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