On an example of Wermer

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0. Introduction

In [W] J. Wermer constructed a compact set $Y \subset \mathbf{C}^2$ which projects to the circle $\{z: |z|=1/2\}$, i.e., if $\pi: \mathbf{C}^2 \to \mathbf{C}$ via $\pi(z, w)=z$, then $\pi(Y)=\{z: |z|=1/2\}$, which has the property that the polynomially convex hull $\hat{Y}$ of Y projects to the disc $\{z: |z| \leqq 1/2\}=\pi(\hat{Y})$ and such that $\hat{Y}-Y$ contains no analytic variety of positive dimension. We show that by suitably choosing the parameters in Wermer's example, we can construct Y so that $\hat{Y} \cap A$ is polar in A for any analytic variety A. We then discuss the consequences of this result. The author would like to thank Professor B. A. Taylor for many valuable discussions on this subject and Professor J. Siciak for pointing out a previous error in Proposition 1.2.

1. Outline of Wermer's construction

We proceed to sketch the details of Wermer's construction, both for the convenience of the reader and also to make later modifications clearer.

Let $a_1, a_2, \ldots$ denote the points in the disc $\{z: |z|<1/2\}$ whose real and imaginary parts are rational (except for statement (1.3) below, we only use the fact that $\{a_i\}$ is a countable dense set in the disc). Form the algebraic functions $B_i(z)=(z-a_1)\ldots(z-a_{i-1})\sqrt{z-a_i}$ $(i=1, 2, \ldots)$.

Given positive constants $c_1, \ldots, c_n>0$, form the algebraic function

$$g_n(z) = \sum_{i=1}^n c_i B_i(z)$$

and let $\sum(c_1, \ldots, c_n)$ denote the subset of the Riemann surface of g_n lying in $\{z: |z| \leqq 1/2\}$; i.e., $\sum(c_1, \ldots, c_n)=\{(z, w): |z| \leqq 1/2, w=w_i, i=1, \ldots, 2^n\}$ where w_i, $i=1, \ldots, 2^n$ are the 2^n values of g_n at z.

Lemma 1.1. (See Lemma 1, [W].) *There exist sequences $\{c_i\}$ and $\{\varepsilon_i\}$ with* $c_{i+1} \leqq$

$c_i/10$ and a sequence of polynomials $\{p_n\}$ in z and w such that

(1) $\{p_n=0\}\cap\{|z|\leqq\frac{1}{2}\}=\sum(c_1, \ldots, c_n),\ n=1, 2, \ldots$

(2) $\{|p_{n+1}|\leqq\varepsilon_{n+1}\}\cap\{|z|\leqq\frac{1}{2}\}\subset\{|p_n|\leqq\varepsilon_n\}\cap\{|z|\leqq\frac{1}{2}\},\ n=1, 2, \ldots$

(3) *If* $|a|\leqq\frac{1}{2}$ *and* $|p_n(a, w)|\leqq\varepsilon_n$, *then there exists* w_n *with* $p_n(a, w_n)=0$ *and* $|w-w_n|<\frac{1}{n}$, $n=1, 2, \ldots$.

Proof. For $i=1$, set $p_1(z, w)=w^2-c_1^2(z-a_1)$; we can choose, e.g., $c_1=\frac{1}{10}$ and $\varepsilon_1=\frac{1}{4}$ so that (1) and (3) are satisfied. Suppose c_i, ε_i, p_i have been chosen for $i=1, \ldots, n$ so that (1)—(3) hold. Let $w_i(z)$, $i=1, \ldots, 2^n$, denote the roots of $p_n(z, .)=0$ (i.e., the values of g_n at z). For $c\geqq 0$, define the polynomial

$$p_c(z, w) = \prod_{i=1}^{2^n}\left[(w-w_i(z))^2-c^2(B_{n+1}(z))^2\right]$$

so that

$$\{p_c(z, w) = 0\}\cap\{|z| \leqq 1/2\} = \sum(c_1, \ldots, c_n, c).$$

Note that

$$p_c = p_n^2+c^2Q_1+\ldots+(c^2)^{2^n}Q_{2^n} \tag{1.1}$$

where $Q_i=Q_i(z, w)$ do not depend on c. From (1.1) it follows that for c sufficiently small,

$$\{|p_c| < \varepsilon_n^2/2\}\cap\{|z| \leqq \tfrac{1}{2}\} \subset \{|p_n| < \varepsilon_n\}\cap\{|z| \leqq \tfrac{1}{2}\}. \tag{1.2}$$

Fix c such that (1.2) holds and $c<c_n/10$; put $c=c_{n+1}$ and $p_{n+1}=p_c$; then choose $\varepsilon_{n+1}<\min(\varepsilon_n^2/2, (1/n+1)^{2^{n+1}})$ so that (2) and (3) hold for $i=1, 2, \ldots, n+1$ and the proof is complete.

With p_n, ε_n, $n=1, 2, \ldots$ as in Lemma 1.1, set

$$X_n = \{|p_n| \leqq \varepsilon_n\}\cap\{|z| \leqq 1/2\}\quad (n = 1, 2, \ldots)\quad\text{and}\quad X = \bigcap_{n=1}^{\infty} X_n.$$

It is the set X in which we are interested; in Wermer's paper, $X=\hat{Y}$ and $Y=X\cap\{|z|=1/2\}$; the rest of the paper consisted of the proof that

$$X \text{ contains no analytic disc.} \tag{1.3}$$

Note from condition (2) and (3) in Lemma 1.1 it follows that

(1.4) $(z, w)\in X$ if and only if $|z| \leqq 1/2$ and there exists a sequence

$$\{(z, w_n)\}\ \text{ with }\ (z, w_n)\in\sum(c_1, \ldots, c_n)\ \text{ and }\ w_n \to w$$

(Lemma 2 in [W]).

To give an idea of the main result in the next section, we show that under mild assumptions on the parameters ε_n, c_n in the construction, the set X intersects each algebraic variety of a special form in a polar set.

Proposition 1.2. *If* $(\varepsilon_n/(c_n^{2^n}))^{1/m_n}\to 0$ *where* $m_n=(2n-1)2^{n-1}=\deg(p_n)$, *then*

(1) *X is pluripolar (as a subset of $\mathbf{C}^2$);*
(2) *$A\cap X$ is polar in A for each algebraic variety A which can be written as the graph of a polynomial in one variable.*

Proof. To show that X is pluripolar, we use the notion of τ-capacity (see, e.g., [LT]). Let $P_n=\{q_n=\sum_{k=0}^n F_k\colon q_n$ is a polynomial of degree n, $\|F_n\|_B=1\}$ where $q_n=F_n+F_{n-1}+\ldots+F_0$ is the decomposition of q_n into the sum of its homogeneous polynomials F_k of degree k, and $\|F_n\|_B=\sup\{|F_n(z,w)|\colon |z|^2+|w|^2\leqq 1\}$. For a compact set $K\subset\mathbf{C}^2$ set $M_n(K)=\inf\{\|q_n\|_K=\sup\{|q_n(z,w)|\colon (z,w)\in K\}\colon q_n\in P_n\}$ and $\tau(K)=\inf_n M_n(K)^{1/n}=\lim_{n\to\infty}M_n(K)^{1/n}$. The limit in the definition of $\tau(K)$ exists; furthermore, $\tau(K)=0$ precisely when K is pluripolar ([LT]). Thus we must show that $\tau(X)=0$. From the construction,

$$p_n(z,w)=\prod_{i=1}^{2^{n-1}}\left[(w-w_i(z))^2-c_n^2(B_n(z))^2\right]$$

where $w_i(z)$, $i=1,\ldots,2^{n-1}$, are the 2^{n-1} values of $g_{n-1}(z)$. Thus

$$p_n(z,w)=c_n^{2^n}(B_n(z))^{2^n}+\ldots=c_n^{2^n}z^{m_n}+R_n(z,w)$$

where $\deg(R_n)<m_n$. Thus $(1/c_n^{2^n})p_n\in P_{m_n}$; since

$$X_n=\{(z,w)\colon |z|\leqq 1/2,\ |p_n(z,w)|/c_n^{2^n}\leqq \varepsilon_n/c_n^{2^n}\},\ \tau(X_n)\leqq(\varepsilon_n/c_n^{2^n})^{1/m_n}.$$

Since $X\subset X_n$, $\tau(X)\leqq\tau(X_n)$ and the result follows.

Case 1. $A=\{(z,w)\colon w=Q_N(z)=\alpha z^N+Q_{N-1}(z)\}$ where $\deg Q_{N-1}\leqq N-1$. Then

$$A\cap X\subset A\cap X_n=\{(z,Q_N(z))\colon |z|\leqq 1/2,\ |p_n(z,Q_N(z))|\leqq\varepsilon_n\}.$$

Now

$$p_n(z,Q_N(z))=$$

$$\prod_{i=1}^{2^{n-1}}\left[(Q_N(z)-w_i(z))^2-c_n^2(B_n(z))^2\right]=\prod_{i=1}^{2^{n-1}}\left[(\alpha z^N+Q_{N-1}(z)-w_i(z))^2-c_n^2(B_n(z))^2\right].$$

Suppose $2N<2n-1=\deg(B_n^2)$. Then $p_n(z,Q_N(z))=c_n^{2^n}z^{m_n}+R_n(z)$ where $\deg R_n<m_n$. Thus $(1/c_n^{2^n})p_n(z,Q_N(z))$ is a monic polynomial of degree m_n; if we let $\mathrm{Cap}(K)$ denote the logarithmic capacity of a compact set $K\subset\mathbf{C}$, it follows that

$$\mathrm{Cap}\{z\colon (z,Q_N(z))\in X\}\leqq\mathrm{Cap}\{z\colon (z,Q_N(z))\in X_n\}\leqq(\varepsilon_n/c_n^{2^n})^{1/m_n}$$

for $n>N+1/2$ and the result follows.

Case 2.

$$A=\{(z,w)\colon z=Q_N(w)=\alpha w^N+Q_{N-1}(w)\}.$$

If $N\neq 0$,

$$A\cap X\subset A\cap X_n=\{(Q_N(w),w)\colon |Q_N(w)|\leqq 1/2,\ |p_n(Q_N(w),w)|\leqq \varepsilon_n\}.$$

Now

$$p_n(Q_N(w),w)=\prod_{i=1}^{2^{n-1}}[(w-w_i(Q_N(w)))^2-c_n^2B_n(Q_N(w))^2]=c_n^{2^n}(\alpha w^N)^{m_n}+R_n(w)$$

where $\deg R_n<Nm_n$. Thus $(1/c_n^{2^n}\alpha^{m_n})p_n(Q_N(w),w)$ is a monic polynomial of degree Nm_n and we obtain that

$$\operatorname{Cap}\{w\colon (Q_N(w),w)\in X\}\leqq(\varepsilon_n/c_n^{2^n}\alpha^{m_n})^{1/Nm_n}=(1/\alpha^{1/N})[(\varepsilon_n/c_n^{2^n})^{1/m_n}]^{1/N}$$

which tends to 0 as $n\to\infty$. If $N=0$, i.e., if we set $z=c=$constant, then $p_n(c,w)=w^{2^n}+R_n(w)$ with $\deg R_n<2^n$, so that

$$\operatorname{Cap}\{w\colon (c,w)\in X\}\leqq\varepsilon_n^{1/2^n}\leqq\varepsilon_n^{1/m_n}\leqq(\varepsilon_n/c_n^{2^n})^{1/m_n}\to 0.$$

2. The main result

To construct X so that $A\cap X$ is polar in A for all analytic varieties A, not just for algebraic varieties, requires more work. The key ingredient is a modification of Lemma 1.1. We retain the notation from the previous section; however, to avoid confusion, $w_i=w_i(z)$ will always denote one of the 2^n values of g_n at z.

Lemma 2.1. *Given any sequence $\{b_i\}$ with $b_{i+1}\geqq b_i\geqq\ldots\geqq 1$, there exist sequences $\{c_i\}$ and $\{\varepsilon_i\}$, and polynomials $\{p_n\}$ in z and w such that*

(1) $\{p_n=0\}\cap\{|z|\leqq\frac{1}{2}\}=\sum(c_1,\ldots,c_n),\quad n=1,2,\ldots$

(2) $\varepsilon_nA_n\to 0$ *where* $A_n=(2^{n+2})^{2^n}$

(3) $\{|p_{n+1}|\leqq\varepsilon_{n+1}\}\cap\{|z|\leqq 1/2\}\subset\{|p_n|\leqq\varepsilon_n\}\cap\{|z|\leqq 1/2\},\quad n=1,2,\ldots$

(4) $\{|p_{n+1}|\leqq\varepsilon_{n+1}^{b_{n+1}}\}\cap\{|z|\leqq 1/2\}\subset\{|p_n|\leqq\varepsilon_n^{b_n}\}\cap\{|z|\leqq 1/2\},\quad n=1,2,\ldots$

(5) *If* $|a|\leqq 1/2$ *and* $|p_n(a,w)|\leqq\varepsilon_n$, *there is a* w_n *with* $p_n(a,w_n)=0$ *and* $|w-w_n|<\dfrac{1}{n}$.

Proof. For $i=1$, set $p_i(z,w)=w^2-c_1^2(z-a_1)$ and choose, e.g., $c_1=1/10$, $\varepsilon_1=1/4$ so that (1) and (5) hold. Suppose c_i, ε_i, p_i have been chosen for $i=1,\ldots,n$ so that (1), (3), (4) and (5) hold, and, say, $\varepsilon_iA_i<1/i$, $i=1,\ldots,n$. For $c\geqq 0$, form

$$p_c(z,w)=\prod_{i=1}^{2^n}[(w-w_i(z))^2-c^2(B_{n+1}(z))^2]=p_n^2+q_n$$

where $q_n = c^2 Q_1 + \ldots + (c^2)^{2^n} Q_{2^n}$ with Q_i independent of c. We have the following:

(2.1) There exists $M_n > 0$ such that

$$\{|p_c| < \varepsilon_n^2/2\} \cap \{|z| \leqq 1/2\} \subset \Delta_{M_n} = \{|z| \leqq 1/2,\ |w| \leqq M_n\} \quad \text{for all} \quad 0 \leqq c \leqq 1.$$

(2.2) For c sufficiently small,

$$\{|p_c| < \varepsilon_n^2/2\} \cap \{|z| \leqq 1/2\} \subset \{|p_n| \leqq \varepsilon_n\} \cap \{|z| \leqq 1/2\}.$$

Furthermore if we assume that $c_1 + \ldots + c_n \leqq 1$,

(2.3)

$$\left|(w - w_i(z))^2 - c^2 (B_{n+1}(z))^2\right| \leqq 4M_n^2 \quad \text{for} \quad (z, w) \in \Delta_{M_n}, \quad 0 \leqq c \leqq 1, \quad i = 1, \ldots, 2^n.$$

This follows from the fact that $|B_i(z)| \leqq 1$ if $|z| \leqq 1/2$, $i = 1, 2, \ldots$; hence, if $c_1 + \ldots + c_n \leqq 1$, then $|w_i(z)| \leqq 1$ if $|z| \leqq 1/2$, $i = 1, \ldots, 2^n$. Thus from (2.3) it follows that $\|q_n\|_{\Delta M_n} \leqq c^2 \|Q_1\|_{\Delta M_n} + \ldots + (c^2)^{2^n} \|Q_{2^n}\|_{\Delta M_n}$ where $\|Q_i\|_{\Delta M_n}$ depends only on n, M_n ($i = 1, \ldots, 2^n$). Choose $c = c_{n+1}$ sufficiently small so that (2.2) holds and

(2.4) $$\|q_n\|_{\Delta M_n} \leqq (1/2)\, \varepsilon_n^{2b_n}.$$

Set $p_{n+1} = p_{c_{n+1}}$. Note that c_{n+1} depends on $c_1, \ldots, c_n$; $\varepsilon_1, \ldots, \varepsilon_n$; n, M_n, and b_n. Choose ε_{n+1} so that

(2.5) $$\varepsilon_{n+1} < \min\left[\varepsilon_n^2/2,\ \left((n+1) A_{n+1}\right)^{-1}\right].$$

Note that (2.5) implies that (5) holds for $n+1$; conditions (1) and (3) are satisfied for $n+1$ by (2.5) and (2.2); and $\varepsilon_{n+1} A_{n+1} < 1/(n+1)$ by (2.5). It remains to verify (4) for $n+1$.

Suppose (z, w) satisfies $|z| \leqq 1/2$ and $|p_{n+1}(z, w)| \leqq \varepsilon_{n+1}^{b_{n+1}}$. Since $b_{n+1} \geqq 1$, $\{|p_{n+1}| \leqq \varepsilon_{n+1}^{b_{n+1}}\} \cap \{|z| \leqq 1/2\} \subset \{|p_{n+1}| \leqq \varepsilon_{n+1}\} \cap \{|z| \leqq 1/2\} \subset \Delta_{M_n}$ by (2.1) since $\varepsilon_{n+1} < \varepsilon_n^2/2$. Thus, from (2.4),

$$|q_n(z, w)| \leqq (1/2)\, \varepsilon_n^{2b_n}$$

and

$$|p_n(z, w)|^2 \leqq |p_{n+1}(z, w)| + |q_n(z, w)| \leqq \varepsilon_{n+1}^{b_{n+1}} + (1/2)\, \varepsilon_n^{2b_n} \leqq \varepsilon_n^{2b_n}$$

since $\varepsilon_{n+1} < \varepsilon_n^2/2$ and $b_{n+1} \geqq b_n$. Thus (4) holds for $n+1$. We remark that as long as each $c_i \leqq 1$,

(2.6) $$\sup\{|p_n(z, w)| \colon |z| \leqq 1,\ |w| \leqq 2^n\} \leqq A_n,$$

since $|w_i(z)| \leqq 2^{n+1}$ so that $|p_n(z, w)| = \prod_{i=1}^{2^n} |w - w_i(z)| \leqq \prod_{i=1}^{2^n} (2^n + 2^{n+1}) < A_n$.

We next need a version of the two-constant theorem (see, e.g. [A]). Let Ω be a bounded strictly pseudoconvex domain in $\mathbf{C}^n$; $P(\Omega)$ will denote the set of plurisubharmonic functions on Ω.

Definition. For a compact subset $K\subset\Omega$, define the relative extremal function $U_K(z)=U_K(\Omega; z)=\sup\{u(z)\colon u\in P(\Omega),\ u<0 \text{ on } \Omega,\ u\leqq -1 \text{ on } K\}$.

If K is not pluripolar, then the uppersemicontinuous regularization $U_K^*(z)=\overline{\lim}_{\xi\to z} U_K(\xi)$ is a negative plurisubharmonic function in Ω satisfying $\lim_{z\to\partial\Omega} U_K^*(z)=0$ and $U_K^*=-1$ on $\hat{K}_\Omega=\{z\in\Omega\colon u(z)\leqq\sup_{\xi\in K} u(\xi),\ u\in P(\Omega)\}$ except perhaps on a pluripolar set (see, e.g., [BT]). Thus if L is a compact subset of Ω properly containing $\hat{K}_\Omega$, it follows that $\alpha=-\sup_{\xi\in L} U_K^*(\xi)$ satisfies $0<\alpha<1$.

Lemma 2.2. (See [A], Lemma 3.2.) *Let K, L be compact subsets of a strictly pseudoconvex domain Ω with $\hat{K}_\Omega\subsetneqq L$ and K not pluripolar. Then for any holomorphic function f in Ω, $|f(z)|\leqq\|f\|_K^{\alpha}\|f\|_\Omega^{1-\alpha}$ for $z\in L$ where $\alpha=-\sup_{\xi\in L} U_K^*(\xi)$.*

Proof. From the definition of U_K, if $u\in P(\Omega)$ then

$$[u(z)-\sup_{\xi\in\Omega} u(\xi)]/[\sup_{\xi\in\Omega} u(\xi)-\sup_{\xi\in K} u(\xi)]\leqq U_K(z)\leqq U_K^*(z),\quad z\in\Omega.$$

Thus if $z\in L$, $u(z)\leqq\alpha\sup_{\xi\in K} u(\xi)+(1-\alpha)\sup_{\xi\in\Omega} u(\xi)$. Apply the above inequality to $u(z)=\log|f(z)|$.

Remark. In the application of Lemma 2.2 below, we only use the case where Ω is a domain in C; in this case, U_K+1 is the harmonic measure of K relative to Ω.

We now state and prove the main theorem, using the same notation as in Lemma 2.1.

Theorem 2.1. *If $b_n\uparrow+\infty$ and the set $X=\bigcap_{n=1}^{\infty} X_n$ is constructed using the parameters in Lemma 2.1, then for any analytic variety $A\subset\mathbf{C}^2$, $A\cap X$ is polar in A.*

Remark. Each connected component of the set of regular points V^0 of a variety V is a complex manifold; in general we say that a subset S of a p-dimensional analytic variety V is pluripolar in V if $S\cap W_i$ is pluripolar in W_i for each connected component W_i of V^0 of dimension p. Since there are countably many such components W_i and since a countable union of (pluri)-polar sets is (pluri)-polar, we may assume that A is connected and can be written as the graph of a holomorphic function in one variable.

Proof. Case 1. $A=\{(z,f(z))\colon f$ is holomorphic in a neighbourhood of $\{|z|<1\}\}$. There exists $M>0$ such that $|f(z)|\leqq M$ if $|z|\leqq 1$. Let

$$K_n=\{z\colon |z|\leqq 1/2,\ |p_n(z,f(z))|\leqq\varepsilon_n\}\quad\text{and}\quad K_n'=\{z\colon |z|\leqq 1/2,\ |p_n(z,f(z))|\leqq\varepsilon_n^{b_n}\}.$$

If we set $K=\{z\colon (z,f(z))\in X\}$, then by (3) in Lemma 2.1, $K_{n+1}\subset K_n$ and $\cap K_n=K$; also, from (1.4) and (4) in Lemma 2.1, $K_{n+1}'\subset K_n'$ and $\cap K_n'=K$.

We prove the theorem by contradiction. If $A\cap X$ is not polar in A, then the set K is not polar. Thus, by the above paragraph, K_n and K_n' are not polar for all n.

Let $p_n^f(z)=p_n(z,f(z))$. Then p_n^f is holomorphic in $|z|<1$. Apply Lemma 2.2 to p_n^f with K_n', $L=\{z\colon |z|\leqq 1/2\}$, and $\Omega=\{z\colon |z|<1\}$. We obtain

$$\left|p_n(z,f(z))\right| \leqq \|p_n^f\|_\Omega^{1-\alpha_n}\|p_n^f\|_{K_n'}^{\alpha_n}$$

for $|z|\leqq 1/2$ where $\alpha_n=-\sup_{|\xi|\leqq 1/2} U_{K_n'}(\xi)$. Since $K_n'\downarrow K$, $U_{K_n}'\uparrow U_K$ and $\alpha_n\downarrow\alpha=-\sup_{|\xi|\leqq 1/2} U_K(\xi)$ with $0<\alpha<1$ since K is not polar. Thus

$$\left|p_n(z,f(z))\right| \leqq \max\left[1, \sup\{|p_n(z,w)|\colon |z|\leqq 1,\ |w|\leqq M\}\right]\cdot \varepsilon_n^{b_n\alpha_n} \leqq$$
$$\leqq \max\left[1, \sup\{|p_n(z,w)|\colon |z|\leqq 1,\ |w|\leqq M\}\right]\cdot \varepsilon_n^{b_n\alpha} \quad \text{for} \quad |z|\leqq 1/2.$$

For n sufficiently large, $M<2^n$, and we obtain, using (2.6), $\left|p_n(z,f(z))\right|\leqq A_n\varepsilon_n^{b_n\alpha}$ for $|z|\leqq 1/2$. Since $A_n\varepsilon_n\to 0$, $\alpha>0$, and $b_n\uparrow+\infty$, it follows that $b_n\alpha\uparrow+\infty$ so that for n sufficiently large,

$$\text{(2.7)} \qquad \left|p_n(z,f(z))\right| \leqq \varepsilon_n \quad \text{for} \quad |z|\leqq 1/2.$$

However, (2.7) says that

$$\{(z,f(z))\colon |z|\leqq 1/2\} \subset \{(z,w)\colon |z|\leqq 1/2, |p_n(z,w)|\leqq\varepsilon_n\}$$

for n sufficiently large and hence

$$\{(z,f(z))\colon |z|\leqq 1/2\} \subset X.$$

This contradicts (1.3), i.e., X contains no analytic disc.

Case 2. $A=\{(f(w),w)\colon f \text{ holomorphic}\}$. Since X is compact, there exists $M>0$ such that $X\subset\Delta_M=\{(z,w)\colon |z|\leqq 1/2,\ |w|\leqq M\}$. Since we are only interested in the part of A that hits X, we may assume $|f(w)|\leqq 1$ if $|w|\leqq 2M$. Let $L_n=\{w\colon |f(w)|\leqq 1/2, \left|p_n(f(w),w)\right|\leqq\varepsilon_n\}$ and $L_n'=\{w\colon |f(w)|\leqq 1/2, \left|p_n(f(w),w)\right| \leqq\varepsilon_n^{b_n}\}$.

As in Case 1, if we let

$$L=\{w\colon (f(w),w)\in X\},$$

then $L_n\downarrow L$, $L_n'\downarrow L$. Again we proceed by contradiction; assume that L is not polar. Then for each n, L_n, L_n' are not polar. Let $p_n^f(w)=p_n(f(w),w)$ so that p_n^f is holomorphic; apply Lemma 2.2 to p_n^f and L_n', $\bar{D}_M=\{w\colon |w|\leqq M\}$, and $D_{2M}=\{w\colon |w|<2M\}$. We obtain

$$\left|p_n(f(w),w)\right| \leqq \|p_n^f\|_{D_{2M}}^{1-\beta_n}\|p_n^f\|_{L_n'}^{\beta_n}$$

for $|w|\leqq M$ where $\beta_n=-\sup_{|w|\leqq M} U_{L_n'}(w)$ (here, $U_{L_n'}(\cdot)=U_{L_n'}(D_{2M};\,.)$). Note that $\beta_n\downarrow\beta=-\sup_{|w|\leqq M} U_L(w)$ where $0<\beta<1$ since L is not polar. Thus

$$\left|p_n(f(w),w)\right| \leqq \max\left[1, \sup\{|p_n(z,w)|\colon |z|\leqq 1, |w|\leqq 2M\}\right]\cdot\varepsilon_n^{\beta b_n}$$

for $|w|\leqq M$. If $2M<2^n$, we obtain, from (1.6), $\left|p_n(f(w),w)\right|\leqq A_n\varepsilon_n^{\beta b_n}$, $|w|\leqq M$. The rest of the proof proceeds as in Case 1.

3. Applications of main theorem

We now discuss some applications and open problems related to the Wermer example. Let K be a compact set in $\mathbf{C}^n$ and let L denote the class of functions u which are plurisubharmonic in $\mathbf{C}^n$ and satisfy $u(x) \leqq \log |x| + O(1)$, $|x| \to \infty$.

If $n=1$ and K is polar, then there always exists a function $u \in L$, $u \not\equiv -\infty$, with

$$K = \{x\colon u(x) = -\infty\}, \tag{3.1}$$

e.g., the Evans' potential for K (see, e.g. [T]). We say that a set K is complete L-polar if there exists $u \in L$ such that (3.1) holds. In $\mathbf{C}^n$, $n>1$, it is not true that an arbitrary pluripolar compact set K is L-complete; a typical non-example occurs if we let K be a compact subset of an analytic variety A with K not pluripolar as a subset of A. Then any $u \in L$ which is $-\infty$ on K is automatically $-\infty$ on (a component of) A.

Given a compact, pluripolar set $K \subset \mathbf{C}^n$, we call $K_p = \cap \{x\colon u(x) = -\infty\}$, where the intersection is taken over $u \in L$ with $u|_K = -\infty$, the L-polar hull of K. In some sense, K_p should be the smallest complete L-polar set containing K, but due to an example of Gamelin and Sibony [GS], K_p is not necessarily complete L-polar. Clearly if K is complete L-polar, then $K = K_p$; the condition that $K = K_p$ is equivalent to the following: given $x_0 \notin K$, there exists $u \in L$, $u \not\equiv -\infty$, with $u|_K = -\infty$ but $u(x_0) \neq -\infty$.

From the above remarks, we see that a necessary condition for a compact pluripolar set K to satisfy $K = K_p$ is that $A \cap K$ be pluripolar in A for all analytic varieties A. Using Theorem 2.1, it follows that this condition is not sufficient.

Proposition 3.1. *There exists a compact pluripolar set $K \subset \mathbf{C}^2$ with*
(1) *$A \cap K$ is polar in A for all analytic varieties A*
(2) *$K \neq K_p$.*

Proof. With X as in Theorem 2.1, we set $K = \{(z, w) \in X\colon \operatorname{Re} z \geqq 0\}$. Condition (1) follows from Theorem 2.1. To verify (2), we show that if $u \in L$, $u \not\equiv -\infty$, with $u|_K = -\infty$, then $u|_X = -\infty$, i.e., $X \subset K_p$.

For fixed z, let $X(z) = \{w\colon (z, w) \in X\}$. Given $u \in L$, the function $U(z) = \sup \{u(z, w)\colon w \in X(z)\}$ defines a subharmonic function in $|z| \leqq 1/2$ (see [R], Proposition 12.6).

If $u|_K = -\infty$, then $U(z) = -\infty$ for $\operatorname{Re} z \geqq 0$; thus $U(z) = -\infty$ on $|z| \leqq 1/2$; i.e., $u(z, w) = -\infty$ for all $(z, w) \in X$.

In all our calculations thus far, in order to show that a set was "small", i.e., polar or pluripolar, we used estimates from above on the size of certain analytic functions on the set. To show a set is not small seems to be a more difficult problem.

For example, one cannot detect "pluripolarity" via intersection with affine subspaces. Kiselman ([K]) has given an example in $\mathbf{C}^2$ of a non-pluripolar set E with the property that every complex line in $\mathbf{C}^2$ hits E in at most four points; we can take $E=\{(z, w)\in\mathbf{C}^2\colon \operatorname{Im}(z+w^2)=\operatorname{Re}(z+w+w^2)=0\}$. However, if A is the analytic variety $A=\{(z, w)\in\mathbf{C}^2\colon z+w^2=0\}$, then $E\cap A$ forms a nonpolar subset of A.

This leads to the following questions:

Question 1. Let K be a compact set in $\mathbf{C}^n$ and suppose $K\cap A$ is pluripolar in A for all analytic varieties A. Must K be pluripolar (as a subset of $\mathbf{C}^n$)?

Question 2. Can a pluripolar set X be constructed à la Wermer which intersects some variety A in a non-polar set? Such an example would be interesting from the viewpoint of analytic multifunctions (see [R]).

Note that the proof of Theorem 2.1 required much more effort than that of Proposition 1.1.

Question 3. Can a set X be constructed a la Wermer which intersects each algebraic variety in a polar set but intersects some transcendental variety in a non-polar set?

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