Zeros of analytic functions of several variables

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1. Introduction

The aim of this note is to give some estimates for the number of zeros of analytic functions of several variables. In particular we will study functions bounded in tube domains and entire functions of finite order. Special attention will be directed to real zeros or zeros that lie close to the reals. A general reference for the corresponding questions in one variable is Boas [1].

In one variable a basic and trivial fact is that the number of real zeros does not exceed the total number of zeros. In section 2 we give a corresponding statement for p-dimensional analytic varieties in $\mathbf{C}^n$, i.e. we compare the p-dimensional measure of the variety intersected with R^n with the $2p$-dimensional measure of the whole variety. This comparison results from the following theorem which is inspired by a result by Lelong [4].

Theorem. *Let K be a compact and convex subset of $\{z=x+iy\in\mathbf{C}^n;\ y=0\}$. Let V be an analytic variety of pure dimension p defined in a neighbourhood of K in $\mathbf{C}^n$. Let*

$$\mu(t) = |V\cap\{z;\ d(z;K)<t\}|,$$

where $d(z;K)$ is the distance from z to K and $|\cdot|$ means $2p$-dimensional measure. Then $\mu(t)/t^p$ is nondecreasing where it is defined.

In section 3 the above results are used to prove a density result for the number of zeros of analytic functions bounded in a radial tube, and in section 4 corresponding questions for entire functions are studied. In section 5 we indicate how these estimates can be applied to discrete sets of uniqueness, and the result is compared to earlier work by Korevaar and Hellerstein [3], and Ronkin [7, 8, 9].

We will now fix some notation which will be used throughout the note.

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$\mathbf{C}^n$ always denotes complex n-dimensional space and is given the orientation determined by the complex structure. The coordinates $z=(z_1, \dots, z_n)$ are sometimes written $z=x+iy$ where $z_j=x_j+iy_j$. The euclidean norm is defined by $|z|^2=\Sigma |z_j|^2$ and $B(z; r)$ denotes the open ball with center z and radius r. The exterior differentiation operator is decomposed $d=\partial+\bar\partial$ and the Kähler form of $\mathbf{C}^n$ $(i/2)\partial\bar\partial|z|^2$, is denoted β. The expression $\beta\wedge\dots\beta$ (p times) is abbreviated β^p and correspondingly for other forms. The volume element of $\mathbf{C}^n$, $dV=\beta^n/n!$, will frequently be used to identify forms of degree 0 and $2n$. I.e. we will write (abusively) $\varphi dV=\varphi$ for functions φ.

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2. A local estimate

Let V be an analytic variety of pure (complex) dimension p (see [2]), defined in a domain in $\mathbf{C}^n$. It was proved by Lelong that V defines a natural current of integration, θ_V, which moreover is closed and positive (see [4]). This current is of bidegree $(n-p; n-p)$ and its trace measure, $\tau_V=\theta_V\wedge\beta^p/p!$, is equal to surface measure on the manifold of regular points of V.

Now let K be a compact and convex subset of R^n which we identify with $\{x+iy\in\mathbf{C}^n; y=0\}$. Put $\Omega(t)=\{z\in\mathbf{C}^n; d(z; K)<t\}$ where $d(z; K)$ is the distance from z to K. Assume θ is a closed positive current of bidegree $(n-p; n-p)$, defined in a neighbourhood of K, and put

$$\mu(t) = \int_{\Omega(t)} \theta\wedge\beta^p/p!$$

We will prove.

Theorem 2.1. *$\mu(t)t^{-p}$ is a nondecreasing function where it is defined.*

Remark. In particular, when $\theta=\theta_V$ is the current of integration on an analytic variety, V, we get estimates for the area of V. It was proved by Lelong that if K is a point, $\mu(t)t^{-2p}$ is increasing (see [4] which is also a general reference for positive currents). However, in general the exponent in Theorem 2.1 can not be increased. Note also that the assumption $K\subseteq R^n$ is essential.

We will need two lemmas.

Lemma 2.2. *Let Ω be an open subset of R^N, defined by $\Omega=\{x; \varrho(x)<0\}$, where ϱ is a piecewise C^1-function with nonvanishing gradient on $\partial\Omega$. Let $\partial\Omega$ be given the* *standard orientation of Stokes theorem, and ω be its suface element. Then, if Σ is the current of integration on $\partial\Omega$ we have $\Sigma = d\varrho|\nabla\varrho|^{-1}\omega$.*

Proof. We only need to check that if

$$\varphi = \psi\, dx_1 \wedge \dots dx_i \wedge \dots dx_N = \psi\, d\hat{x}_i \quad \text{with} \quad \psi \in C_0^\infty, \quad \text{then}$$

(i) $$\int_{\partial\Omega} \varphi = \int_{\partial\Omega} (d\varrho \wedge \varphi)|\nabla\varrho|^{-1}\omega$$

(where we have identified 0-forms and N-forms). But

$$(d\varrho \wedge \varphi)|\nabla\varrho|^{-1}\omega = \sum \frac{\partial\varrho}{\partial x_j} dx_j \wedge \varphi|\nabla\varrho|^{-1}\omega = (-1)^{i-1}\psi\frac{\partial\varrho}{\partial x_i}|\nabla\varrho|^{-1}\omega.$$

Hence (i) says that

$$\int_{\partial\Omega} \psi\, d\hat{x}_i = \int_{\partial\Omega} (-1)^{i-1}\psi n_i \omega,$$

where n_i is the $i:th$ component of the unit outer normal vector field of $\partial\Omega$. This is an elementary formula which can be found e.g. in [13].

Lemma 2.3. *Let θ be a positive form of bidegree $(n-p; n-p)$, and ϱ a C^1-function. Then the measure*

$$m = \theta \wedge (|\nabla\varrho|^2\beta^p - 2ip\,\partial\varrho \wedge \bar{\partial}\varrho \wedge \beta^{p-1})$$

is positive.

Proof. Cauchys inequality shows that $\gamma = |\nabla\varrho|^2\beta - 2i\ \partial\varrho \wedge \bar{\partial}\varrho$ is a positive form. Since $(\partial\varrho \wedge \bar{\partial}\varrho)^2 = 0$ and forms of even degree commute the binomial theorem gives

$$\gamma^p = |\nabla\varrho|^{2p}\beta^p - |\nabla\varrho|^{2p-2}2ip\,\partial\varrho \wedge \bar{\partial}\varrho \wedge \beta^{p-1}.$$

Hence $m = \theta \wedge \gamma^p|\nabla\varrho|^{2-2p}$ (where $\nabla\varrho \neq 0$) and since the product of a positive form and a positive (1, 1)-form is positive, the lemma follows.

Proof of Theorem 2.1. First assume θ is smooth and K a polygon. Put $k(x) = d(x; K)^2$ for $x \in R^n$ and $\varrho(z) = d(z; K)^2 = k(x) + |y|^2$ for $z \in \mathbf{C}^n$. Since K is convex k is a convex, piecewise C^1 function. Put $\Omega = \{z; \varrho(z) < t^2\}$. Stokes formula and Lemma 2.2 give

(i) $$\int_\Omega \theta \wedge \partial\bar{\partial}\varrho \wedge \beta^{p-1} = \int_{\partial\Omega} \theta \wedge \bar{\partial}\varrho \wedge \beta^{p-1} = \int_{\partial\Omega} (\theta \wedge \partial\varrho \wedge \bar{\partial}\varrho \wedge \beta^{p-1})|\nabla\varrho|^{-1}\omega.$$

Since k is convex $i\partial\bar{\partial}k$ is a positive current. Hence $i\partial\bar{\partial}\varrho \geqq i\partial\bar{\partial}|y|^2 = i/2\partial\bar{\partial}|z|^2 = \beta$. Thus (i) together with Lemma 2.3 gives

(ii) $$\int_\Omega \theta \wedge \beta^p \leqq \int_{\partial\Omega} (2p)^{-1}|\nabla\varrho|\theta \wedge \beta^p\omega.$$

Let $\tau=\theta\wedge\beta^p(p!)^{-1}$. Obviously $|\nabla d(\cdot\,;K)|\leqq 1$, hence $|\nabla\varrho|\leqq 2t$ on $\partial\Omega$, so (ii) implies

(iii)
$$\int_\Omega \tau \leqq t/p\int_{\partial\Omega}\tau\omega.$$

Now the right hand side of (iii) is precisely $\mu(t)$ and it is readily seen that

$$\int_{\partial\Omega}\tau\omega \leqq \mu'(t).$$

So (iii) gives the differential inequality $\mu(t)\leqq t/p\,\mu'(t)$ which says that $\mu(t)t^{-p}$ is nondecreasing. This is proved under the assumption that θ is smooth and K a polygon, but approximation gives the general statement.

Next I will study the limit of $\mu(t)t^{-p}$ as t tends to 0. I will assume that $\theta=\theta_V$ is the current of integration on a variety, so that the trace measure is the $2p$-dimensional area of the variety.

Theorem 2.4. *Let K be a compact and convex subset of R^n and V an analytic variety of pure dimension p, defined in a neighbourhood of K. Define μ as in Theorem 2.1. Then*

$$\lim_{t\to 0}\mu(t)t^{-p} \geqq cH_p(V\cap K)$$

where c is a positive constant and H_p is p-dimensional Hausdorff-measure.

Proof. Take $\varepsilon>0$, and select a maximal family of disjoint balls $B(x_j,\varepsilon)$, $j=1,\dots,N_\varepsilon$, with radii ε and centers $x_j\in V\cap K$. By maximality, $V\cap K\subseteqq \cup B(x_j,2\varepsilon)$. By a result of Lelong ([4]), the area of $V\cap B(x_j;\varepsilon)\geqq\sigma_{2p}\varepsilon^{2p}$, where σ_{2p} is the volume of the $2p$-dimensional unit ball. Hence $\mu(\varepsilon)\geqq N_\varepsilon\sigma_{2p}\varepsilon^{2p}$. Letting $\varepsilon\to 0$ we get by the definition of Hausdorff-measure

$$\liminf_{\varepsilon\to 0}\sigma_p N_\varepsilon(2\varepsilon)^p \geqq H_p(K\cap V).$$

Hence

$$\lim_{\varepsilon\to 0}\mu(\varepsilon)\varepsilon^{-p} \geqq \sigma_{2p}\sigma_p^{-1}2^{-p}H_p(K\cap V),$$

which proves the theorem.

Theorems 2.1 and 2.4 imply the

Corollary 2.5. *Let K and V be as above. Then there is a constant c such that for all $\varepsilon>0$,*

$$H_p(K\cap V) \leqq cH_{2p}(\{z;\, d(z;K)<\varepsilon\}\cap V)\varepsilon^{-p}.$$

3. Zeros of bounded functions

Throughout this section K will denote an open cone in R^n with vertex at the origin. Then the tube with base K is defined by $T(K)=\{z\in \mathbf{C}^n; x\in K\}$. We are primarily interested in estimating zero sets of bounded analytic functions in $T(K)$. However, the discussion is more naturally carried out in the setting of plurisubharmonic functions and there is no extra difficulty in assuming only a weaker growth condition.

Definition 3.1. *$P(K)$ is the class of functions u which are plurisubharmonic in $T(K)$ and satisfy*

(i) $u(z) \leqq A|z|+B$ *for some constants A, B and*

(ii) $\limsup_{x\to 0} u(x+iy) \leqq 0$ *for all* $y\in R^n$.

We recall briefly the situation in one variable. Then $T(K)$ is a halfplane, say the right hand one. $P(K)$ consists of subharmonic functions which by the Phragmén—Lindelöf principle satisfy the condition $u(x+iy)\leqq Ax$. Let μ be the Riesz measure (i.e. the Laplacian) of u. Then μ satisfies

(iii) $\int_{|z|>1} \frac{x}{|z|^2}\, d\mu(z) < \infty$ and consequently

(iv) $\lim_{\substack{r\to\infty \\ 1<|z|<r \\ |y|<x}} r^{-1}\, d\mu(z) = 0.$

Condition (iv) is of course strictly weaker than (iii). Its analogue in higher dimensions would be

(v) $\lim_{r\to\infty} \sigma(r) r^{-(2n-1)} = 0$

where $\sigma(r)=\mu(\{z; x\in K, 1<|x|<r, |y|<|x|\})$, for μ the Riesz-measure of a function in $P(K)$. Condition (v) is however not true in general as can be seen from

Example 3.2. Let $K=\{x\in R^2; x_j>0\ j=1, 2\}$ so that $T(K)$ is a product of two halfplanes. Consider either of the functions $u_1(z_1; z_2)=|x_1-x_2|$ or $u_2(z_1; z_2)=\log|e^{-z_1}-e^{-z_2}|$. The Riesz measure μ_1 of u_1 is a constant times Lebesgue measure on the hyperplane $x_1=x_2$, while the Riesz measure of u_2 behaves like μ_1 at infinity. Thus in both cases $\sigma(r)\approx r^{-3}$, so (v) fails.

In order to get an analogue to (iv) in higher dimensions we will restrict our attention to zeros which lie close to the reals. We will prove

Theorem 3.3. *Let $t(r)$ be a function from R_+ to R_+ such that t is increasing and $t(r)r^{-1}\to 0$ as $r\to\infty$. Let $u\in P(K)$ and let K' be another cone such that*

$K' \subset\subset K$ (i.e. K' is compactly included in K). Put

$$\sigma(K'; r) = \mu(\{x+iy; x \in K', 1 < |x| < r, |y| < t(|x|)\}),$$

where μ is the Riesz measure of u. Then

$$\lim_{r\to\infty} \sigma(K'; r) r^{-n} t(r)^{-(n-1)} = 0.$$

For instance $t \equiv 1$ will do in Theorem 3.3. As a limiting case when t gets smaller we have

Theorem 3.4. *Let K, K' be as above, and suppose f is analytic in $T(K)$ and that $\log |f| \in P(K)$. Put*

$$s(K'; r) = H_{n-1}(\{x \in K'; 1 < |x| < r \text{ and } f(x) = 0\})$$

where H_{n-1} is $(n-1)$-dimensional Hausdorff-measure. Then

$$\lim_{r\to\infty} s(K'; r) r^{-n} = 0.$$

The proofs of Theorems 3.3 and 3.4 are based upon the following lemma.

Lemma 3.5. *Let B be a ball, $B \subset\subset K$. Then there is a constant c, not depending on α such that for all sufficiently small $\alpha > 0$*

$$\limsup_{r\to\infty} \mu(\{z; d(z; rB) < r\alpha\}) r^{-(2n-1)} \leqq c\alpha^n$$

(here $rB = \{x; xr^{-1} \in B\}$).

Proof of Theorem 3.3 from the lemma: First we make some reductions. There is no loss of generality in assuming that K' is a sufficiently small circular cone. It then suffices to prove

(vi) $\lim_{r\to\infty} \mu(\{z; d(z; rB) < t(r)\}) r^{-n} t(r)^{-(n-1)} = 0$

for every ball $B \subset\subset K$. This is so because

$$\sigma(K'; r) \leqq \sum_{k=0}^{N(r)} \mu(\{z; d(z; ra^k B) < t(ra^k)\})$$

for a suitable ball B and some $a < 1$, where $N(r)$ is choosen large enough depending on r. Hence

$$\sigma(K'; r) r^{-n} t(r)^{-(n-1)}$$
$$\leqq \sum \mu(\{z; d(z; ra^k B) < t(ra^k)\})(ra^k)^{-n} t(ra^k)^{-(n-1)} a^{kn} = \sum \mu(k; r) a^{kn}.$$

(the last equality is a definition). By (vi) $\mu(k; r)$ tends boundedly to zero as $r \to \infty$ and the theorem follows since $\Sigma a^{kn} < \infty$. To prove (vi) we will use Theorem 2.1 with $p = n-1$ and $\theta = i/2\partial\bar{\partial}u$. Then θ is a positive current since u is plurisubharmonic, and its trace is 4^{-1} times the Riesz measure of u. Now take r_0 so large that

$r > r_0$ implies $t(r) < \varepsilon r$ (which is possible by assumption), and

$$\mu(\{z;\, d(z;\, rB) < \varepsilon r\}) \leqq 2c\varepsilon^n r^{2n-1}$$

(which is possible by Lemma 3.5.). By Theorem 2.1

$$\mu(\{z;\, d(z;\, rB) < t(r)\}) \leqq 2c\varepsilon t(r)^{n-1} r^n$$

for $r > r_0$. This proves the theorem.

Proof of Theorem 3.4: This follows from Theorem 3.3. with $t \equiv 1$, because of Corollary 2.5 (the Riesz measure of $u = \log |f|$ is proportional to surface measure on f: s zerovariety, with multiplicity).

What remains is the proof of Lemma 3.5. This is based upon a comparison between the Riesz measure of u, and that of its indicator function, which we now define.

Definition 3.6. *Let $u \in P(K)$. Put*

$$L(u;\, x) = \limsup_{r \to \infty} u(rx) r^{-1}$$

for $x \in K$. The indicator function L^* of u is defined by

$$L^*(u;\, x) = \limsup_{x' \to x} L(u;\, x')$$

for $x \in K$.

The function $L^*(u;\, x)$ is convex and homogenous of order 1 (cf. [7]). Moreover, it was proved by Ronkin that $u(x+iy) \leqq L^*(u;\, x)$ for all $y \in R^n$ (cf. [10]). For the proof of Lemma 3.5. we fix u and extend the domain of definition of L^* to $T(K)$ by $L^*(z) = L^*(x+iy) = L^*(u;\, x)$. Thus L^* is convex and therefore plurisubharmonic.

Proof of Lemma 3.5: Let η be the Riesz measure of L^*. Put $u_r(z) = u(rz) r^{-1}$, $\mu_r =$ the Riesz measure of u_r and

$$\tilde{\mu}_r(x;\, \tau) = \mu_r(\{z;\, |x-z| < \tau\})$$

Similarily

$$\tilde{\eta}(x;\, \tau) = \eta(\{z;\, |x-z| < \tau\}).$$

Let $x_0 \in B$ be such that $u(\lambda x_0) \not\equiv -\infty$ in the complex halfline $\operatorname{Re} \lambda > 0$. For α small enough Jensens formula gives

$$a_n \int_0^{4\alpha} \mu_r(x_0;\, \tau) \tau^{-(2n-1)}\, d\tau = -u_r(x_0) + \int_{|z-x_0|=4\alpha} u_r\, dS,$$

where dS is normalized surface measure and a_n only depends on the dimension. Hence

$$\tilde{\mu}_r(x_0;\, 3\alpha)\, a_n 4^{-(2n-1)} \alpha^{-(2n-2)} \leqq -u_r(x_0) + \int_{|z-x_0|=4\alpha} u_r\, dS.$$

By the Ahlfors—Heins theorem we can choose a sequence $r_k\to\infty$ such that $r_{k+1}r_k^{-1}\to 1$ and $\lim u_{r_k}(x_0)=L(x_0)$. Moreover, by the above result by Ronkin we have $u_r(z)\leqq L^*(z)$. Hence we get

$$\limsup_{r\to\infty} \mu_r(x_0; 2\alpha)\alpha^{-(2n-2)} \leqq c \left[-L(x_0)+\int_{|z-x_0|=4\alpha} L^*\, dS\right]$$

(here and in the sequel c denotes a constant, not depending on α, not necessarily hw same at each occurrence). Assume $L(x_0)=L^*(x_0)$. Then by another application of Jensens formula we get

$$\text{(vii)}\qquad \limsup_{r\to\infty} \tilde{\mu}_r(x_0; 2\alpha)\alpha^{-(2n-2)} \leqq c\int_0^{4\alpha} \tilde{\eta}(x_0; \tau)\tau^{-(2n-1)}\, d\tau.$$

By a theorem by Lelong ([5]) $L(x_0)=L^*(x_0)$ for almost all $x_0\in K$. Thus we may integrate (vii) with respect to Lebesgue measure, $d\lambda$, over B, and since obviously $\tilde{\mu}_r(x_0; 2\alpha)$ is bounded from above uniformly in $x_0\in B$ and r, Fatou's lemma yields

$$\text{(viii)}\qquad \limsup_{r\to\infty}\int_B \tilde{\mu}_r(x; 2\alpha)\, d\lambda(x)\alpha^{-(2n-2)} \leqq c\int_B d\lambda(x)\int_c^{4\alpha} \tilde{\eta}(x; \tau)\tau^{-(2n-1)}\, d\tau.$$

To estimate the left hand side, note that

$$\int_B \tilde{\mu}_r(x; 2\alpha)\, d\lambda(x) = \int_B d\lambda(x)\int_{|z-x|<2\alpha} d\mu_r(z)$$
$$= \int_{d(z;B)<2\alpha} d\mu_r(z)\int_{x\in B,\, |z-x|<2\alpha} d\lambda(x) \geqq c\int_{d(z;B)<\alpha} d\mu_r(z)\alpha^n.$$

Estimating the right hand side of (viii) in a similar manner we get

$$\int_B d\lambda(x)\int_0^{4\alpha} \tilde{\eta}(x; \tau)\tau^{-(2n-1)}\, d\tau \leqq c\int_0^{4\alpha} \tau^{-(n-1)}\, d\tau\int_{d(z;B)<\tau} d\eta(z).$$

But since L^*, and hence η^*, is independent of y clearly

$$\int_{d(z;B)<\tau} d\eta(z) \leqq c\tau^n.$$

Collecting we get

$$\limsup_{r\to\infty}\int_{d(z;B)<\alpha} d\mu_r(z) \leqq c\alpha^{-n}\alpha^{2n-2}\int_0^{4\alpha} \tau\, d\tau \leqq c\alpha^n.$$

This completes the proof of the lemma.

It seems that Theorem 3.3 is the best possible density result for $P(K)$. However, it is a natural conjecture that further restrictions on $t(r)$ (e.g. $t\equiv 1$ or even $\int^\infty t(r)r^{-2}\, dr<\infty$) would give an analogue to the stronger condition (iii). I have not been able to prove any results in this direction.

4. Zeros of entire functions

Let V be a variety in $\mathbf{C}^n$ of pure dimension p. Let $\sigma(r)$ denote the surface measure of $V\cap B(0; r)$ and $s(r)$ the p-dimensional Hausdorff-measure of the real analytic set $V\cap B(0; r)\cap R^n$. Then Corollary 2.5 implies

Theorem 4.1. *For every* $k>1$ *there is a constant* c_k *such that*

$$s(r) \leqq c_k\sigma(kr)r^{-p}.$$

This result can be compared to sections 7 and 8 of [6].

Next we will study a class of entire functions which is connected with trigonometric approximation.

Definition 4.2. *For* $a=(a_1, \dots, a_n)$, $0\leqq a_j<\infty$, $B(a)$ *denotes the class of entire functions* f *which satisfy*

$$\text{(i)} \qquad \log|f(x+iy)| \leqq \sum a_j|y_j|.$$

We will use two different norms on $\mathbf{C}^n$. As usual, $|\cdot|$ denotes the euclidean norm while $\|x\|=\max(|x_j|)$ for real vectors and $\|z\|=\max(\|x\|; \|y\|)$. For $x_0\in R^n$, $\alpha>0$ and $r>0$ let

$$T(r; x_0; \alpha) = \{z;\ \|x-rx_0\| < r\alpha \text{ and } \|y\| < 1\}.$$

For an entire function f with zero-variety V_1 we put

$$\sigma_f(r; x_0; \alpha) = |V_f\cap T(r; x_0; \alpha)|,$$

where $|\cdot|$ means surface measure with multiplicity.

Theorem 4.3. *Suppose* $f\in B(a)$. *Then*

$$\text{(ii)} \qquad \limsup_{r\to\infty}(2\alpha r)^{-n}\sigma_f(r; x_0; \alpha) \leqq \pi^{-1}2^{n-1}\sum a_j.$$

Remark. For $n=1$ Theorem 4.3 was proved by Levinson (cf. [1]). For arbitrary n, the estimate (ii) with right hand side $e\pi^{-1}2^{n-1}\sum a_j$ was proved by Ronkin (cf [8, 9]). That there is some constant bounding the left hand side of (ii) even for general functions of exponential type, follows easily from Theorem 2.1. However, to get the constant in (ii) (which is best possible), we will use a direct method, quite similar to Ronkin's.

Proof. For $z=(z_1, \dots, z_n)$ let $z'=(z_2, \dots, z_n)$. Also

$$T'(r; x_0; \alpha) = \{z'\in\mathbf{C}^{n-1};\ \|x'-rx_0'\| < r\alpha \text{ and } \|y'\| < 1\}.$$

We will use the fundamental fact: the area of f:s zerovariety V_f with multiplicity is equal to the sum of the areas of V:s projection on the $(n-1)$-dimensional coordinate planes, also counted with multiplicities. So to estimate σ_f (which in the sequel is denoted σ since we keep f fixed), we start by estimating the measure of V_f:s projection on $T'(r; x_0; \alpha)$.

Let $\nu(z; t)$=number of zeros of the function $f(\xi; z')$ in $\{\xi; |\xi-z_1|<t\}$. By Jensens formula and (i)

$$\int_0^{\varepsilon r} \nu(z; t)t^{-1}\,dt \leqq 2\pi^{-1}\varepsilon r a_1+\sum a_j-\log|f(z)| \tag{iii}$$

for $\varepsilon>0$. The idea is now to integrate (iii) with respect to Lebesgue measure $d\lambda$, over $T(r; x_0; \alpha)$. An elementary computation with Fubini's theorem, similar to the proof of Lemma 3.5. gives

$$\int_{T(r;x_0;\alpha)} d\lambda(z)\int_0^{\varepsilon r} \nu(z; t)t^{-1}\,dt \geqq 4\Big(\int_2^{\varepsilon r} \sqrt{t^2-4}\,t^{-1}\,dt\sigma'(r; x_0; \alpha-\varepsilon)\Big)$$

where σ' is the measure of the projection on $T'(r; x_0; \alpha)$. Hence (iii) yields

$$(4\alpha r)^{-n}4\int_2^{\varepsilon r}\sqrt{t^2-4}\,t^{-1}\,dt\sigma'(r; x_0; \alpha-\varepsilon) \tag{iv}$$
$$\leqq 2\pi^{-1}\varepsilon r a_1+\sum a_j-(4\alpha r)^{-n}\int_{T(r;x_0;\alpha)}\log|f|\,d\lambda.$$

Estimating the other projections in the same way, dividing by εr and summing up we get

$$(4\alpha r)^{-n}(\varepsilon r)^{-1}4\int_2^{\varepsilon r}\sqrt{t^2-4}\,t^{-1}\,dt\sigma(r; x_0; \alpha-\varepsilon) \tag{v}$$
$$\leqq 2\pi^{-1}\sum a_j+n(\varepsilon r)^{-1}\sum a_j-(4\alpha)^{-n}\varepsilon^{-1}nr^{-(n+1)}\int_{T(r;x_0;\alpha)}\log|f|\,d\lambda.$$

As $r\to\infty$ the last term on the right hand side tends to zero. Accepting this for the moment and letting first $r\to\infty$ and then $\varepsilon\to 0$ we get

$$\limsup_{r\to\infty}(2\alpha r)^{-n}\sigma(r; x_0; \alpha)\leqq 2^{n-1}\pi^{-1}\sum a_j,$$

which proves the theorem. All that remains is thus to prove that

$$r^{-(n-1)}\int_{T(r;x_0;\alpha)}\log|f|\,d\lambda\to 0 \quad\text{as}\quad r\to\infty$$

for $f\in B(a)$. This is Lemma 1 of [9], but for completeness I will indicate a proof. When $n=1$ it follows since $\log|f|$ is integrable on the real line with the weight $(1+x^2)^{-1}$. For general n it can be proved by writing the integral in polar coordinates and applying the one-dimensional result.

5. Discrete sets of uniqueness

In this section we sketch some applications of the above estimates to discrete sets of uniqueness. The link is provided by the following theorem, due to Rutishauser ([12]), for $n=2$, and to Lelong [4] in general.

Theorem 5.1. *Let B be the unit ball in $\mathbf{C}^n$ and let V be a variety in B of pure dimension $n-1$. Assume $0\in V$. Then the area of V is larger than $\pi^{n-1}/(n-1)!$ (i.e. the volume of the $(2n-2)$-dimensional unit ball).*
Thus Theorem 3.3 can be used to prove

Theorem 5.2. *Let K and $T(K)$ be as in section 3. Let E be a discrete subset of K such that $x_1, x_2\in E\Rightarrow|x_1-x_2|\geqq h>0$ for some constant h. Let $n(r)=\#E\cap B(0;r)$. Assume f is analytic and bounded in $T(K)$, and that f vanishes on E. Then if*

$$\limsup_{r\to\infty} n(r)r^{-n}>0$$

f is identically zero.

Theorem 5.2. was proved by Korevaar and Hellerstein [3], for $n=2$ and E consisting of points with integer coordinates, and by Ronkin [7] in general. We note that Theorem 3.3 implies corresponding results for $E\subset T(K)$ not necessarily consisting of reals, and for sets E not uniformly spearated. We omit the detailed formulations.

Theorem 4.3. also implies (via Theorem 5.1.) a result on discrete sets of uniqueness for the class $B(a)$ of entire functions. However, in this case the precise values of the constants involved are more important, and to get a precise result one has to replace Theorem 5.2 with a corresponding result for cubes instead of the unit ball. We refer to [11], for a discussion of this problem. We only note that Theorem 4.3 together with the result in [11], gives a best possible result for $n=2$, while the case of general n remains open.

Finally we mention that results on discrete sets of uniqueness can be translated (via the Hahn—Banach theorem) to approximation theorems. Thus Theorem 5.2. in connected with approximation with lacunary polynomials, and the consequences of Theorem 4.3. with trigonometric approximation. For this see [3, 7, 8, 9].

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