Cocycles and spectra

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1. Introduction

One-dimensional cocycles arise in ergodic theory in connection with group extensions and velocity changes. Unitary operators are thereby defined in function space that depend on the given dynamical system and the cocycle. In the particular case of an irrational flow on a torus, cocycles presented themselves in the study of the function theory associated with the flow. Thus the cocycles defined on a flow, and their associated unitary operators, are interesting from several points of view. The first question is to decide what kind of spectrum the operators can have. Results so far known seem to show that cocycles leading to singular spectrum are easier to construct than cocycles leading to absolutely continuous spectrum. The main result of this paper is that for aperiodic transformations preserving a probability measure there always exist cocycles whose associated unitary operators have Lebesgue spectrum.

Cocycles in ergodic theory are discussed in [3, 5], and in harmonic analysis in [2].

In order to describe our results more exactly we introduce the concepts and notation that will be used in the paper. T denotes an aperiodic transformation of the standard measure space $(X, \mathbf{B}, \sigma)$ that preserves the probability measure σ. *Aperiodic* means that if $B \in \mathbf{B}$ and $T^n x = x$ for all x in B (n a positive integer), then $\sigma(B) = 0$.

A *cocycle* for T with values in an abelian topological group G is a measurable function $\varphi: X \times Z \to G$ satisfying $\varphi(x, m+n) = \varphi(T^n x, m) + \varphi(x, n)$ almost everywhere on X, for each pair of integers m, n. Evidently φ is determined by the functional equation and the function $\varphi(x) = \varphi(x, 1)$ from X to G, and each measurable function from X to G generates a cocycle. Usually G will be $\mathbf{R}$, the real line, or $\mathbf{T}$, the circle group, and then we speak of additive or multiplicative cocycles, respectively.

A unitary operator $U = U_T$ is defined in $L^2(X)$ by setting $Uf(x) = f(Tx)$.

For any measurable function φ on X with values in $\mathbf{T}$ define $V^\varphi f(x) = \varphi(x)f(Tx)$. We are interested in the spectrum of the unitary operator V^φ in $L^2(X)$.

V^φ seems to be a more general kind of operator than U, but actually V^φ is similar to the operator U defined on another measure space with another transformation, or at least to U restricted to an invariant subspace. Indeed let $X' = X \times \mathbf{T}$, and define the transformation $T'(x, y) = T(x, \varphi(x)y)$. The product of σ with Lebesgue measure in $\mathbf{T}$ is invariant under T', so we can define the associated unitary operator U' in $L^2(X')$. It is easy to verify that $L^2(X')$ is the orthogonal sum of the invariant subspaces

$$(1) \qquad H_n = \{f(x)y^n : f \in L^2(X)\}, \quad (n = 0, \pm 1, \ldots).$$

In H_1 we have $U'(f(x)y) = f(Tx)\varphi(x)y$, so that U' restricted to H_1 is similar to V^φ in $L^2(X)$.

If φ takes values in $Z_2 = \{1, -1\}$, then T can be extended in the same way to a transformation T' in $X \times Z_2$, and V^φ is similar to U' acting in a subspace of $L^2(X \times Z_2)$. A cocycle with values in Z_2 is a *real cocycle*.

Theorem. (a) *For every aperiodic measure-preserving transformation there are real cocycles φ such that V^φ has Lebesgue spectrum.* (b) *If X is a compact abelian group, T an ergodic translation, and σ Haar measure, there is a continuous function φ from X to $\mathbf{T}$ such that V^φ has Lebesgue spectrum.*

The functions φ of (a) are exceptional in some sense, for Katok and Stepin have shown [4] that for certain transformations a dense G_δ of φ lead to V^φ with simple singular spectrum.

Multiplicative cocycles giving absolutely continuous or even Lebesgue spectrum have been constructed implicitly or explicitly in the following cases at least:

(i) T an ergodic translation of a compact connected abelian group (Abramov [1], in connection with the theory of quasi-discrete spectrum)

(ii) T an argodic translation of a compact abelian group that is not totally disconnected (S. Parrott, unpublished thesis)

(iii) T a K-automorphism, or even (using Sinai's theorem [7]) ergodic with positive entropy (Jones and Parry [3]).

This paper presents three methods of constructing cocycles giving absolutely continuous spectrum. Since the methods are different it seems of interest to sketch them all, but some details have been omitted.

2. First construction

T is an aperiodic measure-preserving transformation on the probability measure space $(X, \mathbf{B}, \sigma)$, and U the associated unitary operator in $L^2(X)$. We seek a function $\varphi = \exp im$, where m takes the values $0, \pi$ so that $V = V^\varphi$ has Lebesgue spectrum in $L^2(X)$. We shall find φ so that $\varrho_f(n) = (V^n f, f)$ is a square-summable sequence for each f in a dense subset of $L^2(X)$, proving that the spectrum is absolutely continuous, and $\varrho_f(n)$ decreases very rapidly for some f, making the spectrum Lebesgue.

Let E_j $(j = 1, 2, \ldots)$ be disjoint sets in X with characteristic functions h_j. Form the random set E whose characteristic function is $h = \sum \eta_j h_j$, where the η_j are independent random variables taking the values $0, 1$, each with probability $1/2$. Define $m = \pi h$. We shall choose the sets E_j so that V has Lebesgue spectrum with positive probability.

For any f in $L^2(X)$ and disjoint sets E_j we have $(n = 1, 2, \ldots)$

(2)
$$|\varrho(n)|^2 = \iint f(T^n x)\overline{f(T^n y)}\overline{f(x)}f(y) \exp \pi i \sum_{j=1}^{\infty} \sum_{k=0}^{n-1} \eta_j U^k\big(h_j(x) - h_j(y)\big) \, d\sigma(x, y).$$

Integrating over the probability space gives

(3)
$$\int |\varrho(n)|^2 \, d\omega$$
$$= \iint f(T^n x)\overline{f(T^n y)}\overline{f(x)}f(y) \prod_{j=1}^{\infty} \frac{1}{2}\Big(1 + \exp \pi i \sum_{k=0}^{n-1} U^k\big(h_j(x) - h_j(y)\big)\Big) \, d\sigma(x, y).$$

The product on the right side takes the values $0, 1$, and equals 1 on the set in $X \times X$ consisting of all (x, y) such that

(4)   parity of $\sum_{k=0}^{n-1} U^k h_j(x) =$ parity of $\sum_{k=0}^{n-1} U^k h_j(y)$,   all $j = 1, 2, \ldots$.

Define $\mathscr{T}(x, n)$, the *orbit of length n at x*, to be the set $\{x, Tx, \ldots, T^{n-1}x\}$. Set $a_j^n(x) = 0$ if E_j intersects $\mathscr{T}(x, n)$ in an even number of points, $= 1$ if the intersection contains an odd number of points. Let $a^n(x) = \big(a_1^n(x), a_2^n(x), \ldots\big)$, a sequence of 0's and 1's. Each sequence $a^n(x)$ terminates in 0's, because the finite set $\mathscr{T}(x, n)$ can intersect only finitely many E_j. The condition (4) means simply that $a^n(x) = a^n(y)$.

For each $a = (a_1, a_2, \ldots)$, a sequence of 0's and 1's terminating in 0's, let G_a^n be the set of x in X such that $a^n(x) = a$. For fixed n, $\{G_a^n\}$ is a disjoint covering of X. Evidently $a^n(x) = a = a^n(y)$ if and only if x and y both belong to G_a^n. The measure of this set of (x, y) is $\sigma(G_a^n)^2$. Thus by (3) we have

(5) $\int |\varrho(n)|^2 \, d\omega \leqq \|f\|_\infty^4 \sum_a \sigma(G_a^n)^2$

for any bounded function f, with equality if $f = 1$.

Lemma 1. *Given positive numbers* ε_k *we can find disjoint sets* F_k *in* X *so that*

(6) $$\sigma \bigcup_{|j|<2^k} T^j F_k > 1 - \varepsilon_k \quad (k = 1, 2, \ldots).$$

Proof. For each positive integer k construct a Rohlin tower of height 2^k and residue ε_k: H_j^k $(1 \le j \le 2^k)$ are disjoint sets in X such that $TH_j^k = H_{j+1}^k$ $(1 \le j < 2^k)$, and $\sigma \bigcup_j H_j^k > 1 - \varepsilon_k$. If the sets H_1^k are disjoint we take $F_k = H_1^k$, and then (6) holds with the union merely over $0 \le j < 2^k$.

In general we have to modify H_1^k to obtain F_k. Let $F_1 = H_1^1$ in any case. Take for F_2 the set of x in H_1^2 that are not in F_1, together with $T^{-1}x$ for all x in $F_1 \cap H_1^2$. Then F_2 is disjoint from F_1 because x, $T^{-1}x$ cannot both lie in F_1.

Suppose $F_1, \ldots, F_r$ have been defined. Let F_{r+1} be the set of all $T^p x$ where x is in H_1^{r+1}, and p is the smallest non-negative integer such that $T^p x$ is not in any F_k $(k \le r)$. The existence of such a p, $0 \le p < 2^{r+1}$, has to be verified. For each k, F_k contains at most one point $T^p x$ as p ranges over 2^k consecutive integers. If we exclude 2^{r+1-k} values of p, $0 \le p < 2^{r+1}$, then T^p is not in F_k. The p thus excluded for $k = 1, \ldots, r$ number $2 + 4 + \ldots + 2^r < 2^{r+1}$, so some values of p are left and F_{r+1} has been defined.

A point x in the tower $\{H_j^k\}$ lands in F_k by traveling up or down in the tower, in at most $2^k - 1$ steps. Hence the union in (6) contains all x except some points outside the tower, of measure at most ε_k. This proves the lemma.

We shall apply the lemma in this form: *if* $n \ge 2^{k+1} - 1$, *then* $\mathcal{T}(x, n)$ *intersects* F_k *except for* x *in a set of measure less than* ε_k.

Lemma 2. *Let* H *be a subset of* X, ε *a positive number, and* r *a positive integer. There are disjoint sets* H_k $(k = 1, 2, \ldots)$ *with union* H *such that* $\sigma(H_k) < \varepsilon$, *and* H_k *intersects* $\mathcal{T}(x, r)$ *in at most one point, for each* k *and each* x *in* X.

Proof. For each $n = 1, 2, \ldots$ build a Rohlin tower $\{R_j^n\}$ of height N and residue less than $1/n$. Choose N greater than ε^{-1} and r; then $\sigma(R_j^n) < \varepsilon$ for all n, j and no set R_j^n contains more than one member of $\mathcal{T}(x, r)$ for any x. We take for the sets H_k all the intersections $H \cap R_j^n$ in some particular order, leaving out of H_k the points already in $H_1, \ldots, H_{k-1}$. The H_k cover H aside from a null set, which can be distributed among the H_k without difficulty. The lemma is proved.

Choose sets F_k by Lemma 1, with residues ε_k to be specified later. By Lemma 2 we express each F_k as the disjoint union of sets F_{kl}, each of measure less than $\delta_k 2^{-k-2}$ (the positive numbers δ_k will be chosen later), and intersecting each orbit $\mathcal{T}(x, 2^{k+2})$ in at most one point. The sets E_j needed to prove our theorem are all the sets F_{kl}.

The E_j are disjoint by construction. Each E_j is contained in a unique F_k. If $2^{k+1} \le n < 2^{k+2}$ then first each orbit $\mathcal{T}(x, n)$ intersects F_k, except for x in a set of measure at most ε_k; and then such an orbit intersects a set F_{kl} in at most one point.

We fix n and choose k so $2^{k+1} \leqq n < 2^{k+2}$. The measure of G_a^n is to be estimated for each sequence $a = (a_1, a_2, ...)$. First suppose $a_j = 0$ for each j such that E_j is contained in F_k. Then $a^n(x) = a$ only if $\mathscr{T}(x, n)$ has empty intersection with each such E_j, and thus with their union F_k. This set has measure at most ε_k.

Otherwise $a_j = 1$ for at least one j such that E_j is in F_k. Then G_a^n is contained in the set of all x such that $\mathscr{T}(x, n)$ intersects E_j, which is exactly

$$(7) \qquad\qquad E_j \cup T^{-1}E_j \cup ... \cup T^{-n+1}E_j$$

of measure at most $n\sigma(E_j) < n\delta_k 2^{-k-2} < \delta_k$.

Since $\{G_a^n\}$ is a disjoint covering of X, these estimates give

$$(8) \qquad\qquad \sum_a \sigma(G_a^n)^2 < \varepsilon_k + \delta_k \quad (2^{k+1} \leqq n < 2^{k+2}).$$

Now ε_k, δ_k were arbitrary positive numbers. Hence, using (5),

$$(9) \qquad\qquad \int |\varrho_f(n)|^2 \, d\omega < \gamma_n \quad (n \geqq 4)$$

for every function f bounded by 1, where $\{\gamma_n\}$ is any sequence of positive numbers.

If $\{\gamma_n\}$ is summable, (9) implies that $\{\varrho_f(n)\}$ is almost surely square-summable. The exceptional set depends on f, but we can add up the null sets corresponding to a countable set of bounded functions dense in $L^2(X)$ to conclude that the spectrum of V is almost surely absolutely continuous.

From (9), with $f = 1$, we have with *positive probability*

$$(10) \qquad\qquad |\varrho(n)|^2 < 2^n \gamma_n \quad (n \geqq 4).$$

If $2^n \gamma_n = 0 \, (\exp -\alpha n)$ for some positive α the spectral density function is analytic on an annulus containing the unit circle, and so can vanish only at isolated points. Hence the spectrum of V fills the circle as we wanted to prove.

3. Some perspective

To see clearly what the theorem means we shall show what can be obtained easily from the ergodic theorem. As before, E is a subset of X, $\varphi(x) = -1$ on E, 1 on CE. Define $\varphi_n(x) = \varphi(x)\varphi(Tx)...\varphi(T^{n-1}x)$. Then $\varphi_n(x) = -1$ on E_n, 1 on CE_n, where E_n is the repeated symmetric difference

$$(11) \qquad\qquad E\Delta T^{-1}E\Delta ... \Delta T^{-n+1}E.$$

We define the unitary operator $V = V^\varphi$ and want to study

$$(12) \qquad\qquad (V^n 1, 1) = \int \varphi_n \, d\sigma = 1 - 2\sigma(E_n).$$

Define $X' = X \times Z_2$ and the transformation T' in X'. The ergodic theorem, applied to $f(x, y) = y$ on X', gives

(13)
$$\lim \frac{1}{N} \sum_1^N \varphi_n(x) = \psi(x)$$

where $\psi(Tx) = \varphi(x)\psi(x)$ almost everywhere on X. If T is ergodic, ψ is either identically 0 or almost everywhere different from 0. In the second case $\varphi = U\psi/\psi$, and V is unitarily equivalent to U. This case is not interesting.

Otherwise the limit in (13) is 0, so that $\varphi_n(x)$ tends to 0 on the average. This is in the direction of saying that the quantity (12) tends to 0, but it is not enough — V may still have discrete spectrum, the sequence (12) can still be almost-periodic.

Integrating (13), for the case $\psi = 0$, gives

(14)
$$\lim \frac{1}{N} \sum_1^N \sigma(E_n) = 1/2,$$

and this is all we can show for E without hypothesis. In the last section we proved that given any positive numbers γ_n $(n \geq 4)$ we can find E so that

(15)
$$|\sigma(E_n) - 1/2| < \gamma_n \quad (n \geq 4).$$

(The proof actually gives the result for $n \geq 3$). The gap between (14) and (15) is satisfyingly wide.

4. Second construction

X is a compact abelian group, σ is Haar measure on X, and T is translation by an element e of X. In order for the translation to be ergodic it is necessary and sufficient that e as a function on Γ, the discrete group dual to X, be an isomorphism of Γ into $\mathbf{T}$. Thus we may think of Γ as a subgroup of the discrete circle group, and $e(\exp i\lambda) = \exp i\lambda$ for $\exp i\lambda$ in Γ. We sometimes write λ for $\exp i\lambda$, $-\pi < \lambda \leq \pi$, with addition modulo 2π. Elements of X are x, y; elements of Γ are λ, τ or $\exp i\lambda$, $\exp i\tau$. When λ is a character of X it is written χ_λ.

We want to find a continuous unimodular function φ on X such that V^φ has Lebesgue spectrum. It is known and easy to prove that the spectrum is Lebesgue if it has any absolutely continuous part. Furthermore the absolutely continuous part is non-trivial if $\varrho(n) = (V^n 1, 1)$ is square-summable over the positive *odd* integers, but not always 0 for such n. We shall find φ with this property.

Set $\varphi = \exp im$, where m is to be real and continuous with Fourier series

(16)
$$m(x) \sim \sum_\lambda c_\lambda \chi_\lambda(x).$$

The c_λ will be chosen to be real, $c_{-\lambda} = c_\lambda$, $c_0 = 0 = c_\pi$ (if π is in Γ), and so that (16)

converges absolutely. For a sequence of elements λ_j of Γ, positive and decreasing to 0, we shall choose independent Gaussian random variables $c_{\lambda_j} = c_j = c_{-\lambda_j}$ so that φ almost surely has the required properties. For each j,

$$(17) \qquad \int \exp iuc_j \, d\omega = \exp(-r_j^2 u^2) \quad \text{where} \quad \int c_j^2 \, d\omega = 2r_j^2.$$

($d\omega$ is the probability measure on the space of the random variables.) If $\{r_j\}$ is a summable sequence, then (16) almost surely converges absolutely.

We perform calculations that will show how to choose the λ_j and r_j. For convenience we write χ_j instead of χ_{λ_j}; $\cos \lambda x$ is the real part of $\chi_j(x)$.

Since $\chi_j(x+e) = \chi_j(x) \exp i\lambda_j$ we have

$$(18) \qquad \varphi(T\varphi) \dots (T^{n-1}\varphi) = \exp i \sum_j c_j \chi_j \sum_{k=0}^{n-1} \exp ik\lambda_j.$$

This gives, after some trigonometry, for positive odd n

$$(19) \qquad \varrho(n) = \int \exp i \sum_j c_j \chi_j \left(x + \frac{1}{2}(n-1)e \right) \frac{\sin n\lambda_j/2}{\sin \lambda_j/2} \, d\sigma(x).$$

The translation in x can be suppressed, and we combine terms with opposite indices:

$$(20) \qquad \varrho(n) = \int \exp 2i \sum_{j>0} c_j \cos \lambda_j x \frac{\sin n\lambda_j/2}{\sin \lambda_j/2} \, d\sigma(x).$$

This leads to

$$(21) \quad \int |\varrho(n)|^2 \, d\omega = \iint \prod_{j>0} \exp -4r_j^2 (\cos \lambda_j x - \cos \lambda_j y)^2 \frac{\sin^2 n\lambda_j/2}{\sin^2 \lambda_j/2} \, d\sigma(x, y).$$

If $\{r_j\}$ is merely square-summable this quantity is positive, so $\varrho(n)$ is almost surely not 0 for each (positive, odd) n.

Let α be a positive number such that $(\sin n\lambda/2)/(\sin \lambda/2) > n/2$ for $n\lambda < \alpha$. Then (21) gives

$$(22) \quad \int |\varrho(n)|^2 \, d\omega \leqq \iint \exp -n^2 \sum_{0 < \lambda_j < \alpha/n} r_j^2 (\cos \lambda_j x - \cos \lambda_j y)^2 \, d\sigma(x, y).$$

The factor n^2 makes the integrand small except on the set where the parenthesis with cosines is small. To estimate the size of this set we use the following lemma.

Lemma. *Let η be a small positive number and k a positive integer. There are elements $\lambda_1, \lambda_2, \dots, \lambda_k$ of Γ, positive and as small as we please, such that the set of (x, y) where $|\cos \lambda_j x - \cos \lambda_j y| \leqq \eta$ (each $j=1, \dots, k$) has measure less than $\eta^{k/2}$ in $K \times K$.*

The condition on the λ_j is a kind of independence. The proof of the lemma is omitted.

For each $n=1, 2, \ldots$ let $r=n^{-a}$, $\eta=n^{-b}$ with positive numbers a, b to be specified. Choose $\lambda_1, \ldots, \lambda_k$ in Γ satisfying $0<\lambda_j<\alpha/n$ by the lemma, all distinct, with k independent of n to be given. Finally choose Gaussian random variables $c_{\lambda_1}, \ldots, c_{\lambda_k}$ with variance $2r_n^2$ such that all the variables so chosen for all n form an independent set. Combine all the c_λ into a series (16).

For absolute convergence of (16) we need $a>1$.

The sum in (22) is at least equal to

$$(23) \qquad \frac{n^{1-2(a+b)}}{2(a+b)-1}$$

except on a set of measure at most $n^{-kb/2}$. Hence the right side of (22) is less than

$$(24) \qquad \exp -\varkappa n^{3-2(a+b)} + n^{-kb/2}$$

for some positive $\varkappa$. Thus (16) is absolutely convergent and the sum over positive odd n of the left side of (22) is finite if we have

$$(25) \qquad a > 1, \quad a+b < 3/2, \quad kb > 2.$$

These relations are compatible (k can be as small as 5), and when they are satisfied we have a continuous function φ such that V^φ has Lebesgue spectrum.

The idea of the proof seems to generalize to measure preserving homeomorphisms of compact metric spaces, but we shall not pursue this subject here.

5. Third construction

The construction in a special case of a real cocycle φ giving absolutely continuous spectrum will illustrate the technique and lead to the general result.

For positive integers p, Z_p is the cyclic group of order p: the set $\{0, 1, \ldots, p-1\}$ with addition modulo p. Form the compact abelian group $X=Z_{p_1^2}\times Z_{p_2^2}\times\ldots$ where $p_1, p_2, \ldots$ are the primes. Normalized Haar measure on Borel subsets of X is σ. Define a transformation T in X by setting

$$(26) \qquad T(x_1, x_2, \ldots) = (x_1+1, x_2+1, \ldots).$$

It is easy to check that T is ergodic with respect to σ (this is the case of interest, but the technique could be applied to non-ergodic T as well). Let A_n be the set of x in X with $x_n=0$, and let h_n be the characteristic function of A_n. Define

$$(27) \qquad m(x) = \pi \sum_1^\infty h_n(x).$$

Then m is a non-negative function on X, and summable because each A_n has measure $1/p_n^2$. Our first objective is to prove

If $\varphi = \exp im$, then V^φ has absolutely continuous spectrum in the cyclic space generated by 1.

We shall show that $\varrho(n) = (V^n 1, 1)$ is square-summable over positive n. By definition,

(28) $$\varrho(n) = \int \prod_{k=1}^\infty \exp \pi i \big(h_k(x) + \ldots + h_k(T^{n-1}x) \big) \, d\sigma(x).$$

Each term of the product is measurable with respect to the kth factor algebra of **B**, and so these terms form an independent sequence. Therefore the product in (28) can be carried outside the integral.

If $p_k^2 \geqq n$, the sets $A_k, \ldots, T^{-n+1} A_k$ are disjoint. Denote the union of these sets by B_k^n. Then the integral of the kth factor in (28) is $1 - 2\sigma(B_k^n) = 1 - 2np_k^{-2}$. Thus we have

(29) $$|\varrho(n)| \leqq \prod_{p_k^2 \geqq n} (1 - 2np_k^{-2}) \leqq \exp - 2n \big(\textstyle\sum_{p_k^2 \geqq n} p_k^{-2} \big).$$

From the crude estimate

(30) $$\textstyle\sum_{p_k^2 \geqq n} p_k^{-2} \geqq \varkappa n^{-3/4} \quad \text{(some } \varkappa > 0\text{)}$$

we conclude that

(31) $$|\varrho(n)| \geqq \exp - 2\varkappa n^{1/4},$$

which shows that $\{\varrho(n)\}$ is square-summable over positive n.

Now we shall use a strong form of Rohlin's lemma due to Weiss [6] to make any aperiodic measure preserving transformation resemble this example closely enough to allow us to apply the same method.

Rohlin's lemma, strong form. *If T is an aperiodic measure preserving transformation of $(X, \mathbf{B}, \sigma)$, then for each $\varepsilon > 0$, positive integer n, and finite partition ξ there is a set A in **B** such that $A, T^{-1}A, \ldots, T^{-n+1}A$ are disjoint, $\sigma(\bigcup_{i=0}^{n-1} T^{-i}A) > 1 - \varepsilon$, and $\sigma(A \cap B) = \sigma(A)\sigma(B)$ for all B in ξ. (We say that A is independent of ξ and write $A \perp\!\!\!\perp \xi$.)*

The conclusion can immediately be strengthened to say that $T^{-i}A$ is independent of ξ for $i = 0, \ldots, n-1$; we simply apply the lemma to the partition $\xi \vee T\xi \vee \ldots \vee T^{n-1}\xi$ (the common refinement of $\xi, \ldots, T^{n-1}\xi$).

Now let T be a measure preserving transformation of $(X, \mathbf{B}, \sigma)$, and let $\{q_n\}$ be an increasing sequence of positive integers. We choose sets $A_1, A_2, \ldots$ inductively as follows. Find A_1 so that $A_1, T^{-1}A_1, \ldots, T^{-q_1+1}A_1$ are disjoint with $\sigma(\bigcup_{i=0}^{q_1-1} T^{-i}A_1) > 1 - \sigma(A_1)$, so that $(q_1+1)\sigma(A_1) > 1 > q_1\sigma(A_1)$. Let α_1 be the partition consisting of the $T^{-i}A_1$ ($i = 0, \ldots, q_1-1$) together with the complement of their union. Having chosen $A_1, \ldots, A_n$ we find A_{n+1} so that the sets $T^{-i}A_{n+1}$

$(i=0, \ldots, q_{n+1}-1)$ are disjoint, $(q_{n+1}+1)\sigma(A_{n+1}) > 1 > q_{n+1}\sigma(A_{n+1})$, and so that these q_{n+1} sets are independent of $\alpha_1 \vee \ldots \vee \alpha_n$. We define α_{n+1} to be the partition consisting of $T^{-i}A_{n+1}$ $(i=0, \ldots, q_{n+1}-1)$ together with the complement of their union.

Let $\eta_1, \eta_2, \ldots$ be an independent sequence of random variables with values $0, 1$, each with probability $1/2$. Define the random function

$$(32) \qquad m(x) = \pi \sum_{j=1}^{\infty} \eta_j h_j(x),$$

where h_j is the characteristic function of A_j. If $q_n = n^2$, as we shall assume henceforth, then m is non-negative and summable. We set $\varphi = \exp im$, a random real multiplicative cocycle. Our main result is that $V = V^{\varphi}$ *almost surely has absolutely continuous spectrum.*

We shall prove that the random sequence

$$(33) \qquad \varrho(n) = \int \varphi(x) \ldots \varphi(T^{n-1}x) f(T^n x) \overline{f(x)} \, d\sigma(x)$$

satisfies

$$(34) \qquad \sum_n \int |\varrho(n)|^2 \, d\omega < \infty,$$

so that $\{\varrho(n)\}$ is almost surely square-summable, provided f is a bounded function.

The definition of V is similar to the definition in the first construction (the sets A_j are however not necessarily disjoint), and we can follow the same calculations as far as (3). If f is bounded by 1 we have

$$(35) \qquad \int |\varrho(n)|^2 \, d\omega \leq \iint \prod_{q_j \geq n} \frac{1}{2}\left(1 + \exp \pi i \sum_{k=0}^{n-1} U^k (h_j(x) - h_j(y))\right) d\sigma(x, y),$$

absolute value signs being unnecessary because the integrand takes only the values $0, 1$. For $q_j \geq n$ the sets $A_j, T^{-1}A_j, \ldots, T^{-q_n+1}A_j$ are disjoint. Their union B_j^n thus has measure $n\sigma(A_j)$. Hence the factor with index j in (35) equals 1 on $(B_j^n \times B_j^n) \cup (CB_j^n \times CB_j^n)$, a set in $X \times X$ of measure $n^2\sigma(A_j)^2 + (1 - n\sigma(A_j))^2$, and equals 0 on the complementary set. Furthermore, the product in (35) commutes with the integration, because the sets B_j^n $(q_j \geq n)$ are mutually independent. Thus the right side of (35) can be written down exactly:

$$(36) \qquad \int |\varrho(n)|^2 \, d\omega \leq \prod_{q_j \geq n} \left[n^2\sigma(A_j)^2 + (1 - n\sigma(A_j))^2\right].$$

We increase the right side by restricting the product to $q_j \geq 2n$. For such j, $0 < n\sigma(A_j) < 1/2$. The inequality $x^2 + (1-x)^2 < 1-x$, valid for $0 < x < 1/2$, gives

$$(37) \qquad \int |\varrho(n)|^2 \, d\omega \leq \prod_{q_j \geq 2n} (1 - n\sigma(A_j)).$$

But $n\sigma(A_j) \geq n/(q_j+1)$, so we find

$$(38) \qquad \int |\varrho(n)|^2 \, d\omega \leq \exp{-n} \sum_{q_j \geq 2n} \frac{1}{q_j+1}.$$

If $q_j = j^2$, this leads to

(39) $$\int |\varrho(n)|^2 \, d\omega \leqq \exp - \varkappa n^{1/2} \quad (\varkappa > 0).$$

We conclude that $\{\varrho(n)\}$ is almost surely square-summable. The exceptional set may depend on the bounded function f, but the sequence is almost surely square-summable simultaneously for all f in a dense subset of L^2 so the spectrum of V is absolutely continuous as we wanted to show.

6. Multiplicity

An open problem, of long standing and attributed in [8] to Banach, is this: *Is there a measure preserving transformation T of a probability space $(X, \mathbf{B}, \sigma)$ with simple Lebesgue spectrum in the orthocomplement of the constant functions?* It is not even known whether T can have absolutely continuous spectrum with finite multiplicity. So far as we know, this conjecture has not been proved or disproved: *If U_T has a Lebesgue component in its spectrum, then it occurs with uniform infinite multiplicity.*

The constructive techniques of this paper could perhaps be used to disprove the conjecture. In the other direction, using our cocycles we can construct Z_2-extensions of some discrete spectrum transformations in which an infinite Lebesgue component appears. Precisely,

If T_1, T_2 are measure preserving transformations of $(X_1, \mathbf{B}_1, \sigma_1)$ and $(X_2, \mathbf{B}_2, \sigma_2)$ respectively, with T_2 aperiodic and $L^2(X_1)$ infinite-dimensional, then there is a Z_2-extension of $T = T_1 \times T_2$ with an infinite Lebesgue component in its spectrum.

Let φ be a real cocycle for T_2 such that V^φ has Lebesgue spectrum. We have shown that such a cocycle always exists. Define $S(x_1, x_2, y) = (T_1 x_1, T_2 x_2, \varphi(x_2)y)$, $y = \pm 1$. If $F(x_1, x_2, y) = f(x_1)y$ then

(40) $$(U_S^n F, F) = \int \int f(T_1^n x_1) \overline{f(x_1)} \varphi(x_2) \dots \varphi(T_2^{n-1} x_2) \, d\sigma_1(x_1) \, d\sigma_2(x_2)$$

$$= \int_0^{2\pi} e^{ni\lambda} \, d\mu(\lambda) \int_0^{2\pi} e^{ni\tau} \, d\nu(\tau)$$

$$= \int_0^{2\pi} e^{ni\lambda} \, d\mu * \nu(\lambda).$$

By hypothesis ν is equivalent to Lebesgue measure, and therefore $\mu * \nu$ has the same property. That is, for each f in $L^2(X_1)$ the element $f(x_1)y$ in the product space has Lebesgue spectrum with respect to S. Since $L^2(X_1)$ is infinite dimensional we can find infinitely many non-null functions $f_1, f_2, \dots$ in the space such that the cyclic subspaces $\{U_T^j f_n\}$ $(j = 0, \pm 1, \dots)$ are mutually orthogonal. Then $f_1(x_1)y$,

$f_2(x_1)y$, ... give mutually orthogonal Lebesgue cycles for U_S, and our assertion is proved.

Similarly, one can show: *If T is an ergodic translation of a compact abelian group X that contains a closed infinite subgroup X_0 such that X/X_0 is infinite, then T has a Z_2-extension with infinite Lebesgue spectrum.*

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