On the multiplicity of a stochastic vector process

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1. Introduction

The object of this note is to prove a theorem on the multiplicity of a stochastic vector process. Before stating the theorem, it will be necessary to give some introductory remarks.

We shall be concerned with complex-valued random variables defined on a fixed probability space. Any random variable x considered will be assumed to have zero mean and a finite variance:

$$Ex = 0, \quad E|x|^2 < \infty.$$

It is well known that all random variables satisfying these conditions can be regarded as elements in a Hilbert space, if the inner product of two elements x and is defined by the relation

$$(x, y) = Ex\bar{y}.$$

A stochastic vector process of finite dimensionality q will be denoted

$$\mathbf{x}(t) = \{x_1(t), \ldots, x_q(t)\}$$

where the components $x_n(t)$ are random variables depending on a real parameter t, which may be regarded as representing time. By $H(\mathbf{x})$ we denote the Hilbert space spanned by the random variables $x_n(u)$ for all n and all real u:

$$H(\mathbf{x}) = \mathscr{S}\{x_n(u), n = 1, \ldots, q, -\infty < u < +\infty\}$$

while $H(\mathbf{x}, t)$ is the subspace of $H(\mathbf{x})$ spanned by the same $x_n(u)$ for all $u \leq t$. The projection operator in $H(\mathbf{x})$ with range $H(\mathbf{x}, t)$ will be denoted by P_t.

For any element z of $H(\mathbf{x})$ we write

$$z(t) = P_t z.$$

$z(t)$ will then be the random variable of a stochastic process with orthogonal increments such that

$$Ez(t) = 0, \quad E|z(t)|^2 = F(t),$$

where the variance function $F(t)$ is never decreasing, everywhere continuous to the left provided (A) and (B) below are satisfied, and such that $F(-\infty)=0$, $F(+\infty)=E|z|^2$. The Hilbert space spanned by the $z(u)$ for all real u is a cyclic subspace of $H(\mathbf{x})$ which we denote by $C(z)$, while $C(z, t)$ is the subspace spanned by the $z(u)$ for all $u \leqq t$. It is known that every element of $C(z)$ will be of the form

$$\int_{-\infty}^{\infty} c(t)\, dz(t),$$

where $c(t)$ is a nonrandom function belonging to $L_2 F(t)$, and the stochastic integral is defined as an integral in quadratic mean.

In the set of all z, and in the set of the corresponding variance functions $F(t)$, we introduce a partial ordering by writing $z_1 \succ z_2$ and $F_1 \succ F_2$ whenever F_2 is absolutely continuous with respect to F_1. Every variance function defines in the well-known way a measure on the real axis, and F_2 is absolutely continuous with respect to F_1 if and only if every set of F_1 measure zero is also F_2 measure zero. If simultaneously $F_1 \succ F_2$ and $F_2 \succ F_1$ we say that F_1 and F_2 are equivalent, and also the corresponding z_1 and z_2. All variance functions F equivalent to a given F_1 will be said to form the equivalence class of F_1.

We shall consider a stochastic vector process $\mathbf{x}(t)$ satisfying the following two conditions (A) and (B):

(A) The limits $x_n(t \pm 0)$ exist for all n and t as limits in the norm of $H(\mathbf{x})$, and $x_n(t-0)=x_n(t)$.

(B) The limiting space $H(\mathbf{x}, -\infty)$ contains only the zero element of $H(\mathbf{x})$.

It follows from (A) that the Hilbert space $H(\mathbf{x})$ is separable, and consequently the subspace $H(\mathbf{x}, t)$, which is never decreasing when t increases, has at most an enumerable number of discontinuities. When (B) is satisfied, the $\mathbf{x}(t)$ process is called purely nondeterministic (or linearly regular).

When (A) and (B) are satisfied, it is known that it is possible to find a finite or infinite sequence of elements of $H(\mathbf{x})$

(1) $z_1 \succ z_2 \succ \ldots \succ z_N$

such that the corresponding cyclic subspaces $C(z_n)$ are mutually orthogonal, and such that $H(\mathbf{x})$ is the vector sum of all these orthogonal cyclic subspaces:

(2) $H(\mathbf{x}) = C(z_1) \oplus \ldots \oplus C(z_N).$

The number N and the equivalence classes of the corresponding variance functions $F_1, \ldots, F_N$ are uniquely defined by the given $\mathbf{x}(t)$ process, and N is called the multiplicity of the process. Applying the projection P_t we obtain the relation

$$H(\mathbf{x}, t) = C(z_1, t) \oplus \ldots \oplus C(z_N, t)$$

for every t. As $x_n(t)$ is an element of $H(\mathbf{x}, t)$, this gives the representation

$$x_n(t) = \sum_{r=1}^{N} \int_{-\infty}^{\infty} c_{n,r}(t, u)\, dz_r(u).$$

This shows how the components of the $\mathbf{x}(t)$ process can be regarded as additively built up by N-dimensional innovation elements $\{dz_1(u), \ldots, dz_N(u)\}$ associated with every instant u in the "past and present" from the point of view of the instant t.

The basic facts of multiplicity theory given above are stated here in the form in which they occur in my papers [1—3].

2. A theorem on multiplicity

Theorem. *Let* $\mathbf{x}(t)$ *be a given vector process satisfying* (A) *and* (B), *and suppose that the component* $x_n(t)$, *regarded as a one-dimensional process, has finite multiplicity* N_n. *Then the multiplicity of the* $\mathbf{x}(t)$ *process is* $N \leq \sum_{n=1}^{q} N_n$.

I have stated this theorem without proof in a paper of 1961 [1, p. 265]. Although the theorem has been referred to in the literature [e.g. 4, p. 228], no proof has been published, as far as I know. It may be of some interest to give a complete proof. As stated in my paper just quoted, the proof is slightly more involved than may possibly be expected. The proof will depend on two Lemmas.

Lemma 1. *Let* $z_1, \ldots, z_n$ *and* w *be elements of* $H(\mathbf{x})$ *such that the cyclic subspaces* $C(z_1), \ldots, C(z_n)$ *are mutually orthogonal. Then it is possible to find an element* z_{n+1} *of* $H(\mathbf{x})$ *such that* $C(z_{n+1})$ *is orthogonal to* $C(z_r)$ *for* $r=1, \ldots, n$, *and such that the vector sum*

$$(3) \qquad\qquad C(z_1) \oplus \ldots \oplus C(z_{n+1})$$

includes the cyclic subspace $C(w)$.

Let w_r be the projection of w on $C(z_r)$. Then define z_{n+1} by

$$w = w_1 + \ldots + w_n + z_{n+1},$$
$$P_t w = P_t w_1 + \ldots + P_t w_n + P_t z_{n+1}.$$

Here w_r and $P_t w_r$ belong to $C(z_r)$, while z_{n+1} is orthogonal to $C(z_r)$ for all $r=1, \ldots, n$. It follows [1, p. 259] that $C(z_{n+1})$ is orthogonal to all these $C(z_r)$.

Now every element of $C(w)$ is the limit of a convergent sequence of finite linear combinations of the $P_t w$. According to the above relation every such linear combination is a sum of finite linear combinations of elements of $C(z_1), \ldots, C(z_{n+1})$, and since these subspaces are mutually orthogonal, the limit of any convergent sequence of this kind will be a sum of elements of the same subspaces. Thus the

vector sum (3) includes every element of $C(w)$, and the Lemma is proved. Note that, in the case when w belongs to the vector sum of $C(z_1), \ldots, C(z_n)$, the z_{n+1} defined in the above proof is equal to zero.

It will be seen without difficulty that, by repeated application of this Lemma to the vector process $\mathbf{x}(t)$ satisfying the conditions of the Theorem, we can find $N = \sum_{n=1}^{q} N_n$ elements $z_1, \ldots, z_N$ of $H(\mathbf{x})$, some of which may be equal to zero, such that the corresponding cyclic subspaces are all mutually orthogonal, and the condition (2) will be satisfied. But the condition (1) may not be satisfied, and in order to complete the proof of the Theorem we shall require the following Lemma.

Lemma 2. *If $z_1, \ldots, z_N$ are a finite number of elements of $H(\mathbf{x})$ such that the cyclic subspaces $C(z_n)$ are all mutually orthogonal, and the relation (2) is satisfied, we can find N elements $w_1, \ldots, w_N$ of $H(\mathbf{x})$, some of which may be equal to zero, such that the cyclic subspaces $C(w_n)$ are all mutually orthogonal, and we have*

$$(4) \qquad\qquad\qquad w_1 \succ w_2 \succ \ldots \succ w_N$$

$$(5) \qquad\qquad\qquad H(\mathbf{x}) = C(w_1) \oplus \ldots \oplus C(w_N).$$

This Lemma being proved, it will be seen that the truth of the Theorem follows from the remark made before stating the Lemma.

Consider the variance functions F_n corresponding to the z_n given in the Lemma. The never decreasing function $\mathbf{F}(t) = \sum_{n=1}^{N} F_n(t)$ defines a measure on the real axis, which we shall use throughout the proof of the Lemma. Expressions like 'measurable', 'almost everywhere (a.e.)', etc., will always refer to the $\mathbf{F}$ measure.

We have $\mathbf{F} \succ F_n$ for every n, so that the Radon—Nikodym derivative $F_n' = dF_n/d\mathbf{F}$ will be defined a.e. and measurable. Note that, if t is a discontinuity point of some of the F_n, we shall have $F_n'(t) > 0$ for every F_n which is discontinuous at t, and $F_n'(t) = 0$ for the other F_n. For $n, r = 1, \ldots, N$ we now define a function $g_{n,r}(t)$ by writing for $n \geq r$

$$g_{n,r}(t) = \begin{cases} 1 & \text{if } F_n'(t) \succ 0 \text{ and there are exactly } r-1 \text{ positive} \\ & \text{among } F_1'(t), \ldots, F_{n-1}'(t), \\ 0 & \text{otherwise,} \end{cases}$$

and $g_{n,r}(t) = 0$ for all t when $n < r$. Then $g_{n,r}(t)$ will be defined a.e. and measurable. It will be seen that $g_{n,r}(t)$ has the following three properties a.e.

(6) For given values of n and t, we have $g_{n,r}(t) = 1$ for at most one r.

(7) For given values of r and t, we have $g_{n,r}(t) = 1$ for at most one n.

(8) If $g_{n,r}(t) = 1$ and $n \geq r > 1$, there is exactly one $m < n$ such that $g_{m,r-1}(t) = 1$.

The sum of stochastic integrals

$$w_r = \sum_{n=1}^{N} \int_{-\infty}^{\infty} g_{n,r}(u) \, dz_n(u)$$

is a well-defined element of $H(\mathbf{x})$, and we have

$$w_r(t) = P_t w_r = \sum_{n=1}^{N} \int_{-\infty}^{t} g_{n,r}(u) \, dz_n(u),$$

$$G_r(t) = E|w_r(t)|^2 = \sum_{n=1}^{N} \int_{-\infty}^{t} g_{n,r}(u) \, dF_n(u).$$

We shall now show that the sequence $w_1, \ldots, w_N$ satisfies the conditions of the Lemma.

In order to show that the cyclic subspaces $C(w_1), \ldots, C(w_N)$ are orthogonal, it will be sufficient to show that $w_r(t) \perp w_s(u)$ for $r \neq s$ and all t, u. Clearly we may suppose $t \leq u$, and then have by (6)

$$E w_r(t) \overline{w_s(u)} = \sum_{n=1}^{N} \int_{-\infty}^{t} g_{n,r}(v) g_{n,s}(v) \, dF_n(v) = 0.$$

Further we evidently have, since all the w_r are elements of $H(\mathbf{x})$,

$$H(\mathbf{x}) \supset C(w_1) \oplus \ldots \oplus C(w_N).$$

In order to show that the two members of this relation are identical, we have to show that every element of $H(\mathbf{x})$ belongs to the vector sum of the $C(w_r)$. Now by hypothesis every element of $H(\mathbf{x})$ is of the form

$$y = \sum_{n=1}^{N} \int_{-\infty}^{\infty} c_n(t) \, dz_n(t)$$

where $c_n(t)$ belongs to $L_2 F_n$. By (6) we have for every n and t

$$\sum_{r=1}^{N} g_{n,r}(t) = 1 \quad \text{or} \quad 0.$$

In the set of values of t where this sum is equal to zero, it follows from the definition of $g_{n,r}(t)$ that $F_n'(t) = 0$, so that the integral of $c_n(t) \, dz_n(t)$ over this set is zero. Thus we have

$$y = \sum_{r=1}^{N} \sum_{n=1}^{N} \int_{-\infty}^{\infty} c_n(t) g_{n,r}(t) \, dz_n(t).$$

Now define a function $b_r(t)$ by writing

$$b_r(t) = \begin{cases} c_n(t) & \text{when} \quad g_{n,r}(t) = 1, \\ 0 & \text{when} \quad g_{n,r}(t) = 0. \end{cases}$$

By (7) this $b_r(t)$ is uniquely defined a.e., and we have

$$y = \sum_{r=1}^{N} \int_{-\infty}^{\infty} b_r(t) \sum_{n=1}^{N} g_{n,r}(t) \, dz_n(t) = \sum_{r=1}^{N} \int_{-\infty}^{\infty} b_r(t) \, dw_r(t).$$

As it is easily seen that $b_r(t)$ belongs to $L_2 G_r$, this shows that y is an element of the vector sum of the $C(w_r)$, so that (5) is satisfied.

It remains to show that (4) is also satisfied. We consider the variance functions $G_r(t)$ of the w_r and have to show that $G_r \succ G_{r+1}$ for $r = 1, \ldots, N-1$. Let S be a set of positive measure such that

$$(9) \qquad \int_S dG_r(t) = \sum_{n=1}^{N} \int_S g_{n,r}(t) \, dF_n(t) = 0.$$

It will then be proved that

$$(10) \qquad \int_S dG_{r+1}(t) = \sum_{n=1}^{N} \int_S g_{n,r+1}(t) \, dF_n(t) = 0.$$

In every term of the sum in the last member, the integral extended over that part of S where $g_{n,r+1}(t) = 0$ is obviously zero. If $g_{n,r+1}(t) = 1$, there is by (8) exactly one $m < n$ such that $g_{m,r}(t) = 1$. But it follows from (9) that, for every m, the integral

$$\int dF_m = \int F'_m \, d\mathbf{F}$$

extended over that part of S where $g_{m,r}(t) = 1$, and thus $F'_m(t) \succ 0$, is equal to zero. Consequently this set is a null set for the $\mathbf{F}$ measure, and the integral

$$\int dF_n = \int F'_n \, d\mathbf{F}$$

extended over this set, is also equal to zero. As this holds for all n and for all $m < n$, the relation (10) follows, and the proof of the Lemma is completed.

As already remarked above, the truth of the Theorem immediately follows from the two Lemmas.

References

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