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Jensen measures and analytic multifunctions

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Abstract. This paper describes plurisubharmonic convexity and hulls, and also analytic multifunctions in terms of Jensen measures. In particular, this allows us to get a new proof of Słodkowski's theorem stating that multifunctions are analytic if and only if their graphs are pseudoconcave. We also show that multifunctions with plurisubharmonically convex fibers are analytic if and only if their graphs locally belong to plurisubharmonic hulls of their boundaries. In the last section we prove that minimal analytic multifunctions satisfy the maximum principle and give a criterion for the existence of holomorphic selections in the graphs of analytic multifunctions.

1. Introduction

Analytic multifunctions, introduced by K. Oka in 1934, have found quite a few of applications in recent years. The main driving force of this development was Słodkowski's theorem ([S1], see also Section 7) that characterized analytic multifunctions in many different ways. This result provided applications of analytic multifunctions to operator theory, uniform algebras and other subjects.

One of the most impressive achievements in this area was the new proof of the corona theorem by Berndtsson–Ransford [BR] and Słodkowski [S2]. The proof used three major facts about analytic multifunctions over the unit disk:

(1) every polynomially convex analytic multifunction has a holomorphic selection;

(2) every analytic multifunction is contained in a polynomially convex analytic multifunction with the same boundary values;

(3) there is an analytic multifunction whose boundary values satisfy the conditions of the corona theorem.

For analytic multifunctions over the unit ball in $\mathbf{C}^n$, $n\geq 2$, (1) does not hold, (2) loses its relevance, and only (3) is still true [S2].

Recent years have also seen the surge of interest in Jensen measures (see, for example, [CCW], [CR], [R3] and [P1]). They also happen to be useful in many

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applications. Many notions of complex analysis and pluripotential theory can be expressed and studied in terms of Jensen measures.

This paper uses Jensen measures to study analytic multifunctions and related objects. Another recent approach was developed by Ransford in [R2]. In Section 2 we present basic facts about plurisubharmonic functions and Jensen measures. In many cases it is sensible to replace polynomial convexity and hulls by their plurisubharmonic analogs. Section 3 describes these notions in terms of Jensen measures. Jensen measures are defined through the space of plurisubharmonic functions on a domain. In Section 4 we show that in some cases, for example when the domain is pseudoconvex and the measure is supported by a plurisubharmonically convex compact set, the dependence on the domain is irrelevant. All this preparatory work is needed for the next sections.

In Section 5 we prove Theorem 5.1: a multifunction K over a plurisubharmonically convex set F is analytic if and only if every Jensen measure on F can be lifted to K as a Jensen measure. Our definition of analytic multifunctions follows Aupetit's paper [A]. Słodkowski calls them weakly analytic. Section 6 contains a standard result describing smooth analytic multifunctions. The famous Słodkowski's theorem gets a new proof using Theorem 5.1 in Section 7.

Theorem 8.1 ties together the notions of plurisubharmonical convexity and analytic multifunctions. It states that a multifunction is analytic if and only if its restriction to every line belongs to the plurisubharmonic hull of its boundary values. The last section deals with minimal analytic multifunctions. In particular, we show that such multifunctions satisfy the maximum principle.

2. Plurisubharmonic functions and Jensen measures

Let us denote by λ the measure $d\theta/2\pi$ on the unit circle $\{z=e^{i\theta}: 0\le\theta\le 2\pi\}$. For a complex manifold M we denote by $\mathcal{H}(M)$ the set of all holomorphic mappings f of a neighborhood W_f of the closure $\overline{U}$ of the unit disk $U\subset\mathbb{C}$ into M.

An upper semicontinuous function u on an open set $V\subset\mathbf{C}^n$ is *plurisubharmonic* if for every $z\in V$ and a vector $v\in\mathbf{C}^n$ there is $r>0$ such that

$$u(z)\le\frac{1}{2\pi}\int_0^{2\pi}u(z+e^{i\theta}tv)\,d\theta \tag{1}$$

for all $t<r$.

A probability Borel measure μ on V with compact support is a *Jensen measure* with barycenter $z\in V$ if

$$u(z)\le\int u\,d\mu$$

for every plurisubharmonic function u on V. We will denote the set of all such measures by $J_z(V)$.

For a relatively closed set $K \subset V$ and $z \in V$ let $J_z(K,V)$ be the set of all measures in $J_z(V)$ supported by K. If $z \in K$ we define $J_z(K)$ to be the set of all measures μ such that μ is in $J_z(K,W)$ for every open set W containing K.

We denote by $C(V)$ the space of all continuous functions ϕ on V with the topology defined by the seminorms

$$\|\phi\|_K = \sup_{z \in K} |\phi(z)|,$$

where K runs over all compact sets in V. Any continuous linear functional μ on $C(V)$ can be represented as $\mu(\phi) = \int_K \phi \, d\mu$, where K is a compact set and μ is a non-negative regular Borel measure on K (see [C, Proposition 4.4.1]).

Since every continuous function on a relatively closed set $K \subset V$ can be extended to a continuous function on V, the weak-$*$ topology of $J_z(K,V)$ in $C^*(V)$ coincides with the weak-$*$ topology in $C^*(K)$. In particular, the set $J_z(K,V)$ is metrizable in this topology when K is compact (see [C, Theorem 5.5.1]). Evidently, the sets $J_z(K,V)$ and $J_z(K)$ are convex and weak-$*$ closed in $C^*(V)$.

If $f \in \mathcal{H}(V)$ then we define the measure λ_f as the push-forward of $\lambda = (1/2\pi)\, d\theta$ by f, i.e., $\lambda_f(E) = f_* \lambda(E) = \lambda(f^{-1}(E))$ for any set $E \subset V$. The measure λ_f is a non-negative regular Borel measure on V and if u is plurisubharmonic and $f(0) = z$, then

$$u(z) \le \int u \, d\lambda_f.$$

Thus λ_f is a Jensen measure. We will call such measures *analytic disk measures* or, simply, *disk measures*.

The following two theorems are proved in [P2] and [BS], respectively (see also [CR, Theorem 7.2]).

Theorem 2.1. *If ϕ is an upper semicontinuous function on an open set $V \subset \mathbf{C}^n$, then the function*

$$\mathcal{D}\phi(z) = \inf\left\{ \int \phi \, d\lambda_f : f \in \mathcal{H}(V) \text{ and } f(0) = z \right\}$$

is plurisubharmonic on V.

Theorem 2.2. *The set $J_{z_0}(V)$ is the weak-$*$ closure of the holomorphic measures μ_f with $f(0) = z_0$.*

We will frequently use their corollary.

Corollary 2.3. *If ϕ is an upper semicontinuous function on V, then*

$$\mathcal{D}\phi(z)=\inf\left\{\int \phi\,d\mu:\mu\in J_z(V)\right\}.$$

Proof. If ϕ is continuous then the corollary follows immediately from Theorem 2.2. If ϕ is upper semicontinuous then there is a decreasing sequence of continuous functions ϕ_j converging to ϕ. The functions $\mathcal{D}\phi_j$ form a decreasing sequence of plurisubharmonic functions converging to a plurisubharmonic function $\mathcal{D}\phi$. If we let $u(z)$ denote the right-hand side in the equation, then, clearly, $\mathcal{D}\phi\geq u$. If $\mu\in J_{z_0}(V)$, then

$$\int \phi\,d\mu=\lim_{j\to\infty}\int \phi_j\,d\mu\geq \lim_{j\to\infty}\mathcal{D}\phi_j(z_0)=\mathcal{D}\phi(z_0)$$

and this proves that $u=\mathcal{D}\phi$. □

3. Plurisubharmonic hulls and pseudoconvex sets

If K is a set in V then the *plurisubharmonic hull* of K in V is the set $\widetilde{K}$ of all points $z\in V$ such that $u(z)\leq 0$ whenever a plurisubharmonic functions u on V is less or equal to 0 on K. We call K *plurisubharmonically convex* in V if $\widetilde{K}=K$. If V is a domain of holomorphy and K is compact then "plurisubharmonically convex" means "holomorphically convex". If V is a Runge domain, then K is plurisubharmonically convex if and only if it is polynomially convex.

Theorem 3.1. *A point $z_0\in V$ belongs to the plurisubharmonic hull $\widetilde{K}$ in V of a compact subset K of an open set V if and only if the set $J_{z_0}(K,V)$ is non-empty.*

Proof. If $\mu\in J_{z_0}(K,V)$ and u is a plurisubharmonic function on V such that $u\leq 0$ on K, then $u(z_0)\leq 0$. Thus $z_0\in\widetilde{K}$.

Suppose that there is a point $z_0\in\widetilde{K}$ and $J_{z_0}(K,V)$ is empty. Consider an exhaustion of V by increasing open sets V_j such that $\overline{V}_j\Subset V_{j+1}$ and both z_0 and K belong to V_1. The set $J_{z_0}(\overline{V}_1,V)$ is weak-$*$ compact and therefore there is a neighborhood V_0 of K and $a_1>0$ such that $\overline{V}_0\subset V_1$ and $\mu(V_0)<1-a_1$ for every measure $\mu\in J_{z_0}(\overline{V}_1,V)$.

We take a non-negative continuous function ϕ_1 on V equal to 0 on K and to $1/a_1$ on $V\setminus V_0$. Then

$$\int \phi_1\,d\mu\geq 1$$

for every $\mu\in J_{z_0}(\overline{V}_1, V)$. Let us prove by induction that for all natural j there are non-negative continuous functions ϕ_j on V satisfying the conditions:

(1) $\phi_j=\phi_{j-1}$ on $\overline{V}_{j-1}$;

(2) for every $\mu\in J_{z_0}(\overline{V}_j, V)$

$$\int \phi_j \, d\mu \geq \frac{1}{2}+\frac{1}{2^j}.$$

We have already found ϕ_1. Suppose that the functions ϕ_i have been chosen for all $1\leq i\leq j$. Let us prove that there exists a neighborhood $W\Subset V_{j+1}$ of $\overline{V}_j$ and a number $a_{j+1}>0$ such that for every $\mu\in J_{z_0}(\overline{V}_{j+1}, V)$ either $\mu(\overline{V}_{j+1}\setminus W)\geq a_{j+1}$ or

$$\int \phi_j \, d\mu \geq \frac{1}{2}+\frac{1}{2^{j+1}}.$$

If not, then we can find a sequence of $\mu_k\in J_{z_0}(\overline{V}_{j+1}, V)$ such that $\mu_k(\overline{V}_{j+1}\setminus V_j)\to 0$, as $k\to\infty$, and

$$\int \phi_j \, d\mu_k < \frac{1}{2}+\frac{1}{2^{j+1}}.$$

Since the set $J_{z_0}(\overline{V}_{j+1}, V)$ is weak-$*$ compact we may assume that measures μ_k weak-$*$ converge to $\mu\in J_{z_0}(\overline{V}_j, V)$. Then

$$\int \phi_j \, d\mu = \lim_{k\to\infty} \int \phi_j \, \mu_k \leq \frac{1}{2}+\frac{1}{2^{j+1}}$$

and this contradiction proves our statement.

Let us take a non-negative continuous function ϕ_{j+1} such that $\phi_{j+1}=\phi_j$ on $\overline{V}_j$ and $\phi_{j+1}=1/a_{j+1}$ on $V\setminus W$. Evidently ϕ_{j+1} will satisfy all conditions listed above.

Let $\phi=\lim_{j\to\infty}\phi_j$. The function ϕ is continuous and non-negative on V, equal to 0 on K, and

$$\int \phi \, d\mu \geq \frac{1}{2}$$

for every $\mu\in J_{z_0}(V)$. Thus $\mathcal{D}\phi=0$ on K and $\mathcal{D}\phi(z_0)\geq\frac{1}{2}$ and therefore $z_0\notin\widetilde{K}$. This contradiction proves the theorem. □

Corollary 3.2. *A compact subset K in an open set V is plurisubharmonically convex in V if and only if the barycenter of every Jensen measure μ in V supported by K belongs to K.*

In general, the plurisubharmonic hull of a compact subset K in an open set V need not even be closed. To see this we use an example of J.-E. Fornæss as it is exposed in [K, Example 2.9.4]. In the notation of [K] let K be the union of $\partial D(0,\frac{3}{2})\times\{1/j\}$, $j=2,3,\dots$, and $\partial D(0,\frac{3}{2})\times\{0\}$. Clearly, the union of $D(0,\frac{3}{2})\times\{1/j\}$, $j=2,3,\dots$, belongs to $\widetilde{K}$, but the plurisubharmonic function u from the example is equal to -1 on K and to $-\frac{1}{2}$ on $D(0,1)$.

However, the following form of the Kontinuitätssatz holds.

Corollary 3.3. *An open set V is a pseudoconvex domain if and only if the plurisubharmonic hull of any compact set K in V is compact.*

Proof. To prove that plurisubharmonic hulls are compact we suppose that points $z_j \in \widetilde{K}$ converge to a point $z_0 \in V$. Then there are measures $\mu_j \in J_{z_j}(K,V)$. We may assume that μ_j weak-$*$ converge to a probability measure μ supported by K.

To show that $\mu \in J_{z_0}(K,V)$ we take a plurisubharmonic function u on V and find by [FN] a decreasing sequence of continuous plurisubharmonic functions u_k converging to u. Then

$$u_k(z_0) = \lim_{j\to\infty} u_k(z_j) \le \lim_{j\to\infty} \int u_k\, d\mu_j = \int u_k\, d\mu.$$

Hence

$$u(z_0) = \lim_{k\to\infty} u_k(z_0) \le \lim_{k\to\infty} \int u_k\, d\mu = \int u\, d\mu.$$

Thus $z_0 \in \widetilde{K}$ and $\widetilde{K}$ is closed.

Let ϕ be a continuous plurisubharmonic exhausting function on V. If $K \subset \{z : \phi(z) \le r\}$, then $\widetilde{K}$ also lies in $\{z : \phi(z) \le r\}$. Hence $\widetilde{K}$ is compact.

To prove the converse statement we need to show that for any sequence of mappings $f_j \in \mathcal{H}(V)$ such that $f_j(S)$ lie in a compact set K in V the sets $f_j(U)$ also lie in a compact set. Then the statement follows from the Kontinuitätssatz. But since $f_j(U) \subset \widetilde{K}$, which is compact, this is evident. □

The following theorem tells us that in the case of pseudoconvex open sets we need only one continuous plurisubharmonic function to describe a plurisubharmonically convex set.

Theorem 3.4. *If an open set V is pseudoconvex, then a compact set $K \subset V$ is plurisubharmonically convex in V if and only if there is a continuous plurisubharmonic function u on V that is equal to 0 on K and positive on $V \setminus K$.*

Proof. The "if" part is trivial. To prove the "only if" part, for every point $z \in V \setminus K$ we take a non-negative plurisubharmonic function v_z that is equal to 0 on K and is such that $v_z(z) = a_z > 0$. By [FN] there is a decreasing sequence of continuous plurisubharmonic functions $v_{z,j}$ converging to v_z. Since the functions $v_{z,j}$ converge uniformly to 0 on K, we can find k such that $v_{z,k} < \frac{1}{2} a_z$ on K. Let $u_z = \max\{v_{z,k} - \frac{1}{2} a_z, 0\}$. The function u_z is continuous, equal to 0 on K and $u_z(z) > \frac{1}{2} a_z$.

Let us take a sequence of open sets V_j, $-\infty < j < \infty$, such that: $V_j \Subset V_{j+1}$, $\bigcup_{j=-\infty}^{\infty} V_j = V$ and $\bigcap_{j=-\infty}^{\infty} V_j = K$. Since the sets $\overline{V}_{j+1} \setminus V_j$ are compact we can find finitely many points z_k in $\overline{V}_{j+1}\setminus V_j$ such that the function $u_j=\max_k u_{z_k}$ is positive on $\overline{V}_{j+1}\setminus V_j$. Let $b_j=2^{-|j|}\|u_j\|^{-1}_{\overline{V}_{|j|}}$. Then the function

$$u(z)=\sum_{j=-\infty}^{\infty} b_j u_j(z)$$

is continuous, plurisubharmonic, equal to 0 on K and positive on $V\setminus K$. □

4. The relativity of Jensen measures

Clearly $J_z(K)\subset J_z(K,V)\subset J_z(K,\mathbf{C}^n)$. We are interested in cases when $J_z(K)=J_z(K,V)$. Let us start with a lemma which guarantees the extension of plurisubharmonic functions.

Lemma 4.1. *Suppose that K is a compact set in an open set V such that there is a continuous plurisubharmonic function u on V equal to 0 on K and positive on $V\setminus K$. If v is a plurisubharmonic function defined on a neighborhood $W\subset V$ of K and bounded below on K, then there is a plurisubharmonic function v' on V which coincides with v on K.*

Proof. Suppose that $v\geq A>-\infty$ on K. We take a neighborhood W' of K that compactly belongs to W. Let $B<\infty$ be the supremum of v on $\partial W'$ and let $C>0$ be the infimum of u on $\partial W'$. Then the function

$$u'=2\frac{B-A}{C}u+A$$

is smaller than v on K and greater than v on $\partial W'$. Hence the function v' equal to the maximum of u' and v on W' and to u' on $V\setminus W'$ is plurisubharmonic on V. Moreover, $v'\equiv v$ on K. □

Theorem 4.2. *Suppose that K is a compact set in an open set V such that there is a continuous plurisubharmonic function u on V equal to 0 on K and greater than zero on $V\setminus K$. If a point $z_0\in K$ then $J_{z_0}(K,V)=J_{z_0}(K)$.*

Proof. We have to show that $J_{z_0}(K,V)\subset J_{z_0}(K)$. Let v be a lower bounded plurisubharmonic function on a neighborhood W of K. By Lemma 4.1 there is a plurisubharmonic function v' on V equal to v on K. If $\mu\in J_{z_0}(K,V)$, then

$$\int v\,d\mu=\int v'\,d\mu\geq v'(z_0)=v(z_0).$$

Hence $\mu\in J_{z_0}(K)$.

If v is any plurisubharmonic function on a neighborhood W of K, then we take the sequence of lower bounded plurisubharmonic functions $v_j = \max\{v, -j\}$. Clearly, $v_j \searrow v$. If $\mu \in J_{z_0}(K, V)$, then

$$\int v\, d\mu = \lim_{j\to\infty} \int v_j\, d\mu \geq \lim_{j\to\infty} v_j(z_0) = v(z_0).$$

Hence $\mu \in J_{z_0}(K)$. □

The corollaries below follow immediately from Theorems 3.4 and 4.2.

Corollary 4.3. *If a compact set K in a pseudoconvex domain V is plurisubharmonically convex, then for every point $z \in K$ we have $J_z(K) = J_z(K, V)$.*

Corollary 4.4. *If u is a lower bounded plurisubharmonic function defined on a neighborhood W of a compact plurisubharmonically convex set K in a pseudoconvex domain V, then there is a plurisubharmonic function v on V which coincides with u on K.*

5. Analytic multifunctions

If K' is a set in $\mathbf{C}^n$, then a *multifunction* K on K' is a mapping of K' into the set of non-empty compacts subsets of $\mathbf{C}^m$. For our purposes it is reasonable to identify a mapping K with the set $\{(z, w) \in K' \times \mathbf{C}^m : w \in K(z)\}$, which is the graph of K. We will denote the graph of K also by K. If $N = n + m$ and $z = (z_1, \dots, z_N) \in \mathbf{C}^N$, then we define a projection $p(z) = z' = (z_1, \dots, z_n)$. A set $K \subset \mathbf{C}^N$ is a *multifunction* on K' if $p(K) = K'$ and the sets $K(z') = \{z \in K : p(z) = z'\}$ are compact. If ϕ is a function on K, then we define the function

$$\phi_K(z') = \sup_{z \in K(z')} \phi(z)$$

on K'.

We say that a multifunction K on K' is *upper semicontinuous* if for every open set $W \subset \mathbf{C}^m$ the set $\{z' \in K' : K(z') \subset W\}$ is relatively open in K'. A multifunction K is upper semicontinuous if and only if for every compact set $F \subset K'$ the set $p^{-1}(F) \cap K$ is compact. It is easy to see that if K is an upper semicontinuous multifunction and a function ϕ is upper semicontinuous on K, then the function ϕ_K is also upper semicontinuous.

A function u on a compact set $K \subset \mathbf{C}^N$ is *plurisubharmonic* if u is upper semicontinuous and for every $z \in K$ and $\mu \in J_z(K)$ we have

$$u(z) \leq \int u(w)\, \mu(dw).$$

If F' is a subset of K', then we define the restriction of a multifunction K to F' as the set $F=p^{-1}(F')\cap K$.

An upper semicontinuous multifunction K is *analytic* if for every open set V containing K and every plurisubharmonic function u on V the function u_K is plurisubharmonic. The function u can be assumed to be C^∞ because any plurisubharmonic function can be approximated by such functions.

Theorem 5.1. *Suppose that $K\subset\mathbf{C}^N$ is an upper semicontinuous multifunction on a relatively closed set K' in an open set $V\subset\mathbf{C}^n$. Then K is analytic if $p_*J_w(K)=J_z(K')$ for every $z\in K'$ and every $w\in K(z)$. If K is analytic and K' is the union of relative interiors of compact subsets $K'_j\subset K$ such that the restrictions of K to K'_j are also analytic, then $p_*J_w(K)=J_z(K')$ for every $z\in K'$ and every $w\in K(z)$.*

Proof. To prove the first part we fix a point $z'_0\in K'$, a Jensen measure $\nu\in J_{z'_0}(K')$ and a plurisubharmonic function u on a neighborhood W of K. For some $\varepsilon>0$ we find a point $z_0\in K(z'_0)$ such that $u_K(z'_0)<u(z_0)+\varepsilon$. Take $\mu\in J_{z_0}(K)$ such that $p_*\mu=\nu$. Then

$$u_K(z'_0)-\varepsilon<u(z_0)\le\int u\,d\mu\le\int u_K\,d(p_*\mu)=\int u_K\,d\nu.$$

Since ε is arbitrary we get the plurisubharmonicity of u_K.

To prove the second part we suppose that there is a point $z'_0\in K'$, a Jensen measure $\nu\in J_{z'_0}(K')$ and a point $z_0\in K(z'_0)$ such that the set $p_*J_{z_0}(K)$ does not contain ν.

Let F' be a compact subset of K' containing the support of ν and such that $K_{F'}$ is analytic. The set $F=p^{-1}(F')\cap K$ is compact and, therefore, the set $H=p_*J_{z_0}(F)$ is a weak-$*$ closed convex set in $C^*(F')$. Since $\nu\notin H$, there is a function $\phi\in C(F')$ and $\varepsilon>0$ such that

$$\int\phi\,d\nu<\inf_{\mu\in H}\int\phi\,d\mu-\varepsilon.$$

By the same letter ϕ we will denote a continuous extension of ϕ to V. Let $\psi=\phi\circ p$. By Corollary 2.3 the function

$$u(z)=\inf_{\mu\in J_z(W)}\int\psi\,d\mu$$

is plurisubharmonic on a neighborhood W of F and clearly $u\le\psi$ on W. Since F is analytic the function u_F is plurisubharmonic on F' and $u_F\le\phi$.

We can find a neighborhood W of F such that

$$u(z_0)\ge\inf_{\mu\in J_{z_0}(F)}\int\psi\,d\mu-\frac{\varepsilon}{2}=\inf_{\mu\in H}\int\phi\,d\mu-\frac{\varepsilon}{2}.$$

So

$$\int u_F\, d\nu \le \int \phi\, d\nu < u(z_0) - \frac{\varepsilon}{2} \le u_F(z_0') - \frac{\varepsilon}{2}.$$

But this contradicts the plurisubharmonicity of u_F and, therefore, there is a measure $\mu \in J_{z_0}(F)$ such that $p_* \mu = \nu$. Since, evidently, $J_{z_0}(F) \subset J_{z_0}(K)$, our theorem is proved. □

Now our goal is to find reasonable situations when the condition on the set K' from the second part of Theorem 5.1 holds. We start with a basic lemma.

Lemma 5.2. *Suppose that a set $K \subset \mathbf{C}^N$ is an analytic multifunction on a relatively closed set K' in an open set $V \subset \mathbf{C}^n$. If for a compact set $F' \subset K'$ there exists a continuous non-negative plurisubharmonic function ϕ on V such that $F' = \{z : \phi(z) = 0\}$, then the restriction of K to F' is analytic.*

Proof. Suppose that $F' \subset K'$ is a compact set and ϕ is a continuous non-negative plurisubharmonic function on V such that $F' = \{z : \phi(z) = 0\}$. Let u be a lower bounded plurisubharmonic function on a neighborhood W of $F = p^{-1}(F') \cap K$. Let a be the maximum of u on F. Take a neighborhood W_1 of F compactly contained to $W \cap p^{-1}(V)$ such that $u \le b = a+1$ on $\overline{W}_1$. Then take an open set $W_2' \Subset W_1' = p(W_1)$ containing F'. As in the proof of Lemma 4.1 choose a continuous plurisubharmonic function ϕ_1 on V which is smaller than u_K on F' and greater than b on $W_1' \setminus W_2'$. Let $\psi = \phi_1 \circ p$ and an open set $Y = p^{-1}(V \setminus \overline{W}_2')$. Let v be equal to $\max\{u, \psi\}$ on W_1 and to ψ on Y. The function v is plurisubharmonic on $W_1 \cup Y$ and $v_K = u_F$ on F'. Hence u_F is plurisubharmonic.

If u is not bounded below then let $u_m = \max\{u, -m\}$, $m = 1, 2, \dots$. Then u_F is the limit of a decreasing sequence of plurisubharmonic functions $(u_m)_F$ and, consequently, also plurisubharmonic. □

This lemma has two corollaries that describe most reasonable situations. The first one is applied to analytic multifunctions on open sets.

Corollary 5.3. *If K is an analytic multifunction on an open set $V \subset \mathbf{C}^n$, then its restriction to any compact subset of V is also analytic.*

Proof. Let F' be a compact set in V and let u be a plurisubharmonic function on an open neighborhood W of $F = p^{-1}(F') \cap K$. For $z_0' \in F'$ we take a closed ball $\overline{B} = \overline{B}(z_0', r)$ of radius r centered at z_0' such that $p^{-1}(\overline{B}) \cap K \subset W$. Since the closed ball satisfies the condition of Lemma 5.2, the function $u_{\overline{B}}$ is plurisubharmonic. Since F' can be covered by such balls, u_F is also plurisubharmonic. □

The second corollary has sense when we are talking about, say, analytic varieties in pseudoconvex domains.

Corollary 5.4. *Suppose that a set $K \subset \mathbf{C}^N$ is an analytic multifunction on a relatively closed plurisubharmonically convex set K' in a pseudoconvex domain $V \subset \mathbf{C}^n$. Then K' is the union of relative interiors of compact subsets $K'_j \subset K$ such that the restrictions of K to K'_j are also analytic.*

Proof. Let ψ be a continuous exhausting function of V. The sets

$$F'_j = \{z : \psi(z) \leq r\} \cap K'$$

are plurisubharmonically convex and their interiors exhaust K'. The rest follows from Theorems 3.4 and 5.1. □

To prove that an upper semicontinuous multifunction K over an open set V is analytic it suffices to verify that for every point $z \in V$, every vector $v \in \mathbf{C}^n$ and every point $w \in K(z)$ there is a Jensen measure ν with barycenter at w such that $p_* \nu = \mu$, where μ is the measure $(2\pi)^{-1} d\theta$ on a circle $z + rve^{i\theta}$. That means that μ can be lifted to $J_w(K)$ for all sufficiently small $r > 0$. Note that the support of ν lies over the line $\{z + v\zeta : \zeta \in \mathbf{C}\}$. This implies a corollary.

Corollary 5.5. *An upper semicontinuous multifunction K over an open set $V \subset \mathbf{C}^n$ is analytic if and only if its restriction to any complex line is analytic.*

The following theorem, due to Hartogs, is well known and its proof is brought here to demonstrate how Theorem 5.1 works.

Theorem 5.6. *Let K be an analytic multifunction over an open set $V \subset \mathbf{C}^n$ such that the fibers $K(z)$ are singletons for all $z \in V$. Then K is a graph of a holomorphic mapping.*

Proof. Let $f(z) = K(z)$. Since K is upper semicontinuous, the mapping f is continuous. We take a point $z_0 \in V$ which we assume to be the origin, a vector $v \in \mathbf{C}^m$ and a number $r > 0$ such that the disk $\{z : z = z_0 + \zeta v \text{ and } |\zeta| \leq r\}$ is in V. For the measure $\mu = (2\pi)^{-1} d\theta$ on a circle $\{z : z = z_0 + te^{i\theta} v \text{ and } 0 \leq t \leq r\}$ we denote by ν the lifting of μ to $J_{w_0}(K)$, where $w_0 = f(z_0)$. Clearly, $\nu = f_* \mu$. If $z = (z_1, \dots, z_N)$ and $f = (f_{n+1}, \dots, f_N)$, then by holomorphicity of z_k

$$0 = \int z_k \, d\mu = \frac{1}{2\pi} \int_0^{2\pi} f_k(te^{i\theta} v) e^{i\theta} \, d\theta$$

for all $k > n$. By Morera's theorem f_k and, consequently, f are holomorphic. □

6. Smooth analytic multifunctions

Theorem 5.1 is rather theoretic and hard to use to recognize analytic multifunctions. We now present a result that allows us to do this. This result is known to specialists but we have never met it in the literature in an explicit form.

A domain $D\subset\mathbf{C}^N$ is *strictly n-pseudoconvex* at a point $z\in\partial D$ if there is a C^2 function ϕ defined on a neighborhood V of this point such that $D\cap V=\{z:\phi(z)<0\}$, $\nabla\phi(z)\neq 0$, and there is an n-dimensional complex space in the complex tangent space to ∂D at z, where the Levy form of ϕ is strictly positive.

The open disk in $\mathbf{C}$ centered at z and of radius r is denoted by $U(z,r)$ while its boundary is $S(z,r)$.

Lemma 6.1. *Suppose that $K\subset\mathbf{C}^N$ is an upper semicontinuous multifunction over a domain $V\subset\mathbf{C}^n$. Let $D=p^{-1}(V)\setminus K$. If D is strictly n-pseudoconvex at every point $w\in\partial D$ such that $p(w)\in V$, then K is an analytic multifunction.*

Proof. Let u be a plurisubharmonic function in a neighborhood of K, $z_0\in\partial D$, $p(z_0)=z_0'$ and $u_K(z_0')=u(z_0)$. We may assume that $z_0'=z_0=0$. Let ϕ be a function defined on a neighborhood W of z_0 such that $D\cap W=\{z:\phi(z)<0\}$, $\nabla\phi(z_0)\neq 0$ and the Levy form H of ϕ is strictly positive on a complex n-space T in the tangent space of ∂D at z_0.

It is well known (see [Sh, Section 13.37]) that there is a holomorphic mapping F of the unit ball B of $\mathbf{C}^n$ into $\mathbf{C}^N$ with the following properties: $F(0)=0$, the rank of $F'(0)$ is n, $F'(0)(\mathbf{C}^n)=T$ and $F(B)\subset K$.

Let $G=p\circ F$ and let $A=\{z\in B:G(z)=0\}$. If there is a component A' of A passing through 0 and of positive dimension, then $F(A')$ belongs to the fiber $K_0=K(0)$ and by the maximum principle u is constant on $F(A')$. But $F(z)$ belongs to the interior of K when $z\neq 0$ and, therefore, there is a point $z\in K_0$ that lies in the interior of K and $u(z)=u_K(0)$. For any vector $v\in\mathbf{C}^n$ take $g(\zeta)=z+\zeta v$. Then

$$u_K(0)=u(z)\le\frac{1}{2\pi}\int_0^{2\pi}u(g(se^{i\theta}))\,d\theta\le\frac{1}{2\pi}\int_0^{2\pi}u_K(se^{i\theta}v)\,d\theta$$

for all sufficiently small s.

Let us fix a vector $v\in\mathbf{C}^n$ and let L be the complex line $\{\zeta v:\zeta\in\mathbf{C}\}$. If there are no components A' of A passing through 0 and of positive dimension, then the analytic variety $G^{-1}(L)$ is one-dimensional near the origin. We take a complex locally irreducible curve $C\subset G^{-1}(L)$ passing through the origin. Then $G(z)=g(z)v$ for every $z\in C$, where g is a holomorphic function on C. Reasoning as in [Ch, 6.1] we find a neighborhood W of 0 in C, where g is a k-sheeted covering mapping of $W\setminus\{0\}$ over $U(0,r)\setminus\{0\}$, $r>0$. Then (see [Ch, 6.1]) there is a holomorphic mapping f of the unit disk U onto W such that $g(f(re^{i\theta}))=s(r)e^{ik\theta}$.

The function $\tilde{u}(\xi)=u(F(f(\xi)))$ is subharmonic on U. If $0<s<r$ and $s=s(t)$, $0<t<1$, then $u_K(se^{ik\theta}v)\geq\tilde{u}(te^{i\theta})$. Therefore,

$$u_K(0)=u(0)\leq\frac{1}{2\pi}\int_0^{2\pi}\tilde{u}(te^{i\theta})\,d\theta\leq\frac{1}{2\pi}\int_0^{2\pi}u_K(se^{i\theta}v)\,d\theta.$$

We have proved that for every point $z\in V$ and every vector $v\in\mathbf{C}^n$ the function u_K satisfies the subaveraging inequality (1). Thus u_K is plurisubharmonic and K is an analytic multifunction. □

Corollary 6.2. *Suppose that $K\subset\mathbf{C}^{n+1}$ is an upper semicontinuous multifunction over an open set $V\subset\mathbf{C}^n$ such that all connected components of $D=p^{-1}(V)\setminus K$ are pseudoconvex. Then K is an analytic multifunction.*

Proof. Each connected component D_j of D has a smooth strictly pseudoconvex exhausting functions ϕ_j. Let $K_r=p^{-1}(V)\setminus\bigcup_j\{z:\phi(z)_j>r\}$. By Lemma 6.1, K_r is an analytic multifunction. Hence $K=\bigcap_{r>0}K_r$ is also an analytic multifunction. □

7. Słodkowski's theorem

In this section we prove the part of the seminal Słodkowski's theorem that is most essential for pluripotential theory.

For an upper semicontinuous multifunction K over an open set $V\subset\mathbf{C}^n$ let us introduce the function

$$\delta_K(z)=-\inf_{w\in K(p(z))}\log|z-w|$$

on $D=p^{-1}(V)\setminus K$.

Theorem 7.1. *Let K be an upper semicontinuous multifunction in $\mathbf{C}^{n+1}$ over an open set $V\subset\mathbf{C}^n$. Then K is analytic if and only if the function δ_K is plurisubharmonic.*

Proof. By Corollary 5.5 it suffices to prove the theorem over a domain $V\subset\mathbf{C}$. Then $D\subset\mathbf{C}^2$. For a domain $D\subset\mathbf{C}^2$ the following result is valid: if the function

$$\delta(z,w)=-\inf_{(z,w')\in\partial D}\log|w'-w|$$

is plurisubharmonic on D, then D is pseudoconvex (see [S1]).

So if $\delta_K(z,w)$ is plurisubharmonic, then every connected component of D is pseudoconvex and K is analytic by Corollary 6.2.

To prove that the function $\delta_K(z)$ is plurisubharmonic in D when K is analytic we note that $\delta_K(z,w)$ is upper semicontinuous on D and subharmonic in w. So we have to show that for every point $(z_0,w_0)\in D$, that we assume to be $(0,0)$, and every $v\in\mathbf{C}^2$ such that $p(v)\neq 0$ we have

$$\delta_K(0,0)\le\frac{1}{2\pi}\int_0^{2\pi}\delta_K(rve^{i\theta})\,d\theta$$

when r is sufficiently small. We may assume that $v=(1,a)$. Let us take a point $z_0=(0,b)\in K$ such that $\delta_K(0,0)=-\log|b|$. Let $r_1>0$ be so small that the line $w-az=0$ does not meet K when $|z|\le r_1$. If $\mu=(2\pi)^{-1}d\theta$ is the measure on the circle $(re^{i\theta},0)$, $r<r_1$, then there is a Jensen measure ν supported by K with barycenter at z_0 such that $p_*\nu=\mu$. The restriction K' of K to $U(0,r_1)$ is also an analytic multifunction. The function $u(z,w)=-\log|w-az|$ is plurisubharmonic in a neighborhood of K' and, therefore,

$$\delta_K(0,0)=u(0,b)\le\int u\,d\nu\le\int\delta_K(z,az)\,d\mu.$$

The theorem is proved. □

8. Plurisubharmonically convex analytic multifunctions

In general, an analytic multifunction K need not to be plurisubharmonically convex even if $p(K)$ and all fibers $K(z)$ are plurisubharmonically convex. For example, the multifunction $K=\{(z,w)\in\mathbf{C}^2:|w|^2\le 1+|z|^2\}$ over U is analytic because the domain $(U\times\mathbf{C})\setminus K$ is pseudoconvex. But it is not plurisubharmonically convex.

Also when $n\ge 2$ the polynomial hull of a set F over ∂D need not to be an analytic multifunction. As an example take the graph in $\mathbf{C}^3=\{(z_1,z_2,z_3)\}$ of the function $z_3=\operatorname{Re} z_1$ over the unit ball B in $\mathbf{C}^2$. This set is plurisubharmonically convex in $B\times\mathbf{C}$ because it is the set of zeros of the plurisubharmonic function $|z_3-\operatorname{Re} z_1|^2$. Also note that it is the plurisubharmonic hull of the restriction of the graph to ∂B.

However, the following theorem holds. For points z and v in $\mathbf{C}^n$ and $r>0$ let $U(z,v,r)$ be the disk $\{z+\zeta v:|\zeta|\le r\}$ and let $S(z,v,r)$ be the circle $\{z+\zeta v:|\zeta|=r\}$.

Theorem 8.1. *If an upper semicontinuous multifunction K over an open set $V\subset\mathbf{C}^n$ is analytic, then for every open set W containing K, every $z\in V$ and every $v\in\mathbf{C}^n$ the restriction of K to $U(z,v,r)$ lies in the plurisubharmonic hull in W of the restriction of K to $S(z,v,r)$ when $U(z,v,r)\Subset V$.*

The converse theorem also holds when, additionally, the fibers $K(z)$ are plurisubharmonically convex in $\mathbf{C}^{N-n}$.

Proof. If K is analytic, $z_0 \in V$, $v \in \mathbf{C}^n$ and $U(z_0,v,r) \Subset D$, then we consider the Jensen measure $\mu=(2\pi)^{-1}d\theta$ supported by $S(z,v,r)=\{z+re^{i\theta}v:0\leq\theta\leq 2\pi\}$. By Theorem 5.1, for every $w\in K(z)$ there is a measure $\nu\in J_w(K)$ such that $p_*\nu=\mu$. Hence ν is supported by the restriction of K to $S(z_0,r,v)$. By Theorem 3.1, w belongs to the plurisubharmonic hull in W of this restriction.

To prove the converse statement we consider a continuous plurisubharmonic function u defined on a neighborhood W of K. Fix a point $z_0\in V$. To prove that u_K is plurisubharmonic we may assume that $u>0$ on $K(z_0)$. Since the fiber $K(z_0)$ is plurisubharmonically convex in $\mathbf{C}^{N-n}$, by Theorem 3.4 there is a continuous plurisubharmonic function ϕ on $\mathbf{C}^{N-n}=p^{-1}(z_0)$ equal to 0 on $K(z_0)$ and positive outside. The function $\phi_1(z)=\phi(z-p(z)+z_0)$ is plurisubharmonic and continuous on $\mathbf{C}^N$. Let us take a neighborhood W' of $K(z_0)$ in $\mathbf{C}^{N-n}$ and a closed ball $B\Subset V$ centered at z_0 and of radius r_0 such that $K_B\Subset W''=B\times W'\Subset W$ and $u>0$ on K_B. The function u' equal $\max\{u,c\phi_1\}$ on W'' and to ϕ_1 on $p^{-1}(B)\setminus W''$ is plurisubharmonic on $p^{-1}(B)$ for an appropriate constant $c>0$ and $u'_K\equiv u_K$ on B.

Let L be the intersection of a complex line $\{z_0+\zeta v:\zeta\in\mathbf{C}\}$, with V, and let F be the restriction of K to $S(z_0,v,r)$, $r<r_0$. We take a point $w\in K(z_0)$, where $u_K(z_0)=u'(w)$. Since the restriction of K to $U(z_0,v,r)$ lies in the plurisubharmonic hull of F in W, by Theorem 3.1 there is a Jensen measure $\nu\in J_w(F,W)$. The measure $\mu=p_*\nu$ is also Jensen and supported by $S(z_0,v,r)$. Hence $\mu=(2\pi)^{-1}d\theta$. Now

$$u_K(z_0)=u'_K(z_0)=u'(w)\leq\int u'\,d\nu\leq\int u_K\,d\mu.$$

Thus u_K is plurisubharmonic. □

The following result belongs to T. Ransford [R1]. If V is a domain in $\mathbf{C}^N$ and F is a compact set in ∂V, then the plurisubharmonic hull of F is the set $\widetilde{K}$ of all points $z\in V$ such that $u(z)\leq 0$ whenever a plurisubharmonic function u on a neighborhood of $\overline{V}$ is less or equal to 0 on F. We will need the following well-known fact: if $K\subset V$ is a compact set, $\widetilde{K}$ is the plurisubharmonic hull of K in V, $z\in\widetilde{K}$ and $W\Subset V$ is a neighborhood of z such that $K\subset V\setminus\overline{W}$, then the intersection of the plurisubharmonic hull of $\widetilde{K}\cap\partial W$ with W coincides with $\widetilde{K}\cap W$.

Corollary 8.2. *Let V be a smooth domain in $\mathbf{C}$ and let $F\subset p^{-1}(\partial V)$ be a compact set. Then the plurisubharmonic hull $\widetilde{F}$ of F in $D=p^{-1}(V)$ is an analytic multifunction over V.*

Proof. Clearly, $\widetilde{F}$ is an upper semicontinuous multifunction. If $w\in\widetilde{F}(z)$ belongs to the plurisubharmonic hull of $\widetilde{F}(z)$ in $\mathbf{C}^{N-1}$, then there is a Jensen measure in $J_w(\widetilde{F}(z),\mathbf{C}^{N-1})$ with barycenter at w. Hence $w\in\widetilde{F}(z)$ and the fibers of $\widetilde{F}$ are plurisubharmonically convex.

If $U=U(z,r)\Subset V$ and G is the restriction of $\widetilde{F}$ to $S=S(z,r)$, by Theorem 8.1 we have to show that $\widetilde{F}(z)\subset\widetilde{G}$. But this follows immediately from the fact mentioned above. □

9. Minimal analytic multifunctions

An analytic multifunction over a set F is *minimal* if it does not contain any analytic multifunction over F except itself. For example, graphs of holomorphic functions are minimal. Since the intersection of a decreasing sequence of analytic multifunctions over the same set is an analytic multifunction, every analytic multifunction contains a minimal analytic multifunction.

If K is a relatively closed set in an open set $V\subset\mathbf{C}^n$, z is a point in K and W is a relatively open set in K, then the *harmonic measure* $\omega(z,W,K)$ of W with respect to K is the supremum of $\mu(W)$, where μ runs over all measures in $J_z(K)$. We define $I_z(K)$ as the set of all points $w\in K$ such that $\omega(z,W,K)>0$ for every relatively open neighborhood W of w in K. It follows that $I_z(K)$ is closed in K.

We say that a multifunction K over a domain D satisfies the *maximum principle* if any plurisubharmonic function which is defined on a neighborhood of K and attains its maximum on K at some point $w\in K$ is constant on K.

Theorem 9.1. *Let K be an analytic multifunction over a domain V and let u be a plurisubharmonic function defined on a neighborhood of K. If u attains its maximum A on K at some point $w_0\in K$, then the set $L=\{w\in K:u(w)=A\}$ is an analytic multifunction over V.*

Proof. First, we note that $I_w(K)\subset L$ for every point $w\in L$. Indeed, if a point $z_0\in I_w(K)\setminus L$, then $u(z)<A-c$ for some $c>0$ and z in some neighborhood W of z_0. There is a measure $\mu\in J_w(K)$ such that $\mu(W)>0$. Hence

$$u(w)\le\int u\,d\mu<A-c\mu(W).$$

This contradiction proves the statement.

Secondly, $p(L)=V$. Indeed, for $z\in V$ let us take a measure $\mu\in J_{z_0}(V)$, where $z_0=p(w_0)$, such that $\mu(W)>0$, where W is some neighborhood of z in V. By Theorem 5.1 there is a measure $\nu\in J_{w_0}(K)$ such that $p_*\nu=\mu$. Let $F=\operatorname{supp}\nu$. Clearly, $F\subset I_{w_0}(K)$ and we see that $z\in p(I_{w_0}(K))$. By the statement above, $z\in p(L)$.

And, finally, if $\mu\in J_z(V)$ and $w\in L$ with $p(w)=z$, then by Theorem 5.1 there is a measure $\nu\in J_w(K)$ such that $p_*\nu=\mu$. We want to show that $\nu\in J_w(L)$. It was proved in [P1, Theorem 2.1] that for a measure $\nu\in J_w(K)$ there is a compact set $X\subset K$

such that $\nu \in J_w(X)$ and $\omega(w, W, X) > 0$ for every relative open set $W \subset X$. By the two constants theorem $X \subset L$. Thus $\nu \in J_w(L)$. By Theorem 5.1, L is analytic. □

Corollary 9.2. *If an analytic multifunction is minimal, then it satisfies the maximum principle.*

Corollary 9.3. *Let K be an analytic multifunction over a domain D and let u be a plurisubharmonic function defined on a neighborhood of K. If u attains its maximum on K exactly at one point of each fiber, then K contains a holomorphic selection.*

Proof. The set L where u attains its maximum is an analytic multifunction by Theorem 9.1. By Theorem 5.6 this set is the graph of a holomorphic function. □

References

[A] Aupetit, B., Analytic multifunctions and their applications, in *Complex Potential Theory, (Montreal, QC, 1993)*, NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci. **439**, pp. 1–74, Kluwer, Dordrecht, 1994.

[BR] Berndtsson, B. and Ransford T. J., Analytic multifunctions, the $\bar{\partial}$-equation and a proof of the corona theorem, *Pacific J. Math.* **124** (1986), 57–72.

[BS] Bu, S. Q. and Schachermayer, W., Approximation of Jensen measures by image measures under holomorphic functions and applications, *Trans. Amer. Math. Soc.* **331** (1992), 585–608.

[CCW] Carlehed, M., Cegrell, U. and Wikström, F., Jensen measures, hyperconvexity and boundary behaviour of the pluricomplex Green function, *Ann. Polon. Math.* **71** (1999), 87–103.

[Ch] Chirka, E. M., *Complex Analytic Sets*, Nauka, Moscow, 1985 (Russian). English transl.: Mathematics and its Applications (Soviet Series) **46**, Kluwer, Dordrecht, 1989.

[CR] Cole, B. J. and Ransford, T. J., Jensen measures and harmonic measures, *J. Reine Angew. Math.* **541** (2001), 29–53.

[C] Conway, J. B., *A Course in Functional Analysis*, Springer-Verlag, New York, 1990.

[FN] Fornæss, J. E. and Narasimhan, R., The Levi problem on complex spaces with singularities, *Math. Ann.* **248** (1980), 47–72.

[K] Klimek, M., *Pluripotential Theory*, Oxford Univ. Press, New York, 1991.

[P1] Poletsky, E. A., Analytic geometry on compacta in $\mathbf{C}^n$, *Math. Z.* **222** (1996), 407–424.

[P2] Poletsky, E. A., Disk envelopes of functions, II, *J. Funct. Anal.* **163** (1999), 111–132.

[R1] Ransford, T. J., *Analytic Multivalued Functions*, Ph.D. Thesis, University of Cambridge, Cambridge, 1983.

[R2] Ransford, T. J., A new approach to analytic multifunctions, *Set-Valued Anal.* **7** (1999), 159–194.

[R3] Ransford, T. J., Jensen measures, in *Approximation, Complex Analysis, and Potential Theory, (Montreal, QC, 2000)*, NATO Sci. Ser. II Math. Phys. Chem. **37**, pp. 221–237, Kluwer, Dordrecht, 2001.

[S1] Słodkowski, Z., Analytic set-valued functions and spectra, *Math. Ann.* **256** (1981), 363–386.

[S2] Słodkowski, Z., An analytic set-valued selection and its applications to the corona theorem, to polynomial hulls and joint spectra, *Trans. Amer. Math. Soc.* **294** (1986), 367–377.

[Sh] Shabat, B. V., *Introduction to Complex Analysis, Part* II. *Functions of Several Variables*, Nauka, Moscow, 1985 (Russian). English transl.: Translations of Math. Monographs **110**, Amer. Math. Soc., Providence, RI, 1992.

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