Ark. Mat., **42** (2004), 301–306

Every positive integer is the Frobenius number of an irreducible numerical semigroup with at most four generators

Pedro A. García-Sánchez and José C. Rosales([1])

Abstract. Let g be a positive integer. We prove that there are positive integers n_1, n_2, n_3 and n_4 such that the semigroup $S=\langle n_1,n_2,n_3,n_4\rangle$ is an irreducible (symmetric or pseudo-symmetric) numerical semigroup with $\mathrm{g}(S)=g$.

Introduction

Let $n_1,\ldots,n_p$ be positive integers with $\gcd\{n_1,\ldots,n_p\}=1$ (where as usual gcd stands for greatest common divisor). Then it is not hard to show that there are finitely many elements n that cannot be expressed as $n=a_1n_1+\ldots+a_pn_p$ for some nonnegative integers $a_1,\ldots,a_p$. Translated to numerical semigroups, this is equivalent to say that if we consider the numerical semigroup S generated by $\{n_1,\ldots,n_p\}$, that is, $S=\langle n_1,\ldots,n_p\rangle=\{a_1n_1+\ldots+a_pn_p \mid a_1,\ldots,a_p\in\mathbf{N}\}$, then the set $\mathbf{N}\backslash S$ is finite (where $\mathbf{N}$ denotes the set of nonnegative integers). The maximum of this set is usually known as the *Frobenius number* of S and it is here denoted by $\mathrm{g}(S)$. The problem of determining a general formula for $\mathrm{g}(S)$ in terms of $n_1,\ldots,n_p$ is known as the Frobenius problem, which goes back to [11], where an explicit formula for $p=2$ is given ($\mathrm{g}(\langle n_1,n_2\rangle)=n_1n_2-n_1-n_2$). It can be shown (see [3]) that no general formula of a certain type can be found even for the case $p=3$. A nice survey on the state of the art of the Frobenius problem can be found in [5].

For a given positive integer g, the semigroup $S=\langle g+1,g+2,\ldots,2g-1\rangle=\{0,g+1,\rightarrow\}$ (here the symbol $\rightarrow$ is used to indicate that every $n\in\mathbf{N}$ with $n\geq g+1$ belongs to the set) fulfills the trivial condition $\mathrm{g}(S)=g$. Denote the set of numerical semigroups with Frobenius number g by $\mathcal{S}(g)$. In a recent paper [9] the authors have shown that for every positive integer g, there always exist n_1, n_2 and n_3 such

([1]) This work has been supported by the project BFM2000-1469.

that $\langle n_1, n_2, n_3\rangle \in \mathcal{S}(g)$ (or in other words, $\mathrm{g}(\langle n_1, n_2, n_3\rangle) = g$). Among the elements in $\mathcal{S}(g)$, there are some numerical semigroups that have some relevance in ring theory, since their associated semigroup rings are Gorenstein and Kunz. These are the so called symmetric and pseudo-symmetric numerical semigroups, respectively, and have been characterized in many ways (see for instance [2] and [4]). One of these characterizations states that they are precisely those numerical semigroups in $\mathcal{S}(g)$ that are maximal with respect to set inclusion. Both concepts (symmetric and pseudo-symmetric) can be unified into the single concept of irreducible numerical semigroups (see for instance [7] and [8]). A numerical semigroup is *irreducible* if it cannot be expressed as the intersection of two numerical semigroups properly containing it. If S is a numerical semigroup with $\mathrm{g}(S)$ odd (respectively even), then S is symmetric (respectively pseudo-symmetric) if and only if S is irreducible. In this paper we give an easy procedure to find, for any fixed positive integer g, an irreducible numerical semigroup with at most four generators and having g as Frobenius number. Moreover, four is the least number of generators needed for the general case, even though in some cases an irreducible numerical semigroup with three (or two if g is odd) generators can be found.

Acknowledgement. The authors would like to thank the referee for her/his comments and suggestions.

1. Preliminaries

Let S be a numerical semigroup. We say that $\{n_1, \dots, n_p\} \subset S$ is a *system of generators* of S if $S = \langle n_1, \dots, n_p\rangle$. For $n \in S \setminus \{0\}$ we define the Apéry set of n in S (see [1]) as the set

$$\mathrm{Ap}(S, n) = \{s \in S \mid s - n \notin S\}.$$

It is not difficult to prove that this set has exactly n elements, which are $w_0 = 0$, $w_1, \dots, w_{n-1}$, where w_i is the least element in S congruent with i modulo n. From the definition of $\mathrm{Ap}(S, n)$ it also follows that if $x = y + z \in \mathrm{Ap}(S, n)$ with $y, z \in S$, then both y and z are again in $\mathrm{Ap}(S, n)$. This idea is implicitly used several times in the rest of the paper. Besides, it is not hard to show that $\mathrm{g}(S) + n$ is the greatest element in $\mathrm{Ap}(S, n)$.

The cardinality of the set

$$\mathrm{T}(S) = \{x \in \mathbf{Z} \setminus S \mid x + s \in S \text{ for all } s \in S \setminus \{0\}\}$$

is known as the type of S (see for instance [2] and [4]; as usual $\mathbf{Z}$ denotes the set of integers). Symmetric numerical semigroups are those numerical semigroups of type 1 (this forces $\mathrm{T}(S)=\{\mathrm{g}(S)\}$, since by the definition of the Frobenius number of S, for every $n\in\mathbf{N}\setminus\{0\}$, the element $g+n$ belongs to S). Pseudo-symmetric numerical semigroups are characterized by $\mathrm{T}(S)=\left\{\frac{1}{2}\mathrm{g}(S),\mathrm{g}(S)\right\}$ (see [2] and [4]).

For a given numerical semigroup one can define the order relation $\leq_S$ as follows: for $x,y\in S$, $x\leq_S y$ holds if $y-x\in S$. In [6] it is shown that for every $n\in S\setminus\{0\}$, we have that $\mathrm{T}(S)=\{x-n \mid x\in\max_{\leq_S}\mathrm{Ap}(S,n)\}$ ($\max_{\leq_S}$ stands for maximal elements with respect to $\leq_S$). Hence from [4], we deduce that, with $n\in S\setminus\{0\}$ fixed, S is irreducible if and only if

$$\max_{\leq_S}\mathrm{Ap}(S,n)=\begin{cases}\{\mathrm{g}(S)+n\}, & \text{if } \mathrm{g}(S) \text{ is odd,}\\ \left\{\frac{1}{2}\mathrm{g}(S)+n,\mathrm{g}(S)+n\right\}, & \text{if } \mathrm{g}(S) \text{ is even.}\end{cases}$$

2. Main result

Lemma 1. *Let g be a positive integer. If $2\nmid g$, then*

$$S=\langle 2,g+2\rangle$$

is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.

The proof is trivial.

Lemma 2. *Let g be a positive integer. If $2\mid g$ and $3\nmid g$, then*

$$S=\left\langle 3,\tfrac{1}{2}g+3,g+3\right\rangle$$

is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.

Proof. First note that $\gcd\left\{3,\frac{1}{2}g+3,g+3\right\}=1$, whence S is a numerical semigroup. Also, $\mathrm{Ap}(S,3)=\left\{0,\frac{1}{2}g+3,g+3\right\}$ and thus $\max_{\leq_S}\mathrm{Ap}(S,3)=\left\{\frac{1}{2}g+3,g+3\right\}$. This implies that $\mathrm{g}(S)=g+3-3=g$ and that S is irreducible. $\square$

Lemma 3. *Let g be a positive integer. If $2\mid g$ and $4\nmid g$, then*

$$S=\left\langle 4,\tfrac{1}{2}g+2,\tfrac{1}{2}g+4\right\rangle$$

is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.

Proof. Since $2\mid g$ and $4\nmid g$, we have $\gcd\left\{4,\frac{1}{2}g\right\}=1$. Hence

$$\gcd\left\{4,\tfrac{1}{2}g+2,\tfrac{1}{2}g+4\right\}=1$$

and S is a numerical semigroup. We prove that

$$\mathrm{Ap}(S,4)=\left\{0,\tfrac{1}{2}g+2,\tfrac{1}{2}g+4,g+4\right\}. \tag{1}$$

But this can easily be deduced from the following two facts:

(a) $\frac{1}{2}g+2$ and $\frac{1}{2}g+4$ are minimal in S with odd remainder modulo 4, and are clearly in different classes;

(b) $g+4$ is minimal in S with even (and nonzero) remainder modulo 4.

Since (1) holds, we conclude that $\mathrm{g}(S)=g$ and $\mathrm{T}(S)=\{\frac{1}{2}g,g\}$. □

Lemma 4. *Let g be a positive integer. Assume that $2|g$, $3|g$ and $4|g$. Let α and q be such that $g=3^\alpha q$, with $(3,q)=1$ (and thus $4|q$). Then*

$$S=\langle 3^{\alpha+1}, \tfrac{1}{2}q+3, 3^\alpha\tfrac{1}{4}q+\tfrac{1}{2}q+3, 3^\alpha\tfrac{1}{2}q+\tfrac{1}{2}q+3\rangle$$

is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.

Proof. Let $n_1=3^{\alpha+1}$, $n_2=\frac{1}{2}q+3$, $n_3=3^\alpha\frac{1}{4}q+\frac{1}{2}q+3$ and $n_4=3^\alpha\frac{1}{2}q+\frac{1}{2}q+3$. As $(3,q)=1$, we have $\gcd\{n_1,n_2,n_3,n_4\}=1$. The reader can check that the following equalities hold:

(i) $2n_4=\frac{1}{4}qn_1+n_2+n_3$;

(ii) $2n_3=n_2+n_4$;

(iii) $n_3+n_4=\frac{1}{4}qn_1+2n_2$;

(iv) $(\frac{1}{3}n_1+1)n_2=n_1+n_4$;

(v) $\frac{1}{3}n_1n_2+n_4=(\frac{1}{4}q+1)n_1+n_3$;

(vi) $(\frac{1}{3}n_1-1)n_2+n_3=(\frac{1}{4}q+1)n_1$.

Every element in $\mathrm{Ap}(S,n_1)$ is of the form $an_2+bn_3+cn_4$. By (repeatedly) using (i), (ii) and (iii) we can assume that one of the following three cases holds:

(a) $b=c=0$;

(b) $b=0$ and $c=1$;

(c) $b=1$ and $c=0$.

It follows from (iv) that a is always $\leq\frac{1}{3}n_1=3^\alpha$. In the case (b), it follows from (v) that $a<\frac{1}{3}n_1$, and in the case (c), a must be less than $\frac{1}{3}n_3-1$ in view of (vi). Since $\#\,\mathrm{Ap}(S,n_1)=n_1$, one can easily deduce that

$$\mathrm{Ap}(S,n_1)=\{0,n_2,2n_2,\dots,\tfrac{1}{3}n_1n_2,n_3,n_3+n_2,\dots,n_3+(\tfrac{1}{3}n_1-2)n_2, \\ n_4,n_4+n_2,\dots,n_4+(\tfrac{1}{3}n_1-1)n_2\}.$$

Now we prove that $\max_{\leq_S}\mathrm{Ap}(S,n_1)=\{\frac{1}{3}n_1n_2,(\frac{1}{3}n_1-1)n_2+n_4\}$. From the shape of $\mathrm{Ap}(S,n_1)$ it is clear that

$$\max_{\leq_S}\mathrm{Ap}(S,n_1)\subseteq\{\tfrac{1}{3}n_1n_2,n_3+(\tfrac{1}{3}n_1-2)n_2,n_4+(\tfrac{1}{3}n_1-1)n_2\}.$$

After some simplifications one gets that

$$(\tfrac{1}{3}n_1-1)n_2+n_4-(n_3+(\tfrac{1}{3}n_1-2)n_2)=n_4-n_3+n_2,$$

which in view of (ii) is equal to n_3, which trivially belongs to S. This implies that $n_3+\left(\frac{1}{3}n_1-2\right)\notin\max_{\leq_S}\mathrm{Ap}(S,n_1)$. Since $\frac{1}{3}n_1n_2\in\mathrm{Ap}(S,n_1)$, we have that

$$\tfrac{1}{3}n_1n_2-n_1=3^{\alpha}\tfrac{1}{2}q\notin S.$$

Thus $n_4+\left(\frac{1}{3}n_1-1\right)n_2-\frac{1}{3}n_1n_2=3^{\alpha}\frac{1}{2}q\notin S$. Hence

$$\max_{\leq_S}\mathrm{Ap}(S,n_1)=\left\{\tfrac{1}{3}n_1n_2,\left(\tfrac{1}{3}n_1-1\right)n_2+n_4\right\}.$$

This in particular implies that $\mathrm{g}(S)=\left(\frac{1}{3}n_1-1\right)n_2+n_4-n_1=3^{\alpha}q$ and that $\mathrm{T}(S)=\left\{\frac{1}{2}\mathrm{g}(S),\mathrm{g}(S)\right\}$. Therefore S is irreducible. □

Theorem 5. *Let g be a positive integer. Then there exist $n_1,n_2,n_3,n_4\in\mathbf{N}$ such that $S=\langle n_1,n_2,n_3,n_4\rangle$ is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.*

Proof. If g is not a multiple of 2, then by Lemma 1 we can choose $n_1=2$ and $n_2=n_3=n_4=g+2$. If on the contrary g is a multiple of 2, then we distinguish two cases.

(1) If g is not divisible by 3, then in view of Lemma 2, for $n_1=3$, $n_2=\frac{1}{2}g+3$ and $n_3=n_4=g+3$, the semigroup $S=\langle n_1,n_2,n_3,n_4\rangle$ is an irreducible numerical semigroup with $\mathrm{g}(S)=g$.

(2) If g is divisible by 3, then

(a) either g is not divisible by 4, and thus we can apply Lemma 3 and take $n_1=4$, $n_2=\frac{1}{2}g+2$ and $n_3=n_4=\frac{1}{2}g+4$;

(b) or g is divisible by 4, and in this case we get the desired semigroup by using Lemma 4. □

If one computes the set of irreducible numerical semigroups with Frobenius number 12 as explained in [10], one gets that this set contains only two elements: $\langle 5,8,9,11\rangle$ and $\langle 7,8,9,10,11,13\rangle$. This implies that the bound of four generators obtained in Theorem 5 cannot be improved.

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Received February 11, 2003

Pedro A. García-Sánchez
Departamento de Álgebra
Universidad de Granada
ES-18071 Granada
Spain
email: pedro@ugr.es

José C. Rosales
Departamento de Álgebra
Universidad de Granada
ES-18071 Granada
Spain
email: jrosales@ugr.es