Removable sets of analytic functions satisfying a Lipschitz condition

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1. Introduction

Let E be a compact subset of the complex plane and let Ω be the complement of E with respect to the extended plane. For $0<\alpha\leqq 1$, we denote by $\mathrm{Lip}_\alpha^a(\Omega)$ the set of bounded analytic functions defined on Ω and satisfying a Lipschitz condition of order α, i.e., if $f\in\mathrm{Lip}_\alpha^a(\Omega)$, then $|f(z)-f(w)|\leqq C_f|z-w|^\alpha$ for any z and w in Ω. We denote the union of all $\mathrm{Lip}_\alpha^a(\Omega)$ by Lip_α^a. We say that E is removable for Lip_α^a if the associated $\mathrm{Lip}_\alpha^a(\Omega)$ consists only of the constants.

The problem of characterizing the removable sets of Lip_α^a has been investigated in several papers, for example [1], [2], [4] and [6]. For $0<\alpha<1$, Dolženko has obtained the following result (see [2]). In order that E be removable for Lip_α^a it is necessary and sufficient that the $(1+\alpha)$-dimensional Hausdorff measure $\Lambda_{1+\alpha}(E)=0$.

The limiting case $\alpha=1$ is particularly interesting and is treated in this paper. The main techniques we use here involve extremal problems and singular integrals. We obtain the following characterization for removable sets of Lip_1^a. A compact set E is removable for Lip_1^a if and only if the 2-dimensional Lebesgue measure $m(E)=0$. This is the main result of the present paper. It should be mentioned that the implication $m(E)=0\Rightarrow E$ removable for Lip_1^a is well known (see e.g. Garnett [4], Chapter III, Section § 2.)

Finally, by using the techniques introduced in Section 3, we obtain an additional result concerning singular integrals. This result is included in Section 5.

* This research was partially supported by SUNY Faculty Research Fellowship.

2. Definitions and notations

Let $\{\chi_\varepsilon\}_{\varepsilon>0}$ be an approximate identity, where $\chi_\varepsilon(z)=\chi(z/\varepsilon)/\varepsilon^2$ and χ is in the set $\mathscr{D}(\mathbf{R}^2)$ of infinitely differentiable functions with compact supports. Furthermore, we will assume that χ satisfies the following properties.

(i) $\chi \geqq 0$, $\operatorname{supp}\chi \subset D(0, 1) = \{z\colon |z| \leqq 1\}$.

(ii) χ is radial, i.e., $\chi(re^{i\theta}) = \chi(r)$ for all real θ.

(iii) $\iint \chi(z)\, dm(z) = 1$.

If $1 \leqq p \leqq \infty$ and if $f \in L^p(\mathbf{R}^2)$, we define

$$f_\varepsilon(z) = \chi_\varepsilon * f(z) = \iint \chi_\varepsilon(z-\zeta) f(\zeta)\, dm(\zeta).$$

Similarly, for any finite Borel measure μ, we set

$$\mu_\varepsilon(z) = \chi_\varepsilon * \mu(z) = \iint \chi_\varepsilon(z-\zeta)\, d\mu(\zeta).$$

Now we recall the following standard notations.

$C_0(\mathbf{R}^2)$ = the set of all continuous functions defined on $\mathbf{R}^2$ which vanish at ∞.

$M(\mathbf{R}^2)$ = the set of finite Borel measures defined on $\mathbf{R}^2$.

If E is an arbitrary compact set we define

$C(E)$ = the set of all continuous functions defined on E.

$M(E)$ = the set of all finite Borel measures supported on E.

Consider the direct sums $C(E) \oplus C_0(\mathbf{R}^2)$ and $M(E) \oplus M(\mathbf{R}^2)$. The norms in these spaces are defined respectively as follows.

$$\|(\varphi, \psi)\| = \max\{\|\varphi\|_\infty, \|\psi\|_\infty\}, \quad (\varphi, \psi) \in C(E) \oplus C_0(\mathbf{R}^2).$$

$$\|(\mu, \nu)\| = \|\mu\| + \|\nu\|, \quad (\mu, \nu) \in M(E) \oplus M(\mathbf{R}^2).$$

Then $C(E) \oplus C_0(\mathbf{R}^2)$ is a Banach space and its dual is $M(E) \oplus M(\mathbf{R}^2)$. The terms on the right hand side of the second equality denote the total variations of μ and ν.

We shall also be involved in a particular type of singular integrals defined as follows. If $1 \leqq p < \infty$ and if $f \in L^p(\mathbf{R}^2)$, then we put

$$Bf(z) = \text{P.V.} \iint \frac{f(\zeta)}{(\zeta - z)^2}\, dm(\zeta).$$

Similarly, for any measure $\mu \in M(\mathbf{R}^2)$ we define

$$B\mu(z) = \text{P.V.} \iint \frac{d\mu(\zeta)}{(\zeta - z)^2}.$$

It is well known that these singular integrals exist almost everywhere and, furthermore, there are absolute constants $A_p>0$, $1\leqq p<\infty$, such that

$$\|Bf\|_p \leqq A_p\|f\|_p, \quad f\in L^p(\mathbf{R}^2) \quad (1<p<\infty) \tag{2.1}$$

and

$$m(\{z\colon |B\mu(z)|>\lambda\}) \leqq \frac{A_1\|\mu\|}{\lambda}, \quad \mu\in M(\mathbf{R}^2). \tag{2.2}$$

For proofs of this and further results see [5], [7]. Note that property (2.2) has only been proved for $L^1(\mathbf{R}^2)$-functions. The above extension for measures follows easily by using a standard technique of truncation and convolution.

Finally, we need the following result which is obtained from Green's theorem

$$B\varphi(z) = \iint \frac{\partial\varphi}{\partial z}(\zeta)\frac{dm(\zeta)}{\zeta-z}, \quad \varphi\in\mathscr{D}(\mathbf{R}^2). \tag{2.3}$$

3. Extremal problems

Suppose E is an arbitrary compact set with $m(E)>0$ and $f\in L^1_{\mathrm{loc}}(\mathbf{R}^2)$. We denote by $\Gamma(E)$ the set of all functions $h\in L^\infty(E)$ such that $\|h\|_\infty\leqq 1$ and $\|Bh\|_\infty\leqq 1$ and set

$$\mathscr{C}_f(E) = \sup_{h\in\Gamma(E)}\left|\iint h(z)f(z)\,dm(z)\right|. \tag{3.1}$$

If the set E has a boundary consisting of a finite number of analytic Jordan curves, we denote by $\mathscr{D}(E)$ the set of those functions in $\mathscr{D}(\mathbf{R}^2)$ with support contained in E, and define

$$\mathscr{C}_f^*(E) = \sup_{\varphi\in\mathscr{D}(E)\cap\Gamma(E)}\left|\iint \varphi(z)f(z)\,dm(z)\right|. \tag{3.2}$$

Now we recall the following simple but useful corollary of the Hahn–Banach theorem. If X is a Banach space and M is a subspace of X, then for any $L\in X^*$ we have

$$\sup_{x\in M,\,\|x\|\leqq 1}|L(x)| = \inf_{\mathscr{L}\in M^\perp}\|L+\mathscr{L}\|,$$

and furthermore, there is always an element of $M^\perp$ for which the infimum is attained. For a proof see e.g. Duren [3], Chapter 7. If we apply this result to $X=C(E)\oplus C_0(\mathbf{R}^2)$, $M=\{(\varphi, B\varphi)\colon \varphi\in\mathscr{D}(E)\}$ and $L=(f_E\,dm, 0)$, where f_E is the restriction of f on E, we obtain

$$\mathscr{C}_f^*(E) = \min\{\|f_E\,dm+\mu\|+\|\nu\|\}, \tag{3.3}$$

where the minimum is taken over all elements $(\mu,\nu)\in M(E)\oplus M(\mathbf{R}^2)$ satisfying the relation

$$(3.4)\qquad \int \varphi(z)\,d\mu(z)+\int B\varphi(z)\,d\nu(z) = 0, \quad \varphi\in \mathcal{D}(E).$$

We shall call any such element which minimizes (3.3) an extremal element.

Lemma 3.5. *If the boundary ∂E is a finite union of analytic Jordan curves, then there exists a function $h\in\Gamma(E)$ such that*

$$\mathscr{C}_f^*(E) = \iint h(z)f(z)\,dm(z).$$

Proof. Let $\{\varphi_n\}$ be a sequence contained in $\mathcal{D}(E)\cap\Gamma(E)$ with

$$\iint \varphi_n(z)f(z)\,dm(z)$$

converging to $\mathscr{C}_f^*(E)$. Since $\|\varphi_n\|_\infty\leqq 1$ $(n=1, 2, \ldots)$, we may assume (by passing to a subsequence if necessary) that there exists a function $h\in L^\infty(E)$ such that $\|h\|_\infty\leqq 1$ and $\varphi_n\to h$ in the weak-star topology of $L^\infty(\mathbf{R}^2)$. Now let us consider the convex hull $\mathrm{co}(\{\varphi_n\})$ and let $\{\Phi_n\}$ be a sequence of functions in $\mathrm{co}(\{\varphi_n\})$ converging to h in $L^2(\mathbf{R}^2)$. By property (2.1), $B\Phi_n$ converges to Bh in $L^2(\mathbf{R}^2)$. Since $\|B\Phi_n\|_\infty\leqq 1$ $(n=1, 2, \ldots)$, this implies that $\|Bh\|_\infty\leqq 1$ and $h\in\Gamma(E)$. Hence the lemma is proved.

Lemma 3.6. *Let μ be a finite Borel measure. Then we have the following properties.*

(a) *$\mu_\varepsilon(z)$ converges to $\dfrac{d\mu}{dt}(z)$ almost everywhere.*

(b) *$B\mu_\varepsilon(z)$ converges to $B\mu(z)$ almost everywhere.*

Proof. (a) is well known. Since (b) follows from (2.2) if μ is absolutely continuous, we can suppose μ is singular. We consider a point z where $B\mu(z)$ exists and such that

$$|\mu|(D(z,r))/r^2 \to 0 \quad \text{as} \quad r\to 0.$$

In the following, for convenience, we shall delete the symbol P. V. before singular integrals.

With the aid of Fubini's theorem we obtain

$$B\mu_\varepsilon(z) = \iint\Big(\int \chi_\varepsilon(\zeta-t)\,d\mu(t)\Big)\frac{dm(\zeta)}{(\zeta-z)^2} = \int\Big(\iint\frac{\chi_\varepsilon(\zeta-t)}{(\zeta-z)^2}\,dm(\zeta)\Big)\,d\mu(t).$$

We divide this integral into two parts, over $\{t\colon |t-z|>\varepsilon\}$ and $\{t\colon |t-z|\leqq\varepsilon\}$, and denote the corresponding integrals by $I_1(z)$ and $I_2(z)$. We obtain

$$I_1(z) = \int_{|t-z|>\varepsilon}\Big(\iint\frac{\chi_\varepsilon(\zeta-t)}{(\zeta-z)^2}\,dm(\zeta)\Big)\,d\mu(t) = \int_{|t-z|>\varepsilon}\frac{d\mu(t)}{(t-z)^2},$$

because $1/(\zeta-z)^2$ (as a function of ζ) is analytic in a neighborhood of $D(t, \varepsilon)$. Hence $I_1(z) \to B\mu(z)$ as $\varepsilon \to 0$.

Now, since

$$I_2(z) = \int_{|t-z| \leqq \varepsilon} \left(\iint \frac{\chi_\varepsilon(\zeta-t)}{(\zeta-z)^2} \, dm(\zeta) \right) d\mu(t)$$

$$= \int_{|t-z| \leqq \varepsilon} \left(\iint \frac{\partial \chi_\varepsilon}{\partial z}(\zeta-t) \frac{dm(\zeta)}{(\zeta-z)} \right) d\mu(t)$$

we obtain

$$|I_2(z)| \leqq \frac{2\pi M}{\varepsilon^2} \int_{|t-z| \leqq \varepsilon} d|\mu|(t) = \frac{2\pi M}{\varepsilon^2} |\mu|(D(z, \varepsilon)),$$

where $M = \|\partial\chi/\partial z\|_\infty$. Hence $I_2(z) \to 0$ as $\varepsilon \to 0$ and the lemma follows.

Theorem 3.7. *Suppose that ∂E is a finite union of analytic Jordan curves. If $(\mu, \nu) \in M(E) \oplus M(\mathbf{R}^2)$ satisfies relation* (3.4), *then we have*

$$\frac{d\mu}{dt}(z) = -B\nu(z) \quad \text{a.e. on } E.$$

Proof. Let $E_\varepsilon = \{z \in E: \operatorname{dist}(z, \Omega) > \varepsilon\}$ and consider an arbitrary $\varphi \in \mathscr{D}(E_\varepsilon)$. Then $\varphi_\varepsilon \in \mathscr{D}(E)$, and by (3.4) we have

$$\int \varphi_\varepsilon(z) \, d\mu(z) + \int B\varphi_\varepsilon(z) \, d\nu(z) = 0.$$

Now, with the aid of Fubini's theorem,

$$\int \varphi_\varepsilon(z) \, d\mu(z) = \iint \varphi(z) \mu_\varepsilon(z) \, dm(z)$$

and

$$\int B\varphi_\varepsilon(z) \, d\nu(z) = \iint B\varphi(z) \nu_\varepsilon(z) \, dm(z) = \iint \varphi(z) B\nu_\varepsilon(z) \, dm(z).$$

So

$$\iint \varphi(z) \mu_\varepsilon(z) \, dm(z) + \iint \varphi(z) B\nu_\varepsilon(z) \, dm(z) = 0$$

for all $\varphi \in \mathscr{D}(E_\varepsilon)$. This implies $\mu_\varepsilon(z) = -B\nu_\varepsilon(z)$ on E_ε. Letting ε tend to 0, by Lemma 3.6, we obtain $d\mu/dt\,(z) = -B\nu(z)$ a.e. on E.

4. Removable sets of Lip_1^a.

In this section we prove the result mentioned earlier in the Section 1 concerning removable sets of Lip_1^a.

Theorem 4.1. *Let E be an arbitrary compact set of the complex plane. Then E is removable for* Lip_1^a *if and only if* $m(E) = 0$.

Lemma 4.2. *Suppose* $m(E)>0$. *If* $\{E_n\}$ $(n=1, 2, \ldots)$ *is a decreasing sequence of compact sets such that each* ∂E_n *is a finite union of analytic Jordan curves and* $E=\cap E_n$, *then*

$$\mathscr{C}_f(E) = \lim_{n\to\infty} \mathscr{C}_f^*(E_n).$$

Proof. Let $h\in\Gamma(E)$ and let n be fixed. Then there exists $\varepsilon_0>0$ such that $h_\varepsilon\in\Gamma(E_n)\cap\mathscr{D}(E_n)$ for all $\varepsilon<\varepsilon_0$. Thus

$$\left|\iint h(z)f(z)\,dm(z)\right| = \lim_{\varepsilon\to 0}\left|\iint h_\varepsilon(z)f(z)\,dm(z)\right| \leqq \mathscr{C}_f^*(E_n)$$

for all n. Hence $\mathscr{C}_f(E)\leqq\lim_{n\to\infty}\mathscr{C}_f^*(E_n)$.

Now, by Lemma 3.5, for each n there exists $h_n\in\Gamma(E_n)$ such that

$$\mathscr{C}_f^*(E_n) = \iint h_n(z)f(z)\,dm(z).$$

We see, as in the proof of Lemma 3.5, that h_n converges to a function $h\in\Gamma(E)$ in the weak-star topology. Therefore

$$\lim_{n\to\infty}\mathscr{C}_f^*(E) = \lim_{n\to\infty}\iint h_n(z)f(z)\,dm(z) = \iint h(z)f(z)\,dm(z) \leqq \mathscr{C}_f(E)$$

and the lemma follows.

Lemma 4.3. *Let E be an arbitrary compact set of positive Lebesgue measure and let* $f\in L^1_{\text{loc}}(\mathbf{R}^2)$. *If* $f\not\equiv 0$ *on* E, *then* $\mathscr{C}_f(E)>0$.

Proof. Let $\{E_n\}$ be a sequence of compact sets having the properties mentioned in Lemma 4.2. For each n, let $(\mu_n, \nu_n)\in M(E_n)\oplus M(\mathbf{R}^2)$ be an extremal element of (3.3), so that

$$\mathscr{C}_f^*(E_n) = \|f_{E_n}+\mu_n\|+\|\nu_n\|.$$

By Lemma 4.2, we obtain

$$\mathscr{C}_f(E) = \lim_{n\to\infty}\{\|f_{E_n}+\mu_n\|+\|\nu_n\|\}.$$

Let us assume $\mathscr{C}_f(E)=0$. Then the above equation implies

(iv) $\displaystyle\iint_{E_n}\left|f(z)+\frac{d\mu_n}{dt}(z)\right|dm(z)\to 0$ as $n\to\infty$

(v) $\|\mu_n^s\|\to 0$ as $n\to 0$, where μ_n^s is the singular part of μ_n

(vi) $\|\nu_n\|\to 0$ as $n\to\infty$.

By (iv) $B\nu_{n_k}(z)$ converges to $f(z)$ a.e. on E for some subsequence $\{n_k\}$, because $d\mu_n/dt\,(z)=-B\nu_n(z)$ a.e. on E_n. Furthermore, because of (vi) and (2.2), $B\nu_{n_k}$ converges to 0 in the mean. This implies that $f\equiv 0$ on E. Hence $\mathscr{C}_f(E)$ must be positive if $f\not\equiv 0$ on E.

Proof of Theorem 4.1. As we have mentioned earlier, the implication $m(E)=0\Rightarrow E$ removable for Lip_1^α is well known. However, for the convenience of reference we include a proof of this result.

Suppose then $m(E)=0$ and let $F\in\mathrm{Lip}_1^\alpha(\Omega)$. Let $z\in\Omega$ and choose a sufficiently small $\varepsilon>0$. We cover E by a finite number of squares R_j with center z_j and side r_j such that $z\notin\cup R_j$, $R_j^0\cap R_l^0=\emptyset$ if $j\neq l$ and $\sum r_j^2<\varepsilon$. By Cauchy's integral formula we have

$$F(z)=-\sum\frac{1}{2\pi i}\int_{\partial R_j}\frac{F(\zeta)}{\zeta-z}\,d\zeta+F(\infty),$$

where ∂R_j denotes the boundary of R_j taken in the positive sense. But, if $I_3(z)$ denotes the first term on the right hand side of this equation, then

$$I_3(z)=-\sum\frac{1}{2\pi i}\int_{\partial R_j}\frac{F(\zeta)-F(z_j)}{\zeta-z}\,d\zeta.$$

So

$$|I_3(z)|\leqq\sum\frac{1}{2\pi}\int_{\partial R_j}\frac{|F(\zeta)-f(z_j)|}{|\zeta-z|}\,d|\zeta|\leqq\sum\frac{C_F r_j^2}{d}\leqq\frac{C_F\varepsilon}{d}\to 0\quad\text{as}\quad\varepsilon\to 0,$$

where $d=\mathrm{dist}\,(z,\bigcup_j R_j)$. Hence F is constant.

Now let us assume that $m(E)>0$. According to Lemma 4.3, there exists $h\in\Gamma(E)$ satisfying the property

$$\iint h(z)\,dm(z)>0.$$

Let

$$F(z)=\iint\frac{h(\zeta)}{\zeta-z}\,dm(\zeta).$$

We observe that F is nonconstant, because

$$F'(\infty)=\lim_{z\to\infty}zF(z)=-\iint h(\zeta)\,dm(\zeta)<0.$$

Furthermore, since

$$F_\varepsilon(z)=\iint\frac{h_\varepsilon(\zeta)}{\zeta-z}\,dm(\zeta),$$

we have $\partial F_\varepsilon/\partial z=Bh_\varepsilon=(Bh)_\varepsilon$ and $\partial F_\varepsilon/\partial\bar z=-\pi h_\varepsilon$. It follows that

$$\left\|\frac{\partial F_\varepsilon}{\partial z}\right\|_\infty\leqq 1,\quad\left\|\frac{\partial F_\varepsilon}{\partial\bar z}\right\|_\infty\leqq\pi.$$

Thus, $|F_\varepsilon(z)-F_\varepsilon(w)|\leqq 4(1+\pi)\,|z-w|$ for all z,w and $\varepsilon>0$. Letting ε tend to 0 we obtain the Lipschitz condition $|F(z)-F(w)|\leqq 4(1+\pi)\,|z-w|$, hence the theorem is proved.

5. Estimation of $\mathscr{C}_1(E)$.

The quantity $\mathscr{C}_f(E)$ is particularly interesting when $f\equiv 1$. We obtain in this case the following estimate.

Theorem 5.1. *There exists an absolute constant* $K_1>0$ *such that*

$$\mathscr{C}_1(E) \geqq K_1 m(E) \tag{5.2}$$

for any compact set E with positive Lebesgue measure.

Lemma 5.3. *Let c be a complex number and let* $r>0$. *Then we have the following properties*

(a) $\mathscr{C}_1(E+c) = \mathscr{C}_1(E)$

(b) $\mathscr{C}_1\left(\frac{E}{r}\right) = \frac{1}{r^2}\mathscr{C}_1(E)$.

Proof. (a) is obvious. To prove (b) we associate to each function $h\in L^\infty(E)$ a function $k\in L^\infty(E/r)$, where $k(z)=h(rz)$. Then it is easily seen that $Bk(z)=Bh(rz)$. Thus the mapping $h\to k$ is an 1—1 correspondance between $\Gamma(E)$ and $\Gamma(E/r)$. Furthermore, by changing variable we have

$$\iint_{E/r} k(z)\,dm(z) = \frac{1}{r^2}\iint_E h(z)\,dm(z).$$

Therefore, $\mathscr{C}_1(E)=\mathscr{C}_1(E/r)/r^2$ and (b) is proved.

Proof of Theorem 5.1. Since the two set functions $\mathscr{C}_1$ and m are both homogeneous of degree 2, it is clearly enough to prove (5.2) for an arbitrary compact E with $m(E)=1$. Furthermore, according to Lemma 4.2, it suffices to show that $\mathscr{C}_1^*(E)\geqq K_1$ for any compact set E with $m(E)=1$, and with a boundary consisting of a finite number of analytic Jordan curves. Now, if (μ, ν) is an extremal element of (3.3), then

$$\mathscr{C}_1^*(E) = \|\chi_E+\mu\|+\|\nu\| \geqq \iint_E |1-B\nu(z)|\,dm(z)+\|\nu\|.$$

Let $F=\{z\in E\colon |B\nu(z)|>\frac{1}{2}\}$. By (2.2) $m(F)\leqq 2A_1\|\nu\|$. Hence

$$\iint_E |1-B\nu(z)|\,dm(z) \geqq \iint_{E\setminus F} |1-B\nu(z)|\,dm(z)$$

$$\geqq \frac{1}{2}m(E\setminus F) \geqq \frac{1}{2}(1-2A_1\|\nu\|).$$

Therefore we obtain

$$\mathscr{C}_1^*(E) \geqq \max\left\{\frac{1}{2}-A_1\|\nu\|,\ \|\nu\|\right\} \geqq \frac{1}{2(1+A_1)}.$$

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Received September 19, 1977

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