Two new multivariable generating relations

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Abstract. A multidimensional extension of Bailey's transform is utilised to deduce two new generating relations of quite a general character. These expressions are then specialised to give more practical formulae in terms of Karlsson's generalised Kampé de Fériet functions which embody very many generating relations. A number of interesting special cases are given in an appendix which includes results involving Lauricella polynomials, generalised hypergeometric polynomials and the polynomials of Meixner, Charlier and Laguerre.

1. Introduction and notation

The basis of the results given in this study is a multidimensional generalisation of Bailey's transform as given by Exton [5], p. 139, which extends the discussion mentioned by Bailey [2] and Slater [9], p. 58. This very general formulation of certain multiple series transforms may be stated as follows:

If

$$(1.1)\qquad \beta_{m_1,\dots,m_n} = \sum_{p_1,\dots,p_n=0}^{m_1,\dots,m_n} \alpha_{p_1,\dots,p_n} u_{m_1-p_1,\dots,m_n-p_n} v_{m_1+p_1,\dots,m_n+p_n}$$

and

$$(1.2)\qquad \gamma_{m_1,\dots,m_n} = \sum_{p_1=m_1,\dots,p_n=m_n}^{\infty} \delta_{p_1,\dots,p_n} u_{p_1-m_1,\dots,p_n-m_n} v_{p_1+m_1,\dots,p_n+m_n},$$

then we have, formally,

$$(1.3)\qquad \sum a_{m_1,\dots,m_n}\gamma_{m_1,\dots,m_n} = \sum \beta_{m_1,\dots,m_n}\delta_{m_1,\dots,m_n}.$$

It is understood that α, δ, u and v are functions of $p_1,\dots,p_n$ only, and any questions of convergence must be dealt with in each individual case as appropriate. The symbol $\sum$ without further qualification denotes a multiple summation with the indices of summation $m_1,\dots,m_n$ running over all non-negative integer values.

The following notation will now be employed:

(1.4) $$(a, n) = a(a+1)\dots(a+n) = \Gamma(a+n)/\Gamma(a), \quad (a, 0) = 1,$$

(1.5) $$((a), n) = (a_1, n)(a_2, n)\dots(a_A, n)$$

and (a) denotes the sequence of parameters $a_1, a_2, \dots, a_A$.

The generalised Kampé de Fériet function which figures largely in the subsequent analysis is given by

(1.6) $$F^{A:G}_{B:H}\begin{bmatrix}(a): (g_1); \dots; (g_n); \\ (b): (h_1); \dots; (h_n); \end{bmatrix} x_1, \dots, x_n\Big]$$

$$= \sum \frac{((a), m_1+\dots+m_n)((g_1), m_1)\dots((g_n), m_n)x_1^{m_1}\dots x_n^{m_n}}{((b), m_1+\dots+m_n)((h_1), m_1)\dots((h_n), m_n)m_1!\dots m_n!},$$

see Karlsson [7]. Any exceptional values of the parameters for which any of the expressions do not make sense are tacitly excluded.

Three of the four Lauricella functions are also used and are defined as

(1.7) $$F_A^{(n)}(a, b_1, \dots, b_n; c_1, \dots, c_n; x_1, \dots, x_n)$$

$$= \sum \frac{(a, m_1+\dots+m_n)(b_1, m_1)\dots(b_n, m_n)x_1^{m_1}\dots x_n^{m_n}}{(c_1, m_1)\dots(c_n, m_n)m_1!\dots m_n!},$$

(1.8) $$F_B^{(n)}(a_1, \dots, a_n, b_1, \dots, b_n; c; x_1, \dots, x_n)$$

$$= \sum \frac{(a_1, m_1)\dots(a_n, m_n)(b_1, m_1)\dots(b_n, m_n)x_1^{m_1}\dots x_n^{m_n}}{(c, m_1+\dots+m_n)m_1!\dots m_n!}$$

and

(1.9) $$F_D^{(n)}(a, b_1, \dots, b_n; c; x_1, \dots, x_n)$$

$$= \sum \frac{(a, m_1+\dots+m_n)(b_1, m_1)\dots(b_n, m_n)x_1^{m_1}\dots x_n^{m_n}}{(c, m_1+\dots+m_n)m_1!\dots m_n!}.$$

See Lauricella [8].

A confluent form of either $F_B^{(n)}$ or $F_D^{(n)}$ also occurs. This is

(1.10) $$\Phi_2(a_1, \dots, a_n; c; x_1, \dots, x_n) = \sum \frac{(a_1, m_1)\dots(a_n, m_n)x_1^{m_1}\dots x_n^{m_n}}{(c, m_1+\dots+m_n)m_1!\dots m_n!}.$$

2. The main results

Theorem 1. *If* $C(m_1, \dots, m_n)$ *is any arbitrary function of* $m_1, \dots, m_n$, *then, formally,*

(2.1)

$$(1-t)^{-d} \sum \frac{C(m_1, \dots, m_n)(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)} [t/(1-t)]^{m_1+\dots+m_n}$$

$$= \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)\, t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)\, m_1!\dots m_n!}$$

$$\times \sum_{p_1, \dots, p_n=0}^{\infty} C(p_1, \dots, p_n)(-1)^{p_1+\dots+p_n}(-m_1, p_1)\dots(-m_n, p_n).$$

Proof. In the multidimensional generalisation of Bailey's transform (1.3), put

(2.2) $$\alpha_{m_1, \dots, m_n} = C(m_1, \dots, m_n),$$

(2.3) $$u_{m_1, \dots, m_n} = 1/(m_1! \dots m_n!),$$

(2.4) $$v_{m_1, \dots, m_n} = 1$$

and

(2.5) $$\delta_{m_1, \dots, m_n} = \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)} t^{m_1+\dots+m_n}.$$

From (1.1) and (1.2), we have

(2.6) $$\beta_{m_1, \dots, m_n} = \sum_{p_1, \dots, p_n=0}^{m_1, \dots, m_n} \frac{C(p_1, \dots, p_n)}{(m_1-p_1)!\dots(m_n-p_n)!}$$

$$= \sum_{p_1, \dots, p_n=0}^{m_1, \dots, m_n} C(p_1, \dots, p_n)(-1)^{p_1+\dots+p_n}(-m_1, p_1)\dots(-m_n, p_n)$$

and

(2.7) $$\gamma_{m_1, \dots, m_n} = \sum_{p_1=m_1, \dots, p_n=m_n}^{\infty} \frac{(d, p_1+\dots+p_n)(k_1, p_1)\dots(k_n, p_n)\, t^{p_1+\dots+p_n}}{(k_1+\dots+k_n, p_1+\dots+p_n)(p_1-m_1)!\dots(p_n-m_n)!}$$

$$= \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(d, m_1+p_1+\dots+m_n+p_n)(k_1, m_1+p_1)\dots(k_n, m_n+p_n)\, t^{m_1+p_1+\dots+m_n+p_n}}{(k_1+\dots+k_n, m_1+p_1+\dots+m_n+p_n)\, p_1!\dots p_n!}$$

$$= \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)} t^{m_1+\dots+m_n}$$

$$\times \sum_{p_1, \dots, p_n=0}^{\infty} \frac{(d+m_1+\dots+m_n, p_1+\dots+p_n)(k_1+m_1, p_1)\dots(k_n+m_n, p_n)}{(k_1+m_1+\dots+k_n+m_n, p_1+\dots+p_n)\, p_1!\dots p_n!} t^{p_1+\dots+p_n}.$$

The multiple series of the previous expression may be written as a Lauricella function of the fourth kind

(2.8) $F_D^{(n)}(d+m_1+\dots+m_n, k_1+m_1, \dots, k_n+m_n;\ k_1+m_1+\dots+k_n+m_n;\ t, \dots, t)$

in which all its arguments are equal.

It has been shown by Appell and Kampé de Fériet [1], p. 116 that this case is reducible to a Gauss function

$$F_D^{(n)}(a, b_1, \dots, b_n;\ c;\ x, \dots, x) = {}_2F_1(a, b_1+\dots+b_n;\ c;\ x). \tag{2.9}$$

Hence, if $c=b_1+\dots+b_n$, this may, in turn, be written as the elementary function

$$(1-x)^{-a}. \tag{2.10}$$

It then follows that

$$\gamma_{m_1,\dots,m_n} = (1-t)^{-d}\frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)}[t/(1-t)]^{m_1+\dots+m_n} \tag{2.11}$$

and if (2.6) and (2.11) are substituted into (1.3), Theorem 1 is established.

Theorem 2. *For the arbitrary function* $C(m_1, \dots, m_n)$, *we have, formally,*

$$\begin{aligned} e^t \sum \frac{C(m_1, \dots, m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)} t^{m_1+\dots+m_n} \\ = \sum \frac{(k_1, m_1)\dots(k_n, m_n)t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)m_1!\dots m_n!} \\ \times \sum_{p_1,\dots,p_n=0}^{\infty} C(p_1, \dots, p_n)(-1)^{p_1+\dots+p_n}(-m_1, p_1)\dots(-m_n, p_n). \end{aligned} \tag{2.12}$$

Proof. This is along exactly parallel lines to that of the proof of Theorem 1. In this case, we take

$$\delta_{m_1,\dots,m_n} = \frac{(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)} t^{m_1+\dots+m_n}. \tag{2.13}$$

Theorem 2 is a confluent form of Theorem 1.

3. The expression of the main results in terms of generalised Kampé de Fériet functions

The form of Theorems 1 and 2 is rather too general for many practical purposes in deducing generating relations of various classes of hypergeometric polynomials. More convenient forms may be obtained by putting

(3.1)
$$C(m_1, \dots, m_n) = \frac{((a), m_1+\dots+m_n)((f_1), m_1)\dots((f_n), m_n)(-1)^{m_1+\dots+m_n}x_1^{m_1}\dots x_n^{m_n}}{((h), m_1+\dots+m_n)((g_1), m_1)\dots((g_n), m_n)m_1!\dots m_n!}.$$

By means of (1.6), it is clear that Theorem 1 may be written as

(3.2)

$$(1-t)^{-d} F_{H+1:G}^{A+1:F+1}\left[\begin{matrix}(a), & d & :(f_1), k_1; \ldots; (f_n), k_n; \\ (h), k_1+\ldots+k_n: & (g_1) & ;\ldots; (g_n);\end{matrix}\; x_1 t/(t-1), \ldots, x_n t/(t-1)\right]$$

$$= \sum \frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)\, t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n, m_n+\ldots+m_n)\, m_1!\ldots m_n!}$$

$$\times F_{H:G}^{A:F+1}\left[\begin{matrix}(a): (f_1), -m_1; \ldots; (f_n), -m_n; \\ (h): \quad (g_1) \quad ;\ldots; \quad (g_n) \quad ;\end{matrix}\; x_1, \ldots, x_n\right].$$

The confluent form

$$(3.3) \qquad e^t F_{H+1:G}^{A:F+1}\left[\begin{matrix}(a) & :(f_1), k_1; \ldots; (f_n), k_n; \\ (h), k_1+\ldots+k_n: & (g_1) \;;\ldots;\; (g_n) \;;\end{matrix}\; -x_1 t, \ldots, -x_n t\right]$$

$$= \sum \frac{(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)}\, t^{m_1+\cdots+m_n}$$

$$\times F_{H:G}^{A:F+1}\left[\begin{matrix}(a): (f_1), -m_1; \ldots; (f_n), -m_n; \\ (h): \quad (g_1) \quad ;\ldots; \quad (g_n) \quad :\end{matrix}\; x_1, \ldots, x_n\right]$$

follows similarly from Theorem 2.

Interesting special cases are given in the following Appendix, including results involving the polynomials of Meixner, Charlier and Laguerre.

Appendix

The results of the previous section may be applied to give various generating relations involving known functions.

(i) $F=G=0;\ A=H=1$.

From (3.2), it follows that

$$(A.1) \qquad (1-t)^{-d} F_{2:0}^{2:1}\left[\begin{matrix}a, & d & : k_1; \ldots; k_n; \\ h, k_1+\ldots+k_n: & - & ;\ldots; -;\end{matrix}\; x_1 t/(t-1), \ldots, x_n t/(t-1)\right]$$

$$= \sum \frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)\, m_1!\ldots m_n!}\, t^{m_1+\cdots+m_n}$$

$$\times F_D^{(n)}(a, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

Put $a=k_1+\ldots+k_n$. Then

(A.2) $$(1-t)^{-d}F_D^{(n)}(d, k_1, \ldots, k_n; h; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum\frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_D^{(n)}(k_1+\ldots+k_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

Put $d=k_1+\ldots+k_n$. Then

(A.3) $$(1-t)^{-k_1-\cdots-k_n}F_D^{(n)}(a, k_1, \ldots, k_n; h; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum\frac{(k_1, m_1)\ldots(k_n, m_n)}{m_1!\ldots m_n!}t^{m_1+\cdots+m_n}F_D^{(n)}(a, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

In (A.2) put $d=h$. Then

(A.4) $$(1-t)^{-h}\left(1-\frac{x_1t}{(t-1)}\right)^{-k_1}\cdots\left(1-\frac{x_nt}{(t-1)}\right)^{-k_n}$$
$$=\sum\frac{(h, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_D^{(n)}(k_1+\ldots+k_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

From (3.3), it follows that

(A.5) $$e^tF_{2:0}^{1:1}\left[\begin{matrix}a & : k_1; k_2; \\ h, k_1+\ldots+k_n: & -; -;\end{matrix}\ -x_1t, \ldots, -x_nt\right]$$
$$=\sum\frac{(k_1, m_1)\ldots(k_n, m_n)t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}F_D^{(n)}(a, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

Put $a=k_1+\ldots+k_n$. Then

(A.6) $$e^t\Phi_2^{(n)}(k_1, \ldots, k_n; h; -x_1t, \ldots, -x_nt)$$
$$=\sum\frac{(k_1, m_1)\ldots(k_n, m_n)t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}$$
$$\times F_D^{(n)}(k_1+\ldots+k_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

(ii) $A=G=0$; $H=F=1$.

From (3.2), it follows that

(A.7) $$(1-t)^{-d}F_{2:0}^{1:2}\left[\begin{matrix}d & : f_1, k_1; \ldots; f_n, k_n; \\ h, k_1+\ldots+k_n: & - \ ;\ldots;\ - \ ;\end{matrix}\ x_1t/(t-1), \ldots, x_nt/(t-1)\right]$$
$$=\sum\frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_B^{(n)}(f_1, \ldots, f_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

Put $d=h$. Then

(A.8) $$(1-t)^{-h}F_B^{(n)}(f_1, \ldots, f_n, k_1, \ldots, k_n; k_1+\ldots+k_n; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum \frac{(h, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_B^{(n)}(f_1, \ldots, f_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

Put $d=k_1+\ldots+k_n$. Then

(A.9) $$(1-t)^{-k_1-\cdots-k_n}F_B^{(n)}(f_1, \ldots, f_n, k_1, \ldots, k_n; h; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum \frac{(k_1, m_1)\ldots(k_n, m_n)}{m_1!\ldots m_n!}t^{m_1+\cdots+m_n}F_B^{(n)}(f_1, \ldots, f_n, k_1, \ldots, k_n; h; x_1, \ldots, x_n).$$

From (3.3), it follows that

(A.10) $$e^t F_{2:0}^{0:2}\begin{bmatrix} - & : f_1, k_1; \ldots; f_n, k_n; & \\ h, k_1+\ldots+k_n: & - \;\; ; \ldots; \;\; - \;\; ; & -x_1t, \ldots, -x_nt\end{bmatrix}$$
$$=\sum \frac{(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)}t^{m_1+\cdots+m_n}$$
$$\times F_B^{(n)}(f_1, \ldots, f_n, -m_1, \ldots, -m_n; h; x_1, \ldots, x_n).$$

(iii) $A=G=1$, $H=F=0$.

From (3.2), it follows that

(A.11) $$(1-t)^{-d}F_{1:1}^{2:1}\begin{bmatrix} a, & d & : k_1; \ldots; k_n; & \\ k_1+\ldots+k_n: & & g_1; \ldots; g_n; & x_1t/(t-1), \ldots, x_nt/(t-1)\end{bmatrix}$$
$$=\sum \frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_A^{(n)}(a, -m_1, \ldots, -m_n; g_1, \ldots, g_n; x_1, \ldots, x_n).$$

Put $d=k_1+\ldots+k_n$. Then

(A.12) $$(1-t)^{-k_1-\cdots-k_n}F_A^{(n)}(a, k_1, \ldots, k_n; g_1, \ldots, g_n; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum \frac{(k_1, n_1)\ldots(k_n, m_n)}{m_1!\ldots m_n!}t^{m_1+\cdots+m_n}F_A^{(n)}(a, -m_1, \ldots, -m_n; g_1, \ldots, g_n; x_1, \ldots, x_n).$$

Put $a=k_1+\ldots+k_n$. Then

(A.13) $$(1-t)^{-d}F_A^{(n)}(d, k_1, \ldots, k_n; g_1, \ldots, g_n; x_1t/(t-1), \ldots, x_nt/(t-1))$$
$$=\sum \frac{(d, m_1+\ldots+m_n)(k_1, m_1)\ldots(k_n, m_n)}{(k_1+\ldots+k_n, m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$
$$\times F_A^{(n)}(k_1+\ldots+k_n, -m_1, \ldots, -m_n; g_1, \ldots, g_n; x_1, \ldots, x_n).$$

From (3.3), it follows that

$$\text{(A.14)} \qquad e^t F^{1:1}_{1:1}\left[\begin{matrix} a & : k_1; \dots; k_n; \\ k_1+\dots+k_n: & g_1; \dots; g_n; \end{matrix} -x_1 t, \dots, -x_n t\right]$$

$$= \sum \frac{(k_1, m_1)\dots(k_n, m_n)\, t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)\, m_1!\dots m_n!}$$

$$\times F_A^{(n)}(a, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n).$$

Put $a=k_1+\dots+k_n$. Then

$$\text{(A.15)} \qquad e^t\, {}_1F_1(k_1; g_1; -x_1 t)\dots {}_1F_1(k_n; g_n; -x_n t)$$

$$= \sum \frac{(k_1, m_1)\dots(k_n, m_n)\, t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)\, m_1!\dots m_n!}$$

$$\times F_A^{(n)}(k_1+\dots+k_n, -m_1, \dots, -m_n; g_1, \dots, g_n; x_1, \dots, x_n).$$

(iv) $A=F=G=0,\ H=1$.

From (3.2), it follows that

$$\text{(A.16)} \quad (1-t)^{-d} F^{1:1}_{2:0}\left[\begin{matrix} d & : k_1; \dots; k_n; \\ h, k_1+\dots+k_n: & -; \dots; -; \end{matrix} x_1 t/(t-1), \dots, x_n t/(t-1)\right]$$

$$= \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)\, m_1!\dots m_n!}\, t^{m_1+\dots+m_n}$$

$$\times \Phi_2^{(n)}(-m_1, \dots, -m_n; h; x_1, \dots, x_n)$$

$$\text{(A.17)} \quad = \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)\, t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)(h, m_1+\dots+m_n)}\, L^{(h-1)}_{m_1,\dots,m_n}(x_1, \dots, x_n),$$

where $L^{(h-1)}_{m_1,\dots,m_n}$ is a Laguerre polynomial of several variables introduced by Erdélyi [3]. It is given by

(A.18)

$$L^{(a)}_{m_1,\dots,m_n}(x_1, \dots, x_n) = \frac{(a+1, m_1+\dots+m_n)}{m_1!\dots m_n!}\, \Phi_2^{(n)}(-m_1, \dots, -m_n; a+1; x_1, \dots, x_n).$$

Put $d=h$. Then

$$\text{(A.19)} \quad (1-t)^{-h}\Phi_2^{(n)}(k_1, \dots, k_n; k_1+\dots+k_n; x_1 t/(t-1), \dots, x_n t/(t-1))$$

$$= \sum \frac{(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)}\, t^{m_1+\dots+m_n} L^{(h-1)}_{m_1,\dots,m_n}(x_1, \dots, x_n).$$

Put $d=k_1+\ldots+k_n$. Then

(A.20) $$(1-t)^{-k_1-\cdots-k_n}\Phi_2^{(n)}(k_1,\ldots,k_n;h;x_1t/(t-1),\ldots,x_nt/(t-1))$$

$$=\sum\frac{(k_1,m_1)\ldots(k_n,m_n)}{(h,m_1+\ldots+m_n)}t^{m_1+\cdots+m_n}L^{(h-1)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n).$$

If $n=2$, an equivalent result has been given by Feldheim [6], eq. (66).

From (3.3), it follows that

(A.21) $$e^tF^{0:1}_{2:0}\left[\begin{matrix}- & :k_1;\ldots;k_n; \\ h,k_1+\ldots+k_n & :-;\ldots;-;\end{matrix}\ -tx_1,\ldots,-tx_n\right]$$

$$=\sum\frac{(k_1,m_1)\ldots(k_n,m_n)t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n,m_1+\ldots+m_n)(h,m_1+\ldots+m_n)}L^{(h-1)}_{m_1,\ldots,m_n}(x_1,\ldots,x_n).$$

(v) $F=G=H=0$, $A=1$.

From (3.2), it follows that

(A.22) $$(1-t)^{-d}F^{2:1}_{1:0}\left[\begin{matrix}a, & d & :k_1;\ldots;k_n; \\ k_1+\ldots+k_n & & :-;\ldots;-;\end{matrix}\ x_1t/(t-1),\ldots,x_nt/(t-1)\right]$$

$$\cong\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$

$$\times F^{1:1}_{0:0}\left[\begin{matrix}a: & -m_1;\ldots;-m_n; \\ -: & -\ ;\ldots;\ -\ ;\end{matrix}\ x_1t/(t-1),\ldots,x_nt/(t-1)\right]$$

$$\cong\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)(a,m_1+\ldots+m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)m_1!\ldots m_n!}(-x_1t)^{m_1}\ldots(-x_nt)^{m_n}$$

$$\times\Phi_n^{(2)}(-m_1,\ldots,-m_n;1-a-m_1-\ldots-m_n;-1/x_1,\ldots,-1/x_n)$$

$$\cong\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)}$$

$$\times(x_1t)^{m_1}\ldots(x_nt)^{m_n}L^{(-a-m_1-\cdots-m_n)}_{m_1,\ldots,m_n}(-1/x_1,\ldots,-1/x_n).$$

The $F^{1:1}_{0:0}$ polynomial has been reversed and the symbol $\cong$ is used in place of the equality because the relationship is purely formal.

Put $a=k_1+\ldots+k_n$. Then

(A.23) $$(1-t)^{-d}F^{1:1}_{0:0}\left[\begin{matrix}d:k_1;\ldots;k_n; \\ -:-;\ldots;-;\end{matrix}\ x_1t/(t-1),\ldots,x_nt/(t-1)\right]$$

$$\cong\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)}(x_1t)^{m_1}\ldots(x_nt)^{m_n}$$

$$\times L^{(-k_1-m_1-\cdots-k_n-m_n)}_{m_1,\ldots,m_n}(-1/x_1,\ldots,-1/x_n).$$

Put $d=k_1+\dots+k_n$. Then

(A.24) $$(1-t)^{-k_1-\dots-k_n}F^{1:1}_{0:0}\begin{bmatrix} a: k_1;\dots;k_n; \\ -: -;\dots;-; \end{bmatrix} x_1t/(t-1),\dots,x_nt(t-1)$$

$$\cong \sum \frac{(k_1,m_1)\dots(k_n,m_n)}{(k_1+\dots+k_n,m_1+\dots+m_n)}(x_1t)^{m_1}\dots(x_nt)^{m_n}$$

$$\times L^{(-a-m_1-\dots-m_n)}_{m_1,\dots,m_n}(-1/x_1,\dots,-1/x_n).$$

From (3.3), it follows that

(A.25) $$e^tF^{(n)}_D(a,k_1,\dots,k_n;k_1+\dots+k_n;-x_1t,\dots,-x_nt)$$

$$=\sum \frac{(k_1,m_1)\dots(k_n,m_n)}{(k_1+\dots+k_n,m_1+\dots+m_n)}(x_1t)^{m_1}\dots(x_nt)^{m_n}$$

$$\times L^{(-a-m_1-\dots-m_n)}_{m_1,\dots,m_n}(-1/x_1,\dots,-1/x_n).$$

Put $a=k_1+\dots+k_n$. Then

(A.26) $$e^t(1+x_1t)^{-k_1}\dots(1+x_nt)^{-k_n}$$

$$=\sum \frac{(k_1,m_1)\dots(k_n,m_n)}{(k_1+\dots+k_n,m_1+\dots+m_n)}(x_1t)^{m_1}\dots(x_nt)^{m_n}$$

$$\times L^{(-k_1-m_1-\dots-k_n-m_n)}_{m_1,\dots,m_n}(-1/x_1,\dots,-1/x_n).$$

(vi) $A=H=0$, F and G unspecified.

From (3.2), it follows that

(A.27) $$(1-t)^{-d}F^{1:F+1}_{1:G}\begin{bmatrix} d : (f_1),k_1;\dots;(f_n),k_n; \\ k_1+\dots+k_n: (g_1);\dots;(g_n); \end{bmatrix} x_1t/(t-1),\dots,x_nt/(t-1)$$

$$=\sum \frac{(d,m_1+\dots+m_n)(k_1,m_1)\dots(k_n,m_n)}{(k_1+\dots+k_n,m_1+\dots+m_n)m_1!\dots m_n!}t^{m_1+\dots+m_n}$$

$$\times {}_{F+1}F_G[(f_1),-m_1;(g_1);x_1]\dots{}_{F+1}F_G[(f_n),-m_n;(g_n);x_n].$$

From (3.3), it follows that

(A.28) $$e^tF^{0:F+1}_{1:G}\begin{bmatrix} - : (f_1),k_1;\dots;(f_n),k_n; \\ k_1+\dots+k_n: (g_1);\dots;(g_n); \end{bmatrix} -x_1t,\dots,-x_nt$$

$$=\sum \frac{(k_1,m_1)\dots(k_n,m_n)}{(k_1+\dots+k_n,m_1+\dots+m_n)}t^{m_1+\dots+m_n}$$

$$\times {}_{F+1}F_G[(f_1),-m_1;(g_1);x_1]\dots{}_{F+1}F_G[(f_n),-m_n;(g_n);x_n].$$

(vii) $A=H=0$, $F=G=1$.

From (3.2), it follows that

$$(A.29)\quad (1-t)^{-d}F_{1:1}^{1:2}\left[\begin{matrix} d & :f_1,k_1;\ldots;f_n,k_n; \\ k_1+\ldots+k_n: & g_1\ ;\ldots;\ g_n\ ; \end{matrix}\ x_1t/(t-1),\ldots,x_nt/(t-1)\right]$$

$$=\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$

$$\times{}_2F_1(f_1,-m_1;g_1;x_1)\ldots{}_2F_1(f_n,-m_n;g_n;x_n)$$

$$=\sum\frac{(d,m_1+\ldots+m_n)(k_1,m_1)\ldots(k_n,m_n)t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n,m_1+\ldots+m_n)(g_1,m_1)\ldots(g_n,m_n)m_1!\ldots m_n!}$$

$$\times m_{m_1}(-f_1;g_1;1/(1-x_1))\ldots m_{m_n}(-f_n;g_n;1/(1-x_n)),$$

where $m_n(a;b;z)$ is a Meixner polynomial. See Erdélyi [4], Vol. II, p. 225.

Put $k_i=g_i$. Then

$$(A.30)\quad (1-t)^{-d}F_D^{(n)}(d,f_1,\ldots,f_n;g_1+\ldots+g_n;x_1t/(t-1),\ldots,x_nt/(t-1))$$

$$=\sum\frac{(d,m_1+\ldots+m_n)t^{m_1+\cdots+m_n}}{(g_1+\ldots+g_n,m_1+\ldots+m_n)m_1!\ldots m_n!}$$

$$\times m_{m_1}(-f_1;g_1;1/(1-x_1))\ldots m_{m_n}(-f_n;g_n;1/(1-x_n)).$$

Put $d=g_1+\ldots+g_n$, and obtain effectively the known result

$$(A.31)\quad (1-t)^{-g}(1-xt/(t-1))^{-f}=\sum_{m=0}^{\infty}m_m(-f;g;1/(1-x))t^m/m!.$$

See Erdélyi [4], Vol. II, p. 225.

From (3.3), it follows that

$$(A.32)\quad e^tF_{1:1}^{0:2}\left[\begin{matrix} - & :f_1,k_1;\ldots;f_n,k_n; \\ k_1+\ldots+k_n: & g_1\ ;\ldots;\ g_n\ ; \end{matrix}\ -x_1t,\ldots,-x_nt\right]$$

$$=\sum\frac{(k_1,m_1)\ldots(k_n,m_n)}{(k_1+\ldots+k_n,m_1+\ldots+m_n)(g_1,m_1)\ldots(g_n,m_n)m_1!\ldots m_n!}t^{m_1+\cdots+m_n}$$

$$\times m_{m_1}(-f_1;g_1;1/(1-x_1))\ldots m_{m_n}(-f_n;g_n;1/(1-x_n)).$$

Put $k_i=g_i$. Then

$$(A.33)\quad e^t\Phi_2(f_1,\ldots,f_n;g_1+\ldots+g_n;-x_1t,\ldots,-x_nt)$$

$$=\sum\frac{t^{m_1+\cdots+m_n}}{(g_1+\ldots+g_n,m_1+\ldots+m_n)}$$

$$\times m_{m_1}(-f_1;g_1;1/(1-x_1))\ldots m_{m_n}(-f_n;g_n;1/(1-x_n)).$$

(viii) $A=H=G=0,\ F=1.$

From (3.2), it follows that

$$\text{(A.34)}\quad (1-t)^{-d}F^{1:2}_{1:0}\left[\begin{matrix} d & : f_1, k_1; \dots; f_n, k_n; \\ k_1+\dots+k_n: & - \;;\dots;\; - \;; \end{matrix} x_1t/(t-1), \dots, x_nt/(t-1)\right]$$

$$\cong \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)m_1!\dots m_n!} t^{m_1+\dots+m_n}$$

$$\times {}_2F_0(f_1, -m_1; -; x_1)\dots {}_2F_0(f_n, -m_n; -; x_n)$$

$$\cong \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)m_1!\dots m_n!} t^{m_1+\dots+m_n}$$

$$\times c_{m_1}(-f_1; -1/x_1)\dots c_{m_n}(-f_n; -1/x_n),$$

where $c_n(z; b)$ is a Charlier polynomial (Erdélyi [4], Vol. II, p. 226) given by

$$\text{(A.35)}\quad c_n(z; b) = \frac{\Gamma(1+z)(-b)^{-n}}{\Gamma(1+z-n)} {}_1F_1(-n; z-n+1; b) = {}_2F_0(-z, -n; -; -1/b)$$

by reversion of the series.

From (3.3), it follows that

$$\text{(A.36)}\quad e^t F_B^{(n)}(f_1, \dots, f_n, k_1, \dots, k_n; k_1+\dots+k_n; -x_1t, \dots, -x_nt)$$

$$= \sum \frac{(k_1, m_1)\dots(k_n, m_n)t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)m_1!\dots m_n!} c_{m_1}(-f_1; -1/x_1)\dots c_{m_n}(-f_n; -1/x_n).$$

If $n=1$, we obtain the known result

$$\text{(A.37)}\quad e^t(1+xt)^{-f} = \sum_{m=0}^{\infty} c_m(-f; -1/x),$$

compare Erdélyi [4], Vol. II, p. 226.

(ix) $A=H=F=0$, $G=1$.

From (3.2), it follows that

$$\text{(A.38)}\quad (1-t)^{-d}F^{1:1}_{1:1}\left[\begin{matrix} d & : k_1; \dots; k_n; \\ k_1+\dots+k_n: & g_1; \dots; g_n; \end{matrix} x_1t/(t-1), \dots, x_nt/(t-1)\right]$$

$$= \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)}{(k_1+\dots+k_n, m_1+\dots+m_n)m_1!\dots m_n!} t^{m_1+\dots+m_n}$$

$$\times {}_1F_1(-m_1; g_1; x_1)\dots {}_1F_1(-m_n; g_n; x_n)$$

$$= \sum \frac{(d, m_1+\dots+m_n)(k_1, m_1)\dots(k_n, m_n)t^{m_1+\dots+m_n}}{(k_1+\dots+k_n, m_1+\dots+m_n)(g_1, m_1)\dots(g_n, m_n)} L_{m_1}^{(g_1-1)}(x_1)\dots L_{m_n}^{(g_n-1)}(x_n).$$

Put $k_i = g_i$. Then

(A.39) $$(1-t)^{-d}{}_1F_1\big(d;\, g_1+\ldots+g_n;\, t(x_1+\ldots+x_n)/(t-1)\big)$$
$$= \sum \frac{(d, m_1+\ldots+m_n)\, t^{m_1+\cdots+m_n}}{(g_1+\ldots+g_n, m_1+\ldots+m_n)} L_{m_1}^{(g_1-1)}(x_1)\ldots L_{m_n}^{(g_n-1)}(x_n).$$

If we now put $d=g_1+\ldots+g_n$ we obtain in effect the expression

(A.40) $$(1-t)^{-d}\exp\big(xt/(t-1)\big) = \sum_{m=0}^{\infty} t^m L_m^{(d-1)}(x),$$

a well-known generating relation for the Laguerre polynomial. See Erdélyi [4], Vol. II, p. 189.

From (3.3), it follows that

(A.41) $$e^t F_{1:1}^{0:1}\left[\begin{matrix} - : k_1;\ldots;k_n; \\ k_1+\ldots+k_n: g_1;\ldots;g_n;\end{matrix}\; -x_1t,\ldots,-x_nt\right]$$
$$= \sum \frac{(k_1, m_1)\ldots(k_n, m_n)\, t^{m_1+\cdots+m_n}}{(k_1+\ldots+k_n, m_1+\ldots+m_n)(g_1, m_1)\ldots(g_n, m_n)} L_{m_1}^{(g_1-1)}(x_1)\ldots L_{m_n}^{(g_n-1)}(x_n).$$

Put $k_i = g_i$ and obtain

(A.42) $$e^t{}_0F_1\big(-;\, g_1+\ldots+g_n;\, -t(x_1+\ldots+x_n)\big)$$
$$= \frac{[(x_1+\ldots+x_n)t]^{-g_1/2-\cdots-g_n/2}}{\Gamma(g_1+\ldots+g_n)} J_{g_1+\ldots+g_n-1}\left\{2\sqrt{[(x_1+\ldots+x_n)t]}\right\}$$
$$= \sum \frac{t^{m_1+\cdots+m_n}}{(g_1+\ldots+g_n, m_1+\ldots+m_n)} L_{m_1}^{(g_1-1)}(x_1)\ldots L_{m_n}^{(g_n-1)}(x_n).$$

Put $n=1$, when the known result

(A.43) $$\sum_{m=0}^{\infty} L_m^{(g-1)} t^m/(g, m) = \frac{(xt)^{-g/2-1/2}}{\Gamma(g)} J_g[2\sqrt{xt}]$$

is recovered. See Erdélyi [4], Vol. II, p. 189.

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