Periods and the asymptotics of a diophantine problem: I

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Introduction

Let $P(z_1, z_2)$ be a polynomial with positive coefficients. For positive x define

$$N_1(x) = \sum_{\{m\in\mathbf{N}^2:\, P(m)\leq x\}} 1.$$

A classical diophantine problem is to describe the asymptotic behavior of $N_1(x)$ as $x\to\infty$. More generally, one can introduce a second polynomial with positive coefficients $\varphi(z_1, z_2)$ and define

$$N_\varphi(x) = \sum_{\{m\in\mathbf{N}^2:\, P(m)\leq x\}} \varphi(m).$$

One can also ask about the asymptotic behavior of $N_\varphi(x)$ as $x\to\infty$. This is an example of a "weighted" diophantine problem, each lattice point m weighted by $\varphi(m)$.

The answer to such questions has been given by Sargos [14], as described in theorem A. The analytic method used to study $N_\varphi(x)$ is based upon the functional properties of the series

$$D_P(s, \varphi) = \sum_{m\in\mathbf{N}^2} \frac{\varphi(m)}{P(m)^s}.$$

One knows from [9, 13] that $D_P(s, \varphi)$ is analytic in a halfplane $\operatorname{Re}(s)>B(\varphi)$ and admits a meromorphic extension to $\mathbf{C}$ with rational poles (by [13, theorem 1.2], also cf. [10]) of order at most 2. Order the poles as $\varrho_0(\varphi)>\varrho_1(\varphi)>\dots$. For each j, set

$$\operatorname{Pol}_{s=\varrho_j(\varphi)}\left(\frac{D_P(s,\varphi)}{s}\right) = \sum_{i=1}^{2} \frac{A_{i,j}(\varphi)}{(s-\varrho_j(\varphi))^i}$$

to be the principal part at $\varrho_j(\varphi)$. Define

$$N_j(x) = \begin{cases} x^{\varrho_j(\varphi)} \sum_{i=1}^{2} A_{i,j}(\varphi) \log^{i-1} x & \text{if} \quad \varrho_j(\varphi) \neq 0 \\ \sum_{i=1}^{2} A_{i,j}(\varphi) \log^{i} x & \text{if} \quad \varrho_j(\varphi) = 0. \end{cases}$$

Let D=degree of P. Define the index k by the condition $\varrho_0(\varphi) > \varrho_1(\varphi) \ldots > \varrho_k(\varphi) > \varrho_0(\varphi) - 1/D \geqq \varrho_{k+1}(\varphi) > \ldots$. Sargos has shown [14].

Theorem A.

(0.1) $$N_\varphi(x) = \sum_{j=0}^{k} N_j(x) + O_\varepsilon(x^{\varrho_0(\varphi)-1/D+\varepsilon}).$$

Define the dominant term $\hat{N}_\varphi(x)$ of $N_\varphi(x)$ to equal $N_0(x)$.

In this paper one will always assume, for simplicity, that $\varrho_0(\varphi)$ is a *simple* pole of $D_P(s, \varphi)$. The aim of this paper is to understand the residue $A_{0,1}(\varphi)$ in terms of the asymptotic behavior of periods over certain cycles in the fibers of P located in a neighborhood of infinity. This aim is accomplished by the theorem 1, using a concrete geometric analysis, similar to that used in [11] to give formulae for certain "geometric" roots of a local b-function. One difference here however is that this analysis is carried out in a neighborhood at infinity in a suitable compactification of $\mathbf{C}^2$.

The main technical problem that is solved by theorem 1 has nothing to do, a priori, with the above diophantine problem. This is the following. Define

(0.2) $$I_P(s, \varphi) = \int_{[1,\infty)^2} (1/P)^s \varphi \, dz_1 \, dz_2.$$

It is well-known from [1] that $I_P(s, \varphi)$ is analytic in a halfplane and admits an analytic continuation to $\mathbf{C}$ with poles contained in finitely many decreasing sequences of rational numbers. Let $\varrho(\varphi)$ denote the first pole of $I_P(s, \varphi)$. *Assume* that $\varrho(\varphi)$ is a simple pole.

Problem. *Over an interval* $[\beta, \infty) \subset \mathbf{C} - \{0\}$, *construct a continuous family of* 1-*cycles* γ_t *and analytic family of* 1-*forms* ω_t *satisfying these properties:*

(1) $|\gamma_t| \subset \{P = t\}$ *for all* $t \in [\beta, \infty)$;

(2) ω_t *is a* 1-*form on* $\{P = t\}$ *for all* $t \in [\beta, \infty)$;

(3) *One has the identity*

(0.3) $$\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi) = \mathrm{Pol}_{s=\varrho(\varphi)} \int_\beta^\infty t^{-s} \int_{\gamma_t} \omega_t.$$

This is essentially a 1-variable problem, and can therefore be addressed using standard residue calculus techniques, adapted to a global geometric setting involving a configuration of normally crossing divisors on an algebraic surface.

To connect (0.3) to Dirichlet series, one first makes the following definition. For P, φ two polynomials with positive coefficients one says

Definition 1. The pair (P, φ) is "good" if

(1) $\varphi_0(\varphi)=\varrho(\varphi)$;

(2) $\mathrm{Pol}_{s=\varrho(\varphi)} D_P(s, \varphi)=\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$.

Section 1 states a simple condition for P which insures that (P, φ) is a good pair, cf. theorem B. Thus, the significance of theorem 1 for the diophantine problem discussed above is this. Whenever one knows that (P, φ) is good and the first pole of $I_P(s, \varphi)$ is simple, one concludes that the coefficient of $\hat{N}_\varphi(x)$ is determined by the asymptotics of a certain family of periods. In this way, one explicitly connects $\hat{N}_\varphi(x)$ to the behavior of a Nilsson function whose algebro-geometric significance is by now well understood [6]. In addition, it implies that $\hat{N}_\varphi(x)$ is a "cohomological invariant", in the (limited) sense of corollary 3. One should also note that earlier results of Cassou—Nogues, for certain special cases of P [3, 4], had suggested that a result like theorem 1 could be true for a much larger class of polynomials.

A second application of theorem 1 is given in corollary 4. In [13, theorem 4.9], Sargos gave an "explicit" expression for $A_{0,2}(\varphi)$, in the event $\varrho_0(\varphi)$ is a double pole. His methods do not extend however to give an expression for $A_{0,1}(\varphi)$ when $\varrho_0(\varphi)$ is simple. Using the geometric methods described herein, a similarly "explicit" expression for the residue is given when (P, φ) is good, $\varrho_0(\varphi)\in\mathbf{Z}$, and is a simple pole.

Sections 2—4 give preliminary definitions and constructions needed in the proof of theorem 1. These involve ideas from toroidal geometry (section 2), the regularization of generalized powers à la Gelfand—Shapiro (section 3), and the construction of V-manifolds ala Steenbrink—Varcenko (section 4). The proof of theorem 1 is to be found in section 5. The most important ingredients for the proof of theorem 1 are the precise overlap relations given in lemmas 4 (section 3) and 9 (section 4), as well as the integral representation for $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$ in lemma 7 (section 3). The reader should keep in mind that the only assumptions needed in sections 3—5, unless explicitly noted otherwise, are that P and φ have positive coefficients and $\varrho(\varphi)$ is a simple pole of $I_P(s, \varphi)$. On the other hand, it should be possible to weaken the first hypothesis. This is discussed at the end of section 5.

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1. Good pairs

In [9, sec. 5] a large class of good pairs was determined. In the two variable case there is a very clear defining property for this class, denoted $\mathscr{C}$ below. This is now recalled.

Let $h(z_1, z_2)=\sum_I b_I z^I$ be any polynomial. Set $\operatorname{supp}(h)=\{I: b_I \neq 0\}$. Define the monomial $M(h)=(M_1, M_2)$ where for each $j=1, 2$

(1.1) $\quad M_j = \max\{i_j: i_j \text{ is the } j^{th} \text{ entry of an index } I \text{ for which } b_I \neq 0\}.$

$M(h)$ is called the maximal monomial for h.

To h one also assigns the Newton polygons.

Definition 2. The boundary of the convex hull of $\bigcup_{I\in \operatorname{supp}(h)}(I-\mathbf{R}^2_+)$ is called the Newton polygon of h at infinity. It is denoted $\Gamma_\infty(h)$. The boundary of the convex hull of $\bigcup_{I\in \operatorname{supp}(h)}(I+\mathbf{R}^2_+)$ is called the Newton polygon at the origin and denoted $\Gamma_0(h)$.

Now write $P(z)=\sum_I a_I z^I$, and define

$$\mathscr{C} = \{(P, \varphi)\colon i)\ a_{M(P)} = 0;\quad ii)\ \text{Both } P \text{ and } \varphi \text{ have positive coefficients.}\}$$

Theorem 2 [9] showed

Theorem B. *If $(P, \varphi)\in\mathscr{C}$, then (P, φ) is a good pair.*

Remark 1.

(1) One can even give a description for $\varrho(\varphi)$ in terms of $\Gamma_\infty(P)$. Let $\Delta(d_1, d_2)=\{t\cdot(d_1, d_2)\colon t\geqq 0\}$. Let $\tau(d_1, d_2)$ denote the value t_0 for which $\{t_0\cdot(d_1, d_2)\}=\Gamma_\infty(P)\cap\Delta(d_1, d_2)$. Let $\bar{1}=(1, 1)$. Write

$$\varphi(z_1, z_2) = \sum_{I\in\operatorname{supp}(\varphi)} c_I z^I.$$

Set

$$\tau(\varphi) = \max\{\tau(I)\colon I\in\operatorname{supp}(z_1 z_2\cdot\varphi)\},$$

$$\mathscr{L}(\varphi) = \{I\in\operatorname{supp}(\varphi)\colon \tau(I+\bar{1}) = \tau(\varphi)\},$$

and

$$\varphi_\Gamma(z_1, z_2) = \sum_{I\in\mathscr{L}(\varphi)} c_I z^I.$$

Let $b_1, b_2, \ldots, b_d$ be the integral and primitive covectors dual to the 1-dimensional faces of $\Gamma_\infty(P)$. One has

Proposition 1. *If $(P, \varphi)\in\mathscr{C}$, then*

$$\varrho(\varphi) = 1/\tau(\varphi) = \max\left\{\frac{b_i\cdot I}{M(b_i)}\colon I\in\mathscr{L}(\varphi),\ i = 1, \ldots, d\right\},$$

where $M(b_i)=\max\{J\cdot b_i\colon J\in\operatorname{supp}(P)\}$, $i=1, \ldots, d$.

Proof. This is a straightforward extension of the arguments in [9, pgs. 109—112]. Details are left to the reader. □

(2) The analysis in [ibid] seems to break down if P has a maximal monomial. The description of $\mathrm{Pol}_{s=\varrho(\varphi)} D_P(s, \varphi)$ can then involve a sum of terms including, but not limited to, $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$. This is the only reason why the first condition in the definition of $\mathscr{C}$ is included. □

2. A Toroidal Compactification

To give a useful analytical description of $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$, one needs to be able to write the integrand in as simple a form as possible. To do this one first chooses a compactification of $\mathbf{C}^2$, ι: $\mathbf{C}^2 \hookrightarrow Z$, and then constructs a modification π: $X \to Z$ for which the divisor defined by $\pi^*(P)$ is locally normal crossing.

It is convenient here to take $Z=(P^1\mathbf{C})^2$. The polygon $\Gamma_\infty(P)$ can now be used to construct a smooth toroidal variety X, defined over $\mathbf{Q}$, and modification π: $X \to Z$ which is both proper and locally monomial. π is obtained by dualizing certain cones derived from $\Gamma_\infty(P)$, $\Gamma_\infty(\varphi)$, as discussed in [5], and forming a partition Σ of simplicial cones in $\mathbf{R}^2$. X is covered by a finite union of affine charts $\{X(\sigma)\}$, where to each cone $\sigma \in \Sigma$ one assigns the chart $X(\sigma)$.

There is one chart in Z in which the analysis below is actually carried out. If $\mathbf{C}^2(z)$ denotes the affine chart with the original coordinates (z_1, z_2), let $\mathbf{C}^2(x)$ denote the chart with coordinates (x_1, x_2) and with overlaps $z_i = 1/x_i$, $i=1, 2$. One abuses notation by denoting $X=\pi^{-1}(\mathbf{C}^2(x))$, so that one writes in the following π: $X \to \mathbf{C}^2(x)$.

To the polynomial $\varphi(z_1, z_2)$ one associates a rational 2-form ω_φ on $(P^1\mathbf{C})^2$ so that

(2.1.1) $$\omega_\varphi|_{\mathbf{C}^2(z)} = \varphi \, dz_1 \, dz_2$$

and

$$\omega_\varphi|_{\mathbf{C}^2(x)} = [\varphi(1/x_1, 1/x_2)/(x_1 x_2)^2] \, dx_1 \, dx_2 = \frac{\Phi(x_1, x_2)}{x_1^{m_1} x_2^{m_2}} \, dx_1 \, dx_2,$$

where Φ is a polynomial not divisible by $x_1 x_2$.

In the notation of (1.1), define the polynomial $Q(x_1, x_2)$ by the following formula

(2.1.2) $$R(x_1, x_2) = P(1/x_1, 1/x_2) = \frac{Q(x_1, x_2)}{x_1^{M_1} x_2^{M_2}}.$$

Thus, one has as an identity between analytic functions of s with $\mathrm{Re}(s) \gg 1$,

$$I_P(s,\varphi) = \int_{[0,1]^2} R^s \omega_\varphi .$$

Evidently, one also has that

$$\text{(2.1.3)} \qquad \mathrm{Pol}_{s=\varrho(\varphi)} I_P(s,\varphi) = \mathrm{Pol}_{s=\varrho(\varphi)} \int_{[0,1]^2} R^s \omega_\varphi .$$

If the chart $X(\sigma)$ of X has coordinates (u_1, u_2) then one can write

$$\text{(2.2.1)} \qquad R\circ\pi(u_1,u_2) = u_1^{A(1)} u_2^{A(2)}/Q_\sigma(u), \quad Q_\sigma(0,0) \neq 0$$

$$\pi^*\omega_\varphi = u_1^{B(1)} u_2^{B(2)} \Phi_\sigma(u)\, du_1\, du_2, \quad \Phi_\sigma(0,0) \neq 0.$$

Set $R_\sigma = 1/Q_\sigma$. By [9, remark (5.4.3) pg. 110],

$$\text{(2.2.2)} \qquad A(i) \geqq 0, \quad i = 1, 2.$$

Notation. In the following, $\mathscr{R}$ denotes the strict transform of the divisor $\mathbf{C}^2(x)$ defined by Q. □

An important point is to estimate the location of $\mathscr{R}$ with respect to the preimage of the domain of integration of I_P. To formulate the result, cf. proposition 3, one first introduces the following.

For positive α set

$$\Gamma_\alpha = \{z = x+iy \colon x \geqq 1 \text{ and } |y| \leqq \alpha(x-1)\}.$$

Set $\Gamma(\alpha) = \Gamma_\alpha^2 \cap \mathbf{C}^2(z)$. Using the overlap relation for $\mathbf{C}^2(z) \cap \mathbf{C}^2(x)$, one considers $\Gamma(\alpha) \hookrightarrow \mathbf{C}^2(x)$.

Notation. Denote the "strict transform" of $\Gamma(\alpha)$ by

$$\Gamma^\pi(\alpha) = \text{closure in } X \text{ of } \pi^{-1}\left[\bigcup_{1\geqq\delta>0} \Gamma(a) \cap \{\|x\| \geqq \delta\}\right].$$

One similarly denotes Δ^π as the strict transform of the box $[0,1]^2$ in $\mathbf{C}^2(x)$.

Sargos has shown [13, sec. 5]

Proposition 2. *There exist positive numbers α, c, ε, ϱ with $\varrho \in (0, \pi/2)$ such that*

$$|P(z_1,z_2)| \geqq c\,|z_1 \cdot z_2|^\varepsilon \quad \textit{and} \quad |\mathrm{Arg}\, P(z_1,z_2)| \leqq \varrho$$

for all $(z_1, z_2) \in \Gamma(\alpha)$.

One now sees by [9, pgs. 110—112]

Proposition 3. *There exists α such that for each chart $X(\sigma)$ one has*

$$\mathscr{R} \cap X(\sigma) \cap \Gamma^\pi(\alpha) = \emptyset. \quad \square$$

This result allows one, in the next section, to give an exceedingly simple description of $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s,\varphi)$.

3. An integral expression for $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$

This section contains the most important of the technical preliminaries for theorem 1. It combines the toroidal construction of section 2 with the regularization method for generalized powers, due originally to Gelfand—Shapiro, to give an integral representation for $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$. The final result is given in lemma 7.

Let $\mathscr{D}=\{D_i\}\cup\{\mathscr{R}\}$ denote the divisors corresponding to π and Q. To each $D\in\mathscr{D}$ there are two numbers of interest

(3.1) $$A(D) = \mathrm{ord}_D(R\circ\pi), \quad B(D) = \mathrm{ord}_D\,\pi^*(\omega_\varphi),$$

from which one forms the ratios, for $e=1, 2, \dots$,

$$\varrho(D, e) = -(e+B(D))/A(D).$$

Definition 3. One says that $D'\in\mathscr{D}$ contributes potentially to $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$ if

$$\varrho(D', 1) = \max_{D\in\mathscr{D}} \varrho(D, 1) = \varrho(\varphi).$$

One first observes the

Lemma 1. *If $D\neq\mathscr{R}$ then $D\cap\Gamma^\pi(\alpha)\neq\emptyset$.*

Proof. This is a simple calculation that is based on the fact that π is locally a monomial map, whence defined over $\mathbf{Q}$ (cf. [9, pg. 111, 112]). □

Remark 2. It suffices to use $e=1$ in Definition 3. To see this one uses the reasoning in the proof of theorem 2 [ibid] and the fact that $\varrho(\varphi)$ equals the *largest* pole of $I_P(s, \varphi)$. □

Notation. Set $\mathscr{B}_\varphi=\{D_1, \dots, D_r\}$ to be the set of divisors contributing potentially to $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$.

If $D\in\mathscr{B}_\varphi$ define

(3.2) $$\mathscr{I}(D) = \{D'\in\mathscr{D}-\{D\}\colon D\cap D' \neq \emptyset\},$$

and

$$\mathscr{X}(D) = \{\text{charts } X(\sigma)\colon D\cap X(\sigma) \neq \emptyset\}.$$

Now let $\mathbf{D}\in\mathscr{B}_\varphi$ and $D'\in\mathscr{I}(\mathbf{D})$. Define

(3.3) $$\lambda_\varphi(\mathbf{D}, D') = A(D')\varrho(\varphi)+B(D').$$

A slightly tricky point is the following

Lemma 2. *When $\varrho(\varphi)$ is a simple pole of $I_P(s, \varphi)$, then*

$$\lambda_\varphi(\mathbf{D}, D') > -1 \quad \text{if} \quad \mathbf{D}\in\mathscr{B}_\varphi, \quad D'\in\mathscr{I}(\mathbf{D}).$$

Proof. It is easy to check that it suffices to show the following assertion:

$$\text{If} \quad \mathbf{D}\in\mathscr{B}_\varphi \quad \text{and} \quad D'\in\mathscr{I}(\mathbf{D}), \quad \text{then} \quad D'\notin\mathscr{B}_\varphi.$$

The reason for the implication is that the contribution to the coefficient of $1/(s-\varrho(\varphi))^2$ in $\mathrm{Pol}_{s=\varrho(\varphi)}\, I_P(s,\varphi)$ is supported (as a current) at the intersection points of D', $\mathbf{D}$, whenever both divisors are contained in $\mathscr{B}_\varphi$. It is now easy to see by [9, pg. 117] that the local contribution at a given intersection $D'\cap\mathbf{D}$ must be positive whenever P and φ have positive coefficients. Thus, since $\varrho(\varphi)$ is assumed to be a simple pole, it follows that $\lambda_\varphi(\mathbf{D}, D')\neq -1$. Since $\varrho(\varphi)$ is the largest pole, necessarily $\lambda_\varphi(\mathbf{D}, D')>-1$. □

To the divisor $\mathbf{D}\in\mathscr{B}_\varphi$ there corresponds a unique covector $\mathbf{b}=(b_1, b_2)$ in the partition Σ, used to construct X. One may assume that $\mathbf{b}\neq(0,1),(1,0)$. The reader will be able to modify the arguments below if this assumption does not hold. One can now find two integral covectors $\mathbf{a}=(a_1, a_2)$, $\mathbf{c}=(c_1, c_2)$, which for simplicity can also both be assumed to have positive components, so that

(3.4) (i) $\det\begin{pmatrix}\mathbf{a}\\ \mathbf{b}\end{pmatrix} = \det\begin{pmatrix}\mathbf{b}\\ \mathbf{c}\end{pmatrix} = 1,$

(ii) $\sigma_1 = \langle\mathbf{b}, \mathbf{c}\rangle,\quad \sigma_2 = \langle\mathbf{a}, \mathbf{b}\rangle$ belong to Σ,

and (iii) $\mathbf{a}, \mathbf{c}$ are dual to vertices of $\Gamma_0(Q)$ (cf. Definition 2).

One sets $\mathscr{X}(\mathbf{D})=\{X(\sigma_1), X(\sigma_2)\}$.

Let (u_1, u_2) resp. (v_1, v_2) be the coordinates on $X(\sigma_1)$ resp. $X(\sigma_2)$.

Lemma 3.

i) *In* $X(\sigma_1)\cap X(\sigma_2)$ *one has*

$$u_2 = 1/v_1, \quad u_1 = v_1^{a_1c_2-a_2c_1}v_2.$$

ii) $\mathbf{D}\cap X(\sigma_1) = \{u_1 = 0\}, \quad \mathbf{D}\cap X(\sigma_2) = \{v_2 = 0\}.$

iii) *Let* D_1 *resp.* D_2 *be the divisors of* X *such that*

$$D_1\cap X(\sigma_1) = \{u_2 = 0\} \quad \textit{resp.} \quad D_2\cap X(\sigma_2) = \{v_1 = 0\}.$$

Then $\mathscr{I}(\mathbf{D})=\{\mathscr{R}, D_1, D_2\}$.

iv) $\mathbf{D}\cap\Delta^\pi\cap X(\sigma_1) = [0,\infty) = \mathbf{D}\cap\Delta^\pi\cap X(\sigma_2).$

Proof. These are simple arguments based on the construction of π. Property (iv) follows from the locally monomial nature of π cf. [9, pg. 111] and the fact that no component of $\mathbf{a}$, $\mathbf{c}$ equals zero. A simple modification of (iv) is needed if this property does not hold. These are left to the reader. □

Notation. Set

$$\delta = \deg Q_{\sigma_1}(0, u_2) = \deg Q_{\sigma_2}(v_1, 0), \quad \delta' = \deg \Phi_{\sigma_1}(0, u_2) = \deg \Phi_{\sigma_2}(v_1, 0). \quad \square$$

The following global ("cocycle") property is useful in the global analysis over the divisors in $\mathscr{B}_\varphi$.

Lemma 4. *In the above notation,*

$$\lambda_\varphi(\mathbf{D}, D_1) = -\lambda_\varphi(\mathbf{D}, D_2) - (2+\delta') + \varrho(\varphi)\delta.$$

Proof. For each covector $\xi \in (\mathbf{R}^2_+)^*$ set (cf. (2.1))

$$m_\xi(\Phi) = \min\{y\cdot\xi:\ y\in\Gamma_0(\Phi)\}$$

$$m_\xi(Q) = \min\{y\cdot\xi:\ y\in\Gamma_0(Q)\}.$$

Set

$$\mathscr{K}_\xi(\Phi) = \{y\in\Gamma_0(\Phi):\ y\cdot\xi = m_\xi(\Phi)\},$$

$$\mathscr{K}_\xi(Q) = \{y\in\Gamma_0(Q):\ y\cdot\xi = m_\xi(Q)\}.$$

These are the "first-meet loci" of the polygons in the direction ξ.

One can always construct the partition Σ used to determine X so that

(i) $$K_\mathbf{a}(Q)\cup K_\mathbf{c}(Q)\subset K_\mathbf{b}(Q),$$

(ii) $$\text{and}\quad K_\mathbf{a}(\Phi)\cup K_\mathbf{c}(\Phi)\subset K_\mathbf{b}(\Phi).$$

Let $K_\mathbf{c}(Q)=\{\bar I_1\}$, $K_\mathbf{a}(Q)=\{\bar I_2\}$, $K_\mathbf{c}(\Phi)=\{\bar J_1\}$, $K_\mathbf{a}(\Phi)=\{\bar J_2\}$. One should note that $K_\mathbf{a}(Q)$ and $K_\mathbf{c}(Q)$ need not be distinct. Set $\bar 1=(1, 1)$. Define

$$\alpha = \det\begin{pmatrix}\mathbf{a}\\ \mathbf{c}\end{pmatrix}.$$

An easy calculation shows that $\alpha\mathbf{b}-\mathbf{c}=\mathbf{a}$. By definition one has (using the notation of (2.1))

$$A(D_1) = \mathbf{c}\cdot(M-\bar I_1), \quad B(D_1) = \mathbf{c}\cdot(\bar J_1 - m + \bar 1) - 1,$$
$$A(D_2) = \mathbf{a}\cdot(M-\bar I_2), \quad B(D_2) = \mathbf{a}\cdot(\bar J_2 - m + \bar 1) - 1$$
$$A(\mathbf{D}) = \mathbf{b}\cdot(M-\bar I_k), \quad B(\mathbf{D}) = \mathbf{b}\cdot(\bar J_k - m + \bar 1) - 1, \quad k = 1, 2.$$

One then computes and finds

$$\lambda_\varphi(\mathbf{D}, D_2) = -\lambda_\varphi(\mathbf{D}, D_1) - (2+\delta') + \varrho(\varphi)[\mathbf{c}\cdot(\bar I_2 - \bar I_1)]$$
$$\lambda_\varphi(\mathbf{D}, D_1) = -\lambda_\varphi(\mathbf{D}, D_2) - (2+\delta') + \varrho(\varphi)[\mathbf{a}\cdot(\bar I_1 - \bar I_2)].$$

It is now easy to check that $\delta=\mathbf{a}\cdot(\bar I_1-\bar I_2)=\mathbf{c}\cdot(\bar I_2-\bar I_1)$. This proves the lemma. $\square$

A simple consequence of lemma 4 is

Corollary 1. *The covector* **b** *corresponding to* $\mathbf{D}\in\mathscr{B}_{\varphi}$ *is dual to a line segment of* $\Gamma_0(Q)$.

Proof. If this does not hold, then one checks easily that $\delta=0$. By lemma 4, $\lambda_{\varphi}(\mathbf{D}, D_1)+\lambda_{\varphi}(\mathbf{D}, D_2)=-2-\delta'$. This is not possible however when each $\lambda_{\varphi}(\mathbf{D}, D_i)>-1$. □

In practice it is useful to have a formula for $\lambda_{\varphi}(\mathbf{D}, D_i)$. The following is easily obtained from the above expressions for $A(D_i)$, $B(D_i)$, $A(\mathbf{D})$, $B(\mathbf{D})$.

Lemma 5. *One has*

$$\lambda_{\varphi}(\mathbf{D}, D_1)=\frac{-\det(\bar{J}_1-m+\bar{1};\ M-\bar{I}_1)}{A(\mathbf{D})}-1$$

$$\lambda_{\varphi}(\mathbf{D}, D_2)=\frac{\det(\bar{J}_2-m+\bar{1};\ M-\bar{I}_2)}{A(\mathbf{D})}-1,$$

where the indicated arguments of det *are column vectors.*

Notation. For $\mathbf{D}\in\mathscr{B}_{\varphi}$, $\mathscr{X}(\mathbf{D})=\{X(\sigma_1), X(\sigma_2)\}$, and $\mathscr{I}(\mathbf{D})=\{D_1, D_2, \mathscr{R}\}$, one sets $\mathbf{D}'=\mathbf{D}-(D_1\cup D_2\cup\mathscr{R})$.

The following property is easy to check.

Lemma 6. *Let* $\mathscr{U}$ *be any simply connected subset of* $\mathbf{D}'$. *The two* 1*-forms*

$$\omega_1'=u_2^{\lambda_{\varphi}(\mathbf{D}, D_1)}R_{\sigma_1}(0, u_2)^{\varrho(\varphi)}\Phi_{\sigma_1}(0, u_2)\,du_2|_{\mathscr{U}\cap X(\sigma_1)}$$

$$\omega_2'=-v_1^{\lambda_{\varphi}(\mathbf{D}, D_2)}R_{\sigma_2}(v_1, 0)^{\varrho(\varphi)}\Phi_{\sigma_2}(v_1, 0)\,dv_1|_{\mathscr{U}\cap X(\sigma_2)}$$

patch on $\mathscr{U}\cap X(\sigma_1)\cap X(\sigma_2)$, *to give a global section of* $\Omega^1_{\mathscr{U}}$.

This section is denoted ω' in the following, the open set $\mathscr{U}$ being clear from the context.

For $\mathbf{D}\in\mathscr{B}_{\varphi}$, one now defines, using the above notation,

$$\text{(3.5)}\qquad R(\mathbf{D})=\int_0^{\infty}u_2^{\lambda_{\varphi}(\mathbf{D}, D_1)}R_{\sigma_1}^{\varrho(\varphi)}(0, u_2)\,\Phi_{\sigma_1}(0, u_2)\,du_2$$

$$=\int_0^{\infty}v_1^{\lambda_{\varphi}(\mathbf{D}, D_2)}R_{\sigma_2}^{\varrho(\varphi)}(v_1, 0)\,\Phi_{\sigma_2}(v_1, 0)\,dv_1.$$

Remark 3. Evidently, by lemmas 2, 6, and the fact that no pole of either R_{σ_i} lies on $\mathbf{D}\cap\Delta^{\pi}$, one concludes not only that $R(\mathbf{D})$ is a finite number but also that the second equality in (3.5) is valid. Indeed, one also has the equation

$$R(\mathbf{D}) = \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} u_2^{\lambda_\varphi(\mathbf{D}, D_1)} R_{\sigma_1}^{\varrho(\varphi)} \Phi_{\sigma_1}(0, u_2)\, du_2 \tag{3.6}$$

$$= \lim_{\varepsilon \to 0} \int_{\varepsilon}^{1/\varepsilon} v_1^{\lambda_\varphi(\mathbf{D}, D_2)} \Phi_{\sigma_2}^{\varrho(\varphi)} \Phi_{\sigma_2}(v_1, 0)\, dv_1. \quad \square$$

Define now

$$\mu = \textstyle\sum_{\mathbf{D} \in \mathscr{B}_\varphi} R(\mathbf{D}).$$

One has the basic

Lemma 7.

$$\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi) = \frac{\mu}{s - \varrho(\varphi)}.$$

Proof. One argues as in [9, pg. 116—117]. A sketch is given here. The main point is this. One can cover Δ^n by finitely many open sets $\mathscr{U}_i$, in each of which there are coordinates $(u_1^{(i)}, u_2^{(i)})$ with the following property:

Each $\mathscr{U}_i$ is a subset of one of the charts $X(\sigma)$ (with coordinates (u_1, u_2)), used to construct the variety X defined in section 2. The coordinates $(u_1^{(i)}, u_2^{(i)})$ are then the restrictions to $\mathscr{U}_i$ of (u_1, u_2). One concludes that in *each* $\mathscr{U}_i$

(1) Equations (2.2.1) hold;

(2) The strict transform defining $\mathscr{R}$ is positive on $\mathscr{U}_i \cap \mathbf{R}^2$ by Proposition 2;

(3) The strict transform Φ_σ is positive on $\mathscr{U}_i \cap \mathbf{R}^2$.

Thus, by a partition of unity subordinate to the cover $\mathscr{U}_i$, one can now apply to $I_P(s, \varphi)$ the regularization procedure, described in [7, pgs. 59ff], also cf. [15], carried out in the two variable normal crossing case. When one pieces together the expression for $\mathrm{Pol}_{s=\varrho(\varphi)} I_P(s, \varphi)$, using (2.1.3) and lemmas 2,3,4, and 6, one obtains $\mu/(s - \varrho(\varphi))$. Details are left to the reader. $\square$

Remark 4. As discussed in section 5, a modification of the 1-forms, defined in lemma 6, is needed if each $\lambda_\varphi(\mathbf{D}, D_i) - \varrho(\varphi)\delta$ is an integer. This implies, by lemmas 2, 4, that each $\lambda_\varphi(\mathbf{D}, D_i)$ is a nonnegative integer. Let $\mathscr{U}$ be a simply connected and open subset of $\mathbf{D}'$. Set

$$\omega_1'' = u_2^{\lambda_\varphi(\mathbf{D}, D_1)} R_{\sigma_1}^{\varrho(\varphi)}(0, u_2)\, \Phi_{\sigma_1}(0, u_2) \log(u_2)\, du_2 \quad \text{on} \quad \mathscr{U} \cap \{u_2 \neq 0\}$$

and

$$\omega_2'' = v_1^{\lambda_\varphi(\mathbf{D}, D_2)} R_{\sigma_1}^{\varrho(\varphi)}(v_1, 0)\, \Phi_{\sigma_2}(v_1, 0) \log v_1\, dv_1 \quad \text{on} \quad \mathscr{U} \cap \{v_1 \neq 0\}.$$

These 1-forms patch whenever the principal value of the argument, assuming values in $(-\pi, \pi]$, is used for each ω_i''. With this choice, one denotes the log by Log. The result is a single-valued holomorphic form over $\mathscr{U}$. This section of $\Omega^1_{\mathscr{U}}$ is denoted ω'' in the following. $\square$

4. V-Manifolds

Section 4 discusses the Steenbrink—Varcenko construction of a "V-manifold". Varcenko was the first to emphasize the use of this construction for studying families of periods. Here, an adaptation is given for the case of a rational function which is needed for section 5. The construction is quite similar but it is necessary to be explicit for the proof of theorem 1.

For a divisor $\mathbf{D}\in\mathscr{B}_\varphi$, $\mathscr{I}(\mathbf{D})=\{D_1, D_2, \mathscr{R}\}$, $A(D_i)$ given by (3.1), define

$$E(\mathbf{D}) = l.c.m.\,\{A(\mathbf{D}), A(D_1), A(D_2)\}.$$

Let T_∞ be the chart at infinity in $P^1\mathbf{C}$ with coordinate t_∞. By proposition 2, it suffices to work in a neighborhood of $t_\infty=0$.

The graph of $R(x_1, x_2)$ in $\mathbf{C}^2(x)\times T_\infty$ is defined as

$$G = \{Q(x_1, x_2)\cdot t_\infty - x_1^{M_1}x_2^{M_2} = 0\}.$$

With $\pi\colon X\to\mathbf{C}^2(x)$ the toroidal modification from section 2, set $\pi'=\pi\times\mathrm{id}\colon X\times T_\infty\to\mathbf{C}^2(x)\times T_\infty$ and define

$$\mathscr{G}(\mathbf{D}) = [\text{strict transform of } \{(Q\cdot t_\infty - x_1^{M_1}x_2^{M_2})\circ\pi' = 0\}]\cap[X(\sigma_1)\cup X(\sigma_2)]\times T.$$

One notes that (2.2) implies that the strict transform of $(Q\cdot t_\infty - x_1^{M_1}x_2^{M_2})\circ\pi'$ is defined by

$$\text{(4.1)}\qquad \begin{aligned} &\text{in } X(\sigma_1)\colon R_1'(u, t_\infty) = Q_{\sigma_1}\cdot t_\infty - u_1^{A(D_1)}u_2^{A(\mathbf{D})} = 0,\\ &\text{in } X(\sigma_2)\colon R_2'(v, t_\infty) = Q_{\sigma_2}\cdot t_\infty - v_1^{A(\mathbf{D})}v_2^{A(D_2)} = 0.\end{aligned}$$

Let $\theta\colon \mathscr{G}(\mathbf{D})\to T_\infty$, $pr\colon \mathscr{G}(\mathbf{D})\to X$ be projections. With T a copy of the affine line with coordinate w, set $\nu\colon T\to T_\infty$ to be the map $\nu\colon w\to w^{E(\mathbf{D})}=t_\infty$. Set $\varrho_1\colon \mathscr{G}(\mathbf{D})\times_{T_\infty}T\to\mathscr{G}(\mathbf{D})$ and $\tau\colon \mathscr{G}(\mathbf{D})\times_{T_\infty}T\to T$ to be the projections from the fiber product. Define $\eta\colon \mathscr{N}(\mathbf{D})\to\mathscr{G}(\mathbf{D})\times_{T_\infty}T$ to be the normalization morphism. Define $\varrho=\varrho_1\circ\eta$, $\Theta=\pi\circ pr\circ\varrho$. Figure 1 summarizes these definitions.

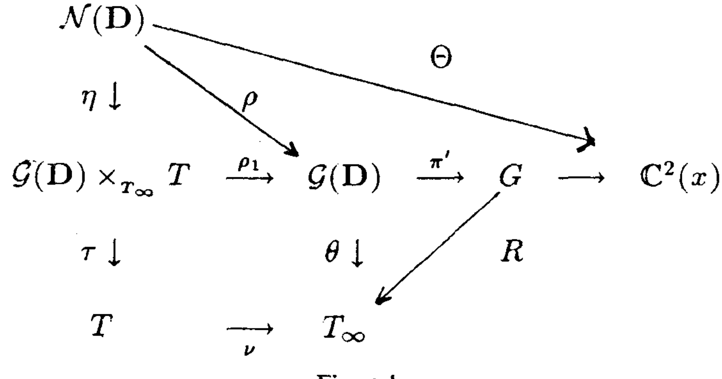

Figure 1

The following proposition describes a local defining equation at any point of $[\mathscr{G}(\mathbf{D})\cap(\mathbf{D}\times T_\infty)]\times_{T_\infty}T$. The proof is straightforward and left to the reader.

Proposition 4. *At each point* $\xi\in[\mathscr{G}(\mathbf{D})\cap(\mathbf{D}\times T_\infty)]\times_{T_\infty}T$ *there exist*

i) *an open neighborhood* $V(\xi)\subset(X(\sigma_1)\cup X(\sigma_2))\times T_\infty$;

ii) *local coordinates* (w_1, w_2, w) *such that one of the following types of equations defines* $V(\xi)\cap[\mathscr{G}(\mathbf{D})\cap(\mathbf{D}\times T_\infty)]\times_{T_\infty}T$.

$$\text{Type (1)}: \; w^{E(\mathbf{D})}-w_1^{A(\mathbf{D})}.$$
$$\text{Type (2)}: \; w^{E(\mathbf{D})}-w_1^{A(\mathbf{D})}w_2^{A(D_1)}.$$
$$\text{Type (3)}: \; w_1 w^{E(\mathbf{D})}-w_2^{A(\mathbf{D})}.$$
$$\text{Type (4)}: \; w^{E(\mathbf{D})}-w_1^{A(D_2)}w_2^{A(\mathbf{D})}.$$

One then says that $\xi\in[\mathscr{G}(\mathbf{D})\cap(\mathbf{D}\times T_\infty)]\times_{T_\infty}T$ is of type (i) if the local defining equation at ξ is of type (i), $i=1, 2, 3, 4$.

Set

$$a(1)=1,\quad a(2)=g.c.d.\,(A(D_1), A(\mathbf{D})),\quad a(3)=1,\quad a(4)=g.c.d.\,(A(\mathbf{D}), A(D_2)),$$

and

$$\alpha_1 = E(\mathbf{D})/A(D_1),\quad \alpha_2 = E(\mathbf{D})/A(D_2), \tag{4.2}$$

One now lifts these considerations to $\mathscr{N}(\mathbf{D})_{sp}$ (the non-singular part of $\mathscr{N}(\mathbf{D})$) as follows. For each $\xi\in[\mathscr{G}(\mathbf{D})\cap(\mathbf{D}\times T_\infty)]$, with corresponding neighborhood $V(\xi)$ as described in proposition 4, one has

Proposition 5. *For* $i=1, 2, 3, 4$, *if* ξ *is of type* (i), *then there exist* $a(i)$ *disjoint open neighborhoods over* $V(\xi)\times_{T_\infty}T$, *denoted* $\mathscr{A}_1(\xi), \ldots, \mathscr{A}_{a(i)}(\xi)$ *with the following properties.*

a) *For each* $p=1, \ldots, a(i)$, *coordinates* $(y_1^{(p)}, y_2^{(p)}, v^{(p)})$ *are defined on each* $\mathscr{A}_p(\xi)\cap\mathscr{N}(\mathbf{D})_{sp}$ *such that if:*

$$i=1 \text{ then } w_1\circ\eta=(y_1^{(p)})^{E(\mathbf{D})/A(\mathbf{D})},\; w_2\circ\eta=y_2^{(p)},\; w\circ\eta=e^{2\pi ip/E(\mathbf{D})}v^{(p)}$$
$$i=2 \text{ then } w_1\circ\eta=(y_1^{(p)})^{E(\mathbf{D})/A(\mathbf{D})},\; w_2\circ\eta=(y_2^{(p)})^{\alpha_1},\; w\circ\eta=e^{2\pi ip/E(\mathbf{D})}v^{(p)}$$
$$i=3 \text{ then } w_1\circ\eta=(y_1^{(p)})^{E(\mathbf{D})},\; w_2\circ\eta=(y_2^{(p)})^{E(\mathbf{D})/A(\mathbf{D})},\; w\circ\eta=e^{2\pi ip/E(\mathbf{D})}v^{(p)}$$
$$i=4 \text{ then } w_1\circ\eta=(y_1^{(p)})^{\alpha_2},\; w_2\circ\eta=(y_2^{(p)})^{E(\mathbf{D})/A(\mathbf{D})},\; w\circ\eta=e^{2\pi ip/E(\mathbf{D})}v^{(p)}.$$

b) *If* $V(\xi)\cap V(\xi')\neq\emptyset$ *the overlap relations between* $\mathscr{A}_j(\xi)$ *and* $\mathscr{A}_k(\xi')$ *are derived from those for* $V(\xi)$, $V(\xi')$ *and the formulae of part* (a).

Proof. This is a straightforward adaptation of the arguments given in [16, pg. 488]. □

In the following, the open sets $\mathscr{A}_j(\xi)$ will be (imprecisely) called "charts". Now define the following subsets of $\mathscr{G}(\mathbf{D})$,

$$E_0 = \{(u, t_\infty)\colon R_1' = u_1 = 0\} \cup \{(v, t_\infty)\colon R_2' = v_2 = 0\},$$

$$E_1 = \{(u, t_\infty)\colon R_1' = u_2 = 0\},$$

$$E_2 = \{(v, t_\infty)\colon R_2' = v_1 = 0\},$$

$$E_3 = \{(u, t_\infty)\colon R_1' = Q_{\sigma_1} = 0\} \cup \{(v, t_\infty)\colon R_2' = Q_{\sigma_2} = 0\}.$$

Set

$$\tilde{\mathbf{D}} = \varrho^{-1}(E_0-(E_1\cup E_2\cup E_3)), \quad \tilde{\mathbf{D}}_i = \varrho^{-1}(E_i), \quad i = 1, 2, \tag{4.3}$$

and

$$\tilde{\mathscr{R}} = \varrho^{-1}(E_3).$$

Lemma 8.

(1) *For any j, ξ one can locally describe $\tilde{\mathbf{D}}'\cap\mathscr{A}_j(\xi)$ as follows.*

a) *If $\varrho(\mathscr{A}_j(\xi))\subset X(\sigma_1)$ then $\tilde{\mathbf{D}}'\cap\mathscr{A}_j(\xi)\subseteqq\{v^{(j)}=y_1^{(j)}=0\}$ and $y_2^{(j)}$ is a local coordinate of $\tilde{\mathbf{D}}'\cap\mathscr{A}_j(\xi)$.*

b) *If $\varrho(\mathscr{A}_j(\xi))\subset X(\sigma_2)$ then $\tilde{\mathbf{D}}'\cap\mathscr{A}_j(\xi)\subseteqq\{v^{(j)}=y_2^{(j)}=0\}$ and $y_1^{(j)}$ is a local coordinate of $\tilde{\mathbf{D}}'\cap\mathscr{A}_j(\xi)$.*

(2) *In any open set of $\mathscr{N}(\mathbf{D})$ disjoint from $\tilde{\mathscr{R}}$, the map $\theta\circ\varrho$ is nonsingular and holomorphic.*

Proof. This is clear from the definitions and construction of $\mathscr{N}(\mathbf{D})$. □

Let $\mathscr{U}=\mathscr{G}(\mathbf{D})-E_3$. The analysis in the next section will be done over $\mathscr{U}$, by means of the *restriction* map ϱ: (an open subset of $\tilde{\mathbf{D}}'$)$\to\mathscr{U}$. This is defined as follows. Set $\theta_i\colon X(\sigma_i)\times T_\infty\to\mathbf{C}^2\times T_\infty$, $i=1, 2$ to be the transformations defined by

$$\theta_1\colon w_1 = u_1\left[\frac{1}{Q_{\sigma_1}(u_1, u_2)}\right]^{1/A(\mathbf{D})}, \quad w_2 = u_2, \; t_\infty = t_\infty$$

$$\theta_2\colon w_1' = v_1, \quad w_2' = v_2\left[\frac{1}{Q_{\sigma_1}(v_1, v_2)}\right]^{1/A(\mathbf{D})}, \quad t_\infty = t_\infty.$$

Then θ_1 resp. θ_2 is well-defined and non-singular on an open neighborhood $\mathscr{W}_1$ of $[(\mathbf{D}\cap X(\sigma_1))\times T_\infty]$ resp. $\mathscr{W}_2$ of $[(\mathbf{D}\cap X(\sigma_2))\times T_\infty]$. Moreover, it is clear that

$$\mathscr{U}\cap\mathscr{W}_1 = \{t_\infty - w_1^{A(\mathbf{D})}w_2^{A(D_1)} = 0\},$$

$$\mathscr{U}\cap\mathscr{W}_2 = \{t_\infty - (u_1')^{A(D_2)}(w_2')^{A(\mathbf{D})} = 0\}.$$

One then lifts $\mathscr{W}_1$ resp. $\mathscr{W}_2$ to an open neighborhood $\mathscr{W}_1'$ resp. $\mathscr{W}_2'$ of $(\mathbf{D}\cap X(\sigma_1))\times_{T_\infty}T$ resp. $(\mathbf{D}\cap X(\sigma_2))\times_{T_\infty}T$ so that

$$(\mathscr{U}\times_{T_\infty}T)\cap\mathscr{W}_1' = \{w^{E(\mathbf{D})}-w_1^{A(\mathbf{D})}w_2^{A(D_1)}=0\}$$

$$(\mathscr{U}\times_{T_\infty}T)\cap\mathscr{W}_2' = \{w^{E(\mathbf{D})}-(w_1')^{A(D_2)}(w_2')^{A(\mathbf{D})}=0\}.$$

Set $\lambda_1=gcd(A(D_1), A(\mathbf{D}))$, $\lambda_2=gcd(A(D_2), A(\mathbf{D}))$. Exactly as in proposition 5, there exist disjoint open sets $\mathscr{A}_1^{(1)}, \ldots, \mathscr{A}_{1\alpha_1}^{(1)}$ resp. $\mathscr{A}_1^{(2)}, \ldots, \mathscr{A}_{2\alpha_2}^{(2)}$ in $\mathscr{N}(\mathbf{D})$ which map onto $(\mathscr{U}\times_{T_\infty}T)\cap\mathscr{W}_1'$ resp. $(\mathscr{U}\times_{T_\infty}T)\cap\mathscr{W}_2'$. In each $\mathscr{A}_i^{(1)}$ resp. $\mathscr{A}_k^{(2)}$ there are defined coordinates on $\mathscr{N}(\mathbf{D})_{sp}$, denoted $(y_1^{(i)}, y_2^{(i)}, v^{(i)})$ resp. $(y_1'^{(k)}, y_2'^{(k)}, v'^{(k)})$, so that the transformation equations of the type given by cases $i=2, 4$ in part (a) of proposition 5 hold.

Select one chart each from $\{\mathscr{A}_i^{(1)}\}$, $\{\mathscr{A}_k^{(2)}\}$. Denote these by $\mathscr{A}_1=\mathscr{A}_1(y_1, y_2, v)$, $\mathscr{A}_2=\mathscr{A}_2(y_1', y_2', v')$. Then define (cf. figure 1)

$$\eta\colon [\mathscr{A}_1\cup\mathscr{A}_2]\cap\tilde{\mathbf{D}}' \to \mathscr{U}\times_{T_\infty}T,$$

by which the sections ω', ω'' (cf. lemma 6, remark 4) can be lifted to an open subset of $\tilde{\mathbf{D}}'$. One sets

$$\eta|_{\mathscr{A}_1\cap\tilde{\mathbf{D}}'}(0, y_2, 0) = (0, y_2^{\alpha_1}, 0) = (u_1, u_2, w)$$

$$\eta|_{\mathscr{A}_2\cap\tilde{\mathbf{D}}'}(y_1', 0, 0) = ((y_1')^{\alpha_2}, 0, 0) = (v_1, v_2, w).$$

Thus, one can give a unique meaning to $\eta^*(\omega')$, $\eta^*(\omega'')$ as single-valued 1-forms on any simply connected subset of $\tilde{\mathbf{D}}'$ disjoint from $\tilde{\mathscr{R}}$. It is useful to have an explicit expression for these forms. To obtain this, define for $i=1, 2$

$$\delta(D_i) = \frac{\lambda_\varphi(\mathbf{D}, D_i)+1}{A(D_i)} = \varrho(\varphi)-\varrho(D_i, 1). \tag{4.4}$$

Lemma 9. *As multi-valued* 1*-forms on* $\tilde{\mathbf{D}}'$ *one has:*

i) $\eta^*(\omega')|_{\mathscr{A}_1\cap\tilde{\mathbf{D}}'} = \alpha_1 y_2^{E(\mathbf{D})\delta(D_1)-1}[R_{\sigma_1}(0, y_2^{\alpha_1})]^{\varrho(\varphi)}\Phi_{\sigma_1}(0, y_2^{\alpha_1})\,dy_2.$

ii) $\eta^*(\omega')|_{\mathscr{A}_2\cap\tilde{\mathbf{D}}'} = -\alpha_2(y_1')^{E(\mathbf{D})\delta(D_2)-1}[R_{\sigma_2}((y_1')^{\alpha_2}, 0)]^{\varrho(\varphi)}\Phi_{\sigma_2}((y_1')^{\alpha_2}, 0)\,dy_1'.$

iii) $\eta^*(\omega'')|_{\mathscr{A}_1\cap\tilde{\mathbf{D}}'} = \alpha_1 y^{E(\mathbf{D})\delta(D_1)-1}[R_{\sigma_1}(0, y_2^{\alpha_1})]^{\varrho(\varphi)}\Phi_{\sigma_1}(0, y_2^{\alpha_1})\,\mathrm{Log}\,(y_2^{\alpha_1})\,dy_2.$

iv) $\eta^*(\omega'')|_{\mathscr{A}_2\cap\tilde{\mathbf{D}}'} = \alpha_2(y_1')^{E(\mathbf{D})\delta(D_2)-1}[R_{\sigma_2}((y_1')^{\alpha_2}, 0)]^{\varrho(\varphi)}\Phi_{\sigma_2}((y_1')^{\alpha_2}, 0)\times$
$\times\,\mathrm{Log}\,((y_1')^{\alpha_2})\,dy_1'.$

Proof. This is a straightforward calculation that is left to the reader to check. □

5. Proof of theorem 1

The basic idea of the proof is to express each summand $R(\mathbf{D})$ of μ (cf. lemma 7) in terms of the Mellin transform of a family of periods in the fibers of P over an interval in $(0, \infty)$. This is accomplished by lifting the expression for $R(\mathbf{D})$ to the divisor $\tilde{\mathbf{D}}'$, defined in (4.3), and then arguing as Varcenko does in [16].

Before stating the theorem, it is helpful to recall certain well-known ideas.

(1) As shown by Verdier [17], the map $P\colon \mathbf{C}^2 \to \mathbf{C}$ is a locally trivial fibration over $\mathbf{C}$ minus a discrete set $\mathscr{B}$ of bifurcation points. Set $\mathbf{C}^* = \mathbf{C} - \mathscr{B}$, and define $P^* = P|_{P^{-1}(\mathbf{C}^*)}$. For $t \in \mathbf{C}^*$ set $X_t = P^{-1}(t)$. Let $\mathbf{H}^1$ denote the flat vector bundle on $\mathbf{C}^*$ with fiber at t equal to the finite dimensional vector space $H^1(X_t, \mathbf{C})$. Let $\mathscr{H}^1 = \mathbf{H}^1 \otimes \mathscr{O}_{\mathbf{C}^*}$ be the sheaf of germs of analytic sections of $\mathbf{H}^1$. Any rational differential 2-form ω, regular on an open set U, determines an analytic section of $\mathscr{H}^1|_{P(U)}$, defined as

$$\sigma_\varphi\colon t \to [\omega/dP|_{X_t \cap U}]$$

where $\omega/dP|_{X_t \cap U} = \operatorname{Res}\left(\frac{\omega}{P-t}\right)\Big|_{X_t \cap U}$.

Similarly, the locally trivial fibration P^* determines a 1-homology bundle $\mathbf{H}_1$ whose fiber at $t \in \mathbf{C}^*$ equals $H_1(X_t, \mathbf{C})$. If t_0 is any point of $\mathbf{C}^*$ and $\gamma_0 \in H_1(X_{t_0}, \mathbf{C})$, one can construct a continuous (multi-valued) section of $\mathbf{H}_1$ by using parallel transport in the fibres of P^*. If $\varkappa$ is a real analytic path from t_0 to ∞ (that is, $t_\infty = 0$) with no self-intersections, then there exists a unique continuous section of $\mathbf{H}_1$ over $\varkappa$ which equals γ_0 at t_0.

(2) It is well-known [6, pg. 113] that if $\omega(t)$ is an analytic section of $\mathscr{H}^1$ and γ_t continuous section of $\mathbf{H}_1$ over a ray $\varkappa$ for which γ_0 is a bounded cycle [12, pgs. 82—83], then the function $I(t) = \int_{\gamma_t} \omega(t)$ is a solution of a differential equation with regular singular point at infinity. Now define the Mellin transform of $I(t)$ as the line integral

$$\mathscr{I}(s) = \int_\varkappa t^{-1} I(t)\, dt.$$

The above remarks imply that $\mathscr{I}(s)$ is analytic in some halfplane $\operatorname{Re}(s) > B$ and is meromorphic in $\mathbf{C}$. There is moreover a connection between the poles of $\mathscr{I}(s)$ and the exponents in the Nilsson function expansion for $I(t)$ in a neighborhood of infinity. This is described in detail in [8, ch. 1].

Notation. From (4.2), set $\zeta_1 = e^{2\pi i/\alpha_1}$, $\zeta_2 = e^{2\pi i/\alpha_2}$, $\zeta_2' = e^{-2\pi i/\alpha_2}$. For a, b positive numbers and $\gamma \in \mathbf{C}^*$, $\gamma[a, b]$ denotes the oriented line segment from γa to γb. For $0 < \varepsilon < 1$, set $a(\varepsilon, i) = \varepsilon^{1/\alpha_i}$, $b(\varepsilon, i) = {}^{-1/\alpha_i}$, $i = 1, 2$. □

Theorem 1. *Assume that P, φ have positive coefficients and $\varrho(\varphi)$ is a simple pole of $I_P(s, \varphi)$. Then for each divisor $\mathbf{D} \in \mathscr{B}_\varphi$, there exist*

i) *An open subset (or "wedge" cf.* (5.7)*)* Ω *of* $\mathbf{C}^*$*, and an analytic section* $\omega_{\mathbf{D}}(\varphi)$ *of* $\mathscr{H}^1$*, defined in* Ω;

ii) *An interval* $[\beta, \infty)\subset\Omega$*, and a continuous section* $\xi(\mathbf{D}, t)$ *of* $\mathbf{H}_1$ *over* $[\beta, \infty)$;

iii) *An explicit constant* $\gamma(\mathbf{D})$ *(cf.* (5.8), (5.11)*) such that*

$R(\mathbf{D}) = \gamma(\mathbf{D})$ [*coefficient of* $(s-\varrho(\varphi))^{-1}$ *in the Laurent expansion for*

$$\int_\beta^\infty t^{-s}\left(\int_{\xi(\mathbf{D},t)}\omega_{\mathbf{D}}(\varphi)\right)dt\Big].$$

Remark 5. An immediate *analytic* corollary of Theorem 1 is the following. Define the function

$$\mathscr{I}(t) = \frac{d}{dt}\int_{\{z\in[1,\infty)^2:\, P(z)\le t\}}\varphi\, dz_1\, dz_2.$$

One knows there exists $\beta'>0$ such that

$$I_P(s,\varphi) = \int_{\beta'}^\infty t^{-s}\mathscr{I}(t)\, dt.$$

By Mellin inversion one now concludes

Corollary 2. *Over the interval* $[\beta', \infty)\subset\Omega$*, defined in remark 5, there exist*

i) *A continuous section* γ_t *of* $\mathbf{H}_1$,

ii) *An analytic section* ω_t *of* $\mathscr{H}^1$,

iii) *A positive number* θ,

such that

$$\mathscr{I}(t) - \int_{\gamma_t}\omega_t = O(t^{\varrho(\varphi)-\theta}) \quad as \quad t\to\infty. \quad \square$$

Proof of theorem 1. The argument divides into two cases. In the notation used in lemma 4 these are

Case 1: At least one $\lambda_\varphi(\mathbf{D}, D_i)-\varrho(\varphi)\delta\notin\mathbf{Z}$.

Case 2: Each $\lambda_\varphi(\mathbf{D}, D_i)-\varrho(\varphi)\delta\in\mathbf{Z}$.

Proof in Case 1. By reindexing if necessary, one may assume that $\lambda_\varphi(\mathbf{D}, D_1)-\varrho(\varphi)\delta\notin\mathbf{Z}$. Define the oriented 1-cycle $\Psi(\varepsilon)=\sum_{i=1}^4\Psi_i(\varepsilon)$ with support in the $\mathscr{A}_1$ chart as follows.

(5.1)

i) $\Psi_1(\varepsilon)$ is the sector of the circle with radius ε^{1/α_1} from $\zeta_1 a(\varepsilon, 1)$ to $a(\varepsilon, 1)$. Parametrize it by $y_2=\varepsilon^{1/\alpha_1}e^{i(2\pi/\alpha_1-\theta)}$, $\theta\in[0, 2\pi/\alpha_1]$.

ii) $\Psi_2(\varepsilon)$ is the segment $[a(\varepsilon, 1), b(\varepsilon, 1)]$. Parametrize it by $y_2=x^{1/\alpha_1}$, $x\in[\varepsilon, \varepsilon^{-1}]$.

iii) $\Psi_3(\varepsilon)$ is the sector of the circle with radius $(1/\varepsilon)^{1/\alpha_1}$ from $b(\varepsilon, 1)$ to $\zeta_1 b(\varepsilon, 1)$. Parametrize it by $y_2 = \varepsilon^{-1/\alpha_1} e^{i\theta}$, $\theta \in [0, 2\pi/\alpha_1]$.

iv) $\Psi_\beta(\varepsilon)$ is the segment $\zeta_1 \cdot [a(\varepsilon, 1), b(\varepsilon, 1)]$. Parametrize it by $y_2 = \zeta_1(-x+\varepsilon+\varepsilon^{-1})^{1/\alpha_1}$, $x \in [\varepsilon, \varepsilon^{-1}]$.

This is sketched below.

An analogous 1-cycle is constructed in the $\mathscr{A}_2$ chart. Denote this by $\Psi'(\varepsilon)$. Of use below is the parametrization of the 1-cycle in the $\mathscr{A}_2$ chart to which $\Psi(\varepsilon)$ is *identified* under the overlap relation $y_2^{\alpha_1} = 1/(y_1')^{\alpha_2}$. This is important because one is dealing with multivalued forms and one needs to keep track of the arguments of the integrands over the cycle in both charts. Define the following 1-cycle $\Xi(\varepsilon) = \sum_{i=1}^4 \Xi(\varepsilon)$, with support in the $\mathscr{A}_2$ chart where

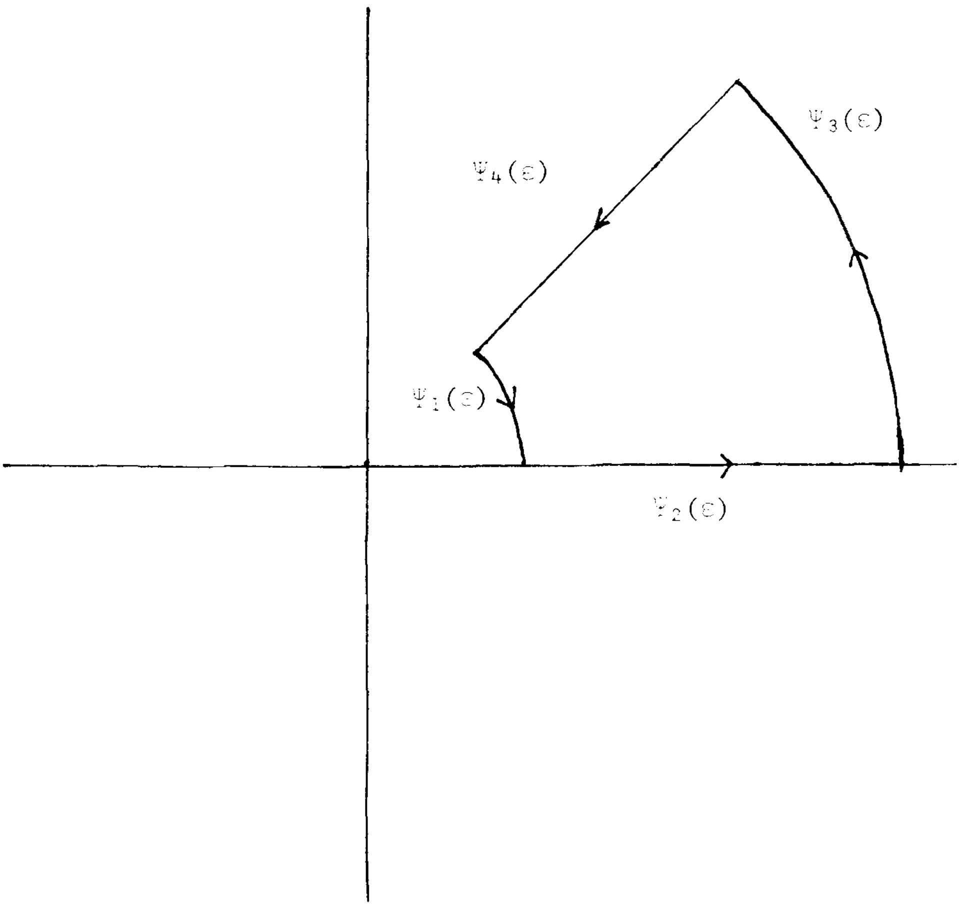

Figure 2

i) $\Xi_1(\varepsilon)$ is the sector of the circle of radius ε^{1/α_1} from $a(\varepsilon, 2)$ to $\zeta_2' a(\varepsilon, 2)$. Parametrize it by $y_1' = \varepsilon^{1/\alpha_2} e^{i(2\pi i/\alpha_2 - \theta)}$, $\theta \in [0, 2\pi/\alpha_2]$.
ii) $\Xi_2(\varepsilon)$ is the segment $\zeta_2'[a(\varepsilon, 2), b(\varepsilon, 2)]$ with parametrization $y_1' = \zeta_2' x^{1/\alpha_2}$, $x \in [\varepsilon, \varepsilon^{-1}]$.
iii) $\Xi_3(\varepsilon)$ is the sector of the circle of radius $\varepsilon^{-1/\alpha_2}$ from $\zeta_2' b(\varepsilon, 2)$ to $b(\varepsilon, 2)$. Parametrize it by $y_1' = \varepsilon^{-1/\alpha_2} e^{i\theta}$, $\theta \in [-2\pi/\alpha_2, 0]$.
iv) $\Xi_4(\varepsilon)$ is the segment $[a(\varepsilon, 2), b(\varepsilon, 2)]$. Parametrize it by $y_1' = (-x + \varepsilon + \varepsilon^{-1})^{1/\alpha_2}$, $x \in [\varepsilon, \varepsilon^{-1}]$.

It is clear that the compact cycle $\Psi(\varepsilon)$, resp. $\Psi'(\varepsilon)$ has support disjoint from $\tilde{\mathscr{R}}$ when ε is sufficiently small. Each is evidently the lift of a "Hankel contour" in $\mathbf{D}'$ to $\mathscr{A}_1$, resp. $\mathscr{A}_2$. When the 1-form $\eta^*(\omega')$ has poles or branch points other than 0, ∞ then $\Psi(\varepsilon)$, resp. $\Psi'(\varepsilon)$ is non-homologous to 0 in $\mathscr{A}_1 \cap \tilde{\mathbf{D}}'$ for ε sufficiently small. On the other hand, since each of the coefficients of Q is positive, $\eta^*(\omega')$ is defined at each point of $|\Psi(\varepsilon)| \cup |\Psi'(\varepsilon)|$ for small ε.

By an abuse of notation, denote by $\eta^*(\omega')(y_2)$ resp. $\eta^*(\omega')(y_1')$ the function of y_2, resp. y_1' which determines the factor of dy_2, resp. dy_1' in $\eta^*(\omega')$ and is defined in lemma 9 part i) resp. part ii).

One now has the following, which is easily checked using lemma 4 and the Argument Principle.

Lemma 10. *Let*

$$\operatorname{Arg} b(\varepsilon, 1) = \operatorname{Arg} a(\varepsilon, 2) = \operatorname{Arg} R_{\sigma_1}^{\varrho(\varphi)} \circ \eta(b(\varepsilon, 1)) = \operatorname{Arg} R_{\sigma_2}^{\varrho(\varphi)} \circ \eta(a(\varepsilon, 2)) = 0.$$

Then the branch of $\eta^*(\omega')(y_2)$, obtained by analytic continuation along $\Psi(\varepsilon)$ and starting at $b(\varepsilon, 1)$, agrees with the branch of $\eta^*(\omega')(y_1')$, obtained by analytic continuation along $\Xi(\varepsilon)$ and starting at $a(\varepsilon, 2)$.

In this sense, one says that the 1-forms $\eta^*(\omega')\big|_{\mathscr{A}_1 \cap \tilde{\mathbf{D}}'}$ and $\eta^*(\omega')\big|_{\mathscr{A}_2 \cap \tilde{\mathbf{D}}'}$ patch along the paths $\Psi(\varepsilon)$, $\Xi(\varepsilon)$ via the overlap relation $y_2^{\alpha_1} = (y_1')^{\alpha_2}$.

In the $\mathscr{A}_1$ resp. $\mathscr{A}_2$ chart one may speak of the oriented region $\Omega_1(\varepsilon)$ resp. $\Omega_2(\varepsilon)$ enclosed by $\Psi(\varepsilon)$ resp. $\Xi(\varepsilon)$. Thus, $\partial\Omega_1(\varepsilon) = \Psi(\varepsilon)$, $\partial\Omega_2(\varepsilon) = \Xi(\varepsilon)$. If $\varepsilon_1 < \varepsilon_2 \ll 1$ define the regions $\mathscr{R}_i$, $\mathscr{S}_i$ by

$$\Omega_1(\varepsilon_1) - \Omega_1(\varepsilon_2) = \mathscr{R}_1 \cup \mathscr{R}_2, \quad \Omega_2(\varepsilon_1) - \Omega_2(\varepsilon_2) = \mathscr{S}_2 \cup \mathscr{S}_1.$$

Here, one has (cf. figure 3)

$$\begin{aligned} \partial\mathscr{R}_1 &= \Psi_1(\varepsilon_1) + \Psi_2'(\varepsilon_1) - \Psi_1(\varepsilon_2) + \Psi_4'(\varepsilon_1), \\ \partial\mathscr{R}_2 &= \Psi_2''(\varepsilon_1) + \Psi_3(\varepsilon_1) + \Psi_4''(\varepsilon_1) - \Psi_3(\varepsilon_2), \\ \partial\mathscr{S}_1 &= \Xi_2'(\varepsilon_1) + \Xi_3(\varepsilon_1) + \Xi_4'(\varepsilon_1) - \Xi_3(\varepsilon_2), \\ \partial\mathscr{S}_2 &= \Xi_4''(\varepsilon_1) + \Xi_1(\varepsilon_1) + \Xi_2''(\varepsilon_1) - \Xi_1(\varepsilon_2). \end{aligned} \tag{5.2}$$

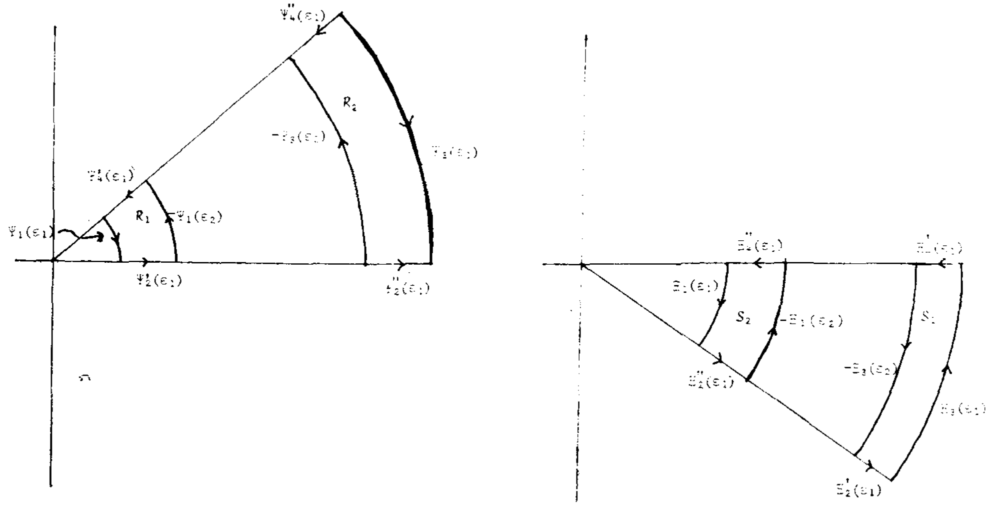

Figure 3

To be precise, one should specify the branches of $\eta^*(\omega')(y_2)$, $\eta^*(\omega')(y_1')$ over the $\mathscr{R}_i$, $\mathscr{S}_i$. In order to insure that the branches patch in the above sense, it suffices to choose the branches of these functions by lemma 10 and restrict them to the paths comprising the loops $\mathscr{R}_i$, $\mathscr{S}_i$ as defined in (5.2).

Now define the oriented closed paths $\gamma_i=\eta(\partial\mathscr{R}_i)\subset$(the u_2 chart), $\gamma_i'=\eta(\partial\mathscr{S}_i)\subset$ (the v_1 chart) for $i=1, 2$. Evidently, one concludes that γ_i patches with γ_i' for $i=1, 2$ via the overlap $u_2=1/v_1$. One now shows

Lemma 11. *For* $i=1, 2$

$$\int_{\gamma_i} \omega_1' = \int_{\gamma_1'} \omega_2' = 0.$$

Proof. The basic point is to observe that if c is any rational exponent then $\int_{\gamma_1} u_2^c du_2=\int_{\gamma_2'} v_1^c dv_1=0$. This follows from a straightforward calculation that is left to the reader. Next, over the regions enclosed by γ_1, resp. γ_2' one can expand into a uniformly convergent series in u_2 resp. v_1 the functions $R_{\sigma_1}^{\varrho(\varphi)}\cdot\Phi_{\sigma_1}(0, u_2)$ resp. $R_{\sigma_2}^{\varrho(\varphi)}\cdot\Phi_{\sigma_2}(v_1, 0)$. This is possible when ε_1, ε_2 are sufficiently small so that no zero of $Q_{\sigma_1}(0, u_2)$ resp. $Q_{\sigma_2}(v_1, 0)$ lies inside or on γ_1 resp. γ_2. Thus, by interchanging integration over each γ_1, γ_2' with summation, one concludes that $\int_{\gamma_1}\omega_1'=\int_{\gamma_2'}\omega_2'=0$. By using the patching of the two branches from lemma 10, one concludes that $\int_{\gamma_2}\omega_1'=\int_{\gamma_2'}\omega_2'=0$, and $\int_{\gamma_1}\omega_1'=\int_{\gamma_1'}\omega_2'=0$. □

Lemma 11 now implies that for $\varepsilon_1 < \varepsilon_2 \ll 1$ one has

$$\int_{\Psi(\varepsilon_1)} \eta^*(\omega') = \int_{\Psi(\varepsilon_2)} \eta^*(\omega'). \tag{5.3}$$

Let S_ε resp. S'_ε denote the circles of radius ε oriented clockwise and centered at the points (cf. lemma 3) $\mathbf{D} \cap D_1$ resp. $\mathbf{D} \cap D_2$. An elementary calculation now shows that

$$\int_{\Psi(\varepsilon)} \eta^*(\omega') = e^{-2\pi i \lambda_\varphi(\mathbf{D}, D_1)} (1 - \varepsilon^{2\pi i (\lambda_\varphi(\mathbf{D}, D_1) - \varrho(\varphi)\delta)}) \times \tag{5.4}$$

$$\times \int_\varepsilon^{1/\varepsilon} u_2^{\lambda_\varphi(\mathbf{D}, D_1)} R_{\sigma_1}^{\varrho(\varphi)}(0, u_2) \cdot \Phi_{\sigma_1}(0, u_2)\, du_2 + \int_{S_\varepsilon} \omega'_1 + \int_{S'_1} \omega'_2.$$

By assumption, each $\lambda_\varphi(\mathbf{D}, D_i) > -1$. Thus, one concludes that $\lim_{\varepsilon\to\infty} \int_{S_\varepsilon} \omega'_1 = \lim_{\varepsilon\to\infty} \int_{S'_\varepsilon} \omega'_2 = 0$. By (5.3) the left side of (5.4) is independent of ε. Let Ψ denote the typical cycle $\Psi(\varepsilon)$. Then, fixing ε on the left side of (5.4), while on the right side letting $\varepsilon \to 0$, one finds, by (3.6), that

$$\int_\Psi \eta^*(\omega') = e^{-2\pi i \lambda_\varphi(\mathbf{D}, D_1)} (1 - \varepsilon^{2\pi i (\lambda_\varphi(\mathbf{D}, D_1) - \varrho(\varphi)\delta)}) R(\mathbf{D}). \tag{5.5.1}$$

An obvious modification of the above arguments uses the typical cycle $\Psi' = \Psi'(\varepsilon)$ in $\mathscr{A}_2$ if $\lambda_\varphi(\mathbf{D}, D_2) - \varrho(\varphi)\delta \notin \mathbf{Z}$ so that

$$\int_{\Psi'} \eta^*(\omega') = e^{-2\pi i \lambda_\varphi(\mathbf{D}, D_2)} (1 - e^{2\pi i (\lambda_\varphi(\mathbf{D}, D_2) - \varrho(\varphi)\delta)}) R(\mathbf{D}). \tag{5.5.2}$$

Because Ψ is disjoint from $\tilde{\mathscr{R}} \cup \tilde{D}_1 \cup \tilde{D}_2$ (cf. (4.3)), there exists an open tubular neighborhood $\mathscr{T}$ of Ψ in $\mathscr{N}(\mathbf{D})$ which remains disjoint from $\tilde{\mathscr{R}} \cup \tilde{D}_1 \cup \tilde{D}_2$. By [16, pg. 489], one sees that this implies that the 2-form $(\tau \circ \eta)^{1+\varrho(\varphi)E(\mathbf{D})} \varrho^* \pi^* \omega_\varphi$ extends to a well-defined 2-form on $\mathscr{T}$. A straightforward calculation (following that in [ibid]) using proposition 4, 5, shows that

$$\frac{E(\mathbf{D})}{A(\mathbf{D})} \eta^*(\omega')\Big|_{\mathscr{T} \cap \tilde{\mathbf{D}}'} = (\tau \circ \eta)^{1+\varrho(\varphi)E(\mathbf{D})} \varrho^* \pi^* \omega_\varphi / d(\tau \circ \eta)\Big|_{\mathscr{T} \cap \tilde{\mathbf{D}}'}.$$

Now, since $\tau \circ \eta$ is defined at each point of $\mathscr{T}$, reasoning as in [ibid] allows one to assert the existence of a continuous family of 1-cycles $\Xi(\mathbf{D}, w)$ satisfying these properties.

(5.6) (a) $|\Xi(\mathbf{D}, w)| \subset (\tau \circ \eta)^{-1}(w)$ for all $w \in \tau \circ \eta(\mathscr{T})$.

(b) $\Xi(\mathbf{D}, 0) = \Psi$.

(c) $\dfrac{E(\mathbf{D})}{A(\mathbf{D})} \displaystyle\int_\Psi \eta^*(\omega') = \lim_{w \to 0} w^{1+\varrho(\varphi)E(\mathbf{D})} \int_{\Xi(\mathbf{D}, w)} \varrho^* \pi^* \omega_\varphi / d(\tau \circ \eta).$

Thus, the wedge alluled to in the statement of theorem 1 is the set

$$\Omega = \nu \circ \tau \circ \eta(\mathscr{T}), \tag{5.7}$$

or viewed in the range of P, it equals the image of this set under the reciprocal mapping $t_\infty \to t = 1/t_\infty$. From the construction and proposition 2, it is clear that the wedge contains a line segment whose pointset equals an interval of the form $[0, \alpha] \subset T_\infty$, for some $\alpha > 0$.

The map $\pi' \circ \varrho$ (cf. figure 1) induces the family of cycles

$$\xi(\mathbf{D}, t_\infty) = \pi' \circ \varrho(\Xi(\mathbf{D}, w))$$

in the fibers $R = t_\infty$. Note that the support of each cycle is contained in $\Theta(\mathscr{T})$, where ω_φ is regular. Arguing exactly as in [ibid] one concludes that for each $w \neq 0$

$$\int_{\Xi(\mathbf{D}, w)} \varrho^* \pi^* \omega_\varphi / d(\tau \circ \eta) = E(\mathbf{D})\, w^{E(\mathbf{D})-1} \int_{\xi(\mathbf{D}, w^{E(\mathbf{D})})} \omega_\varphi / dR.$$

Define

(5.8)
$$\gamma(\mathbf{D}) = \begin{cases} A(\mathbf{D})\, e^{2\pi i \lambda_\varphi(\mathbf{D}, D_1)} (1 - e^{2\pi i (\lambda_\varphi(\mathbf{D}, D_1) - \varrho(\varphi)\delta)})^{-1} & \text{if } \lambda_\varphi(\mathbf{D}, D_1) - \varrho(\varphi)\delta \notin \mathbf{Z} \\ A(\mathbf{D})\, e^{2\pi i \lambda_\varphi(\mathbf{D}, D_2)} (1 - e^{2\pi i (\lambda_\varphi(\mathbf{D}, D_2) - \varrho(\varphi)\delta)})^{-1} & \text{if } \lambda_\varphi(\mathbf{D}, D_2) - \varrho(\varphi)\delta \notin \mathbf{Z}. \end{cases}$$

One concludes the proof of case (1) by the following sequence of identities, each of which is easy and left to the reader to verify. Set $\beta = 1/\alpha$.

$$\begin{aligned} R(\mathbf{D}) &= \gamma(\mathbf{D}) \lim_{w \to 0} w^{(1+\varrho(\varphi))E(\mathbf{D})} \int_{\xi(\mathbf{D}, w^{E(\mathbf{D})})} \omega_\varphi / dR \\ &= \gamma(\mathbf{D}) \lim_{t_\infty \to 0} t_\infty^{1+\varrho(\varphi)} \int_{\xi(\mathbf{D}, t_\infty)} \omega_\varphi / dR \\ &= \gamma(\mathbf{D}) \Big(\text{coefficient of } t_\infty^{-(1+\varrho(\varphi))} \text{ in the asymptotic} \\ &\qquad \text{expansion as } t_\infty \to 0 \text{ of } \int_{\xi(\mathbf{D}, t_\infty)} \omega_\varphi / dR\Big) \\ &= \gamma(\mathbf{D}) \Big(\text{coefficient of } (s - \varrho(\varphi))^{-1} \text{ in the Laurent} \\ &\qquad \text{expansion of } \int_0^\alpha t_\infty^s \Big(\int_{\xi(\mathbf{D}, t_\infty)} \omega_\varphi / dR\Big) dt_\infty\Big) \\ &= \gamma(\mathbf{D}) \Big(\text{coefficient of } (s - \varrho(\varphi))^{-1} \text{ in the Laurent} \\ &\qquad \text{expansion of } \int_\beta^\infty t^{-s} \Big(\int_{\xi(\mathbf{D}, t)} \omega_\varphi / dP\Big) dt\Big). \end{aligned}$$

Proof in case (2).

The arguments of case (1) fail to identify $R(\mathbf{D})$ via $\int_\Psi \eta^*(\omega')$ because the factors $1-e^{2\pi i(\lambda_\varphi(\mathbf{D}, D_i)-\varrho(\varphi)\delta)}$ equal zero for $i=1, 2$. In order to avoid this problem, the 1-form ω'' (cf. remark 4) replaces ω'. On the other hand, one can continue to use the 1-cycles $\Psi(\varepsilon)$. As in case 1 one must specify the branches of the functions (using the same abuse of notation as above) $\eta^*(\omega'')(y_2)$, $\eta^*(\omega'')(y_1')$ along the cycles $\Psi(\varepsilon)$, $\Xi(\varepsilon)$ so as to be able to patch and prove the independence from ε of $\int_{\Psi(\varepsilon)} \eta^*(\omega'')$ for ε sufficiently small.

One observes that case 2 and lemmas 2, 4 imply that each $\lambda_\varphi(\mathbf{D}, D_i) \in \mathbf{N} \cup \{0\}$. Now, the branches of the logarithm factors are defined as in remark 4. Thus, after traversal of $\Psi_3(\varepsilon)$ $\mathrm{Log}\,(y_2^{\alpha_1})$ changes by $2\pi i$. After traversal of $\Xi_1(\varepsilon)$, $\mathrm{Log}\,((y_1')^{\alpha_2})$ changes by $-2\pi i$. By the identification of $y_2^{\alpha_1}$ with $1/(y')_1^{\alpha_2}$, one concludes that the analytic continuations of $\eta^*(\omega'')(y_2)$ along $\Psi(\varepsilon)$ and $\eta^*(\omega'')(y_1')$ along $\Xi(\varepsilon)$ patch via the overlap relation between $\mathscr{A}_1 \cap \tilde{\mathbf{D}}'$ and $\mathscr{A}_2 \cap \tilde{\mathbf{D}}'$. Using the notation from lemma 11, one now concludes

Lemma 12. *For* $i=1, 2$

$$\int_{\gamma_i} \omega_1'' = \int_{\gamma_i'} \omega_2'' = 0.$$

Proof. By the above remark, it suffices to show that $\int_{\gamma_1} \omega_1'' = \int_{\gamma_2'} \omega_2'' = 0$. Using the same idea as in lemma 11 one reduces to showing that for any nonnegative integer n $\int_{\gamma_1} u_2^n \,\mathrm{Log}\,(u_2)\,du_2 = \int_{\gamma_2'} v_1^n \,\mathrm{Log}\,(v_1)\,dv_1 = 0$. With the above discussion, this is a straightforward calculation by an integration by parts that is left to the reader. □

A standard calculation also shows that for any $\varepsilon > 0$

$$\int_{\Psi(\varepsilon)} \eta^*(\omega'') = 2\pi i \int_\varepsilon^{1/\varepsilon} u_1^{\lambda_\varphi(\mathbf{D}, D_1)} R_{\sigma_1}^{\varrho(\varphi)}(0, u_2)\,\Phi_{\sigma_1}(0, u_2)\,du_2 + \int_{S_\varepsilon} \omega_1'' + \int_{S_\varepsilon'} \omega_2''.$$

The nonnegativity of each $\lambda_\varphi(\mathbf{D}, D_i)$ implies

$$\lim_{\varepsilon \to 0} \int_{S_\varepsilon} \omega_1'' = \lim_{\varepsilon \to 0} \int_{S_\varepsilon'} \omega_2'' = 0.$$

Thus, one concludes, as in (5.5) and with the same notation,

$$\int_\Psi \eta^*(\omega'') = 2\pi i R(\mathbf{D}). \tag{5.9}$$

It is clear that the topological situation is the same as case 1. In particular, the neighborhood $\mathscr{T}$ and family of 1-cycles $\Xi(\mathbf{D}, w)$ can be constructed exactly as in case 1.

The one difference concerns the section of $\mathscr{H}^1$. One needs to interpret $\eta^*(\omega'')$ as follows. Define

$$\mathscr{W} = \Theta(\mathscr{T}) \subset \mathbf{C}^2(x). \tag{5.10}$$

It is evident that $\mathscr{W}$ contains the origin. Let $\mathscr{W}^0$ denote the interior of $\mathscr{W}$.

By construction $\operatorname{Log}(y_2)$, $\operatorname{Log}(y_1')$ are analytic and multi-valued on $\mathscr{T}$ which patch on the overlap $\mathscr{A}_1 \cap \mathscr{A}_2 \cap \mathscr{T}$. (By choosing $\mathscr{T}$ sufficiently small this is possible.) Thus $\operatorname{Log}(u_2), \operatorname{Log}(v_1)$ patch on $pr \circ \varrho(\mathscr{T})$. By using the inverse of $\pi'|_{X(\sigma_i)\times T}$, $i=1,2$, which is given by a monomial transformation, one can view $\operatorname{Log}(u_2), \operatorname{Log}(v_1)$ as the composition of the logarithm on $\mathbf{C}^2(x)$ and certain monomials in x_1, x_2 which are defined over $\mathscr{W}^0$ and possess well-defined limits at all points in $\mathscr{W}$. Observe too that these functions patch on the overlap

$$[\pi|_{X(\sigma_1)}(pr\circ\varrho(\mathscr{T}))] \cap [\pi|_{X(\sigma_2)}(pr\circ\varrho(\mathscr{T}))]$$

by construction. So one has constructed in this way a (multi-valued) section $\zeta_{\mathscr{W}}$ of $\mathscr{O}_{\mathbf{C}^2(x)}|_{\mathscr{W}^0}$.

Set $\omega_\varphi(\mathscr{W}) = \omega_\varphi \cdot \zeta_{\mathscr{W}}$. This is a 2-form which is holomorphic and single-valued over $\mathscr{W}^0$. One can then repeat the argument in case 1 word for word, replacing ω_φ by $\omega_\varphi(\mathscr{W})$.

In particular, as in case 1 one has the identity

$$\frac{E(\mathbf{D})}{A(\mathbf{D})}\eta^*(\omega'')|_{\mathscr{T}\cap\tilde{\mathbf{D}}'} = (\tau\circ\eta)^{1+\varrho(\varphi)E(\mathbf{D})}\varrho^*\pi^*\omega_\varphi(\mathscr{W})/d(\tau\circ\eta)|_{\mathscr{T}\cap\tilde{\mathbf{D}}'}.$$

As such one finds that

$$\gamma(\mathbf{D}) = A(\mathbf{D})/2\pi i. \tag{5.11}$$

This completes the proof of theorem 1. □

Remark 6. From lemmas 4, 5 it is clear that for fixed $\bar{I}_1$, $\bar{I}_2$, $\bar{m}$, the set of possible vectors $\bar{J}_1, \bar{J}_2$ for which $\lambda_\varphi(\mathbf{D}, D_1) - \varrho(\varphi)\delta$, $\lambda_\varphi(\mathbf{D}, D_2) - \varrho(\varphi)\delta \in \mathbf{Z}$ corresponds to a *subset* of the positive integral solutions (x_1, y_1), (x_2, y_2) to the pair of congruence equations

$$\det\begin{pmatrix} x_1 - m_1 + 1, & \bar{M} - \bar{I}_1 \\ y_1 - m_2 + 1, & \end{pmatrix} \equiv 0(A(\mathbf{D}))$$

$$\det\begin{pmatrix} x_2 - m_1 + 1, & \bar{M} - \bar{I}_2 \\ y_2 - m_2 + 1, & \end{pmatrix} \equiv 0(A(\mathbf{D})).$$

In this sense one may call a polynomial φ "generic" if case 1 of theorem 1 applies. □

Two corollaries of interest to the diophantine problem discussed in the introduction follow from theorem 1.

Corollary 3. *Assume the following hypotheses are satisfied.*

(1) φ_1, φ_2, P *satisfy the conditions in theorem* 1;
(2) φ_1, φ_2 *are generic polynomials in the sense of remark* 6;
(3) $\varrho(\varphi_1)=\varrho(\varphi_2)$.

Then one concludes the following. If $\sigma_{\varphi_1}=\sigma_{\varphi_2}$ *for all* $t\in\mathbf{C}^*$, *then for all* $x>0$ $\hat{N}_{\varphi_1}(x)=\hat{N}_{\varphi_2}(x)$.

Thus, the dominant term for $N_\varphi(x)$ is a type of cohomological invariant. It would be interesting to understand the implications and extent of this observation.

In general, it is clear from the expression (3.6) that each $R(\mathbf{D})$ is the value of some generalized hypergeometric function. As such, it is not yet possible to detect an "algebraic part" within each $R(\mathbf{D})$, which is presumably a highly transcendental number. On the other hand, if $\varrho(\varphi)\in\mathbf{Z}$ then one is in case (2) of theorem 1. Assume further that both P, φ are defined over a number field K. Here the situation is different. For now one *can* evaluate $\int_{\Psi}\eta^*(\omega'')$ directly via standard residue calculus. Let $\beta_1(\mathbf{D}), \dots, \beta_R(\mathbf{D})$ denote the poles of

$$u_2\cdot(R\circ\pi)^{\varrho(\varphi)}\pi^*(\omega^\varphi)|_{\mathbf{D}}.$$

These are algebraic numbers. One concludes

Corollary 4.

$$2\pi i\,\mathrm{Res}_{s=\varrho(\varphi)}\frac{D_P(s,\varphi)}{s} = \sum_{\mathbf{D}\in\mathscr{B}_\varphi}\sum_{u=1}^{R}\alpha_u(\mathbf{D})\,\mathrm{Log}\,(\beta_u(\mathbf{D})),$$

where each $\alpha_u(\mathbf{D})$ *is an algebraic number that depends explicitly upon the polynomials* P, φ.

Remark 7. The cycles $\xi(\mathbf{D}, t_\infty)$, constructed in theorem 1 have support in $\mathbf{C}^2(x)\cap\mathbf{C}^2(z)$, when $t_\infty\neq 0$, as is easy to check. Thus, each $\xi(\mathbf{D}, t_\infty)$ is a 1-cycle in $X_{1/t_\infty}=\{y\in\mathbf{C}^2(z)\colon P(y)=1/t_\infty\}$ which is non-homologous to zero.

Broughton [2] has studied the homology of the fibers X_τ. Because $n=2$, P is "tame" [ibid] if it is not the square of another polynomial. Thus one has [ibid, thm. 1.2]

Theorem C. *There exists a nonnegative integer* μ *such that for all but at most finitely many* $\tau\in\mathbf{C}$ *one has*

$$H_1(X_\tau, \mathbf{Z}) \simeq \mathbf{Z}^\mu. \tag{5.12}$$

One says τ *is "generic" if* (5.12) *holds.*

For τ generic, let $\{\xi_i(\tau)\}_{i=1}^{\mu}$ be a basis of $H_1(X_\tau, \mathbf{Z})$. By Verdier's fibration theorem [17], one can move the basis to any other generic fiber X_t, $t \neq \tau$, by parallel transport.

In the notation of the proof of theorem 1, Broughton's result implies as an immediate consequence

Corollary 5. *There exist integers* $n_1, \ldots, n_\mu$ *such that for each* $t \in [\beta, \infty)$ *one has*

$$[\xi(\mathbf{D}, 1/t)] = \sum_{i=1}^{\mu} n_i \xi_i(1/t).$$

In this sense one can say that the coefficient of $\hat{N}_\varphi(x)$ is "determined by the global topology" of the fibers of P. □

Concluding remarks. Theorem 1 should be extendable to the case in which P, φ are both non-degenerate with respect to their polygons at infinity, and positive on $[1, \infty)^2$. On the other hand, under these conditions, it is not yet known if $N_\varphi(x)$ admits the description given in theorem A. One only knows a priori that $N_\varphi(x) \sim \hat{N}_\varphi(x)$ (actually one knows slightly more by Freud's tauberian theorem). So, an extension of theorem 1 to the non-degenerate case would still have something to say about the diophantine problem of interest in this paper. However, the case of positive coefficients was emphasized here because one has the strongest result on $N_\varphi(x)$.

References

1. Bernstein, J., The Analytic Continuation of Generalized Functions with Respect to a Parameter, *Functional Analysis and Applications* **6** (1972), 273—285.
2. Broughton, S. A., Milnor Numbers and the Topology of Polynomial Hypersurfaces, *Invent. Math.* **92** (1988), 217—241.
3. Cassou-Nogues, P., Valeurs aux entiers negatifs des séries de Dirichlet associées a un polynome *II*, *Amer. J. of Math.* **106** (1985), 255—299.
4. Cassou-Nogues, P., Valeurs aux entiers negatifs des séries de Dirichlet associées a un polynome *III*, *Amer. J. of Math.* **109** (1987), 71—90.
5. Danilov, V. I. and Khovanski, A. G., Newton Polyhedra and an Algorithm for computing Hodge-Deligne Numbers, *Math. USSR Izv.* **29** (1987), 279—299.
6. Deligne, P., *Equations Différentielles a Points Singuliers Réguliers*, Lecture Notes in Mathematics, vol. 163, Spinger-Verlag, 1970.
7. Gel'fand, I. M. and Shapiro, Z., *Homogeneous functions and their Extensions*, *AMS* Translations series **2**, vol. 8, 1958, pp. 21—87.
8. Igusa, J.-I., *Lectures on Forms of Higher Degree*, Tata Institute Lectures Series, Springer-Verlag, 1978.
9. Lichtin, B., Generalized Dirichlet Series and *b*-functions, *Compositio Math.* **65** (1988), 81—120.
10. Lichtin, B., The Asymptotics of a Lattice Point Problem and $\mathscr{D}$ modules (to appear in Proceedings of Conf. on *D*-modules and Microlocal Geometry, Lisbon, Portugal).

11. LICHTIN, B., Poles of $|f|^{2s}$ and Roots of the B-function, *Arkiv för Matematik* **27** (1989), 283—304.
12. MALGRANGE, B., On the Polynomials of J. N. Bernstein, *Russian Math. Surveys* **29** (1974), 81—88.
13. SARGOS, P., Prolongement Meromorphe des Series de Dirichlet associées a des Fractions Rationnelles de plusieurs variables, *Ann. d'Inst. Fourier* **34** (1984), 83—123.
14. SARGOS, P., Croissance de certaines series de Dirichlet et applications, *Journal für die reine un angewandte Mathematik* **367** (1986), 139—154.
15. VARCENKO, A., Newton Polyhedra and estimation of oscillating integrals, *Functional Analysis and Applications* **10** (1976), 13—38.
16. VARCENKO, A., Asymptotic Hodge Structure in the Vanishing Cohomology, *Math. USSR Izv.* **18** (1982), 469—512.
17. VERDIER, J. L., Stratification de Whitney et Théoreme de Bertini-Sard, *Invent. Math.* **36** (1977), 295—312.

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