On the spectral synthesis property and its application to partial differential equations

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Abstract. Let M be a $(n-1)$-dimensional manifold in $\mathbf{R}^n$ with non-vanishing Gaussian curvature. Using an estimate established in the early work of the author [4], we will improve the known result of Y. Domar on the weak spectral synthesis property by reducing the smoothness assumption upon the manifold M. Also as an application of the method, a uniqueness property for partial differential equations with constant coefficients will be proved, which for some specific cases recovers or improves Hörmander's general result.

1. Introduction

Let $\mathcal{S}(\mathbf{R}^n)$ be the space of Schwartz class functions and $\mathcal{S}'(\mathbf{R}^n)$ be the dual space of $\mathcal{S}(\mathbf{R}^n)$. Given $T\in\mathcal{S}'(\mathbf{R}^n)$, denote its Fourier transform by $\hat{T}$. We know that $L^p(\mathbf{R}^n)\subset\mathcal{S}'(\mathbf{R}^n)$ for $1\leqq p\leqq\infty$. Also as usual we denote the support of T in the distributional sense by $\operatorname{supp}(T)$.

Let $FL^p(\mathbf{R}^n)=\{\hat{f}; f\in L^p(\mathbf{R}^n), \|\hat{f}\|_{FL^p}=\|f\|_{L^p}\}$, $1\leqq p\leqq\infty$.

It is well-known that FL^q is the dual space of FL^p for $\frac{1}{p}+\frac{1}{q}=1$, $1\leqq p<\infty$. For a closed subset M in $\mathbf{R}^n$, we denote

$$I(M) = \{f\in FL^1(\mathbf{R}^n), f(M) = 0\}$$

$$J(M) = \{f\in C_0^\infty(\mathbf{R}^n), f(M) = 0\}$$

$$K(M) = \{f\in C_0^\infty(\mathbf{R}^n), \operatorname{supp}(f)\cap M = \emptyset\}$$

We know that $I(M)$ is a closed ideal in FL^1 and it is obvious that

$$\overline{K(M)} \subseteqq \overline{J(M)} \subseteqq I(M) \quad \text{in } FL^1 \text{ norm.}$$

If $\overline{K(M)}=I(M)$, we say that M is of *spectral synthesis*. If $\overline{J(M)}=I(M)$, we say that M is of *weak spectral synthesis*.

For the sphere S^{n-1} in $\mathbf{R}^n$, $n \geqq 3$, L. Schwartz [12] showed that the spectral synthesis property does not hold and later C. Herz [5] proved the spectral synthesis property for S^1 in $\mathbf{R}^2$. Varopoulos [13] showed that S^{n-1} in $\mathbf{R}^n$, $n \geqq 3$ has the weak spectral synthesis. For a general hypersurface, Y. Domar [2] obtained:

If M is a compact C^∞ $(n-1)$-dimensional manifold in $\mathbf{R}^n$ $(n \geqq 2)$ with non-vanishing Gaussian curvature, then M is of weak spectral synthesis.

We refer the reader to Domar's survey paper [3] for more information on the spectral synthesis property.

The following argument will show that for any positive integer m there is no hope to prove the weak spectral synthesis property for a general compact C^m $(n-1)$-dimensional manifold in $\mathbf{R}^n$ (even with non-vanishing Gaussian curvature).

For a small ball U in $\mathbf{R}^{n-1}$ choose $\psi(x)$ such that at any point in U, ψ is only differentiable up to a finite order $(\geqq m)$. Let $E=\{(x, \psi(x)),\ x \in U\}$ and fix a point $s_0=(x_0, \psi(x_0)) \in E$. Using an affine transformation, we may assume $\nabla\psi(x_0)=(0, \ldots, 0)$. For $f \in C_0^\infty(\mathbf{R}^n)$ vanishing on E, that is, $f(x, \psi(x))=0$ for $x \in U$, we let $H(x)=f(x, \psi(x))$. Then H is identically zero in U and hence by the chain rule we have for $1 \leqq i \leqq n-1$

$$0 = H'_{x_i}(x_0) = f'_{x_i}(x_0, \psi(x_0)) + f'_z(x_0, \psi(x_0) \cdot \psi' x_i(x_0)) = f'_{x_i}(x_0, \psi(x_0)).$$

So f'_{x_i} vanishes at s_0 for $i=1, \ldots, n-1$. If $f'_z(x_0, \psi(x_0)) \neq 0$, then the implicit function theorem implies that ψ is C^∞ smooth at x_0 since f is C^∞ smooth. This contradicts our smoothness assumption upon ψ at x_0 and hence all the first derivatives of f at s_0 must be zero. Since $s_0 \in E$ is arbitrary, we see that all the first derivatives of f vanish on E. By a standard inductive argument we can conclude that for $f \in C_0^\infty(\mathbf{R}^n)$, f vanishing on E implies that all the derivatives of f vanish on E. Now it follows easily from the result in [13] and Schwartz's counter-example for the spectral synthesis that there is no hope to prove the weak spectral synthesis property for this manifold.

Thus it is natural to consider the following property. We let

$$J^m(M) = \{f \in C_0^m(\mathbf{R}^n),\ f(M) = 0\}.$$

If $\overline{J^m(M)}=I(M)$ in FL^1 norm, we say that M is of *m-spectral synthesis*.

As was pointed out by Domar ([2], p. 25, line 1), the method in [2] can also give the result for a manifold with differentiability up to a certain order. Indeed, Domar's method yields for $n \geqq 2$ and $k \geqq 2n+1$ that if M is a compact C^k $(n-1)$-dimensional manifold in $\mathbf{R}^n$ with non-vanishing Gaussian curvature, then M is of k-spectral synthesis.

The C^{2n+1} smoothness assumption in the above result is too strong if we compare it with Domar's result in the case $n=2$ [1].

The main purpose of this article is to report the following result, which follows directly from the basic estimate obtained by the author in [4].

Theorem 1. *Let $k \geqq n+2$. If M is a compact C^k $(n-1)$-dimensional manifold in $\mathbf{R}^n$ with non-vanishing Gaussian curvature, then M is of k-spectral synthesis.*

Then we consider an application of the method to the uniqueness property of some partial differential equations, which is closely related to the unique continuation property and absence of positive eigenvalues of differential operators (see [9]). Let u be a solution of any partial differential equation with constant coefficients. Assume that the support of $\hat{u}$ is contained in a subset M of $\mathbf{R}^n$ with measure zero. If $u \in L^p$, $1 \leqq p \leqq 2$, then from the Hausdorff–Young theorem, $\hat{u}$ is a measurable function in $\mathbf{R}^n$ and hence is zero since M has measure zero. Thus $u=0$ in this case. When $p>2$, we consider the following three cases.

(i) $\triangle u(x)+u(x) = 0, \quad x \in \mathbf{R}^n; \quad \operatorname{supp}(\hat{u}) \subset \{\xi, |\xi| = 1\};$

(ii) $$i\frac{\partial u}{\partial t}+\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_{n-1}^2} = 0;$$

$$\operatorname{supp}(\hat{u}) \subset \{(\xi_1, \ldots, \xi_{n-1}, \xi_n), \xi_n = |\xi_1|^2+\ldots+|\xi_{n-1}|^2\};$$

(iii) $$\frac{\partial^2 u}{\partial t^2}-\left(\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\ldots+\frac{\partial^2 u}{\partial x_{n-1}^2}\right) = 0;$$

$$\operatorname{supp}(\hat{u}) \subset \{(\xi_1, \ldots, \xi_{n-1}, \xi_n), |\xi_n|^2 = |\xi_1|^2+\ldots+|\xi_{n-1}|^2\}.$$

In the cases (i) and (ii), the corresponding hypersurfaces have non-vanishing Gaussian curvature while in the case (iii), the hypersurface is a cone which has vanishing Gaussian curvature everywhere except the vertex. Combining the basic estimate with the Beurling–Pollard technique, we can recover (slightly improve) a general uniqueness result of Hörmander ([7], theorem 2.2) for the first two cases. This is done in our theorem 2. Our theorem 3 deals with the third case, the wave equation, where a group action on the cone is used to obtain the optimal result, which improves Hörmander's result for this specific equation.

The organization of this article is as follows. In section 2, we quote the basic estimate obtained in [4] and some facts from the Beurling–Pollard technique. Section 3 will contain the proof of our theorem 1. Theorems 2 and 3 will be stated and proved together in section 4 and at the end of this section an example will be given to show that the curvature assumption upon M in the basic estimate cannot be removed completely.

2. The basic estimate and the Beurling–Pollard technique

Let $M=\{(x, \psi(x)), x\in U\}$, where U is a small open ball in $\mathbf{R}^{n-1}$, $\psi(x)$ a real-valued function defined on U such that $\psi(x)\in C^k(U)$ for a positive integer k (to be fixed later), and such that the inverse of the Hessian determinant $\left|\left(\frac{\partial^2\psi}{\partial x_i \partial x_j}\right)\right|$ exists and is bounded in U. Let $T\in FL^\infty(\mathbf{R}^n)$ and vanish on $J^k(M)$. As in [4], for all sufficiently small h we can *convolve T with a family of nice functions φ_h along M* to obtain a family of nice measures T_h on M such that the following estimate holds.

Basic estimate (see [4], p. 510). *Let k be a positive integer such that $k\geqq n+2$, and let $(\eta, \xi)\in\mathbf{R}^{n-1}\times\mathbf{R}$. If we set $M_{2,\eta}(\hat{T})(\eta, \xi)=\left(\sup_{r>0}\frac{1}{m(B_r(\eta))}\int_{B_r(\eta)}|\hat{T}(u, \xi)|^2du\right)^{1/2}$, then we have*

$$|\hat{T}_h(\eta, \xi)| \leqq CM_{2,\eta}(\hat{T})(\eta, \xi),$$

where C is independent of η, ξ, and h.

Let $f(x)$ be a measurable complex-valued function on $\mathbf{R}^n$ and m the Lebesgue measure on $\mathbf{R}^n$. For $y>0$, $t>0$, denote

$$\lambda_f(y) = m\{x;\ |f(x)| > y\},\ f^*(t) = \inf\{y;\ \lambda_f(y) \leqq t\}.$$

$f^*(t)$ is called the *non-increasing rearrangement* of $f(x)$.

The Lorentz space $L(p, q)$ is the collection of all f such that $\|f\|_{p,q}<\infty$, where

$$\|f\|_{p,v} = \begin{cases}\left(\frac{q}{p}\int_0^\infty [t^{1/p}f^*(t)]^v \frac{dt}{t}\right)^{1/q}, & 1\leqq p<\infty,\quad 1\leqq q<\infty;\\ \sup_{t>0} t^{1/p}f^*(t), & 1\leqq p\leqq\infty,\quad q=\infty.\end{cases}$$

Lemma 1. *Let M be as above, $T\in\mathcal{S}'(\mathbf{R}^n)$ such that T vanishes on $J^{n+2}(M)$. Then $T=0$, provided $\hat{T}\in L(p, q)$ for $2\leqq p<\frac{2n}{n-1}$, $1\leqq q\leqq\infty$.*

Lemma 2. *Let M be an $(n-1)$-dimensional manifold in $\mathbf{R}^n$ with area. Let $T\in\mathcal{S}'(\mathbf{R}^n)$ with $\operatorname{supp}(T)\subset M$. If $\hat{T}\in L(p, q)$, then T vanishes on $J^{n+2}(M)$ provided*

(i) $2\leqq p\leqq\infty$, $1\leqq q\leqq\infty$, *when* $n=2$;

(ii) $2\leqq p<\infty$, $1\leqq q\leqq\infty$, *when* $n=3$;

(iii) $2\leqq p<\dfrac{2n}{n-3}$, $1\leqq q\leqq\infty$, *when* $n\geqq 4$.

Lemma 3. *Given T, T_h, M as in the basic estimate, we have*

$$\|\hat{T}_h\|_{p,q} \leqq C_{p,q}\|\hat{T}\|_{p,q}$$

for

$$\begin{cases} 2 \leqq p < \dfrac{2n}{n-3}, & 1 \leqq q \leqq \infty, \quad when \quad n > 3, \\ 2 \leqq p < \infty, & 1 \leqq q \leqq \infty, \quad when \quad n = 2, 3. \end{cases}$$

Proof. From lemma 2 above, we see that T vanishes on $J^{n+2}(M)$ if p and q are in the range contained in the condition of this lemma. So the basic estimate yields

$$|\hat{T}_h(\eta, \xi)| \leqq CM_{2,\eta}(\hat{T})(\eta, \xi).$$

But

$$M_{2,\eta}(\hat{T})(\eta, \xi) \leqq C\left(\sup_{r>0} \frac{1}{m(B_r(\eta, \xi))} \int_{B_r(\eta,\xi)} |\hat{T}(y)|^2\, dy\right)^{1/2} = CM_2(\hat{T})(\eta, \xi),$$

so we have $|\hat{T}_h(\eta, \xi)| \leqq CM_2(\hat{T})(\eta, \xi)$ and hence $\|\hat{T}_h\|_{p,q} \leqq C\|M_2(\hat{T})\|_{p,q}$.

It is easy to check that the operator M_2 is sublinear and for any $g \in L^\infty(\mathbf{R}^n)$, we have $\|M_2(g)\|_\infty \leqq \|g\|_\infty$, and for any $g \in L^2(\mathbf{R}^n)$, we have $\|M_2(g)\|_{2,\infty} \leqq \|g\|_2$. Thus the conclusion of this lemma follows directly from the interpolation between Lorentz spaces (see [8]).

Remark 1. The proofs of lemma 1 and lemma 2 are similar to the proof of lemma 1 in [4], but the case $p=2$, $1 \leqq q \leqq \infty$ should be treated more carefully. Also in the basic estimate and all three lemmas, $J^{n+2}(M)$ can be replaced by $J(M)$ without changing the proof.

3. The proof of theorem 1

Proof of theorem 1. We will follow Domar's argument in [1]. The compactness of M implies that we can find $\{E_j, j=1, \ldots, m\} \subset M$ such that $M = \bigcup_{j=1}^m E_j$ and each E_j has the same form as M in the basic estimate. Choose $\varphi_j \in C_0^\infty(\mathbf{R}^n)$ such that $\operatorname{supp}(\varphi_j) \cap M \subset E_j$ and $\sum_{j=1}^m \varphi_j = 1$ in M.

Given $T \in FL^\infty(\mathbf{R}^n)$ vanishing on $J^{n+2}(M)$, which implies $\operatorname{supp}(T) \subseteqq M$, we have

$$T = \left(\sum_{j=1}^m \varphi_j\right) T = \sum_{j=1}^m \varphi_j T = \sum_{j=1}^m T^j.$$

Here $T_j = \varphi_j T$. It is easy to see that $\operatorname{supp}(T^j) \subset E_j$ and $\widehat{T^j} = \hat{\varphi}_j * \hat{T} \in L^\infty(\mathbf{R}^n)$ since $\hat{\varphi}_j \in L^1(\mathbf{R}^n)$. Thus $T^j \in FL^\infty(\mathbf{R}^n)$ and hence as in the basic estimate we can construct $\{T_h^j\}$ for small h such that $|\hat{T}_h^j(\eta, \xi)| \leqq CM_{2,\eta}(\hat{T}^j)(\eta, \xi)$ and hence

$$\|\hat{T}_h^j\|_\infty \leqq C_j \|\hat{T}^j\|_\infty, \quad j = 1, 2, \ldots, m,$$

with C_j independent of h.

By the construction of T_h^j as in [4], we have for $f \in \mathcal{S}(\mathbf{R}^n)$,

$$\langle T_h^j, f\rangle = \langle \Sigma^j * \check{\varphi}_h, f\circ\beta\rangle = \langle \Sigma^j, \varphi_h * f\circ\beta\rangle.$$

Using the Lebesgue dominated convergence theorem, we see that

$$\|\varphi_h * f\circ\beta - f\circ\beta\|_{FL^1(R^{n-1})} = \left\|(\hat{\varphi}(h\xi)-1)\widehat{f\circ\beta}(\eta)\right\|_{L^1(R^{n-1})} \to 0, \quad \text{as} \quad h\to 0.$$

So $\langle T_h^j, f\rangle \to \langle \Sigma^j, f\circ\beta\rangle$, since $\Sigma^j \in FL^\infty(\mathbf{R}^{n-1})$.

From the construction of Σ^j and the assumption $T(J^{n+2}(M))=0$, we have $\langle \Sigma^j, f\circ\beta\rangle = \langle T^j, f\rangle$. This yields $\langle T_h^j, f\rangle \to \langle T^j, f\rangle$, as $h\to 0$.

Now if we let $T_h = \sum_{j=1}^m \varphi_j T_h^j$, then for $f\in\mathscr{S}(\mathbf{R}^n)$,

$$\langle T_h, f\rangle \to \langle T, f\rangle, \quad \text{as} \quad h\to 0.$$

Recall that $\mathscr{S}(\mathbf{R}^n)$ is dense in $FL^1(\mathbf{R}^n)$ and FL^∞ is the dual space of $FL^1(\mathbf{R}^n)$, so for $f\in FL^1(\mathbf{R}^n)$, from the estimate $\|\hat{T}_h\|_\infty \leqq C\|\hat{T}\|_\infty$, we have

$$\langle T_h, f\rangle \to \langle T, f\rangle, \quad \text{as} \quad h\to 0.$$

To prove the theorem, by the Hahn–Banach theorem it is enough to show that given $T\in FL^\infty(\mathbf{R}^n)$ vanishing on $J^{n+2}(M)$, we have $\langle T, f\rangle = 0$ for $f\in FL^1(\mathbf{R}^n)$ vanishing on M.

But T_h is a measure on M absolutely continuous with respect to the area measure of M. We have $a_h(s)\in C(M)$ such that

$$\langle T_h, f\rangle = \int_M a_h(s) f(s)\, ds = 0 \quad \text{for} \quad f\in FL^1(\mathbf{R}^n).$$

Thus for each $F\in FL^1(\mathbf{R}^n)$ vanishing on M, we have $\langle T, f\rangle = \lim_{h\to 0}\langle T_h, f\rangle = 0$.

The proof of theorem 1 is complete.

It seems to the author that the C^{n+2} smoothness assumption in theorem 1 is best possible as indicated by the proof of the basic estimate, which is based on Hörmander's estimate ([6], theorem 1), while Domar bases his argument on Littman's well-known estimate [10], and the C^{2n+1} smoothness assumption is imperative. It should be pointed out that Müller [11] adopted another idea of Domar's developed in [3] to extend our theorem 1 to M with vanishing curvature in a simple way (a cone is a standard example), but the C^{2n+1} smoothness assumption in his argument is still essential. It remains open how to modify Hörmander's estimate in order to reduce the smoothness assumption in Müller's result.

4. The proof of theorem 2 and theorem 3

Theorem 2. *Let u be a solution of the cases* (i), (ii) *in section* 1. *Then* $u=0$ *if* $u\in L(p,q)$ *for some* (p,q) *satisfying* $2\leqq p<\frac{2n}{n-1}$, $1\leqq q\leqq\infty$; *or* $p=\frac{2n}{n-1}$, $1\leqq q<\infty$.

Actually we can prove a stronger result from which theorem 2 follows.

Theorem 2′. *Let M be a C^{n+2} $(n-1)$-dimensional manifold in $\mathbf{R}^n$ with non-vanishing Gaussian curvature and $T\in\mathscr{S}'(\mathbf{R}^n)$ with* $\operatorname{supp}(T)\subset M$. *If $\hat{T}\in L(p,q)$ for some (p,q) satisfying $2\leqq p<\frac{2n}{n-1}$, $1\leqq q\leqq\infty$; or $p=\frac{2n}{n-1}$, $1\leqq q<\infty$, then $T=0$.*

Note that in theorem 2′, M need not be compact. Also from Littman's estimate in [10], it is easy to see that given $\frac{2n}{n-1}<p\leqq\infty$, $1\leqq q\leqq\infty$; or $p=\frac{2n}{n-1}$, $q=\infty$, we can find $T\in\mathscr{S}'(\mathbf{R}^n)$ with $\operatorname{supp}(T)\subset M$ such that $\hat{T}\in L(p,q)$ and $T\neq 0$. Thus the result in theorem 2′ is optimal. Theorem 2 recovers Hörmander's result in [7] for the cases (i) and (ii), and even more, Hörmander's method in [7] cannot cover the case $p=2$, $2<q\leqq\infty$, since he assumes $\hat{T}\in L^2_{\mathrm{loc}}(\mathbf{R}^n)$ and uses the Plancherel theorem.

Proof of theorem 2′. If M is compact, from lemma 2 of section 2, we see that T vanishes on J^{n+2}, so the case $2\leqq p<\frac{2n}{n-1}$, $1\leqq q\leqq\infty$ follows from lemma 1 of section 2. In the following we only consider the case $p=\frac{2n}{n-1}$, $1\leqq q<\infty$, for the compactness argument remains valid for the other cases.

For any open set U in $\mathbf{R}^n$ with $E=U\cap M$ open in M, and any $\varphi(x)\in C_0^\infty(U)$, we let $T_1=\varphi\cdot T$. Then we have $\operatorname{supp}(T_1)\subset E$ and $\hat{T}_1=\hat{\varphi}*\hat{T}$. Thus $\hat{T}_1\in L\left(\frac{2n}{n-1},q\right)$ since $\hat{\varphi}\in L^1(\mathbf{R}^n)\cap L^\infty(\mathbf{R}^n)$ and $\hat{T}\in L\left(\frac{2n}{n-1},q\right)$.

Taking U smaller if necessary, we can choose a coordinate system in $\mathbf{R}^n$ such that $E=(x,\psi(x))$, ψ satisfies the properties needed for us to adopt the basic estimate.

So for a small h, from lemma 3 of section 2, we can find a good measure with $\operatorname{supp}(T_h)\subset E$ such that the corresponding density function $a_h(s)\in C^{n+1}(E)$ and

$$\|\hat{T}_h\|_{2n/n-1,q}\leqq C\|\hat{T}_1\|_{2n/n-1,q}.$$

If $a_h(s)$ is not identically zero on E, Littman's asymptotic estimate in [10] yields

$$\hat{T}_h(\xi)\approx C(1+|\xi|)^{-(n-1)/2}\quad\text{for large}\quad \xi\in\mathbf{R}^n.$$

But the function $(1+|\xi|)^{-(n-1/2)}\notin L\left(\frac{2n}{n-1},q\right)$ for $1\leqq q<\infty$, so we must have $a_h(s)=0$ identically on E, that is $T_h=0$ for all small h. By Titchmarsh's convolution theorem and the definition of T_h, this implies $T_1=0$ and hence $T=0$ since U and φ are arbitrary. This is the end of the proof of theorem 2′.

For simplicity, we state the following theorem in terms of L^p spaces rather than the $L(p,q)$ spaces. Also the case $n=2$ is simple as we can see from lemma 1 and lemma 2 of section 2.

Theorem 3. *If u is a solution of* (iii) *in section* 1 *such that $u\in L^p(\mathbf{R}^n)$ $(n\geqq 3)$ for some p satisfying $1\leqq p\leqq\frac{2(n-1)}{n-2}$, then $u=0$; furthermore, given $\frac{2(n-1)}{n-2}<p\leqq\infty$, we can find a non-zero solution u_p of* (iii) *such that $u_p\in L^p(\mathbf{R}^n)$.*

Remark 2. Hörmander's general result only covers the range of p, $1 \leqq p \leqq \frac{2n}{n-1}$. It is easy to check that $\frac{2n}{n-1} < \frac{2(n-1)}{n-2}$ for $n \geqq 3$.

Proof of theorem 3. For simplicity we only give the proof for the case $n=3$. The case $n \geqq 4$ can be treated similarly, but the $(n-1)$-dimensional rotation group SO_{n-1} acting on $\mathbf{R}^{n-1}$ and the Haar measure on SO_{n-1} are needed. For $n=3$, the hypersurface M is the cone $M=\{(x_1, x_2, x_3);\ x_3^2 = x_1^2 + x_2^2,\ (x_1, x_2) \in \mathbf{R}^2\}$. If $\hat{u}$ is supported at the vertex of the cone M, $(0, 0, 0)$, then the proof is trivial. So we may assume by lemma 2 that $\operatorname{supp}(\hat{u}) \subset M \diagdown (0, 0, 0)$. To prove $\hat{u}=0$, which implies $u=0$, it is enough to show that for any bounded closed ball $\bar{B} \subset \mathbf{R}^n \diagdown (0, 0, 0)$, and any $C_0^\infty(B)$ function φ, we have $\varphi\hat{u}=0$.

But $\widehat{\varphi\hat{u}} = \check{\varphi} * u \in L^p(\mathbf{R}^3)$ since $\check{\varphi} \in L^1(\mathbf{R}^3)$ and $u \in L^p(\mathbf{R}^3)$, so we may assume $\operatorname{supp}(\hat{u})$ is compact. Without loss of generality, we may assume that

$$\operatorname{supp}(\hat{u}) \subset E = \{(r\cos\theta, r\sin\theta, r);\ 0 \leqq \theta \leqq 2\pi,\ 0 < c_1 \leqq r \leqq c_2 < \infty\} \subset M.$$

Denote $\hat{u}$ by T and let $U = \{(\theta, r); 0 \leqq \theta \leqq 2\pi, 1 < r < 2\}$. We define $T_a \in \mathcal{S}'(\mathbf{R}^n)$ by

$$\langle T_a, f\rangle = \left\langle T, \int_U f\left(\frac{r}{s}\cos(\varphi-\theta), \frac{r}{s}\sin(\varphi-\theta), \frac{r}{s}\right) a(s\cos\theta, s\sin\theta, s)\, d\theta \frac{ds}{s}\right\rangle,$$

$$f \in \mathcal{S}(\mathbf{R}^n).$$

Here a is a smooth function in $\mathbf{R}^n$ with compact support. Note that $\frac{2(n-1)}{n-2} < \frac{2n}{n-3}$ for all $n \geqq 3$. We see that T_a is well-defined since $\langle T, f(x, y, z) - f(r\cos\theta, r\sin\theta, r)\rangle = 0$ by lemma 2.

It is not difficult to check again by lemma 2 that T_a is a nice measure on the cone M with the density function $\alpha \in C_0^\infty(M)$,

$$\alpha(\theta, s) = \left\langle T, a\left(\frac{\cos\theta}{s}x + \frac{\sin\theta}{s}y, \frac{\cos\theta}{s}y - \frac{\sin\theta}{s}x, \frac{z}{s}\right)\right\rangle.$$

Now our lemma 2, a linear transformation of (ξ_1, ξ_2, ξ_3), and Minkowski's inequality for integrals yield

$$\left(\int_{\mathbf{R}^3} |\hat{T}_a(\xi)|^p d\xi\right)^{1/p} \leqq C\int_U \left(\int_{\mathbf{R}^3} |\hat{T}(\xi')|^p d\xi'\right)^{1/p} a(s\cos\theta, s\sin\theta, s)\, d\theta \frac{ds}{s}$$

$$\leqq C\left(\int_{\mathbf{R}^3} |\hat{T}(\xi)|^p d\xi\right)^{1/p}.$$

Here C stands for a uniform constant.

Thus we have proved

$$(*) \qquad \|\hat{T}_a\|_p \leqq C\|\hat{T}\|_p \quad \text{for} \quad 1 \leqq p \leqq 4 = \frac{2(3-1)}{3-2}.$$

If the density function α of T_a is not identically zero, we want to show that $\hat{T}_a \in L^p(\mathbf{R}^3)$ if and only if $p>4$. This fact together with the inequality $(*)$ implies both parts of the conclusion in our theorem 3.

Since we may assume that $\operatorname{supp}(\hat{T}_a)$ is very small, we can have small δ_1 and δ_2 such that

$$\operatorname{supp}(\alpha) \subset V$$

$$= \left\{(\theta, r);\ \theta_1 < \theta < \theta_2, \tfrac{1}{2} < r_1 < r < r_2 < 1, 0 < \theta_2 - \theta_1 < \delta_1, 0 < r_2 - r_1 < \delta_2\right\}.$$

We now examine the asymptotic behaviour of

$$\hat{T}_a(\xi) = \int_{r_1}^{r_2}\int_{\theta_1}^{\theta_2} e^{-i(\xi_1 r\cos\theta + \xi_2 r\sin\theta + \xi_3 r)}\alpha(\theta, r)\,d\theta\,\frac{dr}{r}.$$

Noticing that for r fixed, the integral

$$b(\xi_1, \xi_2, r) = \int_{\theta_1}^{\theta_2} e^{-i(\xi_1 r\cos\theta + \xi_2 r\sin\theta)}\alpha(\theta, r)\,d\theta,$$

is nothing but the Fourier transform of a smooth measure on a small piece of a circle (with non-vanishing curvature), if we check the proof in [10] carefully, we see that for ξ_3 sufficiently small, the asymptotic behaviour of $\hat{T}_a(\xi)$ for the fixed ξ_3 is $C\big((1+\xi_1^2+\xi_2^2)^{1/2}\big)^{-1/2}$, where $0<C<\infty$ is is independent of θ, r, ξ_1, ξ_2 and all the small ξ_3. It follows that for a small ξ_3, as a function of ξ_1 and ξ_2, $\hat{T}_a \in L^p(\mathbf{R}^2)$ iff $p>4$. Thus $\hat{T}_a(\xi) \in L^p(\mathbf{R}^3)$ only if $p>4$. The fact that $\hat{T}_a(\xi) \in L^p(\mathbf{R}^3)$ for $p>4$ is a simple consequence of the expression of $\hat{T}_a(\xi)$ and the above argument. This finishes the proof of theorem 3.

Remark 3. Note that in the proof of theorem 3 we only used the rotation and the dilation on the cone, so the whole space $\mathbf{R}^n$ in the condition $u \in L^p(\mathbf{R}^n)$ can be replaced by a solid cone, which is the condition contained in [7].

Now we give an example to show that the curvature assumption upon M in the basic estimate cannot be removed completely.

Example 5.1. Let $x=(x_1, x_2) \in \mathbf{R}^2$ and $x_0 \in \mathbf{R}^2$ with $|x_0|=3$. Let $U_1=\{x;\ |x|<1\}$, $U_2=\{x;\ |x-x_0|<1\}$, $U_1'=\{x;\ |x|<\frac{1}{2}\}$ and $U_2'=\{x;\ |x-x_0|<\frac{1}{2}\}$. Let $U=\{x;\ |x|<5\}$ and choose $\alpha(x) \in C_0^5(U)$ such that $\alpha(x)=1$ on $U_1' \cup U_2'$ and $\alpha(x)=0$ on $U \setminus (U_1 \cup U)$. Define $\psi(x) \in C^5(U)$ by letting $\psi(x)=(2-|x|^2)^{1/2}\alpha(x)$ for $x \in U_1$, $\psi(x)=(x_1^2+x_2^2)^{1/2}\alpha(x)$ for $x \in U_2$, $\psi(x)=0$ for $x \in U \setminus (U_1 \cup U_2)$.

Let $E=\{(x, \psi(x)), x\in U\}$, then E contains a sphere-piece

$$E_1 = \{(x, (2-|x|^2)^{1/2});\ x\in U_2'\}$$

and a cone-piece $E_2=\{(x_1, x_2, (x_1^2+x_2^2)^{1/2});\ (x_1, x_2)\in U_2'\}$. Choose a nice measure T on E with a non-zero smooth density function contained in the piece of the sphere, then from Littman's estimate we have $\hat{T}\in L^p(\mathbf{R}^3)$ for $p>3$. Let $p=4$, then lemma 2 yields that T vanishes on $J^5(E)$. So if the basic estimate is true for E, then we would have from lemma 3 that

$$\|\hat{T}_h\|_4 \leqq C\|\hat{T}\|_4.$$

We can apply lemma 3 several times to make T_h a measure on E such that the C^4 density function $a_h(s)$ is not identically zero on the piece of the cone. Choose $\varphi(x)\in C_0^\infty(\mathbf{R}^3)$ such that φT_h is contained in the piece of the cone and non-zero, then we have since $\varphi\in FL^1(\mathbf{R}^3)$

$$\|\widehat{\varphi T_h}\|_4 = \|\hat{\varphi} * \hat{T}_h\|_4 \leqq C\|\hat{T}_h\|_4.$$

But from theorem 3, we see that $\varphi T_h\neq 0$ and $\widehat{\varphi T_h}\in L^p(\mathbf{R}^3)$ if and only if $p>4$, so we must have $\varphi T_h=0$, which contradicts the choice of φ and T_h.

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