The direct product of a hopfian group with a group with cyclic center

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Abstract

In this paper we continue our study of hopficity begun in [1], [2], [3], [4] and [5]. Let A be hopfian and let B have a cyclic center of prime power order. We improve Theorem 4 of [2] by showing that if B has finitely many normal subgroups which form a chain (we say B is n-normal), then $A \times B$ is hopfian. We then consider the case when B is a p-group of nilpotency class 2 and show that in certain cases $A \times B$ is hopfian.

Some notations and preliminary results

We draw freely from the ideas in [1], [2] and [3]. In the sequel α will designate a surjective endormorphism of $A \times B$. In our proofs we begin by assuming that $A \times B$ is not hopfian so we may assume α is not an isomorphism on A. We use the idea of the homomorphism α_* introduced in [3]. We note that for α_* to exist as introduced in [3], we must have α an isomorphism on B and $B\alpha \cap B = 1$. The symbols a, a_1, $a_2 \ldots$ will designate elements in A.

LEMMA 1. *Suppose* $Z(B)$ *is cyclic of prime power order* p^j *and* $B\alpha^j \cap B = 1$, α^j *an isomorphism on* B, $j = 1, 2$. *Let* $\langle b \rangle = Z(B) = Z$ *and let* $(b^t a)\alpha = b$, *Then if* α *is not an isomorphism on* A, $t \equiv 1 \bmod \mathrm{p}$.

Proof. Let $b\alpha = a_1 b^q$. We claim that $q \equiv 0 \bmod p$. For otherwise $AB\alpha = A \times \langle b \rangle$ so that $B\alpha = \langle b\alpha \rangle \times A \cap B\alpha$. But $B\alpha$ has a cyclic center and cannot decompose nontrivially as a direct product. Thus $B \approx B\alpha = \langle b\alpha \rangle$ which contradicts Theorem 3 of [1]. Now

$$A = (A \times Z)\alpha_* = A\alpha_* \langle a_1 \rangle . \tag{1}$$

Let $\alpha_1 = (a_2 b^f)\alpha$. Hence $(a_2 b^{f-1})\alpha$ is a p:th power and so therefore is $(a_2 b^{f-1})\alpha_*$. This implies $f \equiv 1 \bmod p$, for otherwise if $L = \langle a_2 b^{f-1} \rangle$

$$A = (A \times Z)\alpha_* = (AL)\alpha_* = A\alpha_* L\alpha_* . \tag{2}$$

But since $L\alpha_*$ is generated by a p:th power we may conclude from (1) and (2) that $A = A\alpha_*$ which is impossible. Now we show that $(t, p) = 1$. For if $M = \langle b^t a \rangle$ then

$$(AM)\alpha_* = A\alpha_* M\alpha_* = A\alpha\langle a_1 \rangle = A . \tag{3}$$

But

$$AM = A \times \langle b^t \rangle \tag{4}$$

so that if $p|t$, we could deduce from (3) and (4) that $A = A\alpha_*$.

Since $(t, p) = 1$, we see $\langle b\alpha \rangle \subset A\alpha B \subset A\alpha\langle b\alpha \rangle$ so

$$A\alpha B = A\alpha\langle b\alpha \rangle = A\alpha\langle b \rangle . \tag{5}$$

Also since $(p, q) = p$, we see that $B \cdot A\alpha = A\alpha\langle a_1 \rangle$. Hence we may write $b = a_3\alpha \cdot a_1^r$. But then $a_3\alpha_* = a_1^{1-r}$ so that $(p, 1-r) = p$ or else $A = A\alpha_*$. Now let $c = b^t a a_2^{-1} b^{-f}$. Let P be the subgroup of p:th powers in $Z(A) \times Z(B)$. Note

$$c\alpha^2 = a_3\alpha^2 \bmod P . \tag{6}$$

If $t \not\equiv f \bmod P$, then from (6) we see that $A = (A \cdot \langle c a_3^{-1} \rangle)(\alpha^2)_* = A(\alpha^2)_*$. Hence $t \equiv f \equiv 1$.

Some results

THEOREM 1. *If B is n-normal, $A \times B$ is hopfian.*

Proof. Suppose the theorem is false. Let C be as in [3], p. 182 and let w be a generator for $Z(C)$. Let α be an endomorphism of $A \times C$ which is not an isomorphism on A. Then, as in (5) of Lemma 1, $A\alpha C = A\alpha\langle w \rangle$. But then $C = \langle w \rangle C \cap A\alpha$. Hence $C \subset A\alpha$ or $C = \langle w \rangle$. The former is impossible by Lemma 4 of [2] and the latter is impossible by Theorem 3 of [1].

LEMMA 2. *Let B be a group with generators b, $x_1, y_1, x_2, y_2, \ldots x_n, y_n$ and defining relations of the form:*

$$[x_i, y_i] = b^{p^{j_i}}, \quad x_i^{p^{e_i}} = b^{n_i}, \quad y_i^{p^{e_i}} = b^{m_i}, \quad b^{p^k} = 1 , \tag{7}$$

and all other generators commute. Then $Z(B) = \langle b \rangle$ and any automorphism of $Z(B)$ may be extended to an automorphism of B.

Proof. This is an unpublished result of F. Pickel. For the duration of this paper B will be as in the above Lemma and p will be a prime distinct from 2.

LEMMA 3. *If B is finite we may assume $B \cap B\alpha^n = 1$ for all n sufficiently large.*

Proof. Otherwise as in Lemma 5 of [1], p. 237, we can produce a normal subgroup $B_1 \neq 1$ of B with $B_1\alpha^m = B_1$ for some $m > 0$. Hence α^m is an isomorphism on B. But then $A \cap B\alpha^m = 1$ for $B_1\alpha^m$ must intersect any nontrivial normal subgroup of $B\alpha^m$ nontrivially. Hence $A \times B = A \times B\alpha^m = A\alpha^m B\alpha^m$ and hence A is isomorphic to a homomorphic image of $A\alpha^m$.

THEOREM 2. *If $j_i = r$, $i = 1, 2, \ldots n$ in (7), then $A \times B$ is hopfian.*

Proof. By Lemma 3 we may assume $B \cap B\alpha^i = 1$, $i = 1, 2$. Let $(b^t a)\alpha = b$. From Lemma 3 we see that we may assume $t \not\equiv 1 \bmod p$, for otherwise we may replace α by $\theta\alpha$ where θ is a surjective endomorphism of $A \times B$ which is the identity on A and which acts as a suitable isomorphism on B. By Lemma 1, α is not an isomorphism on B. Let $K = (\text{kernel } \alpha) \cap B$. Then α induces a surjective endomorphism γ of $A + B/K$ onto $A + B/K$ in the natural way. Let $P = \{g^p | g \in B\}$. If $g \in B$, we set $g' = gK$ in $B/K = B_*$ and claim

$$Z(B_*) = \langle b', g_1', g_2', \ldots, g_k' \rangle$$

for suitable g_i in P. To see this, suppose w is a word in the generators of B such that $w' \in Z(B/K)$. Let s be the exponent of y_1 in w. Then $[x_1, w] = [x_1, y_1]^s \in K$. If $(p, s) = 1$ then $[x_1, y_1] = b^{p^r} \in K$. Hence $[x_i, y_i] \in K$ for all i. Hence B_* is abelian contrary to Theorem 3 of [1]. Hence s is divisible by p. Similarly, the exponent of any other x or y symbol in w is divisible by p. Now let

$$b\alpha = b^q a_1 . \tag{8}$$

Since α is not an isomorphism on B, α is not an isomorphism on $Z(B)$ os that $(p, q) = p$. Moreover, in view of (8)

$$A = (A \times \langle b' \rangle)\gamma_* = A\gamma_* \langle a_1 \rangle . \tag{9}$$

Now $(b^{t-1}a)\alpha = b^{1-q}a_1^{-1}$. But then we see, $((b')^{t-1}a)\gamma_* = a_1^{-q}$ and since $t \not\equiv 1 \bmod p$,

$$A = (A \times \langle b' \rangle)\gamma_* = [A\langle (b')^{t-1}a \rangle]\gamma_* = A\gamma_* \langle a_1^q \rangle . \tag{10}$$

Since $(p, q) = p$, (9) and (10) imply $A = A\gamma_*$.

THEOREM 3. *In (7) let $j_1 \leq j_2 \leq \ldots \leq j^{n-1} < j_n < k$ and $e_i = k - j_i$. Suppose further that $m_i \equiv n_i \equiv 0 \bmod p^{j_i}$, $i = 1, 2, \ldots n$. Then $A \times B$ is hopfian.*

Proof. Our notation is as in the previous theorem. If $b^{p^{jn}} \notin K$, as in Theorem 2, we could deduce that $Z(B_*) = \langle b', g_1', g_2' \ldots g_k' \rangle$, $g_i \in P_j$ and we could then proceed exactly as before. We may assume $b^{p^{jn}} \in K$. Let e be the least positive integer such that for $t = p^{je}$, $b^t \in K$. Let

$$F = \langle x_1, y_1, x_2, y_2, \ldots x_{e-1}, y_{e-1}, b \rangle ,$$

$$A_1 = \langle A, x_e, y_e, x_{e+1}, y_{e+1}, \ldots, x_n, y_n \rangle / \langle b^t \rangle, \quad B_1 = F/\langle b^t \rangle .$$

Then the defining relations for B imply $(A \times B)/\langle b^t \rangle = A_1 \times B_1$. Note that A_1 is the direct product of A and a finite abelian group so that A_1 is hopfian by Theorem 3 of [1]. Now α induces an endomorphism γ of $A_1 \times B_1$ in the natural way. If K_1 is the kernel of $\gamma \cap B_1$, and $B_2 = B_1/K_1$, then γ induces a surjective endormorphism θ of $A_1 \times B_2$ in the natural way. Now let $F_1 =$ kernel $\alpha \cap F$ and $K_1 = F_1/\langle b^t \rangle$. Then $B_2 \approx F/F_1$. With the aid of the definition of e we see that the center of F/F_1 is generated by elements $bF_1, g_1F_1 \ldots g_tF_1$ for certain $g_i \in F$ where each g_i is a p:th power in F. We may now mimick the proof of Theorem 3 and obtain $A_1\theta_* = A_1$ a contradiction of the hopficity of A_1.

References

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