On strong Ditkin sets

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Let G be a locally compact abelian group with character group Γ, and let $M(G)$ be the convolution algebra consisting of all bounded regular measures on G. The Fourier transform of a measure μ in $M(G)$ is defined by

$$\hat{\mu}(\gamma) = \int_G (x, -\gamma) d\mu(x) \quad (\gamma \in \Gamma) .$$

We shall regard the group algebra $L^1(G)$ as a closed ideal in $M(G)$ (see [9, p. 16]). For a given closed subset E of Γ, let us denote by:

$I(E) = \{f \in L^1(G) : \hat{f} = 0 \text{ on } E\}$;

$I_0(E) = \{f \in L^1(G) : \hat{f} = 0 \text{ on some neighborhood of } E\}$;

$J(E) =$ the closure of $I_0(E)$,

and, for any measure μ in $M(G)$, define

$$\|\mu\|_E = \sup \{\|f * \mu\| : f \in I_0(E), \ \|f\| \leq 1\} .$$

In other words, $\|\mu\|_E$ is the operator norm of the mapping: $f \to f * \mu$ (from $I_0(E)$ into $L^1(G)$).

Definition 1. (cf. [10] and [8]). We say that a closed subset E of Γ is a *Wik set* if there exists a family $\{\mu_\alpha \in M(G)\}_{\alpha \in A}$ of measures which is directed, in the sense that the index set A is a directed set, such that:

(a) $\sup \{\|\mu_\alpha\|_E : \alpha \in A\} < \infty$;

(b) $\hat{\mu}_\alpha(\gamma) \xrightarrow[\alpha \in A]{} 0$ if $\gamma \in E$, and $\hat{\mu}_\alpha(\gamma) \xrightarrow[\alpha \in A]{} 1$ if $\gamma \in E^c$.

Definition 2. (cf. [10]). E is called a *strong Ditkin set* if there exists a directed family $\{\mu_\alpha \in M(G)\}_{\alpha \in A}$ of measures such that:

(c) $\sup\{\|\mu_\alpha\|_E : \alpha \in A\} < \infty$;

(d) $\hat{\mu}_\alpha = 0$ on some neighborhood of E depending on μ_α $(\alpha \in A)$;

(e) For each f in $I(E)$, $f * \mu_\alpha \xrightarrow[\alpha \in A]{} f$ in the norm of $M(G)$.

It is then trivial that every strong Ditkin set is a Wik set, and we can also show that our definition of a strong Ditkin set coincides with the original one given by Wik if $I(E)$ is separable (see Theorem 2). Note that this last condition on $I(E)$ is always satisfied if G is both σ-compact and metrizable.

In this paper we shall be concerned with the problem of determining all strong Ditkin sets without interior. We shall solve this problem entirely, and thus complete a line of investigation by Wik [10], Rosenthal [7], [8], and Gilbert [4].

THEOREM 1. *Let K be a closed subset of Γ, let $u + \Lambda$ be a coset of an algebraic subgroup Λ of Γ such that $K \cap (u + \Lambda) = \emptyset$, and let $\{\gamma_i\}_i$ be any finite subset of Λ. Then for every $\varepsilon > 0$, there exists a function k in $L^1(G)$ such that:*

(i) $\hat{k} = 1$ *on the set* $\{u + \gamma_i\}_i$, *and* $\hat{k} = 0$ *on some neighborhood of* K;

(ii) $\|k\| \leq 1 + \varepsilon$.

Proof. By translating the sets under consideration, we may assume that $u = 0$. We shall also assume that the set $\{\gamma_i\}_i$ algebraically generates the group Λ, since this assumption has no effect on the hypothesis and the conclusion of our theorem. Thus Λ has the form $\Lambda = \mathbf{Z}^N \times D$ as algebraic groups, where $\mathbf{Z}$ is the group of integers, N a non-negative integer, and D a finite group [5, (A. 27)]. For every positive integer n, denote by

$$\Lambda_n = \{(z_1, \ldots, z_N, d) \in \Lambda : |z_j| \leq n \quad (j = 1, \ldots, N), \quad d \in D\}.$$

It is then easy to see that

$$\operatorname{Card}(\Lambda_n) = (2n+1)^N \operatorname{Card}(D), \quad \text{and} \quad \Lambda_m \pm \Lambda_n = \Lambda_{m+n} \tag{1.1}$$

for all m and n, where, in general, Card (A) denotes the cardinal number of a set A.

Fix now an arbitrary positive integer m so that $\{\gamma_i\}_i \subset \Lambda_m$. Then for every n, there is a symmetric compact neighborhood V_n such that:

$$\text{The sets } \lambda + V_n, \ \lambda \in \Lambda_{m+n}, \text{ are pairwise disjoint;} \tag{1.2}$$

$$K \text{ is disjoint from } \Lambda_{m+2n} + V_n + V_n. \tag{1.3}$$

Setting $C = \Lambda_m$ and $V = \Lambda_n + V_n$ in [9, 2.6.1], we can find a function k_n in $L^1(G)$ such that:

$$k_n = 1 \text{ on } \Lambda_m, \text{ and } \hat{k}_n = 0 \text{ outside } \Lambda_{m+2n} + V_n + V_n\,; \tag{1.4}$$

$$\|k_n\| \leq \{h(\Lambda_{m+n} + V_n)/h(\Lambda_n + V_n)\}^{\frac{1}{2}} \tag{1.5}$$

where h denotes the Haar measure on Γ. It then follows from (1.1), (1.2), and (1.5) that

$$\|k_n\| \leq \{(2m + 2n + 1)^N/(2n + 1)^N\}^{\frac{1}{2}}\,. \tag{1.6}$$

Thus, for every $\varepsilon > 0$, we can take n so that $\|k_n\| < 1 + \varepsilon$. Putting $k = k_n$ for such an n, we see from (1.3) and (1.4) that k satisfies the desired conditions.

THEOREM 2. (cf. [6]). *Let E be any closed subset of Γ, and F the closure of the interior of E, then we have*

$$\|\mu\|_E = \sup\{\|f * \mu\| : f \in I(F),\ \|f\| \leq 1\} \equiv \|\mu\|'_F$$

for every μ in $M(G)$. Thus, in particular, if E has no interior point, we have $\|\mu\|_E = \|\mu\|$ for all μ in $M(G)$.

Proof. Fix μ in $M(G)$ and let $\varepsilon > 0$ be arbitrary. We can choose f in $I(F)$ so that $\|f\| \leq 1$, $\hat{f}$ has compact support, and

$$\|f * \mu\| > \|\mu\|'_F - \varepsilon\,. \tag{2.1}$$

There exists then a trigonometric polynomial P on G such that:

$$\|P\|_\infty \leq 1;\ \ P(x) = \sum_{i=1}^{n} c_i(x, -\gamma_i) \quad (x \in G)\,; \tag{2.2}$$

$$\sum_{i=1}^{n} c_i f(\gamma_i)\hat{\mu}(\gamma_i) > \|f * \mu\| - \varepsilon\,. \tag{2.3}$$

Let K be the intersection of the (compact) support of $\hat{f}$ and the boundary of E, and let Λ be the subgroup of Γ generated by the set $\{\gamma_i\}_1^n$. Then K does not contain any interior point, and Λ is countable; therefore Baire's theorem assures that $K + \Lambda$ has no interior point. Thus, every neighborhood U of 0 in Γ contains an element u with $(u + \Lambda) \cap K = \emptyset$. Theorem 1 applies, and we can find k in $I_0(K)$ such that $\|k\| < 1 + \varepsilon$ and $\hat{k} = 1$ on the set $\{u + \gamma_i\}_1^n$. It is easy to see that

$$k * f \in I_0(E), \text{ and } \|k * f\| \leq 1 + \varepsilon\,, \tag{2.4}$$

which, combined with (2.2), shows

$$\begin{aligned} \Big|\sum_{i=1}^{n} c_i\hat{f}(u + \gamma_i)\hat{\mu}(u + \gamma_i)\Big| &= \left|\int_G (x, -u)P(x)d(k * f * \mu)(x)\right| \\ &\leq \|k * f * \mu\| \leq (1 + \varepsilon)\|\mu\|_E\,. \end{aligned} \tag{2.5}$$

Since U is an arbitrary neighborhood of 0, and since u belongs to U, (2.3) and (2.5) show that

$$(1+\varepsilon)\|\mu\|_E \geq \|f * \mu\| - \varepsilon . \tag{2.6}$$

(Note that the Fourier transform of a measure is continuous.) Combining (2.1) and (2.6), we have

$$(1+\varepsilon)\|\mu\|_E \geq \|\mu\|'_F - 2\varepsilon .$$

Since $\varepsilon > 0$ was arbitrary, this yields the inequality $\|\mu\|_E \geq \|\mu\|'_F$. But, since $I_0(E) \subset I(F)$, the converse inequality $\|\mu\|_E \leq \|\mu\|'_F$ is obvious, and we have proved that $\|\mu\|_E = \|\mu\|'_F$. This completes the proof.

We now introduce some notations. Let C, AS, and CS be the families of all closed subsets, algebraic subgroups, and closed subgroups of Γ, respectively. For any family $\mathcal{F}$ of subsets of Γ, let us denote by $\mathcal{R}(\mathcal{F})$ the smallest Boolean algebra that contains $\mathcal{F}$ and is translation-invariant.

We then have:

THEOREM 3. $\mathcal{R}(CS) = \mathcal{R}(C) \cap \mathcal{R}(AS)$.

We need two lemmas. The first one is due to Cohen [2], and the second one is also essentially contained in [2, p. 225]. We shall prove here only the second one.

LEMMA 4 ([2, p. 223]). *Let Λ_{ij} be a finite collection of cosets of subgroups Λ_i in AS. Then if*

$$f(\gamma) = \sum_{ij} c_{ij} f(\Lambda_{ij}, \gamma)$$

for some constants c_{ij} ($f(\Lambda, \gamma)$ denotes the characteristic function of a set Λ), and B_k are the disjoint sets on which $f(\gamma)$ takes its finite number of values, then there are finitely many subgroups Λ'_i such that $\mathcal{R}(\{B_k\}_k) = \mathcal{R}(\{\Lambda'_i\}_i)$.

LEMMA 5. *Every coset in $\mathcal{R}(C)$ is closed.*

Proof. Let Λ be any coset in $\mathcal{R}(C)$. To prove that Λ is closed, we may assume that Λ is a subgroup, and also, by replacing Γ by $\overline{\Lambda}$, that Λ is dense in Γ. Since Λ is in $\mathcal{R}(C)$, there are finitely many closed sets F_i and open sets G_i in Γ such that

$$\Lambda = \bigcup_i (F_i \cap G_i) .$$

Since Λ is dense in Γ, there is an index i such that the closure of the set $S_i = F_i \cap G_i$ contains a non-empty open set U. Note then that $U \subset \bar{S}_i \subset F_i$. It is also trivial that $U \cap S_i \neq \emptyset$, and so $\emptyset \neq U \cap G_i \subset S_i \subset \Lambda$ which implies that Λ is an open subgroup. Since every open subgroup is closed, this completes the proof.

Proof of Theorem 3. Let E be any set in $\mathcal{R}(C) \cap \mathcal{R}(AS)$. Applying Lemma 4 to the function $f(\gamma) = f(E, \gamma)$, we see that there are finitely many subgroups Λ_i such that $\mathcal{R}(\{E\}) = \mathcal{R}(\{\Lambda_i\}_i)$. Since E is in $\mathcal{R}(C)$, it follows that every Λ_i is in $\mathcal{R}(C)$, and so Lemma 5 assures that every Λ_i is closed. Therefore we have $E \in \mathcal{R}(\{\Lambda_i\}_i) \subset \mathcal{R}(CS)$, and this clearly establishes Theorem 3.

COROLLARY 6 (due to Gilbert [4, Theorem 3.1]). *Every closed set E in $\mathcal{R}(AS)$ has the form*

$$E = \bigcup_i \left[\Lambda_i \cap (\bigcup_j \Lambda_{ij})^c\right], \tag{6.1}$$

where Λ_i and Λ_{ij} are finitely many closed cosets in Γ such that every Λ_{ij} is contained and open in Λ_i with respect to the relative topology of Λ_i.

Proof. Let E be any closed set in $\mathcal{R}(AS)$. It then follows from Theorem 3 that E has the form (6.1), where Λ_i and Λ_{ij} are finitely many closed cosets in Γ such that $\Lambda_i \supset \Lambda_{ij}$. Since E is closed, E is the union of the closures of $\Lambda_i \cap (\bigcup_j \Lambda_{ij})^c$. But all Λ_i and Λ_{ij} are closed cosets, and so that it is easy to check that the closure of $\Lambda_i \cap (\bigcup_j \Lambda_{ij})^c$ is $\Lambda_i \cap (\bigcup'_j \Lambda_{ij})^c$, where $\bigcup'_j \Lambda_{ij}$ denotes the union of those Λ_{ij} that are open in the relative topology of Λ_i (cf. the argument in [9, p. 86]). This establishes the proof.

THEOREM 7. *For every closed set E in Γ without interior, the following three statements are equivalent:*

(i) *E is a strong Ditkin set;*
(ii) *E is a Wik set;*
(iii) *E is of the form* (6.1).

Proof. The implication »(i) implies (ii)» is trivial. Suppose that E is a Wik set. There exists then a directed family $\{\mu_\alpha \in M(G)\}_{\alpha \in A}$ having the properties (a) and (b) in Definition 1. Since E has no interior point, the property (a), together with Theorem 2, yields

$$\sup \{\|\mu_\alpha\| : \alpha \in A\} < \infty . \tag{7.1}$$

Regarding each μ_α as a measure on the Bohr compactification $\bar{G}$ of G, we can conclude from (7.1) and (b) that $f(E, \gamma)$ is the Fourier transform of a measure in $M(\bar{G})$. It follows from Cohen's theorem [1] that E is a member of $\mathcal{R}(AS)$. (Note that the dual group of $\bar{G}$ is Γ with the discrete topology.) Therefore Corollary 6 guarantees that E has the form (6.1), which establishes the implication »(ii) $\Rightarrow$ (iii)».

Finally suppose that (iii) is the case. It is trivial that a finite union of strong Ditkin sets and a translate of a strong Ditkin set are strong Ditkin sets, too. Thus, to prove (i), it suffices to verify that every closed set E of the form

$$E = \Lambda \cap (\bigcup_{j=1}^{n} \Lambda_j)^c \tag{7.2}$$

is a strong Ditkin set, where Λ is a closed subgroup of Γ and each Λ_j is a closed coset in Γ which is open in the relative topology of Λ. Observe then that, for each j, Λ_j corresponds to a point in the quotient group $\Gamma/(\Lambda_j - \Lambda_j)$, and that the subset $[\Lambda/(\Lambda_j - \Lambda_j)] \cap \{\Lambda_j\}^c$ of this group is a closed set which does not contain the »point» Λ_j, since Λ_j is both open and closed in Λ. It follows that there is a measure ν_j in $M(G)$ such that $\hat{\nu}_j = 1$ on Λ_j, $\hat{\nu}_j = 0$ on some open set U_j containing $\Lambda \cap \Lambda_j^c$. Define

$$\nu = (\delta - \nu_1) * \ldots . * (\delta - \nu_n)\,,$$

where δ denotes the Dirac measure at 0 in G. Then $\hat{\nu} = 0$ on $\bigcup_{j=1}^{n} \Lambda_j$ and $\hat{\nu} = 1$ on $U = U_1 \cap \ldots \cap U_n$, which is an open set containing E. Suppose now that $\{f_i\}_i$ is any finite subset of $I(E)$ and $\varepsilon > 0$, then $f_i * \nu$ belongs to $I(\Lambda)$ for every i. It follows from a theorem of Calderon [9, 2.7.2] that there is a measure $\mu' = \mu'(\{f_i\}_i, \varepsilon)$ in $M(G)$ such that $\hat{\mu}' = 1$ on some neighborhood of Λ, $\|\mu'\| < 2$, and $\|f_i * \nu * \mu'\| < \varepsilon$. Setting

$$\mu = \mu(\{f_i\}_i\,, \varepsilon) = \delta - \nu * \mu'$$

we see that $\|\mu\| \leq 1 + 2\|\nu\|$, $\hat{\mu} = 0$ on some neighborhood of E, and that $\|f_i - f_i * \mu\| < \varepsilon$ for all i. Therefore the family $\{\mu(\{f_i\}_i\,, \varepsilon)\}$ has all the required properties (c), (d), and (e) in Definition 2. This proves that E is a strong Ditkin set, and hence (iii) implies (i). The proof is now established.

Remark. The part »(iii) $\Rightarrow$ (i)» of Theorem 7 is (essentially) due to Gilbert [3], although our proof seems to be simpler than his.

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