On the synthesis problem for orbits of Lie groups in $\mathbf{R}^n$

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1.

Let $F_1(\mathbf{R}^n)$ be the subalgebra of all functions in $C_\infty(\mathbf{R}^n)$, which are Fourier transforms of functions in $L_1(\mathbf{R}^n)$. The elements of the dual $PM(\mathbf{R}^n)$ of $F_1(\mathbf{R}^n)$ are called pseudomeasures. Considered as distributions the pseudomeasures are just the Fourier transforms of essentially bounded measurable functions.

For a closed ideal I in $F_1(\mathbf{R}^n)$ the cospectrum of I is defined as the set of common zeros of the functions in I.

For a closed subset E of $\mathbf{R}^n$, we denote by $j(E)$ the smallest, by $k(E)$ the biggest closed ideal with cospectrum E.

It is well known that the quotient algebra

$$r(E) := k(E)/j(E)$$

is a radical algebra.

For certain manifolds it has been proved that $r(E)$ is even nilpotent. E is said to be of spectral synthesis if $r(E)=\{0\}$.

So C. Herz [4] showed for the circle S^1 that $r(S^1)=\{0\}$, and Varopoulos [11] proved more generally that for the $(n-1)$-dimensional sphere S^{n-1} the algebra $r(S^{n-1})$ is nilpotent of degree $\left[\frac{n+1}{2}\right]$. F. Lust [8] discovered that $r(E)=\{0\}$ for each closed orbit of a one-parameter group.

Y. Domar [1] proved that for compact subsets E of $(n-1)$-dimensional submanifolds of $\mathbf{R}^n$ with non-vanishing curvature the algebra $r(E)$ is nilpotent of degree $\left[\frac{n+1}{2}\right]$ provided that E satisfies a certain technical condition. He also gave an example [2] of a C^∞-curve in $\mathbf{R}^2$ without spectral synthesis, thus showing that the curvature condition is essential. In this paper we investigate orbits in $\mathbf{R}^n$ under the action of a general connected subgroup G of $GL(n, \mathbf{R})$. Among other things we shall prove that if E is a closed orbit of dimension m, then $r(E)^{\left[\frac{m}{2}+1\right]}=\{0\}$. This follows easily from a more general theorem (Theorem 1) about certain compact subsets of a general, not necessarily closed orbit $\omega=Gx_0=\{gx_0;\ g\in G\}$, $x_0\in\mathbf{R}^n$, of G in $\mathbf{R}^n$. The result on $r(E)$, E closed, was already conjectured by H. Leptin in [6].

The proof of Theorem 1 follows the line of the proof Domar gave for his main theorem in [1]. We also thank Prof. Domar for his interest and valuable comments on the subjects of this paper.

2.

In the following we denote by G a connected Lie group acting continuously on $\mathbf{R}^n$ by linear transformations.

For $x\in\mathbf{R}^n$ let

$$\omega = Gx$$

be the orbit through x and $H:=H_x$ the stabilizer of x.

Transferring the C^∞-structure of G/H to ω via the canonical mapping we consider ω as a regular submanifold of $\mathbf{R}^n$.

While for closed orbits the topology defined by the C^∞-structure is equal to the topology induced by $\mathbf{R}^n$, this is not true in general. (see Helgason [3], Ch. II, Ex.; Hochschild [5]). Nevertheless both topologies induce the same topology on any compact subset of the manifold ω.

The following definitions carry over the notion of the "restricted cone property" used by Domar in [1] to our situation.

A subset P of G is called an *approximation set*, if the identity e of G lies in the closure of the interior of P.

A compact subset E of ω is said to have the *convolution property*, if for every $x\in E$ there exist a neighbourhood U_x of x in E and an approximation set P_x, such that $P_x\overline{U}_x\subset E$.

Now we can state our main result:

Theorem 1. *If ω is an m-dimensional orbit of G and E a compact subset of ω having the convolution property, then*

$$r(E)^{[m/2+1]} = \{0\}.$$

This theorem contains for $m=1$ the result of F. Lust [8]. Of course for $n>1$ the degree of nilpotency of $r(E)$ may be smaller than $\left[\frac{m}{2}+1\right]$, e.g. $r(E)=\{0\}$ for every flat orbit E. The determination of the exact degree of nilpotency of $r(E)$ would require a Littman type estimate of the Fourier—Stieltjes transforms of measures supported by E (see Littman [7]).

3.

I. Now we shall prove Theorem 1.

In the following we denote by E a compact subset of ω having the convolution property.

The following definitions are essentially due to Domar [1]:

1°. Let $B(E)$ denote the space of all bounded measures on $\mathbf{R}^n$ with support in E. $B(E)$ can be considered as a subspace of $PM(\mathbf{R}^n)$.

2°. For every integer $i\geqq 1$, let $J_i(E)$ denote the space of all test functions in $\mathscr{D}(\mathbf{R}^n)$, which vanish on E together with all partial derivatives up to the order $i-1$.

3°. For every integer $i\geqq 1$, let $C_i(E)$ denote the annihilator of $J_i(E)$ in $PM(\mathbf{R}^n)$.

The closure of $B(E)$ is just the annihilator of $k(E)$ in $PM(\mathbf{R}^n)$, where the closure is taken in the weak* topology $\sigma(PM(\mathbf{R}^n), F_1(\mathbf{R}^n))$. Obviously we have $J_1(E)^i\subset \subset J_i(E)$, hence $\overline{J_1(E)^i}\subset\overline{J_i(E)}$ for every integer $i\geqq 1$.

The following theorem is an easy generalization of Domar's theorem 2.9.4° in [1]:

Theorem 2. *Let M be a smooth, m-dimensional submanifold of $\mathbf{R}^n$ and let E be a compact subset of M. Furthermore suppose that $f\in F_1(\mathbf{R}^n)$ has compact support, $T\in PM(\mathbf{R}^n)$ has its support in E and*

$$|\hat{T}(x)| = \mathcal{O}(|x|^\tau)$$

as $x\to\infty$, where $-\frac{n}{2}<\tau\leqq 0$. Let

$$|f|_{\varepsilon,\infty} := \sup\{|f(x)|;\ \operatorname{dist}(x, E)\leqq 2\varepsilon\}.$$

Then

$$\langle T,f\rangle = \mathcal{O}(\varepsilon^{-\tau-(m/2)}|f|_{\varepsilon,\infty})$$

as $\varepsilon\to 0$.

This result goes back to Beurling, Pollard and Herz. We omit the proof, because the proof given by Domar in [1] can be adopted with only slight changes.

We apply Theorem 2 to functions $f\in J_i(E)$. Using Taylor expansion of f around boundary points of E, we derive easily the estimate

$$|f|_{\varepsilon,\infty} = \mathcal{O}(\varepsilon^i).$$

For $T\in PM(\mathbf{R}^n)$ clearly

$$|\hat{T}(x)| = \mathcal{O}(1) \quad \text{as} \quad |x|\to\infty.$$

Thus Theorem 2 yields

$$\langle T, f\rangle = 0$$

for all $f\in J_{[m/2+1]}(E)$ and $T\in PM(\mathbf{R}^n)$ with supp $T\subset E$.

Consequently, by Hahn—Banach theorem, we have proved

$$\overline{J_{[m/2+1]}(E)} = j(E),$$

because the annihilator of $j(E)$ consists precisely of all pseudomeasures T with supp $T\subset E$.

Remembering that $\overline{J_1(E)}^{[m/2+1]}\subset\overline{J_{[m/2+1]}}$, we have shown:

Corollary 1. $\overline{J_1(E)}^{[m/2+1]}=j(E)$.

To prove $r(E)^{[m/2+1]}=\{0\}$, it is now sufficient to show

$$\overline{J_1(E)} = k(E)$$

or, equivalently,

$$\overline{B(E)} = C_1(E).$$

II. The aim of this section is the proof of the following proposition, which will also finish the proof of Theorem 1:

Proposition 1. *If E is a compact subset of ω which has the convolution property, then every $T\in C_1(E)$ is the weak* limit of a sequence $\{T_\nu\}_\nu$ of measures in $B(E)$.*

We shall use the following two lemmas. The first one is a localisation lemma due to Domar.

Lemma 1. *Assume that every point $x\in E$ has an open neighbourhood $U_x\subset\mathbf{R}^n$ such that every $T\in C_1(E\cap\overline{U}_x)$ is the weak* limit of a sequence $\{T_\nu\}_\nu\subset B(E)$. Then every $S\in C_1(E)$ is the weak* limit of a sequence $\{S_\nu\}_\nu$ of measures in $B(E)$.*

The proof can be found in Domar [1].

Lemma 2. *If E is a compact subset of ω which has the convolution property, then for every $x\in E$ and every neighbourhood V of x in $\mathbf{R}^n$ there exist neighbourhoods V_1 and V_2 of x in $\mathbf{R}^n$ and an approximation set $P\subset G$, such that*

$$\overline{V}_1\subset V_2\subset\overline{V}_2\subset V \quad \textit{and} \quad P\cdot(\overline{V}_1\cap E)\subset V_2\cap E.$$

Proof. There exist an open neighbourhood θ of x in E and an approximation set $P'\subset G$, such that $P'\cdot\bar{\theta}\subset E$. We have $\theta=U\cap E$ for some neighbourhood U of x in $\mathbf{R}^n$. We choose a neighbourhood V_2' of x in E, such that $V_2'\subset\overline{V_2'}\subset V$, and set $V_2:=U\cap V_2'$, hence $\overline{V}_2\subset\overline{U}\cap\overline{V_2'}\subset V$. There exist a neighbourhood W of e in G and a neighbourhood V_x of x in $\mathbf{R}^n$, such that $W\cdot V_x\subset V_2$. Choosing a neighbourhood V_1 of x in $\mathbf{R}^n$ such that $\overline{V_1}\subset V_x$, we get $\overline{V_1}\subset U$. For $P:=P'\cap W$ we easily obtain $P(\overline{V_1}\cap E)\subset V_2\cap E$.

Let $x_0\in E$. Lemma 2 allows us to choose suitable neighbourhoods U_{x_0}, V_2 and V of x_0 in $\mathbf{R}^n$, an approximation set P and an open subset $\Omega\subset\omega$, for which $\Omega\cap E=V\cap E$, such that

$$\overline{U_{x_0}}\subset V_2\subset\overline{V_2}\subset V,\quad P\cdot(\overline{U_{x_0}}\cap E)\subset V_2\cap E,$$

and such that Ω is covered both by a chart defined by the exponential mapping and by a chart (Ω', Ψ), where Ψ is of the form

$$\Psi: x'\to(x', \psi(x')),\ x'\in\Omega'\subset\mathbf{R}^m,\ \psi\in C^\infty(\Omega', \mathbf{R}^{n-m}).$$

Now, by Lemma 1, it suffices to show that every $T\in C_1(E\cap\overline{U_{x_0}})$ is the weak* limit of a sequence $\{T_\nu\}_\nu\subset B(E)$. We shall prove this by using regularisations of pseudomeasures.

The group G acts continuously on $F_1(\mathbf{R}^n)$ by isometries, explicitly

$$f_g(x):=f(g^{-1}x),\quad\text{if}\quad f\in F_1(\mathbf{R}^n),\quad g\in G,\quad x\in\mathbf{R}^n.$$

Let $M(G)$ denote the algebra of bounded Radon measures on G. For $\mu\in M(G)$, $f\in F_1(\mathbf{R}^n)$ let

$$f_\mu:=\int_G f_g\,d\mu(g).$$

It is clear that $F_1(\mathbf{R}^n)$ may be regarded as a Banach $M(G)$-module. Choosing a fixed left Haar measure dg on G we identify a function $f\in L^1(G)$ with the measure $f\,dg$. We define an action of $\mu\in M(G)$ on $PM(\mathbf{R}^n)$ by

$$\langle T_\mu, f\rangle:=\langle T, f_\mu\rangle\quad\text{for}\quad T\in PM(\mathbf{R}^n),\quad f\in F_1(\mathbf{R}^n).$$

If μ_ν is a sequence of positive measures with total mass one, such that $\mu_\nu(\complement U)\to 0$ for every open neighbourhood U of $e\in G$, then

$$T_{\mu_\nu}\to T\quad\text{in the weak* topology for every}\quad T\in PM(\mathbf{R}^n).$$

Now let $T\in C_1(E\cap\overline{U_{x_0}})$, and let P be the approximation set choosen before.

Choose a sequence $\{\varphi_\nu\}_\nu$ of functions in $\mathscr{D}(G)$ such that

$$\varphi_\nu\geqq 0,\ \int\varphi_\nu\,dg=1,\ \operatorname{supp}\varphi_\nu\subset P^{-1}\quad\text{for every}\quad\nu,$$

and such that $\int_{\complement U}\varphi_\nu\,dg\to 0$ for every open neighbourhood U of $e\in G$. Thus we have $T=\lim T_{\varphi_\nu}$.

To prove Proposition 1 and hence Theorem 1 it remains to show that $T_{\varphi_\nu}\in B(E)$.

In the following we restrict our considerations to functions and distributions on $\Omega'\times\mathbf{R}^{n-m}$, because for every $T\in C_1(E\cap\overline{U_{x_0}})$ we have $\operatorname{supp} T\subset\Omega'\times\mathbf{R}^{n-m}$ and $\operatorname{supp} T_{\varphi_\nu}\subset\Omega'\times\mathbf{R}^{n-m}$.

We define a diffeomorphism T: $\Omega'\times\mathbf{R}^{n-m}\to\Omega'\times\mathbf{R}^{n-m}$ by

$$\Gamma(x',x''):=(x',\psi(x')-x'').$$

For $\varphi\in\mathcal{D}(\Omega'\times\mathbf{R}^{n-m})$ we set $\varphi^\Gamma:=\varphi\circ\Gamma$, and for $S\in\mathcal{D}'(\Omega'\times\mathbf{R}^{n-m})$ we define S^Γ by

$$\langle S^\Gamma,\varphi\rangle:=\langle S,\varphi^\Gamma\rangle.$$

Let $T\in C_1(E\cap\overline{U_{x_0}})$; then $\operatorname{supp} T^\Gamma\subset\Omega'\times\{0\}\subset\Omega\times\mathbf{R}^{n-m}$.

For a multi-index $\alpha\in\mathbf{N}^{n-m}$ we denote by $D^\alpha_{x''}$ the partial derivation

$$\frac{\partial^{\alpha_1}}{\partial^{\alpha_1}x''_1}\cdots\frac{\partial^{\alpha_{n-m}}}{\partial^{\alpha_{n-m}}x''_{n-m}}$$

with respect to the decomposition $x=(x',x'')$ for $x\in\mathbf{R}^n$, $x'\in\mathbf{R}^m$, $x''\in\mathbf{R}^{n-m}$.

Then it is wellknown (Schwartz [10]) that for some unique distributions $t_\alpha\in\mathcal{D}'(\Omega')$ with

$(\operatorname{supp} t_\alpha)\times\{0\}\subset\operatorname{supp} T^\Gamma\subset\Psi^{-1}(E\cap\overline{U_{x_0}})\times\{0\}$ we have

$$T^\Gamma=\sum\nolimits_{|\alpha|\leq p}D^\alpha_{x''}(\bar{t}_\alpha),$$

where $\bar{t}_\alpha$ denotes the extension of t_α onto $\Omega'\times\mathbf{R}^{n-m}$, and p denotes the order of T^Γ.

Now choose a multi-index $\beta\in\mathbf{N}^{n-m}$, $0<|\beta|\leq p$, and a test function $\xi\in\mathcal{D}(\mathbf{R}^{n-m})$, $\xi\equiv1$ in a neighbourhood of the origin of $\mathbf{R}^{n-m}$. For $\varrho\in\mathcal{D}(\Omega')$ we define $\varrho_\beta\in\mathcal{D}(\Omega'\times\mathbf{R}^{n-m})$ by

$$\varrho_\beta(x',x''):=\xi(x'')\frac{x''^\beta}{\beta!}\varrho(x').$$

An easy computation shows that $\langle T,\varrho_\beta^\Gamma\rangle=\langle t_\beta,\varrho\rangle$. But $\varrho_\beta^\Gamma\in J_1(E\cap\overline{U_{x_0}})$, hence $\langle T,\varrho_\beta^\Gamma\rangle=0$, and thus we have $\langle t_\beta,\varrho\rangle=0$ for all $\varrho\in\mathcal{D}(\Omega')$. This shows that we have

$$T^\Gamma=\bar{t}_0.$$

Now we define a distribution t on the submanifold Ω of ω by

$$\langle t,\varrho\rangle:=\langle t_0,\varrho\circ\Psi\rangle.$$

Then it is easily seen that T is the extension of t, i.e.

$$\langle T,f\rangle=\langle t,f|_\Omega\rangle\quad\text{for all}\quad f\in\mathcal{D}(\Omega'\times\mathbf{R}^{n-m}).$$

In the following we use another chart constructed from the exponential mapping of the Lie group G.

Let $\mathfrak{g}$ be the Lie algebra of G, $\mathfrak{h}$ the Lie algebra of $H:=\operatorname{Stab}_G x_0$, and let $\mathfrak{m}$ be a subspace of $\mathfrak{g}$ such that $\mathfrak{g}=\mathfrak{m}\oplus\mathfrak{h}$. Choose a basis $X_1,\dots,X_m$ of $\mathfrak{m}$, and let

$\tilde{\Psi}$ be defined by

$$\tilde{\Psi}(s_1, \dots, s_m) := \exp\left(\sum s_i X_i\right) x_0$$

on an open neighbourhood $\tilde{\Omega}$ of the origin in $\mathbf{R}^m$, such that $(\tilde{\Omega}, \tilde{\Psi})$ is a chart for Ω.

Since t is a distribution with compact support contained in Ω, there is an $N \in \mathbf{N}$ and a constant $C > 0$, such that

$$|\langle t, \varrho\rangle| \leqq C \sum_{|\alpha| \leqq N} \sup_{s \in \tilde{\Omega}} |D^\alpha(\varrho \circ \tilde{\Psi})(s)|$$

for all $\varrho \in \mathcal{D}(\Omega)$.

We choose a function $\xi \in \mathcal{D}(\Omega' \times \mathbf{R}^{n-m})$, $\xi \equiv 1$ on a neighbourhood of $\overline{U_{x_0}} \in E$ in $\mathbf{R}^n$. Since $\operatorname{supp} T \subset \overline{U_{x_0}} \cap E$, we get:

$$\begin{aligned} |\langle T_{\varphi_\nu}, f\rangle| &= |\langle T, f_{\varphi_\nu}\rangle| = |\langle T, \xi \cdot f_{\varphi_\nu}\rangle| \\ &= |\langle t, (\xi \cdot f_{\varphi_\nu})|_\Omega\rangle| \\ &\leqq C \sum_{|\alpha| \leqq N} \sup_{s \in \tilde{\Omega} \cap \operatorname{supp} \xi} |D^\alpha(\xi f_{\varphi_\nu} \circ \tilde{\Psi})(s)|. \end{aligned}$$

From this we get, with a new constant $C' > 0$:

$$|\langle T_{\varphi_\nu}, f\rangle| \leqq C' \sum_{|\alpha| \leqq N} \sup_{s \in \tilde{\Omega} \cap \operatorname{supp} \xi} |D^\alpha(f_{\varphi_\nu} \circ \tilde{\Psi})(s)|.$$

Since

$$\begin{aligned} f_{\varphi_\nu} \circ \tilde{\Psi}(s) &= \int f\left(g^{-1}\left(\exp \sum s_i X_i\right) x_0\right) \varphi_\nu(g)\, dg \\ &= \int f(g^{-1} x_0)\, \varphi_\nu\left(\left(\exp \sum s_i X_i\right) g\right) dg, \end{aligned}$$

we have

$$D_s^\beta(f_{\varphi_\nu} \circ \tilde{\Psi})(s) = \int f(g^{-1} x_0)\, D_s^\beta \varphi_\nu\left(\left(\exp \sum s_i X_i\right) g\right) dg,$$

hence

$$|D_s^\beta(f_{\varphi_\nu} \circ \tilde{\Psi})(s)| \leqq \sup_{x \in \omega} |f(x)| \int \left|D_s^\beta \varphi_\nu\left(\left(\exp \sum s_i X_i\right) g\right)\right| dg.$$

Since the functions $s \to \int |D_s^\beta \varphi_\nu ((\exp \sum s_i X_i) g)|\, dg$ are continuous, we finally have the estimate

$$|\langle T_{\varphi_\nu}, f\rangle| \leqq C \sup_{x \in \omega} |f(x)| \leqq C |f|_\infty$$

with a constant C independent of f. Hence T_{φ_ν} is a Radon measure. Furthermore $\operatorname{supp} T_{\varphi_\nu} \subset (\operatorname{supp} \varphi_\nu)^{-1} \cdot (\operatorname{supp} T) \subset P \cdot (\overline{U_{x_0}} \cap E) \subset E$, and thus we have proved that T_{φ_ν} is contained in $B(E)$.

4.

In this section we shall prove that each point $x_0 \in \omega$ has a local base consisting of compact sets which have the convolution property.

Let V be a finite dimensional vector space. A closed set $K \subset V$ is said to have the *restricted cone property* at a point $y_0 \in K$, if there exist a neighbourhood U of

y_0 in V and a cone P defined by

$$P := \{y \in V;\ (1-\varkappa)|y| \leqq \langle y, y_1 \rangle \leqq \varkappa\},$$

where $0<\varkappa<1$, $y_1 \in V$, $|y_1|=1$, such that $y+P \subset K$ for every $y \in \overline{K \cap U}$ (see Domar [1]).

In the next lemma we reformulate the restricted cone property in a form which will be more useful in the following:

Lemma 3. *A compact set $K \subset V$ has the restricted cone property at every point $x \in K$, if and only if there exists a constant $C>0$, such that for every $v \in K$ there exist a neighbourhood U of v in V and a cone P, such that*

$$\operatorname{dist}(y+z, V \setminus K) \geqq C|y| \quad \textit{for all} \quad y \in P,\ z \in \overline{U \cap K}.$$

Proof. Let $P_\varkappa = \{v \in V;\ (1-\varkappa)|v| \leqq \langle v, y_1 \rangle \leqq \varkappa\}$, $0<\varkappa<1$, $|y_1|=1$. For fixed y_1, there exists a $C_\varkappa>0$, such that $\operatorname{dist}(y, V \setminus P_\varkappa) \geqq C_\varkappa |y|$ for all $y \in P_{\varkappa/2}$, as geometrical considerations show.

Now let $v \in K$. There exist a neighbourhood U_v of v in V and a cone $P_\varkappa$, depending on v, such that $z+P_\varkappa \subset K$ for every $z \in \overline{K \cap U_v}$. Then we have

$$\begin{aligned} \operatorname{dist}(z+y, V \setminus K) &\geqq \operatorname{dist}(z+y, z+V \setminus P_\varkappa) \\ &= \operatorname{dist}(y, V \setminus P_\varkappa) \\ &\geqq C_\varkappa |y| \end{aligned}$$

for every $z \in \overline{K \cap U_v}$, $y \in P_{\varkappa/2}$.

Using an obvious compactness argument we easily find a $C>0$ with $C_\varkappa \geqq C$ uniformly on K. This proves Lemma 3.

Now we choose an inner product $\langle \cdot, \cdot \rangle$ on the Lie algebra $\mathfrak{g}$ of G. Let $\mathfrak{m}$ denote the orthogonal complement of $\mathfrak{h}$ with respect to the chosen inner product. Then

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}.$$

Let π denote the canonical homomorphism from G onto G/H.

Proposition 2. *If K is a compact subset in $\mathfrak{m}$, which has the restricted cone property at every point $z \in K$ with respect to $\mathfrak{m}$, then there exists a $t_0>0$, such that $\pi(\exp tK)$ has the convolution property for every $0<t \leqq t_0$.*

Proof. We denote by φ_1 (resp. φ_2) the canonical coordinates of the first (resp. second) kind, i.e. for $X = X_1 + X_2$, $X_1 \in \mathfrak{h}$, $X_2 \in \mathfrak{m}$, we have $\varphi_1(X) = \exp X$, $\varphi_2(X) = \exp X_1 \exp X_2$.

There is a neighbourhood U of $0 \in \mathfrak{g}$, such that φ_i, $i=1, 2$, maps U homeomorphically onto a neighbourhood of $e \in G$. Also we may assume that $\pi \circ \exp|_{\mathfrak{m}}$ maps $U \cap \mathfrak{m}$ homeomorphically onto an open set in G/H. If U is sufficiently small, by the Campbell—Hausdorff formula there exists a constant $C_1>0$, such that

$$\exp Y \exp Z = \exp\big(Y+Z+\varepsilon(Y,Z)\big) \quad \text{for all} \quad Y, Z\in U,$$

where ε is a function from $U\times U$ to $\mathfrak{g}$, which satisfies the estimate

$$|\varepsilon(Y,Z)| \leqq C_1|Y||Z|.$$

Furthermore, by Lemma 3, there exist a constant $C>0$, bounded open subsets U_j, $j=1, \dots, k$, in $\mathfrak{m}$ and cones $P'_j\subset\mathfrak{m}$, such that

$$K\subset\bigcup_j U_j, \quad P'_j+\overline{U_j\cap K}\subset K,$$

and

$$\operatorname{dist}(Y'+Z, \mathfrak{m}\setminus K)\geqq C|Y'| \quad \text{for all} \quad Y'\in P'_j$$

and $Z\in U_j\cap K$.

Now we choose $r, r'\in\mathbf{R}$, $0<r'<r\leqq\min\left\{\frac{C}{26C_1}, \frac{1}{2C_1}\right\}$, such that $B_r:=\{X\in\mathfrak{g};\ |X|<r\}\subset U$ and $\varphi_2^{-1}(\varphi_1(B_{r'})^3)\subset B_r$. Furthermore we fix a $t_0>0$, such that

$$t_0\cdot\big(\bigcup_j U_j\big)\subset B_r \quad \text{and} \quad t_0\cdot P'_j\subset B_{r'}.$$

In the following we shall show that $\pi(\exp tK)$ has the convolution property in G/H for every $0<t<t_0$.

Considering tK, tU_j instead of K, U_j, we may assume $t=1$. We define $\tilde{K}:=\pi(\exp K)$, $\tilde{U}_j:=\pi(\exp U_j)$ and, if $P'_j=\{X\in\mathfrak{m};\ (1-\varkappa_j)|X|\leqq\langle X, Y_j\rangle\leqq\varkappa_j\}$, let $P_j:=\{X\in\mathfrak{g};\ (1-\varkappa_j)|X|\leqq\langle X, Y_j\rangle\leqq\varkappa_j\}$, and define $Q_j:=\exp P_j$. Q_j is an approximation set.

We shall prove that $Q_j\cdot\overline{\tilde{U}_j\cap\tilde{K}}\subset\tilde{K}$, from which it follows, that $\tilde{K}$ has the convolution property.

For some $Y\in P_j$ and $Z\in\overline{U_j\cap K}$ let

$$y=\exp Y\in Q_j, \quad \tilde{z}=\pi(\exp Z)\in\overline{\tilde{U}_j\cap\tilde{K}}.$$

We have $Y=R+S$ for some $R\in\mathfrak{h}$ and $S\in\mathfrak{m}$.

Obviously $S\in P'_j$.

We get $y\tilde{z}=\pi(y\cdot\exp Z)=\pi\big(\exp(R+S)\exp Z\exp(-R)\big)$, and

$\exp(R+S)\exp Z\exp(-R)=\exp\big(S+Z+\varepsilon(R+S, Z)+\varepsilon(R+S+Z+\varepsilon(R+S,Z), -R)\big)$. Moreover

$$\exp(R+S)\exp Z\exp(-R)\in\varphi_1(B_{r'})^3\subset\varphi_2(B_r),$$

that means: there exist $V\in\mathfrak{m}$ and $T\in\mathfrak{h}$, such that

$$\exp(R+S)\exp Z\exp(-R) = \exp(S+Z+V)\exp T =$$
$$= \exp\big(S+Z+V+T+\varepsilon(S+Z+V, T)\big).$$

From this it follows that

$$\varepsilon(R+S,Z)+\varepsilon\big(R+S+Z+\varepsilon(R+S,Z),\,-R\big) = V+T+\varepsilon(S+V+Z,T).$$

An easy estimate gives:

$$(*)\qquad \big|\varepsilon(R+S,Z)+\varepsilon\big(R+S+Z+\varepsilon(R+S,Z),\,-R\big)\big| \leqq 4C_1 r|Y|$$

and

$$(**)\qquad |\varepsilon(S+Z+V,T)| \leqq C_1|S+Z+V||T| \leqq C_1|S+Z+V+T||T| \leqq C_1 r|T|.$$

We write $\varepsilon(S+Z+V,T)=\varepsilon_{\mathfrak{k}}+\varepsilon_{\mathfrak{m}}$, $\varepsilon_{\mathfrak{k}}\in\mathfrak{k}$, $\varepsilon_{\mathfrak{m}}\in\mathfrak{m}$. Then, using (*) and (**),

$$|T| \leqq |T+\varepsilon_{\mathfrak{k}}|+|\varepsilon_{\mathfrak{k}}| \leqq |T+V+\varepsilon(S+Z+V,T)|+|\varepsilon(S+Z+V,T)|$$
$$\leqq 4C_1 r|Y|+\tfrac{1}{2}|T|,$$

hence $|T|\leqq 8C_1 r|Y|$.

Thus we get

$$|V| \leqq |V+T+\varepsilon(S+Z+V,T)|+|T|+|\varepsilon(S+Z+V,T)|$$
$$\leqq 4C_1 r|Y|+8C_1 r|Y|+C_1 r|Y|$$
$$\leqq \frac{C}{2}|Y|.$$

We may assume that $\varkappa_j<\frac{1}{3}$ for every j. Then $|Y|\leqq\frac{3}{2}|S|$, hence

$$|V| \leqq \tfrac{3}{4}C|S|.$$

Thus we get $S+Z+V\in K$, since $\operatorname{dist}(S+Z,\mathfrak{m}\setminus K)\geqq C|S|$.

This gives the desired result

$$Y\tilde{Z} = \pi\big(\exp(S+Z+V)\exp T\big) = \pi\big(\exp(S+Z+V)\big)\in\tilde{K}.$$

5.

The results of the preceding sections can be applied to the case of closed orbits to get the following theorem:

Theorem 2. *If $\omega\subset\mathbf{R}^n$ is a closed m-dimensional orbit under a linear action of a connected Lie group G, then*

$$r(\omega)^{[m/2+1]} = \{0\}.$$

Proof. For every point $x \in \omega$ there exists, by Proposition 2, a compact neighbourhood U_x of x in ω which has the convolution property. By Theorem 1 we know that $r(U_x)^{[m/2+1]} = \{0\}$.

Because ω is closed in $\mathbf{R}^n$, U_x is also a neighbourhood of x in the topology induced by $\mathbf{R}^n$ on ω. This shows that $r(\omega)^{[m/2+1]} = \{0\}$, since the nilpotency of $r(\omega)$ is a local property as in the case of Wiener sets (see Reiter [9], Chap. II).

References

1. Domar, Y., On the spectral synthesis problem for $(n-1)$-dimensional subsets of $\mathbf{R}^n$. *Ark. Mat.* **9** (1971), 23—37.
2. Domar, Y., A C^∞-curve of spectral non-synthesis. *Mathematika* **24** (1977), 189—192.
3. Helgason, S., *Differential Geometry and Symmetric Spaces.* New York, 1962.
4. Herz, C. S., Spectral synthesis for the circle. *Ann. of Math.* **68** (1958), 709—712.
5. Hochschild, G., *The structure of Lie groups.* San Francisco, 1965.
6. Leptin, H., Lokalkompakte Gruppen mit symmetrischen Algebren. *Symposia Math.* **22** (1977), 267—281.
7. Littman, W., Fourier transforms of surface-carried measures and differentiability of surface averages. *Bull. Amer. Math. Soc.* **69** (1963), 766—770.
8. Lust, F., Le probleme de la synthese et de la p-finesse pour certain orbites des groupes lineares dans $A_p(\mathbf{R}^n)$. *Studia Math.* **39** (1971), 17—28.
9. Reiter, H., *Classical harmonic analysis and locally compact groups.* Oxford, 1968.
10. Schwartz, L., *Theorie des distributions.* Paris, 1966.
11. Varopoulos, N. Th., Spectral synthesis on spheres. *Proc. Cambridge Phil. Soc.* **62** (1966), 379—387.

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