Oscillating kernels that map H^1 into L^1

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0. Introduction

Let Ω and f be two Lebesgue measurable functions on $\mathbf{R}^n$. Then the equation

$$\Omega * f(x) = \int_{\mathbf{R}^n} \Omega(x-t) f(t)\, dt$$

(whenever the integral exists) defines the convolution transform of Ω and f. In that case, we set $Tf = \Omega * f$. W. B. Jurkat and myself have been working on the (L^p, L^q) mapping problem for the kernels $\Omega(t) = b(t) e^{ia(t)}$, that is to determine all pairs (p, q) for which $\|Tf\|_q \leqq B\|f\|_p$ for $f \in L_0^\infty$ and B is a positive constant independent of f. For example also see [2], [4], [5], [6], [7], [9], [11], [13], [14], [15], [16], [17], [18], [19], [20]. Since $(Tf)^\wedge = \Omega^\wedge f^\wedge$, where $\Omega^\wedge$ is the fourier transform of Ω, this problem is closely related to the corresponding multiplier problem.

This paper represents a first step in solving the (L^p, L^q) mapping problem stated above. Here I show, (see Theorems 3 and 4 in **3**) that for a general class of functions $a(t)$ the kernels

$$K(t) = \frac{e^{ia(t)}}{1+|t|}, \quad t \in \mathbf{R} \tag{0}$$

map H^1 into L^1 continuously. As an application of these results it is shown in the Corollary in **3** (also see [11; Theorem 3]) that the functions $a(t) = |t|^a$, $0 < a$, $a \neq 1$ belong to this class. In Theorem 6 (in **4**) it is shown that for the functions $a(t) = t(\log|t|)^\eta$, $0 < \eta < 1$, the kernels defined by (0) do not map H^1 into L^1 continuously; although, these latter kernels do map L^p into L^p for all $1 < p < \infty$ [4; Cor. 1.16]. This should indicate to the reader how delicate these results are.

This work also resolved a question that I had been carrying around for some time. Does the class of kernels $K(t)$ where $|K(t)| \leqq B|t|^{-1}$ and $\|K^\wedge\|_\infty < \infty$ have the same mapping properties? And as explained above some of these kernels map H^1 into L^1 and others do not.

Another problem that has interested me for a while is to fully understand the complex method as first discovered by Hirschman—Stein [21] and developed further by A. P. Calderón [1] and then by Macias [12] (the Macias result is the one we use here, see **1**). What I believe may be true is,

"If $\{\Omega_x\}_{0\leqq x\leqq 1}$ is a real analytic family with

$$\|\Omega_0*f\|_1\leqq B\|f\|_{H^1}\quad and\quad \|(\Omega_1)^\wedge\|_\infty<\infty,$$

then

$$\|\Omega_x*f\|_p\leqq B\|f\|_p\quad for\quad \frac{1}{p}=1-\frac{x}{2}$$

and $0<x\leqq 1$, B a positive constant independent of f".

At this point I know how to solve the problem above when Ω_0 is a regular kernel (definition is in **2**) and Ω_1 satisfies a straightforward type of integral condition. This last result shall be done in a subsequent work. The Theorem 5 (in **4**) plays a key role in obtaining this result.

Recently, Per Sjölin [19] has independently solved some of these mapping problems.

1. Preliminaries

In this paper, I am concerned with determining those kernels $\Omega(t)$ that map H^1 into L^1. Earlier work with W. B. Jurkat [8; Thm. I] suggests I should assume that $|\Omega(t)|\leqq B|t|^{-1}$, $t\in\mathbf{R}$. And so I consider kernels defined by (0), i.e. $K(t)=\dfrac{e^{ia(t)}}{1+|t|}$ where $a(t)$ is a real-valued function ($t\in\mathbf{R}$). In this paper, I only discuss the cases where $a(t)$ stays bounded away from zero as $t\to\infty$ and for which $\dfrac{e^{ia(t)}-1}{t}$ is locally integrable. Thus, with these kernels the bad behavior occurs only at infinity. It follows from this that we could assume that $K(t)=0$ for $t<0$ (or for $t>0$) since for these kernels there is no cancellation across the origin. And so in the next step we set $K(t)=k(t)g(t)$ with $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$ for $t\geqq 0$. Now this paper is concerned with giving conditions on $a(t)$, $k(t)$ and $g(t)$ that imply $K(t)$ maps H^1 into L^1. For example, see Theorems 3, 4 (in **3**) and Lemmas 7, 8 and 9 (in **5**).

The result which is most helpful to us is this result of R. Macias [12] on analytic families of linear operators. We just need the following special case of his result.

Theorem A. *Suppose $\{T_z\}$ is an analytic family of convolution operators ($T_zf=\Omega_z*f$) satisfying*

$$\|\Omega_{iy}*f\|_1\leqq A_0(y)\|f\|_{H^1}\quad and\quad \|\Omega_{1+iy}\|_{2,2}=A_1(y)<\infty$$

for all $y\in(-\infty,\infty)$ *and* $f\in L_0^\infty(R)$, *where* $\log A_j(y)\leqq c_j e^{d_j|y|}$, $c_j>0$, $0<d_j<\pi$, $j=0,1$. *If* $\frac{1}{p}=1-\frac{x}{2}$ $(0<x\leqq 1)$, *then*

$$\|\Omega_x * f\|_p \leqq B\|f\|_p$$

where B *depends on* c_0, c_1, d_0, d_1, x, *but is independent of* f.

Let $Tf=\Omega * f$, we set $\|\Omega\|_{2,2}=\sup_{\|f\|_2\leqq 1}\|Tf\|_2$ where $f\in L_0^\infty$, and when $\|\Omega\|_{2,2}<\infty$ we say that $\Omega\in L_2^2$.

The letter c generally stands for an absolute constant. The letters $B_1, B_2, \dots$ stand for positive constants and we use the letter B generically.

By H^1, see [3; 591—3], we mean the set of all f for which $f(t)=\sum\lambda_j b_j(t)$ where $\sum|\lambda_j|<\infty$ and $b_j(t)$ is a $(1,2)$ atom. And $\|f\|_{H^1}=\inf\sum|\lambda_j|$, where the infimum ranges over all such decompositions of f.

The function b is said to be a $(1,2)$ atom if

(i) the support of b is contained in an interval I,

(ii) $\int|b(t)|^2\,dt\leqq|I|^{-1}$

and

(iii) $\int b(t)\,dt=0$.

2. General results for regular kernels that map H^1 into L^1

Here, we give sufficient conditions on K in order to show that

$$\|K*f\|_1 \leqq B\|f\|_{H^1}, \tag{1}$$

B a positive constant independent of f. In order to prove (1) it suffices to prove that for every $(1,2)$ atom $b(t)$ with support in $[0,|I|]$

$$\|K*b\|_1 \leqq B, \tag{1'}$$

B a positive constant independent of b.

We begin with the following.

Definition. We say that a kernel K is regular if

$$K(t) = k(t)g(t) \tag{2}$$

$$|g(u)| \leqq B_1|g(t)| \quad \text{for} \quad \frac{|t|}{2}\leqq|u|\leqq 2|t|, \quad \forall t\in\mathbf{R}, \tag{3}$$

$$\sup_{u\neq 0}\int_{|t|\geqq 2|u|} dt\,|k(t-u)-k(t)|\,|g(t)| \leqq B_2, \tag{4}$$

and

$$K\in L_2^2. \tag{5}$$

We should add that among our applications we have the kernels

$$K(t)=\frac{e^{i|t|^a}}{1+|t|}\quad \text{with}\quad a>0,\ a\neq 1 \tag{6}$$

and

$$K(t)=\frac{e^{it(\log|t|)^\eta}}{(1+|t|)\log^\delta(2+|t|)}\quad \text{where}\quad 0<\eta<1\ \text{ and }\ 0\leqq\delta\leqq\frac{1}{2}(1-\eta). \tag{7}$$

In [11; proof of Theorem 3], we prove that the kernels in (6) are regular. In the proof of the Corollary in **3** of this paper, we also prove that these kernels are regular. At the end of **4**, we prove that the kernels in (7) are regular. Also, the Calderón—Zygmund—Hörmander kernels are regular and to see that you take $K(t)=k(t)$ and $g(t)\equiv 1$.

Let $b(t)$ stand throughout for a $(1,1)$ atom with support in $[0,|I|]$.

Theorem 1. *If K is regular, $K=kg$, and*

$$\int_{|x|\geqq 2|I|}|k(x)|\,|g*b(x)|\,dx\leqq B_3 \tag{8}$$

where B_3 is a positive constant independent of all $(1,2)$ atoms $b(t)$ (with support in $[0,|I|]$). Then,

$$\|K*f\|_1\leqq B\|f\|_{H^1},$$

where $f\in H^1$ and B is a positive constant independent of f. In fact, B depends only on $B_1, B_2, B_3, \|K\|_{2,2}$.

In a Lemma (that appears in the proof of Theorem 5) we prove that for regular kernels $K, K=kg$, condition (8) is also a necessary condition for K to map H^1 into L^1.

Lemma 2. *Let K be regular. Then for any atom b (supported in $[0,|I|]$) we get*

$$\int_{|x|\geqq 2|I|}dx\,|K*b(x)|\leqq B+|I|^{-1}\int_{|x|\geqq 2|I|}dx\,|k*\chi_I(x)\,g*b(x)|$$

where B is a positive constant that is independent of b and $|I|$.

Proof of Lemma 2. We note that

$$K*b(x)=\int k(x-t)\,g(x-t)\,b(t)\,dt-|I|^{-1}\int dt\,k(x-t)\,\chi_I(t)\,g*b(x)+|I|^{-1}k*\chi_I(x)\,g*b(x). \tag{9}$$

Hence,

$$\int_{|x|\geqq 2|I|}dx\,|K*b(x)|\leqq\int_{|x|\geqq 2|I|}dx\left|\int dt\big(k(x-t)-k(x)\big)\left\{g(x-t)\,b(t)-\frac{\chi_I(t)}{|I|}g*b(x)\right\}\right|+|I|^{-1}\int_{|x|\geqq 2|I|}|k*\chi_I(x)g*b(x)|\,dx,$$

but

(10)
$$\int_{|x|\geqq 2|I|} dx \left|\int dt(k(x-t)-k(x))\left\{g(x-t)b(t)-\frac{\chi_I(t)}{|I|}g*b(x)\right\}\right|$$
$$\leqq \int dt\,|b(t)|\int_{|x|\geqq 2|I|} dx\,|k(x-t)-k(x)|\,|g(x-t)|$$
$$+\int du\,|b(u)|\int dt\,\frac{\chi_I(t)}{|I|}\int_{|x|\geqq 2|I|} dx|k(x-t)-k(x)|\,|g(x-u)|$$

now using (3) and (4) (since K is regular) we get the proof of the lemma.

Now to give the proof of Theorem 1.

Proof of Theorem 1. It suffices to prove the Theorem for (1, 2) atoms where the support of $b\subset[0, |I|]$.

Since $K\in L^2_2$ this implies

$$\int_{|x|\leqq 2|I|} |K*b(x)|\,dx \leqq (2|I|)^{1/2}\|K*b\|_2 \leqq (2|I|)^{1/2}\|b\|_2\|K\|_{2,2} \leqq B.$$

Thus, we are left with estimating

$$\int_{|x|\geqq 2|I|} dx\,|K*b(x)|.$$

From Lemma 2, we get that

$$\int_{|x|\geqq 2|I|} dx\,|K*b(x)| \leqq B+|I|^{-1}\int_{|x|\geqq 2|I|} dx\,|k*\chi_I(x)\,g*b(x)|$$

and hence

(11)
$$|I|^{-1}\int_{|x|\geqq 2|I|} dx\,|k*\chi_I(x)\,g*b(x)|$$
$$\leqq |I|^{-1}\int_{|x|\geqq 2|I|} dx\left|\int dt(k(x-t)-k(x))\chi_I(t)\,g*b(x)\right|$$
$$+\int_{|x|\geqq 2|I|} dx\,|k(x)|\,|g*b(x)|$$
$$\leqq |I|^{-1}\int dt\chi_I(t)\int du\,|b(u)|\int_{|x|\geqq 2|I|} dx\,|k(x-t)-k(x)|\,|g(x-u)|$$
$$+\int_{|x|\geqq 2|I|} dx\,|k(x)|\,|g*b(x)|.$$

The proof of the Theorem follows since K is regular and by (8).

Remark. In the special case where K is regular and $K(t)=k(t)$ (i.e. $g(t)\equiv 1$) then these kernels reduce to the Calderón—Zygmund—Hörmander kernels. In this case the assumption (8) is easily satisfied since $\int g=0$.

In this paper we are concerned with determining those regular kernels which map H^1 into L^1. We shall study those kernels defined by (0) and for the most part set $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$, $t\geqq 0$. Note $K(t)=k(t)g(t)$, this forces us to assume that $a''(t)$ exists for most t's and that $a''(t)$ has isolated zeros. Furthermore, $a(t)$ is a real-valued function of the real variable t.

We shall also assume for g that

$$\sup_{a,b,x}\left|\int_a^b g(t)\,e^{-itx}\,dt\right| \leqq B_4. \tag{12}$$

Let us discuss this condition (12). In an earlier paper [10; Lemma 3] when $a(t)=|t|^a$, $a>0$, $a\neq 1$, we showed (12) holds. In [16; § 3] it was shown when $a(t)=t(\log|t|)^\eta$, $0<\eta<1$, $\delta=\frac{1}{2}(1-\eta)$ that (12) holds. In **5** of this paper we give explicit conditions on $a(t)$ in order that (12) holds. Note that when $a(t)=\sin t$ then (12) fails.

So in order to prove the kernels in (0) map H^1 into L^1, we use the decomposition defined above and then determine conditions on $a(t)$ which force $K(t)$ to be regular and also for which condition (8) holds. Then we apply Theorem 1.

3. Sufficient conditions for regular kernels to map H^1 into L^1

In this section we shall give explicit conditions on $k(t)$ and $g(t)$ in order that (8) of Theorem 1 holds, note $K(t)=k(t)g(t)$. Once again I remind you from the discussion at the end of **2**, we only consider kernels $K(t)$ defined by (0) and we assume $g(t)$ satisfies (12). For the most part, we shall set $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$, $t\geqq 0$.

Let's begin with the following.

Theorem 3. *Let* $K(t)=\dfrac{e^{ia(t)}}{1+|t|}=k(t)g(t)$ *where* $g(t)$ *satisfies* (12) *(*$a(t)$ *is real-valued). Furthermore, if*

$$\int_{|t|\geqq 1}|k'(t)|\,dt\leqq B_5 \quad \textit{and} \quad \lim_{t\to\infty}k(t)=0 \tag{13}$$

and K *satisfies* (3), (4) *and*

$$\int_{2\leqq|t|}|k(t)|^2\,dt\leqq B_6. \tag{14}$$

And for some function h

$$\int_{2s\leqq|t|\leqq h(s)}|k(t)|\,|g(t-u)-g(t)|\,dt\leqq B_7, \quad 0<|u|\leqq s\leqq 1, \tag{15}$$

and

$$\sup_{|u|\leqq 1}\frac{1}{|u|}\int_{h(u)\leqq|t|}dt\,|k(t)|^2\leqq B_8. \tag{16}$$

Then,

$$\|K*f\|_1\leqq B\|f\|_{H^1}$$

where $f \in H^1$ and B is a positive constant independent of f. In fact, B depends only on $B_1, B_2, \|K\|_{2,2}, \|g\|_{2,2}, B_5, B_6, B_7, B_8$.

Proof. Since $g(t)$ satisfies (12) we get $(a<-1<1<b)$

$$\int_a^b K(t)\,e^{-itx}\,dt = \int_{|t|\le 1} \chi_{[a,b]}(t)K(t)\,e^{-itx}\,dt + \int_{|t|\ge 1} \chi_{[a,b]}Ke^{-itx}\,dt$$

and

$$\Big|\int_a^b K(t)\,e^{-itx}\,dt\Big| \le \int_{|t|\le 1} |K(t)|\,dt + \int_1^b dt\,|k'(t)|\,\Big|\int_t^b dv\,g(v)\,e^{-ivx}\Big|$$

$$+\int_a^{-1} dt\,|k'(t)|\,\Big|\int_t^{-1} dv g(v)\,e^{-ivx}\Big|$$

$$+|k(1)|\,\Big|\int_1^b g(v)\,e^{-ivx}\,dv\Big| + |k(-1)|\,\Big|\int_a^{-1} dv g(v)\,e^{-ivx}\Big|$$

and by (13) we get that $K \in L^2_2$. Since K satisfies (3), (4), then we get that K is regular. Now we shall prove K satisfies (8).

For $|I|\ge 1$ we get (note b has support in $[0, |I|]$)

$$\int_{|x|\ge 2|I|} dx\,|k(x)|\,|g*b(x)| \le \Big\{\int_{|x|\ge 2|I|} dx\,|k(x)|^2\Big\}^{1/2}\|g*b\|_2$$

$$\le B_6^{1/2}(2|I|)^{1/2}\|g\|_{2,2}\|b\|_2,$$

since g satisfies (12), b is a (1, 2) atom and (14), we are through with this case.

For $|I|\le 1$ we get,

$$\int_{|x|\ge 2|I|} dx\,|k(x)|\,|g*b(x)| \le \Big(\int_{2|I|\le|x|\le h(|I|)} + \int_{|x|\ge h(|I|)}\Big)\,|k(x)|\,|g*b(x)|\,dx$$

$$\le \int_{2|I|\le|x|\le h(|I|)} dx\,|k(x)|\,\Big|\int du\,b(u)\big(g(x-u)-g(x)\big)\Big|$$

$$+\Big\{\int_{h(|I|)\le|x|} dx\,|k(x)|^2\Big\}^{1/2}\|g*b\|_2$$

$$\le \int du\,|b(u)|\int_{2|I|\le|x|\le h(|I|)} dx\,|k(x)|\,|g(x-u)-g(x)|$$

$$+B_8^{1/2}|I|^{1/2}\|g\|_{2,2}\|b\|_2$$

$$\le B_7 + B|I|^{1/2}\|g\|_{2,2}\|b\|_2 \le B,$$

since g satisfies (12), by (15), (16) and since b is a (1, 2) atom. And now by Theorem 1 we get our result.

Theorem 4. *Let* $K(t)=\dfrac{e^{ia(t)}}{1+|t|}=k(t)g(t)$ *where* $g(t)$ *satisfies* (12) *($a(t)$ is real-valued). Furthermore, if K satisfies* (3), (4), (13) *and*

$$\sup_{|u|\le 1}\int_{|t|\ge 2|u|} dt\,|k(t)|\,|g(t-u)-g(t)| \le B_9, \tag{17}$$

and for some function h,

(18) $$\sup_{|u|\geqq 1}\frac{1}{|u|}\int_{2|u|\leqq|t|\leqq h(u)} dt\,|k(t)|^2 \leqq B_{10},$$

and

(19) $$\int_{h(s)\leqq|t|} dt\,|k(t)|\,|g(t-u)-g(t)| \leqq B_{11} \quad |u|\leqq s,\ s\geqq 1.$$

Then, $\|K*f\|_1 \leqq B\|f\|_{H^1}$, where f is any function in H^1 and B is a positive constant independent of f. In fact, B depends only on $B_1, B_2, \|K\|_{2,2}, \|g\|_{2,2}, B_5, B_9, B_{10}, B_{11}$.

Proof. Just as in the proof of Theorem 3, we get that $K\in L_2^2$ and since K satisfies (3), (4) we get that K is regular. Now to show that K satisfies (8).

We note that for $|I|\geqq 1$,

$$\begin{aligned}\int_{|x|\geqq 2|I|} dx\,|k(x)|\,|g*b(x)| &\leqq \Big(\int_{2|I|\leqq|x|\leqq h(|I|)} + \int_{h(|I|)\leqq|x|}\Big) dx\,|k(x)|\,|g*b(x)|\\ &\leqq \Big\{\int_{2|I|\leqq|x|\leqq h(|I|)} dx\,|k(x)|^2\Big\}^{1/2}\|g*b\|_2\\ &\quad+\int du\,|b(u)|\int_{h(|I|)\leqq|x|} dx\,|k(x)|\,|g(x-u)-g(x)|\\ &\leqq B_{10}^{1/2}|I|^{1/2}\|g\|_{2,2}\|b\|_2 + B_{11} \leqq B,\end{aligned}$$

since b is a (1, 2) atom and K satisfies (18) and (19). Now for the case $|I|\leqq 1$ we get,

$$\begin{aligned}\int_{|x|\geqq 2|I|} dx\,|k(x)|\,|g*b(x)| &\leqq \int_{|x|\geqq 2|I|} dx\,|k(x)|\,\Big|\int du\,\big(g(x-u)-g(x)\big)b(u)\Big|\\ &\leqq \int du\,|b(u)|\int_{|x|\geqq 2|I|} dx\,|k(x)|\,|g(x-u)-g(x)|\\ &\leqq B.\end{aligned}$$

Now Theorem 4 follows by Theorem 1.

The kernels $K(t)=\dfrac{e^{i|t|^a}}{1+|t|}$, $a>0, a\neq 1$ satisfy either Theorem 3 ($a>1$) or Theorem 4 ($0<a<1$) this was done in an earlier work; in this case, we take $h(t)=2|t|^{\frac{1}{1-a}}$ in these Theorems. We shall do that proof here again in the Corollary.

Corollary. *Let $K(t)=\dfrac{e^{i|t|^a}}{1+|t|}$, $0<a, a\neq 1$. Then $\|K*f\|_1\leqq B\|f\|_{H^1}$, $f\in H^1$ and B is a positive constant independent of f.*

Proof. Here we set $g(t)=(1+|t|)^{\frac{a}{2}-1}e^{i|t|^a}$ and $k(t)=\dfrac{1}{(1+|t|)^{a/2}}$. Now that (3) holds for all $a>0$ is clear and that (4) holds for all $a>0$, we note that

$$\left|\frac{d}{dt}k(t)\right| = \frac{a}{2(1+|t|)^{1+\frac{a}{2}}} \quad \text{for} \quad t\in\mathbf{R},\ t\neq 0.$$

And hence,

$$\int_{|t|\geqq 2|u|} dt\,|k(t)-k(t-u)|\,|g(t)| \leqq B_2.$$

Hence all these kernels satisfy (3) and (4).

Now suppose $a>1$. Now we show that K satisfies Theorem 3 (we take $h(u)=2|u|^{\frac{1}{1-a}}$). We show that $g(t)$ satisfies (12) for $a>1$ in Example 7′, 8′ in **5**. That $k(t)$ satisfies (13), (14) and (16) $\left(h(u)=2|u|^{\frac{1}{1-a}}\right)$ can be seen by inspection. We note that

$$|g'(t)| \leqq \frac{\left|1-\frac{a}{2}\right|}{(1+|t|)^{2-\frac{a}{2}}} + \frac{a}{|t|^{1-a}(1+|t|)^{1-\frac{a}{2}}} \tag{20}$$

and for $|u|\leqq 1$

$$\int_{2|u|\leqq|t|\leqq 2|u|^{\frac{1}{1-a}}} dt\,\frac{1}{(1+|t|)^{a/2}}\,|g(t-u)-g(t)|$$

$$\leqq B|u|\left\{\int_{2|u|\leqq|t|\leqq 2}\frac{dt}{(1+|t|)^2}+\int_{2\leqq|t|\leqq 2|u|^{\frac{1}{1-a}}}\frac{dt}{|t|^{1-a}(1+|t|)}\right\}$$

$$\leqq B.$$

Now suppose $0<a<1$. Here, we show K satisfies Theorem 4 (we take $h(u)=2|u|^{\frac{1}{1-a}}$). We show that $g(t)$ satisfies (12) for $0<a<1$ in example 9′ in **5**. That $k(t)$ satisfies (18) can be seen by inspection. From (20), it follows that (17) holds.

And for $|u|\geqq 1$, again by (20)

$$\int_{|t|\geqq 2|u|^{\frac{1}{1-a}}} dt\,|k(t)|\,|g(t-u)-g(t)| \leqq B|u|\int_{|t|\geqq 2|u|^{\frac{1}{1-a}}}\frac{dt}{(1+|t|)|t|^{1-a}}$$

$$\leqq B.$$

Now the proof of the Corollary is complete.

4. A regular kernel that does *not* map H^1 into L^1

In this section we shall show that the kernels in (7) do not map H^1 into L^1 continuously. I have studied these kernels in earlier papers [15], [16]. The proof that these kernels in (7) are regular will be done at the end of this section.

Let us begin with the following Theorem.

Theorem 5. *Suppose* $K_{iy}(t)=k_{iy}(t)g(t)$ *for* $-\infty<y<\infty$ *and* $K_0(t)\equiv K(t)=k(t)g(t)$ $(k_0(t)\equiv k(t))$. *Assume also that* K *is a regular kernel and* $|K(t)|\leqq B(1+|t|)^{-1}$. *Here, we let* c *be a non-zero real constant. If*

$$\text{(i)} \qquad \|K_{iy}\|_{2,2}\leqq M(y) \quad \text{for} \quad -\infty<y<\infty,$$

$$\text{(ii)} \qquad k_{iy}(t)=\frac{k(t)}{(1+|t|)^{icy}} \quad \text{for} \quad -\infty<y<\infty.$$

Then, if $\|K*f\|_1\leqq B\|f\|_{H^1}$ *for all* $f\in H^1$, *then*

$$\|K_{iy}*f\|_1\leqq B'M_1(y)\|f\|_{H^1},$$

where B' *is a positive constant indep. of* f *and* y *and* $M_1(y)=M(y)+1+|y|$.

Proof. It's enough to prove the Theorem for (1, 2) atoms b with support in $[0, |I|]$. We note first,

$$\text{(21)} \qquad \int_{|x|\leqq 2|I|}|K_{iy}*b(x)|\,dx\leqq(2|I|)^{1/2}\|K_{iy}*b\|_2$$
$$\leqq B|I|^{1/2}\|K_{iy}\|_{2,2}\|b\|_2\leqq BM(y).$$

We now show,

Lemma. *If* K *is a regular kernel, and if* b *is a* (1,2) *atom with support in* $[0, |I|]$, *then*

$$\int_{|x|\geqq 2|I|}dx\,|k(x)|\,|g*b(x)|\leqq B+\int_{|x|\geqq 2|I|}|K*b(x)|\,dx.$$

Proof of Lemma.

By (9) we note that

$$|I|^{-1}k*\chi_I(x)\,g*b(x)$$
$$=K*b(x)-\int dt\big(k(x-t)-k(x)\big)\Big\{g(x-t)\,b(t)-\frac{\chi_I(t)}{|I|}\,g*b(x)\Big\}$$

hence,

$$\text{(22)} \qquad |I|^{-1}\int_{|x|\geqq 2|I|}|k*\chi_I(x)|\,|g*b(x)|\,dx$$
$$\leqq\int_{|x|\geqq 2|I|}|K*b(x)|\,dx$$
$$+\int_{|x|\geqq 2|I|}dx\int dt\,|k(x-t)-k(x)|\,\Big|g(x-t)b(t)-\frac{\chi_I(t)}{|I|}\,g*b(x)\Big|$$
$$\leqq B+\int_{|x|\geqq 2|I|}|K*b(x)|\,dx$$

just as in (10) since K is regular. But,

$$|I|^{-1}\int_{|x|\geq 2|I|} dx\left|\int dt\, k(x-t)\chi_I(t)\, g*b(x)\right|$$
$$=|I|^{-1}\int_{|x|\geq 2|I|} dx\left|\int dt\big(k(x-t)-k(x)+k(x)\big)\chi_I(t)\, g*b(x)\right|$$
$$\geq |I|^{-1}\int_{|x|\geq 2|I|} dx\left\{\left|\int dt k(x)\chi_I(t)\, g*b(x)\right|\right.$$
$$\left.-\left|\int dt\big(k(x-t)-k(x)\big)\chi_I(t) g*b(x)\right|\right\}$$

hence,

$$\int_{|x|\geq 2|I|} dx\,|k(x)|\,|g*b(x)| \tag{23}$$
$$\leq |I|^{-1}\int_{|x|\geq 2|I|} dx\,|k*\chi_I(x)|\,|g*b(x)|$$
$$+|I|^{-1}\int dt\,\chi_I(t)\int_{|x|\geq 2|I|} dx|k(x-t)-k(x)|\,|g*b(x)|.$$

Since K is regular and from (22) and (23) we get the lemma.

Now back to the proof of Theorem 5. Again just as in (9) we get that

$$K_{iy}*b(x)=\int k_{iy}(x-t)\, g(x-t)\, b(t)\, dt-|I|^{-1}\int dt k_{iy}(x-t)\chi_I(t)\, g*b(x)$$
$$+|I|^{-1}\int dt k_{iy}(x-t)\chi_I(t)\, g*b(x)$$

hence

$$\int_{|x|\geq 2|I|}|K_{iy}*b(x)|\, dx$$
$$\leq \int_{|x|\geq 2|I|} dx\int dt|k_{iy}(x-t)-k_{iy}(x)|\left|g(x-t)\, b(t)-\frac{\chi_I(t)}{|I|}\, g*b(x)\right|$$
$$+|I|^{-1}\int_{|x|\geq 2|I|} dx\,|k_{iy}*\chi_I(x)|\,|g*b(x)|.$$

Note that from (iii) we get that

$$k_{iy}(x-t)-k_{iy}(x)=\frac{\big(k(x-t)-k(x)\big)}{(1+|x-t|)^{icy}}+k(x)\left(\frac{1}{(1+|x-t|)^{icy}}-\frac{1}{(1+|x|)^{icy}}\right)$$

since K is regular and $|K(t)|\leq B(1+|t|)^{-1}$ we get

$$\int_{|x|\geq 2|I|}|K_{iy}*b(x)|\, dx\leq 2B_1B_2+2(1+|y|)B_1B_2$$
$$+|I|^{-1}\int_{|x|\geq 2|I|} dx\,|k_{iy}*\chi_I(x)|\,|g*b(x)|$$
$$\leq B(1+|y|)+|I|^{-1}\int_{|x|\geq 2|I|} dx\int_I dt\,|k_{iy}(x-t)-k_{iy}(x)|\,|g*b(x)|$$
$$+\int_{|x|\geq 2|I|} dx\,|k_{iy}(x)|\,|g*b(x)|.$$

Hence,

$$\int_{|x|\geqq 2|I|} |K_{iy} * b(x)|\, dx \leqq B(1+|y|) + \int_{|x|\geqq 2|I|} dx\, |k(x)|\, |g * b(x)|. \tag{24}$$

Now from (24), (21) and the lemma we get our result.

I believe a better appreciation of Theorem 5 will occur after we apply it to the kernels in (7). We begin by setting $(z=x+iy)$

$$K_z(t;\,\delta) = \frac{e^{it(\log|t|)^\eta}}{(1+|t|)^{1/2+(1-z)/2}(\log(2+|t|))^\delta}, \quad 0<\eta<1 \tag{25}$$

and $0\leqq\delta\leqq\frac{1}{2}(1-\eta)$.

Theorem 6. *There exists a regular kernel that does* **not** *map H^1 into L^1 continuously. More precisely, the kernels in* (7) *do not map H^1 into L^1 continuously.*

Proof. Now for the kernels defined by (25) we see that $K(t;\,\delta)$ $\big($set $K(t;\,\delta)\equiv K_0(t;\,\delta)\big)$ is a regular kernel. Just take $k(t;\,\delta)=(1+|t|)^{-1/2}(\log(2+|t|))^{-\delta}$ with $K(t;\,\delta)=k(t;\,\delta)g(t)$ and note that $\left|\frac{d}{dt}k(t;\,\delta)\right|\leqq B(1+|t|)^{-3/2}$, $|g(t)|\leqq(1+|t|)^{-1/2}$, as long as $\delta\geqq 0$. We note that $K\in L_2^2$ is clear.

Now assume that $K(t;\,\delta)$ maps H^1 into L^1 (δ fixed) that implies (by the Lemma contained in the proof of Theorem 5) that

$$\int_{|x|\geqq 2|I|} dx\, |k(x;\,\delta)|\, |g * b(x)| \leqq B,$$

B a positive constant independent of b and $|I|$. Now consider the kernel $K\big(t;\,\frac{1}{2}(1-\eta)\big)$, where we set $K(t)\equiv K\big(t;\,\frac{1}{2}(1-\eta)\big)$ and $k(t)\equiv k\big(t;\,\frac{1}{2}(1-\eta)\big)$. Hence,

$$\int_{|x|\geqq 2|I|} dx\, |k(x)|\, |g * b(x)| \leqq B$$

since $|k(t)|\leqq|k(t;\,\delta)|$ for $0\leqq\delta\leqq\frac{1}{2}(1-\eta)$ and all t. Hence by Theorem 1 we get that K maps H^1 into L^1 continuously. Also the conditions (ii) and (iii) of Theorem 5 are satisfied by $K_{iy}(t)$ $\big(\equiv K_{iy}(t;\,\frac{1}{2}(1-\eta))\big)$ since here we have $k_{iy}(t)=\dfrac{k(t)}{(1+|t|)^{-iy/2}}$ with $K_{iy}(t)=k_{iy}(t)g(t)$. In [16; formula (11)], I showed for $\delta=\frac{1}{2}(1-\eta)$

$$\|K_{1+iy}\chi_{[a,b]}\|_{2,2} \leqq B(1+|y|). \tag{26}$$

$K_{iy}(t) = \dfrac{K_{1+iy}(t)}{(1+|t|)^{1/2}}$ this implies that

$$\|K_{iy}\|_{2,2} \leqq B(1+|y|). \tag{27}$$

Now it follows from (27) that condition (i) of Theorem 5 is also satisfied by $K_{iy}(t)$ and hence that implies (by Theorem 5) that

$$\|K_{iy} * f\|_1 \leqq B(1+|y|)\|f\|_{H^1}.$$

Hence from (26) and the estimate above we get by Theorem A of Macias that

$$\|K_x * f\|_p \leqq B\|f\|_p \quad \text{for} \quad \frac{1}{p} = 1-\frac{x}{2} \quad (0<x<1),$$

this positive constant B is independent of f. (Note this family $\{K_z\}$ defined by (25) is an analytic family). But by [15; Thm. 3] I know that $K_x\big(t; \frac{1}{2}(1-\eta)\big)$ does not map L^p into L^p strongly as long as $1<p\leqq\dfrac{2}{1+\eta}$. Hence we get our contradiction.

I would like to point out that in [15; Theorem 2] it was shown that $K_x\big(t; \frac{1}{2}(1-\eta)\big)$ does map L^p into L^p weakly for $\dfrac{1}{p}=1-\dfrac{x}{2}$.

5. When does (12) hold?

As I have explained in **1**, we could assume that $K(t)=0$ for $t<0$ (or for $t>0$). And so we have $K(t)=k(t)g(t)$ and $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$ for $t\geqq 0$ and $g(t)=0$ elsewhere. In this section, we shall get a partial answer to when $g(t)$ satisfies (12). In particular, our results will imply that g satisfies (12) when $a(t)=|t|^a$, $a>0, a\neq 1$ and $a(t)=t\,(\log|t|)^\eta$, $\eta>0$.

We assume throughout that $|a''(t)|$ is positive outside a compact set $C=[0, M]$. For a given x let t_x denote the point where $a'(t_x)=x$, of course when such a solution exists. Of course, the arguments that we use here also apply to the cases when $a'(t)=x$ has a finite number of solutions.

We begin with the case whereby $t_x\notin C$ and here we set $\delta_x=|a''(t_x)|^{-1/2}$ and $P_x=[t_x-\delta_x, t_x+\delta_x]$ (note $t_x\notin C$). Also, we assume

$$\sup_{x\in\mathbf{R}}\ \sup_{v,u,t\in P_x} \frac{|a''(1+t)|+|a''(v)|}{|a''(u)|} \leqq B_{14} \tag{28}$$

sup is only over those x's where $t_x\notin C$.

We use the following conventions. When we write

$$g(t,x)\downarrow \quad \text{for} \quad t>u,$$

that means either,

(i) $g(t, x)\geqq 0$ for $t>u$ and $g(t, x)$ is decreasing as a function of t for $t>u$; or,

(ii) $-g(t, x)\geqq 0$ for $t>u$ and $-g(t, x)$ is decreasing as a function of t for $t>u$.

And similarly for the notation $g(t, x)\uparrow$ for $t<u$.

Also for each t_x, δ_x $(t_x\notin C)$

$$\frac{|a''(1+t)|^{1/2}}{a'(t)-x}\downarrow \quad \text{for} \quad t>t_x+\delta_x. \tag{29}$$

Furthermore, let $t_x \in C$ or suppose no t_x exists, then we assume

$$\frac{|a''(1+t)|^{1/2}}{a'(t)-x}\downarrow \quad \text{for} \quad t>2M \tag{30}$$

and

$$\sup_x \frac{|a''(1+2M)|^{1/2}}{|a'(2M)-x|} \leqq B_{15}, \tag{31}$$

where the sup is over all those x's where either $t_x \in C=[0, M]$ or no t_x exists.

Lemma 7. *Assume* $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$ *for* $t \geqq 0, g \in L_{\text{loc}}, |a''(t)|$ *is positive outside some compact set,* $a(t)$ *satisfies* (28), (29), (30), (31) *and*

$$\frac{|a'(1+t)|^{1/2}}{x-a'(t)}\uparrow \quad \textit{for} \quad M<t<t_x-\delta_x. \tag{32}$$

Then $a(t)$ *satisfies* (12) *and hence* $g \in L_2^2$.

Proof. First suppose that t_x exists and $t_x \notin C$. So consider,

$$\begin{aligned}&\int_{t_x+\delta_x}^{\infty} \chi_{[a,b]}(t)g(t)e^{-itx}\,dt+\int_{t_x-\delta_x}^{t_x+\delta_x} \chi_{[a,b]}(t)g(t)e^{-itx}\,dt\\ &+\int_M^{t_x-\delta_x} \chi_{[a,b]}(t)g(t)e^{-itx}\,dt+\int_0^M \chi_{[a,b]}(t)g(t)e^{-itx}\,dt\\ &=\mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}.\end{aligned}$$

Since $g \in L_{\text{loc}}$, we get that $|\mathrm{IV}| \leqq \int_0^M |g(t)|\,dt$. To do III, we get by (32) (δ_x is defined above)

$$\begin{aligned}|\mathrm{III}| &= \left|\int_M^{t_x-\delta_x} \chi_{[a,b]}(t)\frac{|a''(1+t)|^{1/2}}{(x-a'(t))}(x-a'(t))e^{ia(t)}e^{-itx}\,dt\right|\\ &\leqq \frac{|a''(1+t_x-\delta_x)|^{1/2}}{|a'(t_x)-a'(t_x-\delta_x)|} \leqq B_{14}\end{aligned}$$

by (28) and the mean value theorem.

To do II, using (28) we get

$$|\mathrm{II}| \leqq B_{14}|a''(t_x)|^{1/2}\delta_x \leqq B.$$

To do I, by (29) (and just as in III above)

$$|\mathrm{I}| \leqq \frac{|a''(1+t_x+\delta_x)|^{1/2}}{|a'(t_x+\delta_x)-a'(t_x)|} \leqq B_{14}$$

again by (28) and the mean value theorem.

Now suppose $t_x \in [0, M]$ or t_x does not exist. We look at

$$\int_0^{2M} g(t)\chi_{[a,b]}(t)e^{-itx}\,dt + \int_{2M}^{\infty} g(t)\chi_{[a,b]}(t)e^{-itx}\,dt = \mathrm{I} + \mathrm{II}.$$

$|\mathrm{I}| \leqq \int_0^M |a''(1+t)|^{1/2}\,dt$. And by (30) we get that

$$|\mathrm{II}| \leqq \frac{|a''(1+2M)|^{1/2}}{|a'(2M)-x|} \quad \text{and by (31) we get that } |\mathrm{II}| \leqq B_{15}$$

and hence the lemma is proven.

Example 7′. *The functions* $a(t) = |t|^a$, $a \geqq 2$ *satisfy* (12).

Proof. We take $C = [0, 1]$ and show that for $x > 0$ that these functions satisfy Lemma 7. For $x > 0$, then $t_x = \left(\frac{x}{a}\right)^{\frac{1}{a-1}}$, now $t_x \notin C$ implies $x > a$ and in this case $\delta_x = \big(a(a-1)\big)^{-1/2}\left(\frac{a}{x}\right)^{\frac{a-2}{2(a-1)}}$. We first note that (28) holds. We also note that

$$\frac{(1+t)^{(a-2)/2}}{at^{a-1}-x} > 0 \quad \text{and is decreasing for } t > t_x$$

hence (29) holds. And similarly we get that (32) holds. We note that $t_x \in C$ implies $0 < x < a$ and then to show that (30) and (31) holds is routine.

Now for the case $x < 0$, we consider

$$\int_0^2 dt g(t)e^{-itx} + \int_2^{\infty} \frac{(1+t)^{a/2-1}e^{it^a}e^{-itx}(at^{a-1}-x)\,dt}{t^{a/2-1}t^{a/2}\left(a + \frac{|x|}{t^{a-1}}\right)}$$

and note that $\int_0^2 |g(t)|\,dt \leqq B$. For the second term, keeping in mind for fixed x

$$\int_2^{\infty} \dots = \lim_{n\to\infty} \int_2^n \dots,$$

and observing that

$$\frac{(1+t)^{a/2-1}}{t^{a-1}} \quad \text{and} \quad \left(a + \frac{|x|}{t^{a-1}}\right)$$

are positive and decreasing for $t \geqq 2$, we get by two applications of the second mean value theorem for integrals that

$$\left|\int_2^n \frac{(1+t)^{a/2-1}}{t^{a-1}} \frac{1}{\left(a+\frac{|x|}{t^{a-1}}\right)} e^{it^a}e^{-itx}(at^{a-1}-x)\,dt\right|$$

$$\leqq B\left|\int_{\xi_1(x)}^{\xi_2(x)} (at^{a-1}-x)e^{it^a}e^{-itx}\,dt\right| \leqq B,$$

B a positive constant independent of n and x. And now the proof of the Example is complete.

Now before I begin the next lemma, let me point out some things. We let c denote a positive constant with $c<1$, for most applications we could take $c=1/2$. For $t_x \notin C=[0, M]$ we shall assume $M<\mu_x \leqq ct_x \leqq t_x-\delta_x$, μ_x being some positive constant depending on x. In most cases $\mu_x=ct_x=\frac{1}{2}t_x$.

Lemma 8. *Suppose* $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$, $t\geqq 0$, $|a''(t)|$ *is positive for* $t>M$ *and* $|a''(t)|\downarrow$ *for* $t>M$ *(for some* M*). And suppose* $a(t)$ *satisfies* (28), (29), (30) *and* (31) *and* $g\in L_{\mathrm{loc}}$. *If for each* $t_x>M$ $(0<c<1)$

$$\frac{1}{x-a'(t)}\uparrow \quad \textit{for} \quad M<t<ct_x, \tag{33}$$

and for some μ_x *(as above)*

$$\sup_{x\in\mathbf{R}}\left(\frac{1}{|x-a'(\mu_x)|}+\frac{|a''(1+\mu_x)|^{1/2}}{|x-a'(ct_x)|}\right)\leqq B_{16}, \tag{34}$$

and

$$\sup_{x\in\mathbf{R}}\ \sup_{ct_x\leqq u,t\leqq t_x-\delta_x}\frac{|a''(1+t)|}{|a''(u)|}\leqq B'_{14}, \tag{28'}$$

and the $\sup_{x\in\mathbf{R}}(\ldots)$ in (28′), (33) and (34) is over those x's whereby $t_x>M$. Then $a(t)$ satisfies (12) and hence $g\in L_2^2$.

Remark. In most of our applications we could take $c=1/2$ and replace (34) by

$$\sup_{x\in\mathbf{R}}\frac{1}{|a'(\frac{1}{2}t_x)-x|}\leqq B'_{15} \tag{34'}$$

sup is taken over x's whereby $t_x>M$. And so for those $a(t)$ satisfying the conditions of the lemma with (34′) in place of (34) we would get that $a(t)$ satisfies (12).

Proof of Lemma 8. For $t_x>M$ we note that

$$\begin{aligned}\int_0^\infty \chi_{[a,b]}(t)g(t)e^{-itx}\,dt &= \int_{t_x+\delta_x}^{\infty}\cdots+\int_{t_x-\delta_x}^{t_x+\delta_x}\cdots+\int_{ct_x}^{t_x-\delta_x}\cdots\\ &\quad+\int_{\mu_x}^{ct_x}\cdots+\int_M^{\mu_x}\cdots+\int_0^M\cdots\\ &= \mathrm{I}+\mathrm{II}+\mathrm{III}+\mathrm{IV}+\mathrm{V}+\mathrm{VI}.\end{aligned}$$

The estimates for I, II and VI follow just as in the proof of Lemma 7.

Now to estimate III, we note that since $|a''(t)|\downarrow$ and $|a''(t)|$ is positive for $t>M$ that implies $a'(t)$ is either increasing or decreasing for $t>M$ and hence Van der

Corput applies and we get

$$|\mathrm{III}| = \Big|\int_{ct_x}^{t_x-\delta_x} \chi_{[a,b]}(t)\,|a''(1+t)|^{1/2} e^{ia(t)}\, e^{-itx}\,dt\Big|$$

$$\leqq B\frac{|a''(1+ct_x)|^{1/2}}{|a''(t_x-\delta_x)|^{1/2}} \leqq BB'_{14},$$

by (28′). Now to estimate IV and V, we note that

$$\mathrm{IV}+\mathrm{V} = \Big(\int_{\mu_x}^{ct_x}+\int_{M}^{\mu_x}\Big)\chi_{[a,b]}(t)\frac{|a''(1+t)|^{1/2}}{(x-a'(t))}\big(x-a'(t)\big)e^{ia(t)}\,e^{-itx}\,dt,$$

now by (33), (34) (and $|a''(t)|\downarrow$) we get

$$|\mathrm{IV}|+|\mathrm{V}| \leqq B\Big(\frac{1}{|x-a'(ct_x)|}\,|a''(1+\mu_x)|^{1/2}+\frac{1}{|x-a'(\mu_x)|}\Big)$$

$$\leqq BB_{16},$$

and now the proof is complete.

Let me add (as stated in the Remark) that by (34′) it follows that

$$|\mathrm{IV}+\mathrm{V}| = \Big|\int_{M}^{ct_x}\dots\Big| \leqq B'_{15}$$

and so (34′) could be used in place of (34).

For the cases where either $t_x\in C$ or no t_x exists, the proof follows just as in Lemma 7.

Example 8′. *The functions* $a(t)=|t|^a$, $1<a<2$ *and* $a(t)=t\,(\log|t|)^\eta$ $(\eta>0)$ *satisfy* (12).

Proof. In the cases where $a(t)=|t|^a$ we take $M=4^{2/a}\dfrac{1}{\big(a(a-1)\big)^{1/a}}$ and set $C=[0, M]$. And now these functions $|t|^a$ satisfy the hypothesis of Lemma 8 with $c=1/2$ and (34′) in place of (34).

Now we consider the cases where $a(t)=t\,(\log t)^\eta$, $t>0$ and $0<\eta<1$. The cases where $\eta\geqq 1$ also satisfy Lemma 8 but the proof is slightly different (a little easier) and so we will stay with the cases where $0<\eta<1$.

Now since we are assuming that $a(t)=0$ for $t<0$ (which is purely a technicality) we note $a'(t)=x$ only has solutions when $x>0$. And so for $x>0$ (and x large), we take $\mu_x=e^{\left(\frac{x}{2}\right)^{1/\eta}}$, $ct_x=\frac{1}{2}\,e^{-1/\eta}e^{x^{1/\eta}}$ and note that $a'(t_x)=x$ implies $t_x\approx e^{x^{1/\eta}}$ and

$$\delta_x = |a''(t_x)|^{-1/2} \approx x^{(1-\eta)/(2\eta)}\,e^{(x^{1/\eta})/2}.$$

Now I need to check that the set $P_x=[t_x-\delta_x, t_x+\delta_x]$ "makes good sense" and also that $ct_x \leqq t_x-\delta_x$. But we note that

$$t_x-\delta_x \approx e^{x^{1/\eta}}\left(1-\frac{x^{(1-\eta)/(2\eta)}}{e^{(x^{1/\eta})/2}}\right)$$

and so there is an x_η so that

$$\frac{x^{(1-\eta)/(2\eta)}}{e^{(x^{1/\eta})/2}} \leqq 1-\frac{3}{4}e^{-1/\eta}$$

for $x \geqq x_\eta$. And so we choose x_η large enough so that the above estimates are realized. And then we choose $M=e^{(2x_\eta)^{1/\eta}}$. Then we select $C=[0, M]$ and note that $g(t)$ satisfies Lemma 8 when $t\in[M, \infty)$.

I shall outline this argument. To show that (28) and (29) holds is clear and the proof will be omitted.

Now to see (30) we note that $t_x \leqq M$ or no t_x exists implies $t_x \leqq e^{(2x_\eta)^{1/\eta}}$ and thus $x \leqq 2x_\eta$ (that very last inequality is also valid for negative x's). And we note that

$$\frac{|a''(t)|^{1/2}}{a'(t)-x} \approx \frac{1}{t^{1/2}(\log t)^{(1-\eta)/2}} \frac{1}{(\log t)^\eta - x}$$

is positive and decreasing for $t>2M$ and so (30) is satisfied.

In order to see (31) we note that

$$\sup_{x\leqq 2x_\eta} \frac{|a''(1+2M)|^{1/2}}{|a'(2M)-x|} \approx \sup_{x\leqq 2x_\eta} \frac{1}{(2M)^{1/2}(\log 2M)^{(1-\eta)/2}} \frac{1}{(\log 2M)^\eta - x} = O\left(\frac{x_\eta^{(1-\eta)/2\eta}}{e^{x_\eta^{1/\eta}}}\right)$$

and so (31) is satisfied.

Now that (33) holds is clear and since $\mu_x=e^{(x/2)^{1/\eta}}$ then in order to see the rest of (34) we note ($t_x \geqq 2M$ implies $x \geqq 2x_\eta$)

$$\sup_{x\geqq 2x_\eta} \frac{|a''(1+\mu_x)|^{1/2}}{|x-a'(ct_x)|} \leqq B \sup_{x\geqq 2x_\eta} \frac{x^{(1-\eta)/(2\eta)}}{e^{(x^{1/\eta})/2}} \leqq B.$$

And of course $|a''(t)|\downarrow$ for $t>M$ and so we are finished.

Lemma 9. *Suppose* $g(t)=|a''(1+t)|^{1/2}e^{ia(t)}$, $t\geqq 0$ *where* $|a''(t)|$ *is positive outside some compact set,* $g\in L_{\text{loc}}$, $a(t)$ *satisfies* (28), (28′), (29), (30) *and* (31). *Also,* $\dfrac{|a''(1+t)|^{1/2}}{a'(t)}\downarrow$ *for* $t>M$ *and for* $t_x \notin C$

(33′) $$\frac{1}{\dfrac{x}{a'(t)}-1}\uparrow \quad \textit{for} \quad M<t\leqq \tfrac{1}{2}t_x$$

and

(34″) $$\sup_{x\in\mathbf{R}}\ \sup_{M\leqq t\leqq t_x/2} \frac{1}{\left|1-\dfrac{x}{a'(t)}\right|} \leqq B'_{15}$$

where the outside sup *is over those* x's *where* $t_x\notin C$. *Then* $a(t)$ *satisfies* (12) *and hence* $g\in L_2^2$.

Proof. The argument here follows very closely to the argument found in Lemma 8, except for the term IV. To do IV here, we note that,

$$|\mathrm{IV}| = \left|\int_M^{t_x/2} \chi_{[a,b]}(t)\frac{|a''(1+t)|^{1/2}}{a'(t)} \frac{(x-a'(t))}{\left(\dfrac{x}{a'(t)}-1\right)} e^{-itx}e^{ia(t)}\,dt\right|.$$

Now using (33′), (34″) and $\dfrac{|a''(1+t)|^{1/2}}{a'(t)}\downarrow$ for $t>M$, we get that $|\mathrm{IV}|\leqq B$ and now the proof is complete.

Example 9′. *The functions* $a(t)=|t|^a$, $0<a<1$ *satisfy* (12).

Proof. Let $c_a=4^{\frac{2}{a}}\dfrac{1}{(a(1-a))^{1/a}}$, then we note that

$$\left|\int_0^{c_a} g(t)e^{-itx}\,dt\right| \leqq \int_0^{c_a}|g(t)|\,dt.$$

And for $t\in[c_a,\infty)$ we see that $g(t)$ satisfies Lemma 8, here we take $C=[0,c_a]$.

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