The additive groups of local rings

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1. Introduction

All groups considered in this paper are abelian, with addition as the group operation. A ring R is said to be local if R is a ring with unity, and if R possesses a unique maximal ideal, i.e., the ideal of non-units in R.

Necessary and sufficient conditions will be obtained for a torsion group G to be the additive group of a local ring. Necessary conditions will be given for a non-torsion free group to be the additive group of a local ring.

Notations

$\mathbf{Z}$ = the ring of integers.
$\mathbf{Z}(n)$ = a cyclic group of order n, n a positive integer.
Q = the additive group of the field of rational numbers.
Q^* = $Q-\{0\}$.
Q_p = $\{a/b \mid a, b \in Z,\ p \nmid a,\ p \nmid b\}$, p a prime.
$\mathbf{F}_p$ = a field of order p, p a prime.
G = a group
G_t = the torsion part of G.
G_p = the p-primary component of G, p a prime.
$G[n] = \{x \in G \mid nx = 0\}$, n a positive integer.
R = a ring.
R^+ = the additive group of R.

Definition. A group G is said to be local if there exists a local ring R with $R^+ = G$.

2. The main results

Lemma 2.1. *G is the additive group of a simple ring if and only if either:*
1) $G \cong \bigoplus_\alpha Q$ *or*
2) $G = \bigoplus_\alpha \mathbf{Z}(p)$, *$\alpha$ an arbitrary cardinal, p a fixed prime.*

Proof. Let R be a simple ring with $R^+=G$. For every prime p, pR is an ideal in R. If $pR=0$ for some prime p, then $G=\bigoplus_\alpha \mathbf{Z}(p)$, [1, Theorem 8.5]. If $pR=R$ for every prime p, then G is divisible, and so G is nil [1, Theorem 120.3]. Hence every subgroup of G_t is an ideal in R, and so $G_t=0$. Therefore $G\simeq\bigoplus_\alpha Q$, [1, Theorem 23.1].

Conversely, any group G of the form 1) or 2) is the additive group of a field.

Lemma 2.2. *Let R be a local ring with maximal ideal M, and set of units U.*

1) If R^+ is not a torsion group, and if $(R/M)^+\simeq\bigoplus_\alpha Q$, then $Q^*\subseteqq U$.
2) If R^+ is not a torsion group, and if $(R/M)^+=\bigoplus_\alpha \mathbf{Z}(p)$, then $Q^p\subseteqq U$.
3) If R^+ is a torsion group, then $\mathbf{F}_p\subseteqq U$.

Proof 1) $1+M$ is torsion free in $(R/M)^+$, and so $n=n\cdot 1\notin M$ for every nonzero integer n. Hence n and $\frac{1}{n}$ belong to U, and so $\frac{n}{m}=n(\frac{1}{m})\in U$ for arbitrary nonzero integers n, m.

2) Follows from the same argument as above, assuming n and m to be relatively prime to p.

3) Again follows from the same argument, plus the fact that $p(1+M)=M$, i.e., $p\in M$.

Lemma 2.3. *Let R be a local ring with maximal ideal M, and $R^+=G$.*

1) If $(R/M)^+\simeq\bigoplus_\alpha Q$, then G is torsion free.
2) If $(R/M)^+=\bigoplus_\alpha \mathbf{Z}(p)$, then G_t is a p-primary group.

Proof 1) Suppose that $(R/M)^+\simeq\bigoplus_\alpha Q$. It suffices to show that $G_q=0$ for every prime q. Let $x\in G_q$, $|x|=q^k$. By Lemma 2.2, $q^k, q^{-k}\in R$, and so $x=q^{-k}\cdot q^k x=0$.

2) Suppose that $(R/M)^+=\bigoplus_\alpha \mathbf{Z}(p)$. Let q be a prime $q\neq p$, and let $x\in G_q$, $|x|=q^k$. By Lemma 2.2, $q^k, q^{-k}\in R$, and so $x=q^{-k}\cdot q^k x=0$.

Theorem 2.4. *Let G be a torsion group. G is local if and only if $G=\bigoplus_{k=1}^n \bigoplus_{\alpha_k} \mathbf{Z}(p^k)$, p a prime, n a positive integer, α_k an arbitrary cardinal, $k=1, \dots, n$.*

Proof 1) Let R be a local ring with $R^+=G$. Let $|1|=n$. Clearly $nx=0$ for all $x\in G$. Hence G is bounded, and so G is a direct sum of cyclic groups [1, Theorem 17.2]. By Lemma 2.3, G is p-primary, and so $G=\bigoplus_{k=1}^n \bigoplus_{\alpha_k} \mathbf{Z}(p^k)$.

2) Let $G=\bigoplus_{k=1}^n \bigoplus_{\alpha_n} \mathbf{Z}(p^k)$. Put $H=\bigoplus_{\alpha_n} \mathbf{Z}(p^n)$. If α_n is infinite, there exists a local ring T, with $T^+=H$, [1, Lemma 122.3].

If $\alpha_n=r<\infty$, then $H=(a_1)+\dots+(a_r)$, $|a_i|=p^n$, $i=1, \dots, M$. Let T be the ring with additive group H determined by the products $a_1a_j=a_ja_1=a_j$, and $a_ia_j=a_ja_i=pa_1$ for $i\neq 1, j\neq 1$; $i,j=1, \dots, r$. Then T is a local ring with unique maximal ideal $(pa_1)\oplus a_2\oplus\dots\oplus(a_r)$.

In either case the unity $e \in T$ is an element of a basis for H, i.e. $H=(e)\oplus_{i\in I}(a_i)$. Let $L=\oplus_{k=1}^{n-1}\oplus_{\alpha_k}\mathbf{Z}(p^k)$, and let $\{b_j | j\in J\}$ be a basis for L. Define $eb_j=b_je=b_j$, and $b_jb_k=b_kb_j=a_ib_j=b_ja_i=0$ for all $i\in I$; $j, k\in J$. Define the product of elements in H in accordance with the multiplication in T. These products determine a ring structure R with additive group G, and unity e. Let N be the maximal ideal in T. Then $M=N\oplus L$ is the unique maximal ideal in R.

Theorem 2.5. *G is the additive group of a local ring R with maximal ideal M such that $(R/M)^+=\oplus_\alpha Q$, α an arbitrary cardinal, if and only if $G\cong\oplus Q$.*

Proof. Let R be a local ring with maximal ideal M, and $(R/M)^+\cong\oplus_\alpha Q$. Let $x\in G$ and let n be a positive integer. By Lemma 2.2, n is a unit in R. Hence $x=n(\frac{1}{n}x)$. Therefore G is divisible. G is torsion free by Lemma 2.3, and so $G\cong\oplus Q$, [1, Theorem 23.1].

Conversely, if $G\cong\oplus Q$, then G is the additive group of a field.

Theorem 2.6. *Let R be a local ring with maximal ideal M. If $(R/M)^+=\oplus_\alpha\mathbf{Z}(p)$, and if R^+ is not a torsion group, then $R^+=H\oplus K$, H a divisible group, and K homogeneous of type $(\infty, \dots, 1, \infty, \dots)$ with 1 at the p-th component.*

Proof. Let q be a prime, $q\neq p$. By Lemma 2.2, $q, q^{-1}\in R$. Hence for every $x\in R^+$, $x=q(q^{-1}x)$, and so R^+ is q-divisible. Let H be the maximal divisible subgroup of G. Then $G=H\oplus K$, K homogeneous of type $(\infty, \dots, 1, \infty, \dots)$ with 1 at the p-th component.

Theorem 2.7. *Let R be a Noetherian local ring. Then $R+=H\oplus\oplus_{k=1}^{n}\oplus_{\alpha_k}\mathbf{Z}(p^k)$ n a positive integer, p a prime, α_k an arbitrary cardinal $k=1, \dots, n$, and H torsion, free. If R^+ is mixed, then $pH\neq H$.*

Proof. R_t^+ is p-primary for some prime p by Lemma 2.3. Now $R^+[p]\subseteqq R^+[p^2]\subseteqq\dots$ is an ascending chain of ideals in R. Hence $R_t^+=G[p^n]$ for some positive integer n. Therefore $R^+=H\oplus_{k=1}^{n}\oplus_{\alpha_k}\mathbf{Z}(p^n)$, α_k an arbitrary cardinal, $k=1, \dots, n$, and H torsion free [1, Theorem 17.2 and Theorem 27.5].

Let M be the maximal ideal in R. If R^+ is a mixed group, then R_t^+ and $p^nR=p^nH$ are proper ideals in R. Hence $p^nH\subseteqq M$, and $R_t^+\subseteqq M$. If $p^nH=H$, then $H\oplus R_t^+=R^+\subseteqq M$, a contradiction. Hence $pH\neq H$.

References

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