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Factorization of generalized theta functions in the reducible case

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Introduction

One of the problems in algebraic geometry motivated by conformal field theory is to study the behaviour of moduli space of semistable parabolic bundles on a curve and its generalized theta functions when the curve degenerates to a singular curve. Let X be a smooth projective curve of genus g, and $\mathcal{U}_X$ be the moduli space of semistable parabolic bundles on X, one can define canonically an ample line bundle $\Theta_{\mathcal{U}_X}$ (the theta line bundle) on $\mathcal{U}_X$ and the global sections $H^0(\Theta^k_{\mathcal{U}_X})$ are called generalized theta functions of order k. These definitions can be extended to the case of a singular curve. Thus, when X degenerates to a singular curve X_0, one may ask the question how to determine $H^0(\Theta^k_{\mathcal{U}_{X_0}})$ by generalized theta functions associated to the normalization $\widetilde{X}_0$ of X_0. The so called fusion rules suggest that when X_0 is a nodal curve the space $H^0(\Theta^k_{\mathcal{U}_{X_0}})$ decomposes into a direct sum of spaces of generalized theta functions on moduli spaces of bundles over $\widetilde{X}_0$ with new parabolic structures at the preimages of the nodes. These factorizations and the Verlinde formula were treated by many mathematicians from various points of view. It is obviously beyond my ability to give a complete list of contributions. According to [Be], there are roughly two approaches: infinite and finite. I understand that those using stacks and loop groups are infinite approaches, and working in the category of schemes of finite type is a finite approach. Our approach here should be a finite one.

When X_0 is irreducible with one node, a factorization theorem was proved in [NR] for rank two and generalized to arbitrary rank in [Su]. By this factorization, one can principally reduce the computation of generalized theta functions to the case

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of genus zero with many parabolic points. In order to have an induction machinery for the number of parabolic points, one should prove a factorization result when X_0 has two smooth irreducible components intersecting at a node x_0. This was done for rank two in [DW1] and [DW2] by an analytic method. In this paper, we adopt the approach of [NR] and [Su] to prove a factorization theorem for arbitrary rank in the reducible case.

Let $I=I_1\cup I_2\subset X$ be a finite set of points and $\mathcal{U}_X^I$ be the moduli space of semistable parabolic bundles with parabolic structures at the points $\{x\}_{x\in I}$. When X degenerates to $X_0=X_1\cup X_2$ and the points in I_j ($j=1,2$) degenerate to $|I_j|$ points $x\in I_j\subset X_j\setminus\{x_0\}$, we have to construct a degeneration $\mathcal{U}_{X_0}:=\mathcal{U}_{X_1\cup X_2}^{I_1\cup I_2}$ of $\mathcal{U}_X^I$ and a theta line bundle $\Theta_{\mathcal{U}_{X_0}}$ on it. Fixing a suitable ample line bundle $\mathcal{O}(1)$ on X_0, we construct the degeneration as a moduli space of 'semistable' parabolic torsion free sheaves on X_0 with parabolic structures at the points $x\in I_1\cup I_2$, and define the theta line bundle $\Theta_{\mathcal{U}_{X_0}}$ on it. Our main observation here is that we need a 'new semistability' (see Definition 1.3) to construct the correct degeneration of $\mathcal{U}_X^I$. But in the whole paper, this 'new semistability' is simply called semistable. It should not cause any confusion since our 'new semistability' coincides with Seshadri's semistability in [Se] when $I=\emptyset$, and coincides with the semistability of [NR] when X_0 is irreducible.

Let $\pi\colon\widetilde{X}_0\to X_0$ be the normalization of X_0 and $\pi^{-1}(x_0)=\{x_1,x_2\}$. Then for any $\mu=(\mu_1,\dots,\mu_r)$ with $0\le\mu_r\le\dots\le\mu_1\le k-1$, we can define $\vec{a}(x_j)$, $\vec{n}(x_j)$ and α_{x_j} ($j=1,2$) by using μ (see Notation 3.1). Let

$$\mathcal{U}_{X_j}^{\mu}:=\mathcal{U}_{X_j}(r,\chi_j^{\mu},I_j\cup\{x_j\},\{\vec{n}(x),\vec{a}(x)\}_{x\in I_j\cup\{x_j\}},k)$$

be the moduli space of s-equivalence classes of semistable parabolic bundles E of rank r on X_j and Euler characteristic $\chi(E)=\chi_j^{\mu}$, together with parabolic structures of type $\{\vec{n}(x)\}_{x\in I\cup\{x_j\}}$ and weights $\{\vec{a}(x)\}_{x\in I\cup\{x_j\}}$ at the points $\{x\}_{x\in I\cup\{x_j\}}$, where χ_j^{μ} is defined in Notation 3.1 and may be nonintegers. Thus we define $\mathcal{U}_{X_j}^{\mu}$ to be empty if χ_j^{μ} is not an integer. Let

$$\Theta_{\mathcal{U}_{X_j}^{\mu}}:=\Theta(k,l_j,\{\vec{n}(x),\vec{a}(x),\alpha_x\}_{x\in I_j\cup\{x_j\}},I_j\cup\{x_j\})$$

be the theta line bundle. Then our main result is the following theorem.

Factorization theorem. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_{X_0},\Theta_{\mathcal{U}_{X_0}})\cong\bigoplus_{\mu}H^0(\mathcal{U}_{X_1}^{\mu},\Theta_{\mathcal{U}_{X_1}^{\mu}})\otimes H^0(\mathcal{U}_{X_2}^{\mu},\Theta_{\mathcal{U}_{X_2}^{\mu}}),$$

where $\mu=(\mu_1,\dots,\mu_r)$ runs through the integers $0\le\mu_r\le\dots\le\mu_1\le k-1$.

Section 1 is devoted to the construction of the moduli space $\mathcal{U}_{X_0}$ by generalizing Simpson's construction, and the construction of the theta line bundle on it. Then we determine the number of irreducible components of the moduli space and proving the nonemptyness of them (see Proposition 1.4). In Section 2, we sketch the construction of the moduli space $\mathcal{P}$ of generalized parabolic sheaves (abbreviated GPS) and construct an ample line bundle on it. Then we introduce and study the s-equivalence of GPSes (see Proposition 2.5), which will be needed in studying the normalization of $\mathcal{U}_{X_0}$. In Section 3, we construct and study the normalization $\mathcal{P}\to\mathcal{U}_{X_0}$, and then prove the factorization theorem (Theorem 3.1). As a byproduct, we recover the main results of [NS] (see Corollary 3.1 and Remark 3.1). They have used triples in [NS] instead of GPSes.

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1. The moduli space of parabolic sheaves

Let X_0 be a reduced projective curve over $\mathbf{C}$ with two smooth irreducible components X_1 and X_2 of genus g_1 and g_2 meeting at only one point x_0, which is the node of X_0. We fix a finite set I of smooth points on X_0 and write $I=I_1\cup I_2$, where $I_i=\{x\in I\,|\,x\in X_i\}$ $(i=1,2)$.

Definition 1.1. A coherent $\mathcal{O}_{X_0}$-module E is called *torsion free* if it is purely of dimension one, namely, for all nonzero $\mathcal{O}_{X_0}$-submodules $E_1\subset E$, the dimension of $\operatorname{supp} E_1$ is one.

A coherent sheaf E is torsion free if and only if E_x has depth one at every $x\in X_0$ as an $\mathcal{O}_{X_0,x}$-module. Thus E is locally free over $X_0\setminus\{x_0\}$.

Definition 1.2. We say that a torsion free sheaf E over X_0 has a *quasi-parabolic structure* of type $\vec{n}(x)=(n_1(x),\dots,n_{l_x+1}(x))$ at $x\in I$, if we choose a flag of subspaces

$$E|_{\{x\}}=F_0(E)_x\supset F_1(E)_x\supset\dots\supset F_{l_x}(E)_x\supset F_{l_x+1}(E)_x=0$$

such that $n_j(x)=\dim(F_{j-1}(E)_x/F_j(E)_x)$. If, in addition, a sequence of integers called the *parabolic weights*

$$0<a_1(x)<a_2(x)<\dots<a_{l_x+1}(x)<k$$

are given, we say that E has a *parabolic structure* of type $\vec{n}(x)$ at x, with weights $\vec{a}(x):=(a_1(x),\dots,a_{l_x+1}(x))$. The sheaf E is also simply called a *parabolic sheaf*, whose *parabolic Euler characteristic* is defined as

$$\operatorname{par}\chi(E):=\chi(E)+\frac{1}{k}\sum_{x\in I}\sum_{i=1}^{l_x+1}n_i(x)a_i(x).$$

We will fix an ample line bundle $\mathcal{O}(1)$ on X_0 such that $\deg\mathcal{O}(1)|_{X_i}=c_i>0$ $(i=1,2)$, for simplicity, we assume that $\mathcal{O}(1)=\mathcal{O}_{X_0}(c_1y_1+c_2y_2)$ for two fixed smooth points $y_i\in X_i$. For any torsion free sheaf E, $P(E,n):=\chi(E(n))$ denotes its *Hilbert polynomial*, which has degree one. We define the *rank* of E to be

$$r(E):=\frac{1}{\deg\mathcal{O}(1)}\lim_{n\to\infty}\frac{P(E,n)}{n}.$$

Let r_i denote the rank of the restriction of E to X_i $(i=1,2)$, then

$$P(E,n)=(c_1r_1+c_2r_2)n+\chi(E)\quad\text{and}\quad r(E)=\frac{c_1}{c_1+c_2}r_1+\frac{c_2}{c_1+c_2}r_2.$$

Notation 1.1. We say that E is a *torsion free sheaf of rank* r on X_0 if $r_1=r_2=r$, otherwise it will be said to be of rank (r_1,r_2). In this paper we will fix the parabolic data $\{\vec{n}(x)\}_{x\in I}$, $\{\vec{a}(x)\}_{x\in I}$ and the integers $\chi=d+r(1-g)$, l_1+l_2, k and

$$\alpha:=\{0\le\alpha_x<k-a_{l_x+1}(x)+a_1(x)\}_{x\in I}$$

such that

$$(*)\qquad \sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I}\alpha_x+r(l_1+l_2)=k\chi.$$

where $d_i(x)=a_{i+1}(x)-a_i(x)$ and $r_i(x)=n_1(x)+\dots+n_i(x)$. We will choose c_1 and c_2 such that $l_1=c_1(l_1+l_2)/(c_1+c_2)$ and $l_2=c_2(l_1+l_2)/(c_1+c_2)$ become integers.

Definition 1.3. With the fixed parabolic data in Notation 1.1, and for any torsion free sheaf F of rank (r_1,r_2), let

$$m(F):=\frac{r(F)-r_1}{k}\sum_{x\in I_1}(a_{l_x+1}(x)+\alpha_x)+\frac{r(F)-r_2}{k}\sum_{x\in I_2}(a_{l_x+1}(x)+\alpha_x).$$

If F has parabolic structures at the points $x\in I$, then the *modified parabolic Euler characteristic* and *slope* of F are defined as

$$\operatorname{par}\chi_m(F):=\operatorname{par}\chi(F)+m(F)\quad\text{and}\quad\operatorname{par}\mu_m(F):=\frac{\operatorname{par}\chi_m(F)}{r(F)}.$$

A parabolic sheaf E is *semistable* (resp. *stable*) for $(k, \alpha, \vec{a})$ if, for any subsheaf $F \subset E$ such E/F is torsion free, one has, with the induced parabolic structure,

$$\operatorname{par}\chi_m(F) \leq \frac{\operatorname{par}\chi_m(E)}{r(E)} r(F) \quad (\text{resp. } <).$$

Remark 1.1. When the curves are irreducible, then $m(F)=0$, the above semistability is independent of the choice of α and coincides with Seshadri's semistability of parabolic torsion free sheaves.

In this section we will only consider torsion free sheaves of rank r with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$, and construct the moduli space of semistable parabolic sheaves. Let $\mathcal{W}=\mathcal{O}_{X_0}(-N)$ and $V=\mathbf{C}^{P(N)}$, we consider the Quot scheme

$$\operatorname{Quot}(V\otimes\mathcal{W}, P)(T) = \left\{ \begin{array}{l} T\text{-flat quotients } V\otimes\mathcal{W} \longrightarrow E \longrightarrow 0 \text{ over} \\ X_0 \times T \text{ with Hilbert polynomial } P \end{array} \right\}.$$

and let $\mathbf{Q} \subset \operatorname{Quot}(V\otimes\mathcal{W}, P)$ be the open set

$$\mathbf{Q}(T) = \left\{ \begin{array}{l} V\otimes\mathcal{W} \longrightarrow E \longrightarrow 0, \text{ with } R^1 p_{T*}(E(N)) = 0 \text{ and such} \\ \text{that } V\otimes\mathcal{O}_T \longrightarrow p_{T*}E(N) \text{ induces an isomorphism} \end{array} \right\}.$$

Thus we can assume (Lemma 20 of [Se], p. 162) that N is chosen large enough so that every semistable parabolic torsion free sheaf with Hilbert polynomial P and parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$, weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$ appears as a quotient corresponding to a point of $\mathbf{Q}$.

Let $\widetilde{\mathbf{Q}}$ be the closure of $\mathbf{Q}$ in the Quot scheme and $V\otimes\mathcal{W}\to\mathcal{F}\to 0$ be the universal quotient over $X_0\times\widetilde{\mathbf{Q}}$ and $\mathcal{F}_x$ be the restriction of $\mathcal{F}$ to $\{x\}\times\widetilde{\mathbf{Q}}\cong\widetilde{\mathbf{Q}}$. Let $\operatorname{Flag}_{\vec{n}(x)}(\mathcal{F}_x)\to\widetilde{\mathbf{Q}}$ be the relative flag scheme of type $\vec{n}(x)$, and

$$\mathcal{R} = \prod_{x\in I}{}_{\widetilde{\mathbf{Q}}} \operatorname{Flag}_{\vec{n}(x)}(\mathcal{F}_x) \to \widetilde{\mathbf{Q}}$$

be the product over $\widetilde{\mathbf{Q}}$. A (closed) point $(p, \{p_{r(x)}, p_{r_1(x)}, \cdots, p_{r_{l_x}(x)}\}_{x\in I})$ of $\mathcal{R}$ is by definition given by a point $V\otimes\mathcal{W}\xrightarrow{p} E\to 0$ of the Quot scheme, together with quotients

$$\{V\otimes\mathcal{W} \xrightarrow{p_{r(x)}} Q_{r(x)}, V\otimes\mathcal{W} \xrightarrow{p_{r_1(x)}} Q_{r_1(x)}, \dots, V\otimes\mathcal{W} \xrightarrow{p_{r_{l_x}(x)}} Q_{r_{l_x}(x)}\}_{x\in I},$$

where $r_i(x)=\dim(E_x/F_i(E)_x)=n_1(x)+\ldots+n_i(x)$, and

$$Q_{r(x)}:=E_x,\quad Q_{r_1(x)}:=\frac{E_x}{F_1(E)_x},\quad \ldots,\quad Q_{r_{l_x}(x)}:=\frac{E_x}{F_{l_x}(E)_x}.$$

For large enough m, we have an $SL(V)$-equivariant embedding

$$\mathcal{R}\hookrightarrow \mathbf{G}=\mathrm{Grass}_{P(m)}(V\otimes W_m)\times\mathbf{Flag},$$

where $W_m=H^0(\mathcal{W}(m))$, and **Flag** is defined to be

$$\mathbf{Flag}=\prod_{x\in I}\{\mathrm{Grass}_{r(x)}(V)\times\mathrm{Grass}_{r_1(x)}(V)\times\ldots\times\mathrm{Grass}_{r_{l_x}(x)}(V)\},$$

which maps a point $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I})$ of $\mathcal{R}$ to the point

$$(V\otimes W_m\xrightarrow{g}U,\{V\xrightarrow{g_{r(x)}}U_{r(x)},V\xrightarrow{g_{r_1(x)}}U_{r_1(x)},\cdots,V\xrightarrow{g_{r_{l_x}(x)}}U_{r_{l_x}(x)}\}_{x\in I})$$

of $\mathbf{G}$, where

$$\begin{aligned} g&:=H^0(p(m)), & U&:=H^0(E(m)),\\ g_{r(x)}&:=H^0(p_{r(x)}(N)), & U_{r(x)}&:=H^0(Q_{r(x)}),\\ g_{r_i(x)}&:=H^0(p_{r_i(x)}(N)), & U_{r_i(x)}&:=H^0(Q_{r_i(x)}),\quad i=1,\ldots,l_x.\end{aligned}$$

For any rational number l satisfying $c_il=l_i+c_ikN$ $(i=1,2)$, we give $\mathbf{G}$ the polarization (using the obvious notation)

$$\frac{l}{m-N}\times\prod_{x\in I}\{\alpha_x,d_1(x),\ldots,d_{l_x}(x)\},$$

and we have a straightforward generalization of [NR, Proposition A.6] whose proof we omit.

Proposition 1.1. *A point $(g,\{g_{r(x)},g_{r_1(x)},\cdots,g_{r_{l_x}(x)}\}_{x\in I})\in\mathbf{G}$ is stable (resp. semistable) for the action of $SL(V)$, with respect to the above polarization (we refer to this from now on as GIT-stability), if and only if for all nontrivial subspaces $H\subset V$ we have (with $h=\dim H$)*

$$\frac{l}{m-N}(hP(m)-P(N)\dim g(H\otimes W_m))+\sum_{x\in I}\alpha_x(rh-P(N)\dim g_{r(x)}(H))$$
$$+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)h-P(N)\dim g_{r_i(x)}(H))<0\quad (\textit{resp. }\le 0).$$

Notation 1.2. Given a point $(p, \{p_{r(x)}, p_{r_1(x)}, \dots, p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$, and a subsheaf F of E we denote the image of F in $Q_{r_i(x)}$ (resp. $Q_{r(x)}$) by $Q^F_{r_i(x)}$ (resp. $Q^F_{r(x)}$). Similarly, given a quotient $E \xrightarrow{T} \mathcal{G} \to 0$, set $Q^{\mathcal{G}}_{r_i(x)} := Q_{r_i(x)} / \operatorname{Im}\ker T$ (resp. $Q^{\mathcal{G}}_{r(x)} := Q_{r(x)} / \operatorname{Im}\ker T$).

Proposition 1.2. *Suppose $(p, \{p_{r(x)}, p_{r_1(x)}, \dots, p_{r_{l_x}(x)}\}_{x\in I}) \in \mathcal{R}$ is a point such that E is torsion free. Then E is stable (resp. semistable) if and only if for every subsheaf $0 \neq F \neq E$ we have*

$$\frac{l}{m-N}(\chi(F(N))P(m) - P(N)\chi(F(m))) + \sum_{x\in I} \alpha_x (r\chi(F(N)) - P(N)h^0(Q^F_{r(x)}))$$
$$+ \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)(r_i(x)\chi(F(N)) - P(N)h^0(Q^F_{r_i(x)})) < 0 \quad (\textit{resp. } \leq 0).$$

Proof. For any subsheaf F let $\mathrm{LHS}(F)$ denote the left-hand side of the above inequality. Assume first that E/F is torsion free and that F is of rank (r_1, r_2), thus $h^0(Q^F_{r(x)}) = r_i$, $h^0(Q^F_{r_i(x)}) = \dim(F_x / F_x \cap F_i(E)_x)$ for $x \in I_i$ $(i=1,2)$ and $\chi(F(m)) = (c_1 r_1 + c_2 r_2)(m-N) + \chi(F(N))$. Let $n_i^F(x) := \dim(F_x \cap F_{i-1}(E)_x / F_x \cap F_i(E)_x)$. As

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x) = r\sum_{x\in I} a_{l_x+1}(x) - \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x) n_i(x).$$

$$\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) \dim \frac{F_x}{F_x \cap F_i(E)_x} = r_1 \sum_{x\in I_1} a_{l_x+1}(x) + r_2 \sum_{x\in I_2} a_{l_x+1}(x)$$
$$- \sum_{x\in I}\sum_{i=1}^{l_x+1} a_i(x) n_i^F(x),$$

we have

$$\mathrm{LHS}(F) = \Bigl(\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x) + r\sum_{x\in I}\alpha_x + rc_1 l + rc_2 l\Bigr)\Bigl(\chi(F) - \frac{r(F)}{r}\chi\Bigr)$$
$$+ P(N)\Bigl(r(F)\sum_{x\in I}\alpha_x + \frac{r(F)}{r}\sum_{x\in I}\sum_{i=1}^{l_x} d_i(x) r_i(x)\Bigr)$$
$$- P(N)\Bigl(r_1\sum_{x\in I_1}\alpha_x + r_2 \sum_{x\in I_2}\alpha_x + \sum_{x\in I}\sum_{i=1}^{l_x} d_i(x)\dim\frac{F_x}{F_x\cap F_i(E)_x}\Bigr)$$
$$= kP(N)\Bigl(\operatorname{par}\chi_m(F) - \frac{r(F)}{r}\operatorname{par}\chi_m(E)\Bigr).$$

Thus the inequality implies the (semi)stability of E, and the (semi)stability of E implies the inequality for subsheaves F such that E/F is torsion free.

Suppose now that E is (semi)stable and F is any nontrivial subsheaf, let τ be the torsion of E/F and $F'\subset E$ such that $\tau=F'/F$ and E/F' are torsion free. Then we have $\mathrm{LHS}(F')\leq 0$ and, if we write $\tau=\tilde{\tau}+\sum_{x\in I}\tau_x$, then

$$\begin{aligned}
\mathrm{LHS}(F)-\mathrm{LHS}(F') &= -(c_1+c_2)rlh^0(\tau)\\
&\quad -\sum_{x\in I}\alpha_x(rh^0(\tau)+P(N)(h^0(Q^F_{r(x)})-h^0(Q^{F'}_{r(x)})))\\
&\quad -\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)h^0(\tau)+P(N)(h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)})))\\
&= -h^0(\tau)\Big(\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I}\alpha_x+rc_1l+rc_2l\Big)\\
&\quad +P(N)\Big(\sum_{x\in I}\alpha_xh^0(\tau_x)-\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)}))\Big)\\
&\leq -kP(N)h^0(\tau)+P(N)\sum_{x\in I}\alpha_xh^0(\tau_x)+P(N)\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(\tau_x)\\
&= -kP(N)h^0(\tilde{\tau})-P(N)\sum_{x\in I}(k-\alpha_x-a_{l_x+1}(x)+a_1(x))h^0(\tau_x)\\
&\leq 0,
\end{aligned}$$

where we have used $h^0(Q^F_{r(x)})-h^0(Q^{F'}_{r(x)})=-h^0(\tau_x)$, the assumption about $\{\alpha_x\}$ and $h^0(Q^F_{r_i(x)})-h^0(Q^{F'}_{r_i(x)})\geq -h^0(\tau_x)$. $\square$

Lemma 1.1. *There exists $M_1(N)$ such that for $m\geq M_1(N)$ the following holds. Suppose $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I})\in\mathcal{R}$ is a point which is GIT-semistable then for all quotients $E\xrightarrow{T}\mathcal{G}\to 0$ we have*

$$h^0(\mathcal{G}(N))\geq\frac{1}{k}\Big((c_1+c_2)r(\mathcal{G})l+\sum_{x\in I}\alpha_xh^0(Q^{\mathcal{G}}_{r(x)})+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\Big).$$

In particular, E is torsion free and $V\to H^0(E(N))$ is an isomorphism.

Proof. Let

$$H=\ker\{V\xrightarrow{H^0(p(N))}H^0(E(N))\longrightarrow H^0(\mathcal{G}(N))\}.$$

and $F\subset E$ be the subsheaf generated by H. Since all these F are in a bounded family, $\dim g(H\otimes W_m)=h^0(F(m))=\chi(F(m))$ for m large enough. Thus there exists $M_1(N)$ such that for $m\geq M_1(N)$ the inequality of Proposition 1.1 implies (with $h=\dim H$)

$$(c_1+c_2)l(rh-r(F)P(N))+\sum_{x\in I}\alpha_x(rh-P(N)h^0(Q^F_{r(x)}))$$
$$+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)h-P(N)h^0(Q^F_{r_i(x)}))\leq 0,$$

where we used that $g_{r(x)}(H)=h^0(Q^F_{r(x)})$ and $g_{r_i(x)}(H)=h^0(Q^F_{r_i(x)})$. Now using the inequalities

$$h\geq P(N)-h^0(\mathcal{G}(N)),$$
$$r-r(F)\geq r(\mathcal{G}),$$
$$r-h^0(Q^F_{r(x)})\geq h^0(Q^{\mathcal{G}}_{r(x)}),$$
$$r_i(x)-h^0(Q^F_{r_i(x)})\geq h^0(Q^{\mathcal{G}}_{r_i(x)}),$$

we get the inequality

$$h^0(\mathcal{G}(N))\geq\frac{1}{k}\left((c_1+c_2)r(\mathcal{G})l+\sum_{x\in I}\alpha_x h^0(Q^{\mathcal{G}}_{r(x)})+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right).$$

Now we show that $V\to H^0(E(N))$ is an isomorphism. That it is injective is easy to see: let H be its kernel, then $g(H\otimes W_m)=0$, $g_{r(x)}(H)=0$ and $g_{r_i(x)}(H)=0$, one sees that $h=0$ from Proposition 1.1. To see it being surjective, it is enough to show that one can choose N such that $H^1(E(N))=0$ for all such E. If $H^1(E(N))$ is nontrivial, then there is a nontrivial quotient $E(N)\to L\subset\omega_{X_0}$ by Serre duality, and thus

$$h^0(\omega_{X_0})\geq h^0(L)\geq(c_1+c_2)N+B,$$

where B is a constant independent of E, we choose N such that $H^1(E(N))=0$ for all GIT-semistable points.

Let $\tau=\operatorname{Tor}E$, $\mathcal{G}=E/\tau$ and applying the inequality, noting that $h^0(\mathcal{G}(N))=P(N)-h^0(\tau)$, $h^0(Q^{\mathcal{G}}_{r(x)})=r-h^0(Q^{\tau}_{r(x)})$ and $h^0(Q^{\mathcal{G}}_{r_i(x)})=r_i(x)-h^0(Q^{\tau}_{r_i(x)})$, we have

$$kh^0(\tau)\leq\sum_{x\in I}(\alpha_x+a_{l_x+1}(x)-a_1(x))h^0(\tau_x),$$

by which one can conclude that $\tau=0$ since $\alpha_x<k-a_{l_x+1}(x)+a_1(x)$. □

Proposition 1.3. *There exist integers $N>0$ and $M(N)>0$ such that for $m\geq M(N)$ the following is true. A point $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I})\in\mathcal{R}$ is GIT-stable (resp. GIT-semistable) if and only if the quotient E is torsion free and a stable (resp. semistable) sheaf and the map $V\to H^0(E(N))$ is an isomorphism.*

Proof. If $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I})\in\mathcal{R}$ is GIT-stable (GIT-semistable), by Lemma 1.1, E is torsion free and $V\to H^0(E(N))$ is an isomorphism. For any subsheaf $F\subset E$ with E/F torsion free, let $H\subset V$ be the inverse image of $H^0(F(N))$ and $h=\dim H$, we have $\chi(F(N))P(m)-P(N)\chi(F(m))\leq hP(m)-P(N)h^0(F(m))$ for $m>N$ (note that $h^1(F(N))\geq h^1(F(m))$). Thus

$$\begin{aligned}
&\frac{l}{m-N}(\chi(F(N))P(m)-P(N)\chi(F(m)))+\sum_{x\in I}\alpha_x(r\chi(F(N))-P(N)h^0(Q^F_{r(x)}))\\
&\quad+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)\chi(F(N))-P(N)h^0(Q^F_{r_i(x)}))\\
&\leq\frac{l}{m-N}(hP(m)-P(N)\dim g(H\otimes W_m))+\sum_{x\in I}\alpha_x(rh-P(N)\dim g_{r(x)}(H))\\
&\quad+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)h-P(N)\dim g_{r_i(x)}(H))
\end{aligned}$$

since $g(H\otimes W_m)\leq h^0(F(m))$, $g_{r(x)}(H)\leq h^0(Q^F_{r(x)})$ and $g_{r_i(x)}(H)\leq h^0(Q^F_{r_i(x)})$ (the inequalities are strict when $h=0$). By Propositions 1.1 and 1.2, E is stable (resp. semistable) if the point is GIT-stable (resp. GIT-semistable).

The proof of the other direction is similar to [NR], one can prove the similar Lemma A.9 and Lemma A.12 of [NR] by just modifying the notation. $\square$

One can imitate [Se] (Théorème 12, p. 71) to show that given a semistable parabolic sheaf E, there exists a filtration of E

$$0=E_{n+1}\subset E_n\subset\ldots\subset E_2\subset E_1\subset E_0=E$$

such that E_i/E_{i+1} $(0\leq i\leq n)$ are stable parabolic sheaves with the constant slope $\operatorname{par}\mu_m(E)$, and the isomorphic class of semistable parabolic sheaves

$$\operatorname{gr}E:=\bigoplus_{i=0}^{n}\frac{E_i}{E_{i+1}}$$

is independent of the filtration. Two semistable parabolic sheaves E and E' are called s-equivalent if $\operatorname{gr}E\cong\operatorname{gr}E'$.

Theorem 1.1. *For given data in Notation 1.1 satisfying* $(*)$, *there exists a reduced, seminormal projective scheme*

$$\mathcal{U}_{X_0} := \mathcal{U}_{X_0}(r, \chi, I_1 \cup I_2, \{\vec{n}(x), \vec{a}(x), \alpha_x\}_{x \in I}, \mathcal{O}(1), k),$$

which is the coarse moduli space of s-equivalence classes of semistable parabolic sheaves E *of rank* r *and Euler chracteristic* $\chi(E)=\chi$ *with parabolic structures of type* $\{\vec{n}(x)\}_{x\in I}$ *and weights* $\{\vec{a}(x)\}_{x\in I}$ *at the points* $\{x\}_{x\in I}$. *The moduli space* $\mathcal{U}_{X_0}$ *has at most* $r+1$ *irreducible components.*

Proof. Let $\mathcal{R}^{ss}$ ($\mathcal{R}^{s}$) be the open set of $\mathcal{R}$ whose points correspond to semistable (stable) parabolic sheaves on X_0. Then, by Proposition 1.3, the quotient

$$\varphi: \mathcal{R}^{ss} \longrightarrow \mathcal{U}_{X_0} := \mathcal{R}^{ss} /\!\!/ SL(V)$$

exists as a projective scheme. That $\mathcal{U}_{X_0}$ is reduced and seminormal follow from the properties of $\mathcal{R}^{ss}$ (see [F], [Se] and [Su]).

Consider the dense open set $\mathcal{R}_0 \subset \mathcal{R}^{ss}$ consisting of locally free sheaves. For each $F \in \mathcal{R}_0$, let F_1 and F_2 be the restrictions of F to X_1 and X_2. We have

$$0 \longrightarrow F_1(-x_0) \longrightarrow F \longrightarrow F_2 \longrightarrow 0. \tag{1.1}$$

By the semistability of F and $\operatorname{par}\chi_m(F_1) + \operatorname{par}\chi_m(F_2) = \operatorname{par}\chi_m(F) + r$, we have

$$\frac{c_1}{c_1+c_2}\operatorname{par}\chi_m(F) \le \operatorname{par}\chi_m(F_1) \le \frac{c_1}{c_1+c_2}\operatorname{par}\chi_m(F) + r,$$
$$\frac{c_2}{c_1+c_2}\operatorname{par}\chi_m(F) \le \operatorname{par}\chi_m(F_2) \le \frac{c_2}{c_1+c_2}\operatorname{par}\chi_m(F) + r.$$

Let for $j=1,2$, χ_j denote $\chi(F_j)$ and

$$n_j = \frac{1}{k}\left(\sum_{x\in I_j}\sum_{i=1}^{l_x} d_i(x) r_i(x) + r\sum_{x\in I_j}\alpha_x + r l_j\right). \tag{1.2}$$

We can rewrite the above inequalities into

$$n_1 \le \chi_1 \le n_1 + r \quad \text{and} \quad n_2 \le \chi_2 \le n_2 + r. \tag{1.3}$$

There are at most $r+1$ possible choices of (χ_1, χ_2) satisfying (1.3) and $\chi_1+\chi_2=\chi+r$, each of the choices corresponds to an irreducible component of $\mathcal{U}_{X_0}$. $\square$

For any χ_1 and χ_2 satisfying (1.3), let $\mathcal{U}_{X_1}$ (resp. $\mathcal{U}_{X_2}$) be the moduli space of semistable parabolic bundles of rank r and Euler characteristic χ_1 (resp. χ_2), with parabolic structures of type $\{\vec{n}(x)\}_{x\in I_1}$ (resp. $\{\vec{n}(x)\}_{x\in I_2}$) and weights $\{\vec{a}(x)\}_{x\in I_1}$ (resp. $\{\vec{a}(x)\}_{x\in I_2}$) at the points $\{x\}_{x\in I_1}$ (resp. $\{x\}_{x\in I_2}$). Then we have the following result.

Proposition 1.4. *Suppose that $\mathcal{U}_{X_1}$ and $\mathcal{U}_{X_2}$ are nonempty. Then there exists a semistable parabolic vector bundle E on X_0, with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$, such that*

$$E|_{X_1}\in\mathcal{U}_{X_1} \quad \textit{and} \quad E|_{X_2}\in\mathcal{U}_{X_2}.$$

Moreover, if $n_1\le\chi_1<n_1+r$ and $n_2\le\chi_2<n_2+r$, then E is stable whenever one of E_1 and E_2 is stable.

Proof. For any $E_1\in\mathcal{U}_{X_1}$ and $E_2\in\mathcal{U}_{X_2}$, one can glue them by any isomorphism at x_0 into a vector bundle E on X_0 with the described parabolic structures at the points $\{x\}_{x\in I}$ such that $E|_{X_1}=E_1$ and $E|_{X_2}=E_2$. We will show that E is semistable.

For any subsheaf $F\subset E$ of rank (r_1,r_2) such that E/F is torsion free, we have the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0 \\ & & \big\downarrow & & \big\downarrow & & \big\downarrow & & \\ 0 & \longrightarrow & E_1(-x_0) & \longrightarrow & E & \longrightarrow & E_2 & \longrightarrow & 0, \end{array} \tag{1.4}$$

where F_2 is the image of F under $E\to E_2\to 0$ and F_1 is the kernel of $F\to F_2\to 0$. One easily sees that F_1 and F_2 are torsion free sheaves of rank $(r_1,0)$ and $(0,r_2)$. From the diagram (1.4), we have the equalities

$$\begin{aligned}
&\frac{\operatorname{par}\chi_m(E)}{r}-\frac{\operatorname{par}\chi_m(F)}{r(F)}\\
&\quad=\frac{\operatorname{par}\chi_m(E_1(-x_0))}{r}-\frac{\operatorname{par}\chi_m(F_1)}{r(F)}+\frac{\operatorname{par}\chi_m(E_2)}{r}-\frac{\operatorname{par}\chi_m(F_2)}{r(F)}\\
&\quad=\frac{a_1r_1\operatorname{par}\chi_m(E_1(-x_0))-r\operatorname{par}\chi_m(F_1)+a_2r_2\operatorname{par}\chi_m(E_1(-x_0))}{r(F)r}\\
&\qquad+\frac{a_2r_2\operatorname{par}\chi_m(E_2)-r\operatorname{par}\chi_m(F_2)+a_1r_1\operatorname{par}\chi_m(E_2)}{r(F)r}\\
&\quad=\frac{r_1}{r(F)}(\operatorname{par}\mu_m(E_1(-x_0))-\operatorname{par}\mu_m(F_1))+\frac{r_2}{r(F)}(\operatorname{par}\mu_m(E_2)-\operatorname{par}\mu_m(F_2))\\
&\qquad+\frac{a_2(r_2-r_1)\operatorname{par}\chi_m(E_1(-x_0))+a_1(r_1-r_2)\operatorname{par}\chi_m(E_2)}{r(F)r}\\
&\quad=\frac{r_1}{r(F)}(\operatorname{par}\mu(E_1(-x_0))-\operatorname{par}\mu(F_1))+\frac{r_2}{r(F)}(\operatorname{par}\mu(E_2)-\operatorname{par}\mu(F_2))\\
&\qquad+\frac{(r_1-r_2)\left(\dfrac{c_1}{c_1+c_2}\operatorname{par}\chi_m(E)+r-\operatorname{par}\chi_m(E_1)\right)}{r(F)r};
\end{aligned}$$

where we used the notation $a_1 := c_1/(c_1+c_2)$ and $a_2 := c_2/(c_1+c_2)$. The last equality follows since

$$\frac{m(E_1(-x_0))}{r} - \frac{m(F_1)}{r_1} = 0 \quad \text{and} \quad \frac{m(E_2)}{r} - \frac{m(F_2)}{r_2} = 0.$$

Similarly, if we use the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_2 & \longrightarrow & F & \longrightarrow & F_1 & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & E_2(-x_0) & \longrightarrow & E & \longrightarrow & E_1 & \longrightarrow & 0, \end{array}$$

we get the equality

$$\begin{aligned} \frac{\operatorname{par}\chi_m(E)}{r} - \frac{\operatorname{par}\chi_m(F)}{r(F)} &= \frac{r_2}{r(F)}(\operatorname{par}\mu(E_2(-x_0)) - \operatorname{par}\mu(F_2)) \\ &\quad + \frac{r_1}{r(F)}(\operatorname{par}\mu(E_1) - \operatorname{par}\mu(F_1)) \\ &\quad + \frac{(r_2-r_1)\left(\dfrac{c_2}{c_1+c_2}\operatorname{par}\chi_m(E) + r - \operatorname{par}\chi_m(E_2)\right)}{r(F)r}. \end{aligned}$$

Thus we always have the inequality

$$\frac{\operatorname{par}\chi_m(E)}{r} - \frac{\operatorname{par}\chi_m(F)}{r(F)} \geq 0$$

and the equality implies that $r_1 = r_2$ and that E_1 and E_2 are both unstable. This proves the proposition. □

By a family of parabolic sheaves of rank r and Euler characteristic χ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$ parametrized by T, we mean a sheaf $\mathcal{F}$ on $X_0 \times T$, flat over T and torsion free with rank r and Euler characteristic χ on $X_0 \times \{t\}$ for every $t \in T$, together with, for each $x \in I$, a flag

$$\mathcal{F}_{\{x\}\times T} = F_0(\mathcal{F}_{\{x\}\times T}) \supset F_1(\mathcal{F}_{\{x\}\times T}) \supset \ldots \supset F_{l_x}(\mathcal{F}_{\{x\}\times T}) \supset F_{l_x+1}(\mathcal{F}_{\{x\}\times T}) = 0$$

of subbundles of type $\vec{n}(x)$ and weights $\vec{a}(x)$. Let $\mathcal{Q}_{\{x\}\times T,i}$ denote the quotients $\mathcal{F}_{\{x\}\times T}/F_i(\mathcal{F}_{\{x\}\times T})$, then we define a line bundle $\Theta_{\mathcal{F}}$ on T to be

$$(\det R\pi_T\mathcal{F})^k \otimes \bigotimes_{x\in I}\left\{(\det \mathcal{F}_{\{x\}\times T})^{\alpha_x} \otimes \bigotimes_{i=1}^{l_x}(\det \mathcal{Q}_{\{x\}\times T,i})^{d_i(x)}\right\} \otimes \bigotimes_{j=1}^{2}(\det \mathcal{F}_{\{y_j\}\times T})^{l_j},$$

where π_T is the projection $X_0 \times T \to T$, and $\det R\pi_T\mathcal{F}$ is the determinant bundle defined as

$$\{\det R\pi_T\mathcal{F}\}_t := \{\det H^0(X, \mathcal{F}_t)\}^{-1} \otimes \{\det H^1(X, \mathcal{F}_t)\}.$$

Theorem 1.2. *On the moduli space $\mathcal{U}_{X_0}$, there is a unique ample line bundle $\Theta_{\mathcal{U}_{X_0}}=\Theta(k,l_1,l_2,\vec{a},\vec{n},\alpha,I)$ such that for any given family $\mathcal{F}$ of semistable parabolic sheaves parametrized by T, we have $\phi_T^*\Theta_{\mathcal{U}_{X_0}}=\Theta_{\mathcal{F}}$, where ϕ_T is the induced map $T\to\mathcal{U}_{X_0}$.*

Proof. By using the descend lemma (see Lemma 1.2 below), we will show that the line bundle $\Theta_{\mathcal{R}^{ss}}:=\Theta_{\mathcal{E}}$ on $\mathcal{R}^{ss}$ descends to the required ample line $\Theta_{\mathcal{U}_{X_0}}$, where $\mathcal{E}$ is a universal quotient over $X_0\times\mathcal{R}^{ss}$.

We know that the stabilizer $\mathrm{stab}(q)=\lambda\,\mathrm{id}$ for $q\in\mathcal{R}^s$, which acts on $\Theta_{\mathcal{R}^{ss}}$ via

$$\lambda^{-k\chi+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I}\alpha_x+r(l_1+l_2)}=\lambda^0=1.$$

If $q\in\mathcal{R}^{ss}\setminus\mathcal{R}^s$ has a closed orbit, we know that

$$\mathcal{E}_q=m_1E_1\oplus m_2E_2\oplus\ldots\oplus m_tE_t,$$

with $\mathrm{par}\,\mu_m(E_j)=\mathrm{par}\,\mu_m(\mathcal{E}_q)$, which means that (assuming E_j to be of rank (r_1,r_2))

$$-k\chi(E_j)+r_1\sum_{x\in I_1}\alpha_x+r_2\sum_{x\in I_2}\alpha_x+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)\dim\frac{E_{j,x}}{E_{j,x}\cap F_i(E)_x}+r_1l_1+r_2l_2=0.$$

Thus $(\lambda_1\,\mathrm{id}_{m_1},\ldots,\lambda_t\,\mathrm{id}_{m_t})\in\mathrm{stab}(q)=GL(m_1)\times\ldots\times GL(m_t)$ acts trivially on $\Theta_{\mathcal{R}^{ss}}$, which implies that $\mathrm{stab}(q)$ acts trivially on $\Theta_{\mathcal{R}^{ss}}$ and thus descends to a line bundle $\Theta_{\mathcal{U}_{X_0}}$ having the required universal property.

To show the ampleness of $\Theta_{\mathcal{U}_{X_0}}$, noting that $\det R\pi_{\mathcal{R}^{ss}}\mathcal{E}(N)$ is trivial and

$$\det R\pi_{\mathcal{R}^{ss}}\mathcal{E}=(\det\mathcal{E}_{y_1})^{c_1N}\otimes(\det\mathcal{E}_{y_2})^{c_2N}\otimes\det R\pi_{\mathcal{R}^{ss}}\mathcal{E}(N),$$

we see that the restriction of the polarization to $\mathcal{R}^{ss}$ is

$$(\det R\pi_{\mathcal{R}^{ss}}\mathcal{E}(m))^{l/(m-N)}\otimes\bigotimes_{x\in I}\left\{(\det\mathcal{E}_x)^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(\det\mathcal{Q}_x)^{d_i(x)}\right\}=\Theta_{\mathcal{R}^{ss}}.$$

Thus, by general theorems for GIT, some power of $\Theta_{\mathcal{R}^{ss}}$ descends to an ample line bundle, which implies that some power of $\Theta_{\mathcal{U}_{X_0}}$ is ample. $\square$

Lemma 1.2. *Let G be a reductive algebraic group and V a scheme with G-action. Suppose that there exists a good quotient $\pi: V\to V/\!/G$. Then a vector bundle E with G-action over V descends to $V/\!/G$ if and only if the stabilizer* $\mathrm{stab}(y)$ *of y acts trivially on E_y for any $y\in V$ with closed orbit.*

It is known that for any torsion free sheaf F of rank (r_1,r_2) on X_0 there are integers a, b and c such that

$$F_{x_0}\cong\mathcal{O}^a_{X_0,x_0}\oplus\mathcal{O}^b_{X_1,x_0}\oplus\mathcal{O}^c_{X_2,x_0}.$$

where a, b and c are determined uniquely and satisfy

$$r_1 = a+b, \quad r_2 = a+c \quad \text{and} \quad \dim(F_{x_0} \otimes k(x_0)) = a+b+c.$$

Thus we can define $\mathbf{a}(F) := a$ for any torsion free sheaf F on X_0, and we have the following result.

Lemma 1.3. *Let $0 \to G \to F \to E \to 0$ be an exact sequence of torsion free sheaves on X_0. Then*

$$\mathbf{a}(F) \geq \mathbf{a}(G) + \mathbf{a}(E).$$

Proof. This is clear by counting the dimension of their fibres at x_0. $\square$

Let $\mathcal{R}_a = \{F \in \mathcal{R} \mid F \otimes \widehat{\mathcal{O}}_{x_0} = \widehat{\mathcal{O}}_{x_0}^{\oplus a} \oplus m_{x_0}^{\oplus (r-a)}\}$, and $\widehat{\mathcal{W}}_i = \mathcal{R}_0 \cup \mathcal{R}_1 \cup \ldots \cup \mathcal{R}_i$ (which are closed in $\mathcal{R}$) endowed with their reduced scheme structures. The subschemes $\widehat{\mathcal{W}}_i$ are $SL(n)$-invariant, and yield closed reduced subschemes of $\mathcal{U}_X$. It is clear that

$$\mathcal{R} \supset \widehat{\mathcal{W}}_{r-1} \supset \widehat{\mathcal{W}}_{r-2} \supset \ldots \supset \widehat{\mathcal{W}}_1 \supset \widehat{\mathcal{W}}_0 = \mathcal{R}_0,$$
$$\mathcal{U}_X \supset \mathcal{W}_{r-1} \supset \mathcal{W}_{r-2} \supset \ldots \supset \mathcal{W}_1 \supset \mathcal{W}_0.$$

Let $q_0 \in \mathcal{R}$ be a point corresponding to a torsion free sheaf $\mathcal{F}_0$ such that

$$\mathcal{F}_0 \otimes \mathcal{O}_{X_0, x_0} \cong m_{x_0}^{r-a_0} \oplus \mathcal{O}_{X_0, x_0}^{a_0}.$$

We consider the variety

$$Z = \{(X, Y) \in M(r-a_0) \times M(r-a_0) \mid X \cdot Y = Y \cdot X = 0\},$$

and its subvarieties $Z' = \{(X, Y) \in Z \mid \operatorname{rk} X + \operatorname{rk} Y \leq a\}$. Then the reduced coordinate ring of Z is

$$\mathbf{C}[Z] := \frac{\mathbf{C}[X, Y]}{(XY, YX)},$$

where $X := (x_{ij})_{(r-a_0)\times(r-a_0)}$ and $Y := (y_{ij})_{(r-a_0)\times(r-a_0)}$ (see Lemma 4.8 of [Su]), and Z' is a union of reduced subvarieties of Z (see the proof of Theorem 4.2 in [Su]). Thus we can sum up the arguments of [NS] and [Su] (see also [F]) into a lemma.

Lemma 1.4. *The varieties Z and Z' are the local models of R and $\widehat{\mathcal{W}}_a$, respectively, at the point q_0. More precisely, there are some integers s and t such that*

$$\widehat{\mathcal{O}}_{\mathcal{R}, q_0}[[u_1, \ldots, u_s]] \cong \widehat{\mathcal{O}}_{Z, (0,0)}[[v_1, \ldots, v_t]],$$
$$\widehat{\mathcal{O}}_{\widehat{\mathcal{W}}_a, q_0}[[u_1, \ldots, u_s]] \cong \widehat{\mathcal{O}}_{Z', (0,0)}[[v_1, \ldots, v_t]].$$

In particular, $\mathcal{W}_a$ $(0 \leq a \leq r)$ are reduced and seminormal.

2. The moduli space of generalized parabolic sheaves

Let $\pi\colon \widetilde{X}_0 \to X_0$ be the normalization of X_0 and $\pi^{-1}(x_0)=\{x_1,x_2\}$, then $\widetilde{X}_0$ is a disjoint union of X_1 and X_2, any coherent sheaf E on $\widetilde{X}_0$ is determined by a pair (E_1,E_2) of coherent sheaves on X_1 and X_2. We call as before that E is of rank (r_1,r_2) if E_i has rank r_i on X_i $(i=1,2)$ and define the rank of E to be

$$r(E):=\frac{c_1r_1+c_2r_2}{c_1+c_2}.$$

We can also define similarly the modified parabolic Euler characteristic $\operatorname{par}\chi_m(E)$ if E has parabolic structures at the points $x\in\pi^{-1}(I)$ (we will identify I with $\pi^{-1}(I)$, and note that $m(E)$ defined in Definition 1.3 is only dependent on r_1 and r_2 since $\mathcal{O}(1)$, α and $\vec{a}(x)$ are fixed).

Definition 2.1. A generalized parabolic sheaf of rank (r_1,r_2) (abbreviated GPS)

$$\mathbf{E}:=(E,E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q)$$

on $\widetilde{X}_0$ is a coherent sheaf E on $\widetilde{X}_0$, torsion free of rank (r_1,r_2) outside $\{x_1,x_2\}$ with parabolic structures at the points $\{x\}_{x\in I}$, together with a quotient $E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q$. A morphism $f\colon (E,E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q)\to(E',E'_{x_1}\oplus E'_{x_2}\xrightarrow{q'}Q')$ of GPSes is a morphism $f\colon E\to E'$ of parabolic sheaves, which maps $\ker q$ into $\ker q'$.

We will consider the generalized parabolic sheaf (E,Q) of rank $r_1=r_2=r$ and $\dim Q=r$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points of $\pi^{-1}(I)$, and we will call it a GPS of rank r. Furthermore, by a family of GPSes of rank r over T, we mean

(1) a rank r sheaf $\mathcal{E}$ on $\widetilde{X}_0\times T$ flat over T and locally free outside $\{x_1,x_2\}\times T$;
(2) a locally free rank r quotient $\mathcal{Q}$ of $\mathcal{E}_{x_1}\oplus\mathcal{E}_{x_2}$ on T;
(3) a flag bundle $\mathrm{Flag}_{\vec{n}(x)}(\mathcal{E}_x)$ on T with given weights for each $x\in I$.

Definition 2.2. A GPS (E,Q) is called semistable (resp. stable), if for every nontrivial subsheaf $E'\subset E$ such that E/E' is torsion free outside $\{x_1,x_2\}$, we have, with the induced parabolic structures at the points $\{x\}_{x\in I}$,

$$\operatorname{par}\chi_m(E')-\dim Q^{E'}\le \operatorname{rk}E'\frac{\operatorname{par}\chi_m(E)-\dim Q}{\operatorname{rk}E}\quad (\text{resp. } <),$$

where $Q^{E'}=q(E'_{x_1}\oplus E'_{x_2})\subset Q$.

Let χ_1 and χ_2 be integers such that $\chi_1+\chi_2-r=\chi$, and fix, for $i=1,2$, the polynomials $P_i(m)=c_irm+\chi_i$ and $\mathcal{W}_i=\mathcal{O}_{X_i}(-N)$, where $\mathcal{O}_{X_i}(1)=\mathcal{O}(1)|_{X_i}=\mathcal{O}_{X_i}(c_iy_i)$.

Write $V_i=\mathbf{C}^{P_i(N)}$ and consider the Quot schemes $\mathrm{Quot}(V_i\otimes\mathcal{W}_i, P_i)$. Let $\widetilde{\mathbf{Q}}_i$ be the closure of the open set

$$\mathbf{Q}_i=\left\{\begin{array}{l}V_i\otimes\mathcal{W}_i\longrightarrow E_i\longrightarrow 0, \text{ with } H^1(E_i(N))=0 \text{ and such}\\ \text{that } V\longrightarrow H^0(E_i(N)) \text{ induces an isomorphism}\end{array}\right\}.$$

We have the universal quotient $V_i\otimes\mathcal{W}_i\to\mathcal{F}^i\to 0$ on $X_i\times\widetilde{\mathbf{Q}}_i$ and the relative flag scheme

$$\mathcal{R}_i=\prod_{x\in I_i}{}_{\widetilde{\mathbf{Q}}_i}\mathrm{Flag}_{\vec{n}(x)}(\mathcal{F}^i_x)\longrightarrow\widetilde{\mathbf{Q}}_i.$$

Let $\mathcal{E}^i$ be the pullback of $\mathcal{F}^i$ to $X_i\times\mathcal{R}_i$ and

$$\varrho\colon\widetilde{\mathcal{R}}=\mathrm{Grass}_r(\mathcal{E}^1_{x_1}\oplus\mathcal{E}^2_{x_2})\longrightarrow\mathcal{R}_1\times\mathcal{R}_2.$$

Then we see that, for N large enough, every semistable GPS appears as a point of $\widetilde{\mathcal{R}}$. To rewrite $\mathcal{R}_1\times\mathcal{R}_2$ so that it unifies the R in the last section, let $V=V_1\oplus V_2$, $\mathcal{F}=\mathcal{F}^1\oplus\mathcal{F}^2$ and $\mathcal{E}=\mathcal{E}^1\oplus\mathcal{E}^2$. We have

$$\mathcal{R}_1\times\mathcal{R}_2=\prod_{x\in I}{}_{\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2}\mathrm{Flag}_{\vec{n}(x)}(\mathcal{F}_x)\longrightarrow\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2. \tag{2.1}$$

Note that $V_1\otimes\mathcal{W}_1\oplus V_2\otimes\mathcal{W}_2\to\mathcal{F}\to 0$ is a $\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2$-flat quotient with Hilbert polynomial $P(m)=P_1(m)+P_2(m)$ on $\widetilde{X}_0\times(\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2)$, we have for m large enough a G-equivariant embedding

$$\widetilde{\mathbf{Q}}_1\times\widetilde{\mathbf{Q}}_2\hookrightarrow\mathrm{Grass}_{P(m)}(V_1\otimes W_1^m\oplus V_2\otimes W_2^m),$$

where $W_i^m=H^0(\mathcal{W}_i(m))$ and $G=(GL(V_1)\times GL(V_2))\cap SL(V)$.

A (closed) point $(p=p_1\oplus p_2, \{p_{r(x)}, p_{r_1(x)}, \dots, p_{r_{l_x}(x)}\}_{x\in I})$ of $\mathcal{R}_1\times\mathcal{R}_2$ by the expression of (2.1) is given by points $V_i\otimes\mathcal{W}_i\xrightarrow{p_i}E^i\to 0$ of the Quot schemes $(i=1,2)$, together with quotients (if we write $\mathcal{V}_{\widetilde{X}_0}=V_1\otimes\mathcal{W}_1\oplus V_2\otimes\mathcal{W}_2$ and $E=E^1\oplus E^2$)

$$\{\mathcal{V}_{\widetilde{X}_0}\xrightarrow{p_{r(x)}}Q_{r(x)}, \mathcal{V}_{\widetilde{X}_0}\xrightarrow{p_{r_1(x)}}Q_{r_1(x)}, \dots, \mathcal{V}_{\widetilde{X}_0}\xrightarrow{p_{r_{l_x}(x)}}Q_{r_{l_x}(x)}\}_{x\in I},$$

where

$$r_i(x)=\dim\frac{E_x}{F_i(E)_x}=n_1(x)+\dots+n_i(x),$$

and $Q_{r(x)}:=E_x$, $Q_{r_1(x)}:=E_x/F_1(E)_x, \dots, Q_{r_{l_x}(x)}:=E_x/F_{l_x}(E)_x$. The morphisms $p_{r(x)}$ and $p_{r_j(x)}$ $(j=1,\dots,l_x)$ are defined to be

$$p_{r(x)}\colon\mathcal{V}_{\widetilde{X}_0}\xrightarrow{p}E\longrightarrow E_x,\quad p_{r_j(x)}\colon\mathcal{V}_{\widetilde{X}_0}\xrightarrow{p_{r(x)}}Q_{r(x)}=E_x\longrightarrow\frac{E_x}{F_j(E)_x}.$$

Thus we have a G-equivariant embedding

$$\mathcal{R}_1\times\mathcal{R}_2\hookrightarrow \mathrm{Grass}_{P(m)}(V_1\otimes W_1^m\oplus V_2\otimes W_2^m)\times\mathbf{Flag},$$

where **Flag** is defined to be

$$\mathbf{Flag}=\prod_{x\in I}\{\mathrm{Grass}_{r(x)}(V)\times\mathrm{Grass}_{r_1(x)}(V)\times\ldots\times\mathrm{Grass}_{r_{l_x}(x)}(V)\},$$

which maps a point $(p=p_1\oplus p_2,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I})$ of $\mathcal{R}_1\times\mathcal{R}_2$ to the point

$$(H^0(\mathcal{V}_{\widetilde{X}_0}(m))\xrightarrow{g}U,\{V\xrightarrow{g_{r(x)}}U_{r(x)},V\xrightarrow{g_{r_1(x)}}U_{r_1(x)},\cdots,V\xrightarrow{g_{r_{l_x}(x)}}U_{r_{l_x}(x)}\}_{x\in I})$$

of $\mathrm{Grass}_{P(m)}(V_1\otimes W_1^m\oplus V_2\otimes W_2^m)\times\mathbf{Flag}$, where

$$\begin{aligned} g&:=H^0(p(m)), & U&:=H^0(E(m)),\\ g_{r(x)}&:=H^0(p_{r(x)}(N)), & U_{r(x)}&:=H^0(Q_{r(x)}),\\ g_{r_j(x)}&:=H^0(p_{r_j(x)}(N)), & U_{r_j(x)}&:=H^0(Q_{r_j(x)}), \quad j=1,\ldots,l_x.\end{aligned}$$

Finally, we get a G-equivariant embedding

$$\widetilde{\mathcal{R}}\hookrightarrow\mathbf{G}'=\mathrm{Grass}_{P(m)}(V_1\otimes W_1^m\oplus V_2\otimes W_2^m)\times\mathbf{Flag}\times\mathrm{Grass}_r(V_1\oplus V_2)$$

as follows: a point of $\widetilde{\mathcal{R}}$ is given by a point of $\mathcal{R}_1\times\mathcal{R}_2$ together with a quotient $E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q$, then the above embedding maps $E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q$ to

$$g_G:=H^0(q(N))\colon V_1\oplus V_2=H^0(\mathcal{V}_{\widetilde{X}_0}(N))\longrightarrow H^0(E(N))\longrightarrow E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q.$$

Given $\mathbf{G}'$ the polarization (using the obvious notation)

$$\left\{\frac{l}{m-N}\times\prod_{x\in I}\{\alpha_x,d_1(x),\ldots,d_{l_x}(x)\}\right\}\times k,$$

we have the analogue of Proposition 1.1, whose proof (we refer to Proposition 1.14 and 2.4 of [B], or Lemma 5.4 of [NS]) is a modification of Theorem 4.17 in [N] since our group G here is different from that of [N].

Proposition 2.1. *A point $(g,\{g_{r(x)},g_{r_1(x)},\cdots,g_{r_{l_x}(x)}\}_{x\in I},g_G)\in\mathbf{G}'$ is stable (resp. semistable) for the action of G, with respect to the above polarization (we refer to this from now on as GIT-stability), if and only if for all nontrivial subspaces $H\subset V$, where $H=H_1\oplus H_2$ and $H_i\subset V_i$ $(i=1,2)$, we have (with $h=\dim H$ and $\overline{H}:=H_1\otimes W_1^m\oplus H_2\otimes W_2^m$)*

$$\begin{aligned}\frac{l}{m-N}(hP(m)-P(N)\dim g(\overline{H}))&+\sum_{x\in I}\alpha_x(rh-P(N)\dim g_{r(x)}(H))\\ &+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)h-P(N)\dim g_{r_i(x)}(H))\\ &+k(rh-P(N)\dim g_G(H))<0\quad (\textit{resp. } \le 0).\end{aligned}$$

Proposition 2.2. *Suppose $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I},q)\in\widetilde{\mathcal{R}}$ is a point such that E is torsion free outside $\{x_1,x_2\}$. Then $\mathbf{E}=(E,E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q)$ is stable (resp. semistable) if and only if for every subsheaf $0\neq F\neq E$ we have (using Notation 1.2)*

$$\begin{aligned}\frac{l}{m-N}&(\chi(F(N))P(m)-P(N)\chi(F(m)))+\sum_{x\in I}\alpha_x(r\chi(F(N))-P(N)h^0(Q^F_{r(x)}))\\&+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)(r_i(x)\chi(F(N))-P(N)h^0(Q^F_{r_i(x)}))\\&+k(r\chi(F(N))-P(N)\dim Q^F)<0\quad(\textit{resp. }\le 0).\end{aligned}$$

Proof. For a subsheaf $F\subset E$ such that E/F is torsion free outside $\{x_1,x_2\}$, by the same computation as in Proposition 1.2, we have

$$\mathrm{LHS}(F)=kP(N)\left(\mathrm{par}\,\chi_m(F)-\dim Q^F-r(F)\frac{\mathrm{par}\,\chi_m(E)-r}{r}\right).$$

Thus $\mathbf{E}$ is stable (resp. semistable) if and only if $\mathrm{LHS}(F)<0$ (resp. ≤ 0) for the required F. If E/F has torsion outside $\{x_1,x_2\}$, then $\mathrm{LHS}(F)<0$. □

Lemma 2.1. *There exist N and $M_1(N)$ such that for $m\ge M_1(N)$ the following holds. Suppose $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I},q)\in\widetilde{\mathcal{R}}$ is a point which is GIT-semistable then for all quotients $E\xrightarrow{T}\mathcal{G}\to 0$ we have (with $Q^{\mathcal{G}}:=Q/q(\ker T)$)*

$$h^0(\mathcal{G}(N))\ge\frac{1}{k}\left((c_1+c_2)r(\mathcal{G})l+\sum_{x\in I}\alpha_xh^0(Q^{\mathcal{G}}_{r(x)})+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)h^0(Q^{\mathcal{G}}_{r_i(x)})\right)+h^0(Q^{\mathcal{G}}).$$

In particular, E is torsion free outside $\{x_1,x_2\}$, q maps the torsion on $\{x_1,x_2\}$ to Q injectively and $V\to H^0(E(N))$ is an isomorphism.

Proof. The proof of Lemma 1.1 goes through with obvious modifications except that we cannot assume that the sheaves E are torsion free at x_1 and x_2. To see it clearly, we write out the proof of E being torsion free outside $\{x_1,x_2\}$.

Let $\tau=\mathrm{Tor}\,E$ and $\mathcal{G}=E/\tau$. We note that $h^0(\mathcal{G}(N))=P(N)-h^0(\tau)$, $h^0(Q^{\mathcal{G}}_{r(x)})=r-h^0(Q^{\tau}_{r(x)})$ and $h^0(Q^{\mathcal{G}}_{r_i(x)})=r_i(x)-h^0(Q^{\tau}_{r_i(x)})$. The above inequality gives

$$kh^0(\tau)\le k\dim Q^{\tau}+\sum_{x\in I}(\alpha_x+a_{l_x+1}(x)-a_1(x))h^0(\tau_x),$$

by which one can conclude that $\tau=0$ outside $\{x_1,x_2\}$ and $h^0(\tau_{x_1}\oplus\tau_{x_2})-\dim Q^{\tau}=0$ since $\alpha_x<k-a_{l_x+1}(x)+a_1(x)$. In particular, q maps the torsion on $\{x_1,x_2\}$ to Q injectively. □

Remark 2.1. The proof of Lemma 1.1 and Lemma 2.1 actually implies that one can take N large enough such that for a GIT-semistable point the sheaf E involved satisfies the condition $H^1(E(N)(-x-x_1-x_2))=0$ for any $x\in X_0$, which implies that $E(N)$ and $E(N)(-x_1-x_2)$ are generated by global sections and $H^0(E(N))\to E(N)_{x_1}\oplus E(N)_{x_2}$ is surjective. Conversely, it is easy to prove that every semistable GPS will satisfy the above conditions if N is large enough.

Proposition 2.3. *There exist integers $N>0$ and $M(N)>0$ such that for $m\geq M(N)$ the following is true. A point $(p,\{p_{r(x)},p_{r_1(x)},\cdots,p_{r_{l_x}(x)}\}_{x\in I},q)\in\widetilde{\mathcal{R}}$ is GIT-stable (resp. GIT-semistable) if and only if the quotient E is torsion free outside $\{x_1,x_2\}$, $\mathbf{E}=(E,q)$ is a stable (resp. semistable) GPS and the map $V\to H^0(E(N))$ is an isomorphism.*

Proof. The proof is the same as that of Proposition 1.3 with some obvious modifications in the notation.

Notation 2.1. Define $\mathcal{H}$ to be the subscheme of $\widetilde{\mathcal{R}}$ parametrizing the generalized parabolic sheaves $\mathbf{E}=(E,E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q)$ satisfying

(1) $\mathbf{C}^{P(N)}\cong H^0(E(N))$, and $H^1(E(N)(-x_1-x_2-x))=0$ for any $x\in\widetilde{X}_0$;

(2) $\mathrm{Tor}\,E$ is supported on $\{x_1,x_2\}$ and $(\mathrm{Tor}\,E)_{x_1}\oplus(\mathrm{Tor}\,E)_{x_2}\hookrightarrow Q$.

Let $\widetilde{\mathcal{R}}^{ss}$ ($\widetilde{\mathcal{R}}^s$) be the open set of $\widetilde{\mathcal{R}}$ consisting of the semistable (stable) GPS, then it is clear that
$$\widetilde{\mathcal{R}}^{ss}\overset{\text{open}}{\hookrightarrow}\mathcal{H}\overset{\text{open}}{\hookrightarrow}\widetilde{\mathcal{R}}.$$

We will introduce the so called s-equivalence of GPSes later, in Definition 2.6. It is also known that $\mathcal{H}$ is reduced, normal and Gorenstein with only rational singularities (see Proposition 3.2 and Remark 3.1 in [Su]).

Theorem 2.1. *For given data in Notation 1.1 satisfying $(*)$ and χ_1 and χ_2 with $\chi_1+\chi_2-r=\chi$, there exists an irreducible, Gorenstein, normal projective variety $\mathcal{P}_{\chi_1,\chi_2}$ with only rational singularities, which is the coarse moduli space of s-equivalence classes of semistable GPSes (E,Q) on $\widetilde{X}_0$ of rank r and $\chi(E_j)=\chi_j$ $(j=1,2)$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$.*

Proof. The existence of the moduli space and its projectivity follow from Proposition 2.3, the other properties follow from the corresponding properties of $\mathcal{H}$ and the fact that $\widetilde{\mathcal{R}}^{ss}\subset\mathcal{H}$ if N is large enough. □

Recall that we have the universal quotient $\mathcal{E}^1$ on $X_1\times\mathcal{R}_1$, flat over $\mathcal{R}_1$, and torsion free of rank r outside $\{x_1\}$ with Euler characteristic χ_1, together with, for each $x\in I_1$, a flag
$$\mathcal{E}^1_{\{x\}\times\mathcal{R}_1}=F_0(\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})\supset F_1(\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})\supset\ldots\supset F_{l_x}(\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})\supset F_{l_x+1}(\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})=0$$

of subbundles of type $\vec{n}(x)$ and weights $\vec{a}(x)$. Let $\mathcal{Q}_{x,i}=\mathcal{E}^1_{\{x\}\times\mathcal{R}_1}/F_i(\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})$. We can define a line bundle $\Theta_{\mathcal{R}_1}$ on $\mathcal{R}_1$ as

$$(\det R\pi_{\mathcal{R}_1}\mathcal{E}^1)^k\otimes\bigotimes_{x\in I_1}\left\{(\det\mathcal{E}^1_{\{x\}\times\mathcal{R}_1})^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(\det\mathcal{Q}_{x,i})^{d_i(x)}\right\}\otimes(\det\mathcal{E}^1_{\{y_1\}\times\mathcal{R}_1})^{l_1}.$$

Similarly, we can define the line bundle $\Theta_{\mathcal{R}_2}$ on $\mathcal{R}_2$ and the G-line bundle

$$\Theta_{\widetilde{\mathcal{R}}}:=\varrho^*(\Theta_{\mathcal{R}_1}\otimes\Theta_{\mathcal{R}_2})\otimes(\det\mathcal{Q})^k$$

on $\widetilde{\mathcal{R}}$, where $\varrho^*(\mathcal{E}^1_{x_1}\oplus\mathcal{E}^2_{x_2})\to\mathcal{Q}\to 0$ is the universal quotient on $\widetilde{\mathcal{R}}$. One can check that $\Theta_{\widetilde{\mathcal{R}}}$ is the restriction of ample polarization used to linearize the action of G. Thus some power of $\Theta_{\widetilde{\mathcal{R}}}$ descends to an ample line bundle on $\mathcal{P}_{\chi_1,\chi_2}$. In fact, we have the following result.

Lemma 2.2. *The line bundle $\Theta_{\widetilde{\mathcal{R}}^{ss}}$ descends to an ample line bundle $\Theta_{\mathcal{P}_{\chi_1,\chi_2}}$ on $\mathcal{P}_{\chi_1,\chi_2}$.*

Proof. The proof is similar to the proof of Theorem 1.2, we only make a remark here. If (E,Q) is a semistable GPS of rank r and (E',Q') a sub-GPS of (E,Q) with

$$\operatorname{par}\chi_m(E')-\dim Q'=r(E')\frac{\operatorname{par}\chi_m(E)-\dim Q}{r},$$

we have (assuming that E' is of rank (r_1,r_2))

$$\begin{aligned}&-k\chi(E')+r_1\sum_{x\in I_1}\alpha_x+r_2\sum_{x\in I_2}\alpha_x+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)\dim\frac{E'_x}{E'_x\cap F_i(E_x)}+r_1l_1+r_2l_2\\&+k\dim Q'=\frac{-k\chi+r\sum_{x\in I}\alpha_x+\sum_{x\in I}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r(l_1+l_2)}{r}r(E')=0.\quad\square\end{aligned}$$

Notation 2.2. Let $\mathcal{R}_{1,F}\subset\mathcal{R}_1$ (resp. $\mathcal{R}_{2,F}\subset\mathcal{R}_2$) be the open set of points corresponding to the vector bundles on X_1 (resp. X_2), and $\widetilde{\mathcal{R}}_F=\varrho^{-1}(\mathcal{R}_{1,F}\times\mathcal{R}_{2,F})$, then

$$\varrho:\widetilde{\mathcal{R}}_F\longrightarrow\mathcal{R}_{1,F}\times\mathcal{R}_{2,F}$$

is a grassmannian bundle over $\mathcal{R}_{1,F}\times\mathcal{R}_{2,F}$, and $\widetilde{\mathcal{R}}_F\subset\mathcal{H}$. We define

$$R^1_{F,a}:=\{(E,Q)\in\widetilde{\mathcal{R}}_F\mid E_{x_1}\longrightarrow Q\text{ has rank }a\},$$

and $\widehat{\mathcal{D}}_{F,1}(i):=R^1_{F,0}\cup\ldots\cup R^1_{F,i}$, which have the natural scheme structures. The subschemes $R^2_{F,a}$ and $\widehat{\mathcal{D}}_{F,2}(i)$ are defined similarly. Let $\widehat{\mathcal{D}}_1(i)$ and $\widehat{\mathcal{D}}_2(i)$ be the Zariski closure of $\widehat{\mathcal{D}}_{F,1}(i)$ and $\widehat{\mathcal{D}}_{F,2}(i)$ in $\widetilde{\mathcal{R}}$. Then they are reduced, irreducible and G-invariant closed subschemes of $\widetilde{\mathcal{R}}$, thus inducing the closed subschemes $\mathcal{D}_1(i)_{\chi_1,\chi_2}$ and $\mathcal{D}_2(i)_{\chi_1,\chi_2}$ of $\mathcal{P}_{\chi_1,\chi_2}$. Clearly, we have (for $j=1,2$) that

$$\widetilde{\mathcal{R}} \supset \widehat{\mathcal{D}}_j(r-1) \supset \widehat{\mathcal{D}}_j(r-2) \dots \supset \mathcal{D}_j(1) \supset \widehat{\mathcal{D}}_j(0),$$
$$\mathcal{P}_{\chi_1,\chi_2} \supset \mathcal{D}_j(r-1)_{\chi_1,\chi_2} \supset \mathcal{D}_j(r-2)_{\chi_1,\chi_2} \supset \dots \supset \mathcal{D}_j(1)_{\chi_1,\chi_2} \supset \mathcal{D}_j(0)_{\chi_1,\chi_2}.$$

Lemma 2.3. *The schemes $\mathcal{H}$, $\widehat{\mathcal{D}}_j(a)$ and $\widehat{\mathcal{D}}_1(a)\cap\widehat{\mathcal{D}}_2(b)$ are reduced and normal with rational singularities. In particular, $\mathcal{P}_{\chi_1,\chi_2}$, $\mathcal{D}_j(a)_{\chi_1,\chi_2}$ and $\mathcal{D}_1(a)_{\chi_1,\chi_2}\cap \mathcal{D}_2(b)_{\chi_1,\chi_2}$ are reduced and normal with rational singularities.*

Proof. This is a copy of Proposition 3.2 in [Su] and the proof there goes through. □

Let (E,Q) be a semistable GPS of rank r with $E=(E_1,E_2)$ and $\chi_j=\chi(E_j)$ $(j=1,2)$. Then, by the definition of semistability, we have (for $j=1,2$) that

$$\operatorname{par}\chi_m(E_j)-\dim Q^{E_j} \leq \frac{c_j}{c_1+c_2}(\operatorname{par}\chi_m(E)-r).$$

Recall that $\chi_1+\chi_2-r=\chi$ and

$$n_j=\frac{1}{k}\left(\sum_{x\in I_j}\sum_{i=1}^{l_x} d_i(x)r_i(x)+r\sum_{x\in I_j}a_x+rl_j\right), \quad j=1,2.$$

We can rewrite the above inequality into

$$\begin{aligned} n_1+r-\dim Q^{E_2} \leq \chi(E_1) \leq n_1+\dim Q^{E_1},\\ n_2+r-\dim Q^{E_1} \leq \chi(E_2) \leq n_2+\dim Q^{E_2}. \end{aligned} \tag{2.2}$$

Thus, for fixed χ, the moduli space of s-equivalence classes of semistable GPSes (E,Q) on $\widetilde{X}_0$ of rank r and $\chi(E)=\chi+r$ with parabolic structures of type $\{\vec{n}(x)\}_{x\in I}$ and weights $\{\vec{a}(x)\}_{x\in I}$ at the points $\{x\}_{x\in I}$ is the disjoint union

$$\mathcal{P}:=\coprod_{\chi_1+\chi_2=\chi+r}\mathcal{P}_{\chi_1,\chi_2},$$

where χ_1, χ_2 satisfy the inequalities

$$n_1\leq\chi(E_1)\leq n_1+r \quad\text{and}\quad n_2\leq\chi(E_2)\leq n_2+r.$$

Notation 2.3. The ample line bundles $\{\Theta_{\mathcal{P}_{\chi_1,\chi_2}}\}$ determine an ample line bundle $\Theta_{\mathcal{P}}$ on $\mathcal{P}$, and for any $0\leq a\leq r$, we define the subschemes

$$\mathcal{D}_1(a):=\coprod_{\chi_1+\chi_2=\chi+r}\mathcal{D}_1(a)_{\chi_1,\chi_2}\quad\text{and}\quad\mathcal{D}_2(a):=\coprod_{\chi_1+\chi_2=\chi+r}\mathcal{D}_2(a)_{\chi_1,\chi_2}.$$

We will simply write $\mathcal{D}_1:=\mathcal{D}_1(r-1)$ and $\mathcal{D}_2:=\mathcal{D}_2(r-1)$.

In order to introduce a sheaf theoretic description of the so called s-equivalence of GPSes, we enlarge the category by considering all of the GPSes including the case $r(E)=0$, and also assume that $|I|=0$ for simplicity.

Definition 2.3. A GPS (E,Q) is called semistable (resp. stable), if

(1) when $\operatorname{rk}E>0$, then for every nontrivial subsheaf $E'\subset E$ such that E/E' is torsion free outside $\{x_1,x_2\}$, we have, with the induced parabolic structures at the points $\{x\}_{x\in I}$,

$$\operatorname{par}\chi_m(E')-\dim Q^{E'}\leq\operatorname{rk}E'\frac{\operatorname{par}\chi_m(E)-\dim Q}{\operatorname{rk}E}\quad(\text{resp. }<),$$

where $Q^{E'}=q(E'_{x_1}\oplus E'_{x_2})\subset Q$;

(2) when $\operatorname{rk}E=0$, then $E_{x_1}\oplus E_{x_2}=Q$ (resp. $E_{x_1}\oplus E_{x_2}=Q$ and $\dim Q=1$).

Definition 2.4. If (E,Q) is a GPS and $\operatorname{rk}E>0$, we set

$$\mu_G[(E,Q)]=\frac{\deg E-\dim Q}{\operatorname{rk}E}.$$

It is useful to think of an m-GPS as a sheaf E on $\widetilde{X}_0$ together with a map $\pi_*E\to{}_{x_0}Q\to 0$ and $h^0({}_{x_0}Q)=m$. Let K_E denote the kernel of $\pi_*E\to Q$.

Definition 2.5. Given an exact sequence

$$0\longrightarrow E'\longrightarrow E\longrightarrow E''\longrightarrow 0$$

of sheaves on $\widetilde{X}_0$, and $\pi_*E\to Q\to 0$, a generalized parabolic structure on E, we define the generalized parabolic structures on E' and E'' via the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \pi_*E' & \longrightarrow & \pi_*E & \longrightarrow & \pi_*E'' & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & Q' & \longrightarrow & Q & \longrightarrow & Q'' & \longrightarrow & 0.
\end{array}$$

The first horizontal sequence is exact because π is finite, Q' is defined as the image in Q of π_*E' so that the first vertical arrow is onto, Q'' is defined by demanding that the second horizontal sequence is exact, and finally the third vertical arrow is onto by the snake lemma. We will write

$$0\longrightarrow(E',Q')\longrightarrow(E,Q)\longrightarrow(E'',Q'')\longrightarrow 0$$

whose meaning is clear.

Proposition 2.4. *Fix a rational number μ. Then the category $\mathcal{C}_\mu$ of semistable GPSes (E,Q) such that* $\operatorname{rk} E=0$ *or*, $\operatorname{rk} E>0$ *with* $\mu_G[(E,Q)]=\mu$, *is an abelian, artinian, noetherian category whose simple objects are the stable GPSes in the category.*

One can conclude, as usual, that given a semistable GPS (E,Q) it has a Jordan–Hölder filtration, and the associated graded GPS $\operatorname{gr}(E,Q)$ is uniquely determined by (E,Q).

Definition 2.6. Two semistable GPSes (E_1,Q_1) and (E_2,Q_2) are said to be s-equivalent if they have the same associated graded GPSes, namely,

$$(E_1,Q_1)\sim(E_2,Q_2) \quad\Longleftrightarrow\quad \operatorname{gr}(E_1,Q_1)\cong\operatorname{gr}(E_2,Q_2).$$

Remark 2.2. Any stable GPS (E,Q) with $\operatorname{rk} E>0$ must be locally free (i.e., E is locally free), and two stable GPSes are s-equivalent if and only if they are isomorphic.

Proposition 2.5. *Every semistable (E',Q') with* $\operatorname{rk} E'>0$ *is s-equivalent to a semistable (E,Q) with E locally free. Moreover,*

(1) *if E' has torsion of dimension t at x_2, then (E',Q') is s-equivalent to a semistable (E,Q) with E locally free and*

$$\operatorname{rank}(E_{x_1}\longrightarrow Q)\leq \dim Q-t;$$

(2) *if (E,Q) is a semistable GPS with E locally free and*

$$\operatorname{rank}(E_{x_1}\longrightarrow Q)=a,$$

then (E,Q) is s-equivalent to a semistable (E',Q') such that

$$\dim(\operatorname{Tor} E')_{x_2}=\dim Q-a.$$

The roles of x_1 and x_2 in the above statements can be reversed.

Proof. We prove (1) first. For given $(E',Q')\in\mathcal{C}_\mu$ with $\operatorname{rk} E'>0$, there is an exact sequence

$$0\longrightarrow(E_1',Q_1')\longrightarrow(E',Q')\longrightarrow(E_2',Q_2')\longrightarrow 0$$

such that (E_2',Q_2') is stable and $\mu_G[(E_2',Q_2')]=\mu$ if $\operatorname{rk} E_2'>0$. It is clear that

$$\operatorname{gr}(E',Q')=\operatorname{gr}(E_1',Q_1')\oplus(E_2',Q_2').$$

When $\operatorname{rk} E_2'>0$, then E_2' has to be locally free and E_1' has the same torsion as E'. Thus if $\operatorname{rk} E_1'>0$, there is (by using induction over the rank) $(E_1,Q_1)\in\mathcal{C}_\mu$ with E_1 locally free and

$$\operatorname{rank}(E_{1,x_1}\longrightarrow Q_1)\leq\dim Q_1-t$$

such that $\mathrm{gr}(E_1,Q_1)=\mathrm{gr}(E_1',Q_1')$. One can check that

$$(E,Q):=(E_1\oplus E_2',Q_1\oplus Q_2')\in\mathcal{C}_\mu$$

is s-equivalent to (E',Q') and

$$\mathrm{rank}(E_{x_1}\longrightarrow Q)\leq\dim Q-t.$$

If $\mathrm{rk}\,E_1'=0$, then $\mathrm{gr}(E',Q')=(E_2',Q_2')\oplus\mathrm{gr}(\mathrm{Tor}\,E',\mathrm{Tor}\,E')$. Thus (E',Q') satisfies (up to an s-equivalence) the exact sequence

$$0\longrightarrow(\widetilde{E}',\widetilde{Q}')\longrightarrow(E',Q')\longrightarrow({}_{x_2}\mathbf{C},\mathbf{C})\longrightarrow 0,$$

where $(\widetilde{E}',\widetilde{Q}')\in\mathcal{C}_\mu$ has torsion of dimension $t-1$ at x_2. This is the typical case we treated in Lemma 2.5 of [Su], and we will indicate later how to get our stronger statement by the construction of [Su].

When $\mathrm{rk}\,E_2'=0$ and $\dim(\mathrm{Tor}\,E_1')_{x_2}<t$, then (E_2',Q_2') has to be $({}_{x_2}\mathbf{C},\mathbf{C})$, which is again the above typical case we will treat. If $\dim(\mathrm{Tor}\,E_1')_{x_2}=t$, by repeating the above procedures for (E_1',Q_1'), we will reduce the proof, after a finite number of steps, to the above cases again since $\dim Q_1'$ decreases strictly. All in all, we are reduced to treating the typical case

$$0\longrightarrow(\widetilde{E}',\widetilde{Q}')\longrightarrow(E',Q')\longrightarrow({}_{x_2}\mathbf{C},\mathbf{C})\longrightarrow 0,$$

where $(\widetilde{E}',\widetilde{Q}')\in\mathcal{C}_\mu$ and $\dim(\mathrm{Tor}\,\widetilde{E}')_{x_2}=t-1$.

By using induction over t, there exists $(\widetilde{E},\widetilde{Q})\in\mathcal{C}_\mu$ with $\widetilde{E}$ locally free such that $\mathrm{gr}(\widetilde{E},\widetilde{Q})=\mathrm{gr}(\widetilde{E}',\widetilde{Q}')$ and

$$\mathrm{rank}(\tilde{q}_1:\widetilde{E}_{x_1}\longrightarrow\widetilde{Q})\leq\dim\widetilde{Q}-(t-1),$$

where $\tilde{q}_1$ and $\tilde{q}_2$ are the induced maps by $\tilde{q}$: $\widetilde{E}_{x_1}\oplus\widetilde{E}_{x_2}\to\widetilde{Q}$. Since $({}_{x_2}\mathbf{C},\mathbf{C})$ is stable, we have

$$\mathrm{gr}(E',Q')=\mathrm{gr}(\widetilde{E},\widetilde{Q})\oplus({}_{x_2}\mathbf{C},\mathbf{C}).$$

Let $K_2=\ker(\tilde{q}_2:\widetilde{E}_{x_2}\to\widetilde{Q})$. Choosing a Hecke modification h: $\widetilde{E}\to E$ at x_2 (see Remark 1.4 of [NS]) such that $\widetilde{K}_2:=\ker h_{x_2}\subset K_2$ and $\dim\widetilde{K}_2=1$, we get the extension

$$0\longrightarrow\widetilde{E}\xrightarrow{h}E\xrightarrow{\gamma}{}_{x_2}\mathbf{C}\longrightarrow 0.$$

Let $Q=\widetilde{Q}\oplus\mathbf{C}$ and $E_{x_2}=h_{x_2}(\widetilde{E}_{x_2})\oplus V_1$ for a subspace V_1. We define a morphism f: $E_{x_1}\oplus E_{x_2}\to Q$ such that $E_{x_1}\to Q$ to be

$$E_{x_1}\xrightarrow{h_{x_1}^{-1}}\widetilde{E}_{x_1}\xrightarrow{\tilde{q}_1}\widetilde{Q}\hookrightarrow Q$$

and $E_{x_2}\to Q$ to be

$$E_{x_2}=h_{x_2}(\widetilde{E}_{x_2})\oplus V_1\xrightarrow{(\bar{h}_{x_2}^{-1},\gamma_{x_2})}\frac{\widetilde{E}_{x_2}}{\widetilde{K}_2}\oplus\mathbf{C}\xrightarrow{(\bar{q}_2,\mathrm{id})}\widetilde{Q}\oplus\mathbf{C}=Q.$$

where $\bar{h}_{x_2}:\widetilde{E}_{x_2}/\widetilde{K}_2\cong h_{x_2}(\widetilde{E}_{x_2})$ and $\tilde{q}_2:\widetilde{E}_{x_2}/\widetilde{K}_2\to\widetilde{Q}$ (note that $\widetilde{K}_2\subset K_2$). Thus the diagram

$$\begin{array}{ccccccccc} & & \widetilde{E}_{x_1}\oplus\widetilde{E}_{x_2} & \xrightarrow{(h_{x_1},h_{x_2})} & E_{x_1}\oplus E_{x_2} & \xrightarrow{(0,\gamma_{x_2})} & \mathbf{C} & \longrightarrow & 0\\ & & {\scriptstyle q}\downarrow & & {\scriptstyle f}\downarrow & & \| & & \\ 0 & \longrightarrow & \widetilde{Q} & \longrightarrow & \widetilde{Q}\oplus\mathbf{C} & \longrightarrow & \mathbf{C} & \longrightarrow & 0\end{array}$$

commutes. One checks that f is surjective by this diagram, and thus

$$0\longrightarrow(\widetilde{E},\widetilde{Q})\longrightarrow(E,Q)\longrightarrow({}_{x_2}\mathbf{C},\mathbf{C})\longrightarrow 0.$$

It is easy to see that $(E,Q)\in\mathcal{C}_\mu$ is s-equivalent to (E',Q') and

$$\operatorname{rank}(E_{x_1}\longrightarrow Q)=\operatorname{rank}(\widetilde{E}_{x_1}\longrightarrow\widetilde{Q})\leq\dim Q-t.$$

To prove (2), let $q: E_{x_1}\oplus E_{x_2}\to Q$ and $Q=q_1(E_{x_1})\oplus\mathbf{C}^{\dim Q-a}$. Take the projection $Q\xrightarrow{p}\mathbf{C}^{\dim Q-a}$ and define

$$\widetilde{E}:=\ker(\gamma: E\longrightarrow E_{x_2}\xrightarrow{q_2}Q\xrightarrow{p}{}_{x_2}\mathbf{C}^{\dim Q-a}).$$

We get a semistable $(\widetilde{E},\widetilde{Q})\in\mathcal{C}_\mu$ ($\widetilde{Q}$ being the kernel of p) such that

$$0\longrightarrow(\widetilde{E},\widetilde{Q})\longrightarrow(E,Q)\longrightarrow({}_{x_2}\mathbf{C}^{\dim Q-a},\mathbf{C}^{\dim Q-a})\longrightarrow 0$$

is an exact sequence in $\mathcal{C}_\mu$. Thus (E,Q) is s-equivalent to

$$(E',Q'):=(\widetilde{E}\oplus{}_{x_2}\mathbf{C}^{\dim Q-a},\widetilde{Q}\oplus\mathbf{C}^{\dim Q-a})$$

by Lemma 2.4 below. $\square$

Lemma 2.4. *Given an $(E,Q)\in\mathcal{C}_\mu$, if there is an exact sequence*

$$0\longrightarrow(E_1,Q_1)\longrightarrow(E,Q)\longrightarrow(E_2,Q_2)\longrightarrow 0$$

such that $(E_2,Q_2)\in\mathcal{C}_\mu$, then

$$\operatorname{gr}(E,Q)=\operatorname{gr}(E_1,Q_1)\oplus\operatorname{gr}(E_2,Q_2).$$

In particular, (E,Q) is s-equivalent to $(E_1\oplus E_2,Q_1\oplus Q_2)$.

Proof. Since $(E_2,Q_2)\in\mathcal{C}_\mu$, there exists an exact sequence

$$0\longrightarrow(E_2',Q_2')\longrightarrow(E_2,Q_2)\longrightarrow(E_2'',Q_2'')\longrightarrow 0$$

such that $(E_2'',Q_2'')\in\mathcal{C}_\mu$ is stable. Thus

$$\operatorname{gr}(E_2,Q_2)=\operatorname{gr}(E_2',Q_2')\oplus(E_2'',Q_2'').$$

On the other hand, if we define $(\widetilde{E},\widetilde{Q})$ by the exact sequence

$$0\longrightarrow(\widetilde{E},\widetilde{Q})\longrightarrow(E,Q)\xrightarrow{g}(E_2'',Q_2'')\longrightarrow 0,$$

where $g\colon (E,Q)\to(E_2,Q_2)\to(E_2'',Q_2'')$, then we have an exact sequence

$$0\longrightarrow(E_1,Q_1)\longrightarrow(\widetilde{E},\widetilde{Q})\longrightarrow(E_2',Q_2')\longrightarrow 0,$$

and $(E_2',Q_2')\in\mathcal{C}_\mu$. By using induction over $\operatorname{rk} E_2$ and $h^0(E_2)$ when $\operatorname{rk} E_2=0$, we have

$$\operatorname{gr}(\widetilde{E},\widetilde{Q})=\operatorname{gr}(E_1,Q_1)\oplus\operatorname{gr}(E_2',Q_2').$$

Now the lemma is clear. □

3. The factorization theorem

Recall that $\pi\colon\widetilde{X}_0\to X_0$ is the normalization of X_0 and $\pi^{-1}(x_0)=\{x_1,x_2\}$. Given a GPS $(E,E_{x_1}\oplus E_{x_2}\xrightarrow{q}Q)$ on $\widetilde{X}_0$, we define a coherent sheaf $\phi(E,Q):=F$ by the exact sequence

$$0\longrightarrow F\longrightarrow\pi_*E\longrightarrow{}_{x_0}Q\longrightarrow 0,$$

where we use ${}_xW$ to denote the skyscraper sheaf supported at $\{x\}$ with fibre W, and the morphism $\pi_*E\to{}_{x_0}Q$ is defined as

$$\pi_*E\longrightarrow\pi_*E|_{\{x_0\}}={}_{x_0}(E_{x_1}\oplus E_{x_2})\xrightarrow{q}{}_{x_0}Q.$$

It is clear that F is torsion free of rank (r_1,r_2) if and only if (E,Q) is a GPS of rank (r_1,r_2) and satisfying

$$\text{(T)}\qquad(\operatorname{Tor}E)_{x_1}\oplus(\operatorname{Tor}E)_{x_2}\xhookrightarrow{q}Q.$$

In particular, the GPS in $\mathcal{H}$ in this way gives torsion free sheaves of rank r with the natural parabolic structures at the points of I.

Lemma 3.1. *Suppose that (E,Q) satisfy condition* (T), *and let $F=\phi(E,Q)$ be the associated torsion free sheaf on X_0.*

(1) *If E is a vector bundle and the maps $E_{x_i}\to Q$ are isomorphisms, then F is a vector bundle.*

(2) *If F is a vector bundle on X_0, then there is a unique (E,Q) such that $\phi(E,Q)=F$. In fact, $E=\pi^*F$.*

(3) *If F is a torsion free sheaf, then there is an (E,Q), with E a vector bundle on $\widetilde{X}_0$, such that $\phi(E,Q)=F$ and $E_{x_2}\to Q$ is an isomorphism. The rank of the map $E_{x_1}\to Q$ is a if and only if $F\otimes\widehat{\mathcal{O}}_{x_0}\cong\widehat{\mathcal{O}}_{x_0}^{\oplus a}\oplus m_{x_0}^{\oplus(r-a)}$. The roles of x_1 and x_2 can be reversed.*

(4) *Every torsion free rank r sheaf F on X_0 comes from an (E,Q) such that E is a vector bundle.*

Proof. The proof is similar to the proof of Lemma 4.6 of [NR] and Lemma 2.1 of [Su]. □

Lemma 3.2. *Let $F=\phi(E,Q)$, then F is semistable if and only if (E,Q) is semistable. Moreover,*

(1) *if (E,Q) is stable, then F is stable;*

(2) *if F is a stable vector bundle, then (E,Q) is stable.*

Proof. For any subsheaf $E'\subset E$ such that E/E' is torsion free outside $\{x_1,x_2\}$, the induced GPS $(E',Q^{E'})$ defines a subsheaf $F'\subset F$ by

$$0\longrightarrow F'\longrightarrow \pi_*E'\longrightarrow {}_{x_0}Q^{E'}\longrightarrow 0.$$

It is clear that $\mathrm{par}\,\chi_m(F')=\mathrm{par}\,\chi_m(E')-\dim Q^{E'}$, thus F semistable implies (E,Q) semistable. Note that E may have torsion and thus (E,Q) may not be stable even if F is stable (for instance, taking E' to be the torsion subsheaf). In fact, (E,Q) is stable if and only if F is a stable vector bundle.

Next we prove that if (E,Q) is stable (semistable), then F is stable (semistable). For any subsheaf $F'\subset F$ such that F/F' is torsion free, we have canonical morphisms $\pi^*F'\to\pi^*F\to\pi^*\pi_*E\to E$. Let E' be the image of π^*F'. One has the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F' & \longrightarrow & \pi_*E' & \longrightarrow & {}_{x_0}Q^{E'} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \pi_*E & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F/F' & \longrightarrow & \pi_*(E/E') & \longrightarrow & {}_{x_0}(Q/Q^{E'}) & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

which implies that E/E' is torsion free outside $\{x_1,x_2\}$ (since F/F' is torsion free),

$$\mathrm{par}\,\chi_m(F')=\mathrm{par}\,\chi_m(E')-\dim Q^{E'}\quad\text{and}\quad \mathrm{par}\,\chi_m(F)=\mathrm{par}\,\chi_m(E)-\dim Q.$$

Thus, noting that $\mathrm{rk}\,E'=\mathrm{rk}\,F'$ and $\mathrm{rk}\,E=\mathrm{rk}\,F$, one proves the lemma. □

Lemma 3.3. *Let (E,Q) be a semistable GPS with E locally free and $F=\phi(E,Q)$ be the associated torsion free sheaf. Then (E,Q) is s-equivalent to a semistable (E',Q') such that E' has torsion of dimension $\dim Q-\mathbf{a}(\mathrm{gr}\,F)$.*

Proof. We prove the lemma by induction over the length of $\operatorname{gr} F$ (the number of components of stable sheaves of $\operatorname{gr} F$). For any torsion free sheaf F, we have a canonical exact sequence $0 \to F \to \pi_* \widetilde{E} \to \widetilde{Q} \to 0$, where $\widetilde{E} = \pi^* F / \operatorname{Tor} \pi^* F$ and $\dim \widetilde{Q} = \mathbf{a}(F)$. If $F = \phi(E, Q)$ with E locally free, then we have the commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \pi_* \widetilde{E} & \longrightarrow & {}_{x_0}\widetilde{Q} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F & \longrightarrow & \pi_* E & \longrightarrow & {}_{x_0}Q & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & \pi_* \tau & \longrightarrow & {}_{x_0}Q_3 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0, & &
\end{array}$$

where $\tau = E/\widetilde{E}$, $Q_3 = Q/\widetilde{Q}$ and the map $\pi_* \tau \to {}_{x_0}Q_3$ is defined such that the diagram is commutative, which has to be an isomorphism. This gives an exact sequence $0 \to (\widetilde{E}, \widetilde{Q}) \to (E, Q) \to (\tau, Q_3) \to 0$ in $\mathcal{C}_\mu$, thus (E, Q) is s-equivalent to $(\widetilde{E} \oplus \tau, \widetilde{Q} \oplus Q_3)$ and $\dim \tau = \dim Q - \mathbf{a}(F)$. In particular, the lemma is true when $\operatorname{gr} F$ has length one. For the general case, there exists an exact sequence $0 \to F_1 \to F \to F_2 \to 0$ with F_2 stable and $\operatorname{par} \mu_m(F_2) = \operatorname{par} \mu_m(F)$. Consider

$$\text{(3.1)} \qquad \begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & F_1 & \longrightarrow & F & \longrightarrow & F_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_* E_1 & \longrightarrow & \pi_* \widetilde{E} & \longrightarrow & \pi_* E_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & {}_{x_0}Q_1 & \longrightarrow & {}_{x_0}\widetilde{Q} & \longrightarrow & {}_{x_0}Q_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

where $E_1=\pi^*F_1/\operatorname{Tor}\pi^*F_1$, $\dim Q_1=\mathbf{a}(F_1)$, $E_2=\widetilde{E}/E_1$ and $Q_2=\widetilde{Q}/Q_1$. The first two vertical sequences are the canonical exact sequences determined by F_1 and F. The third vertical sequence is defined by demanding that the diagram commutes, which has to be exact. Using $\operatorname{par}\mu_m(F_2)=\operatorname{par}\mu_m(F)$, it is easy to see that $\mu_G[(E_2,Q_2)]=\mu_G[(\widetilde{E},\widetilde{Q})]$ and (E_2,Q_2) is semistable (since F_2 is stable). Thus $\operatorname{gr}(\widetilde{E},\widetilde{Q})=\operatorname{gr}(E_1,Q_1)\oplus\operatorname{gr}(E_2,Q_2)$. On the other hand, (E_1,Q_1) is semistable with E_1 locally free and $F_1=\phi(E_1,Q_1)$. By the induction, there exists an $(E_1',Q_1')\in\mathcal{C}_\mu$ such that $\operatorname{gr}(E_1,Q_1)=\operatorname{gr}(E_1',Q_1')$ and $\dim\operatorname{Tor}E_1'=\dim Q_1-\mathbf{a}(\operatorname{gr}F_1)$. Thus (E,Q) is s-equivalent to $(E',Q'):=(E_1'\oplus E_2\oplus\tau,Q_1'\oplus Q_2\oplus Q_3)$. One checks that $\dim\operatorname{Tor}E_2=\mathbf{a}(F)-\mathbf{a}(F_1)-\mathbf{a}(F_2)$ by restricting the diagram (3.1) to the point x_0 and counting the dimension of the fibres (the first two vertical sequences remaining exact). Therefore (note that $\dim Q_1=\mathbf{a}(F_1)$) E' has torsion of dimension

$$\dim Q_1-\mathbf{a}(\operatorname{gr}F_1)+\dim\operatorname{Tor}E_2+\dim\tau=\dim Q-\mathbf{a}(\operatorname{gr}F_1)-\mathbf{a}(F_2),$$

which equals $\dim Q-\mathbf{a}(\operatorname{gr}F)$ since $\operatorname{gr}F=\operatorname{gr}(F_1)\oplus F_2$. □

Consider the family $\varrho^*\mathcal{E}=(\varrho^*\mathcal{E}^1,\varrho^*\mathcal{E}^2)$ of GPSes over $\widetilde{\mathcal{R}}^{ss}$ with the universal quotient $\varrho^*(\mathcal{E}^1_{x_1}\oplus\mathcal{E}^2_{x_2})\to\mathcal{Q}$. Using the finite morphism

$$\pi\times I_{\widetilde{\mathcal{R}}^{ss}}:\widetilde{X}_0\times\widetilde{\mathcal{R}}^{ss}\longrightarrow X_0\times\widetilde{\mathcal{R}}^{ss},$$

we can define a family $\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}$ of semistable sheaves (Lemma 3.2) on X_0 by the exact sequence

$$0\longrightarrow\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}\longrightarrow(\pi\times I_{\widetilde{\mathcal{R}}^{ss}})_*(\varrho^*\mathcal{E})\longrightarrow{}_{x_0}\mathcal{Q}\longrightarrow 0. \tag{3.2}$$

Since $\varrho^*\mathcal{E}$ is flat over $\widetilde{\mathcal{R}}^{ss}$ and $\mathcal{Q}$ locally free on $\widetilde{\mathcal{R}}^{ss}$, $\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}$ is a flat family over $\widetilde{\mathcal{R}}^{ss}$. Thus we have a morphism

$$\phi_{\widetilde{\mathcal{R}}^{ss}}:\widetilde{\mathcal{R}}^{ss}\longrightarrow\mathcal{R}^{ss}\longrightarrow\mathcal{U}_{X_0}$$

such that $\phi^*_{\widetilde{\mathcal{R}}^{ss}}\Theta_{\mathcal{U}_{X_0}}=\Theta_{\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}}$ by Theorem 1.2.

Lemma 3.4. *The morphism $\phi_{\widetilde{\mathcal{R}}^{ss}}$ induces a morphism*

$$\phi_{\mathcal{P}_{\chi_1,\chi_2}}:\mathcal{P}_{\chi_1,\chi_2}\longrightarrow\mathcal{U}_{X_0}$$

such that $\phi^*_{\mathcal{P}_{\chi_1,\chi_2}}\Theta_{\mathcal{U}_{X_0}}=\Theta_{\mathcal{P}_{\chi_1,\chi_2}}$.

Proof. The proof is clear, we just remark that one can compute $\Theta_{\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}}=\Theta_{\widetilde{\mathcal{R}}^{ss}}$ by the exact sequence (3.2) defining the sheaf $\mathcal{F}_{\widetilde{\mathcal{R}}^{ss}}$. □

Let $\mathcal{U}_{\chi_1,\chi_2}$ be the image of $\mathcal{P}_{\chi_1,\chi_2}$ under the morphism $\phi_{\mathcal{P}_{\chi_1,\chi_2}}$, then $\mathcal{U}_{\chi_1,\chi_2}$ is an irreducible component of $\mathcal{U}_{X_0}$ and $\phi_{\mathcal{P}_{\chi_1,\chi_2}}$ is a finite morphism since it pulls back an ample line bundle to an ample line bundle. We will see that

$$\phi_{\mathcal{P}_{\chi_1,\chi_2}}: \mathcal{P}_{\chi_1,\chi_2}\setminus\{\mathcal{D}_1,\mathcal{D}_2\} \longrightarrow \mathcal{U}_{\chi_1,\chi_2}\setminus\mathcal{W}_{r-1}$$

is an isomorphism. Thus $\phi_{\mathcal{P}_{\chi_1,\chi_2}}$ is the normalization of $\mathcal{U}_{\chi_1,\chi_2}$. We have clearly the morphism

$$\phi := \coprod_{\chi_1+\chi_2=\chi+r} \phi_{\mathcal{P}_{\chi_1,\chi_2}}: \mathcal{P} \longrightarrow \mathcal{U}_{X_0}.$$

which is the normalization of $\mathcal{U}_{X_0}$. We copy Proposition 2.1 from [Su].

Proposition 3.1. *With the above notation and denoting $\mathcal{D}_1(r-1)$, $\mathcal{D}_2(r-1)$, $\mathcal{W}_{r-1}$ by $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{W}$, we have*

(1) $\phi: \mathcal{P}\to\mathcal{U}_{X_0}$ *is finite and surjective, and* $\phi(\mathcal{D}_1(a))=\phi(\mathcal{D}_2(a))=\mathcal{W}_a$;

(2) $\phi(\mathcal{P}\setminus\{\mathcal{D}_1\cup\mathcal{D}_2\})=\mathcal{U}_{X_0}\setminus\mathcal{W}$, *and induces an isomorphism on* $\mathcal{P}\setminus\{\mathcal{D}_1\cup\mathcal{D}_2\}$;

(3) $\phi|_{\mathcal{D}_1(a)}: \mathcal{D}_1(a)\to\mathcal{W}_a$ *is finite and surjective*;

(4) $\phi(\mathcal{D}_1(a)\setminus\{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)\})=\mathcal{W}_a\setminus\mathcal{W}_{a-1}$, *and ϕ induces an isomorphism on* $\mathcal{D}_1(a)\setminus\{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)\}$;

(5) $\phi: \mathcal{P}\to\mathcal{U}_{X_0}$ *is the normalization of* $\mathcal{U}_{X_0}$;

(6) $\phi|_{\mathcal{D}_1(a)}: \mathcal{D}_1(a)\to\mathcal{W}_a$ *is the normalization of* $\mathcal{W}_a$;

(7) $\phi(\mathcal{D}_1(a)\cap\mathcal{D}_2)=\mathcal{W}_{a-1}$, *and $\mathcal{W}_{a-1}$ is the nonnormal locus of* $\mathcal{W}_a$.

Proof. In proving (4), we used Lemma 2.6 of [Su] to show that ϕ induces a morphism

$$\phi: \mathcal{D}_1(a)\setminus\{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)\} \longrightarrow \mathcal{W}_a\setminus\mathcal{W}_{a-1}.$$

But Lemma 2.6 in [Su] is not correct, we have to prove it without using the lemma (also to fix the gap in [Su]). We will use $[\cdot]$ to denote the s-equivalence classes of the objects we are considering. For any $[(E,Q)]\in\mathcal{D}_1(a)\setminus\{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)\}$, we can assume that E is a vector bundle by Proposition 2.5, and $E_{x_2}\to Q$ is an isomorphism since $[(E,Q)]\notin\mathcal{D}_2$. Thus $\phi(E,Q)=F\in\widehat{\mathcal{W}}_a\setminus\widehat{\mathcal{W}}_{a-1}$ by Lemma 3.1(3). We need to show that $[F]\notin\mathcal{W}_{a-1}$. If this is not so, then F is s-equivalent to a semistable torsion free sheaf $F'\in\widehat{\mathcal{W}}(a-1)$ and (by Lemma 1.3)

$$a-1\geq \mathbf{a}(F')\geq \mathbf{a}(\operatorname{gr} F')=\mathbf{a}(\operatorname{gr} F).$$

On the other hand, by Lemma 3.3, (E,Q) is s-equivalent to a semistable (E',Q') with $\dim \operatorname{Tor} E'=r-\mathbf{a}(\operatorname{gr} F)$. By Proposition 2.5(1), E' has no torsion at x_1 since $[E',Q')]=[(E,Q)]\notin\mathcal{D}_2$. Hence, by Proposition 2.5(1) again, (E',Q') is s-equivalent to a GPS $(\widetilde{E},\widetilde{Q})$ with $\widetilde{E}$ locally free and

$$\operatorname{rank}(\widetilde{E}_{x_1}\longrightarrow\widetilde{Q})\leq \mathbf{a}(\operatorname{gr} F)=\mathbf{a}(\operatorname{gr} F')\leq\mathbf{a}(F')\leq a-1.$$

We get the contradiction $[(E,Q)]=[(\widetilde{E},\widetilde{Q})]\in\mathcal{D}_1(a-1)$. Thus ϕ induces a morphism

$$\phi\colon \mathcal{D}_1(a)\backslash\{\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1)\}\longrightarrow \mathcal{W}_a\backslash\mathcal{W}_{a-1}.$$

The argument in [Su] for the other statements goes through using [B], only (7) is in doubt. This can be seen as follows, the fact $\phi(\mathcal{D}_1(a)\cap\mathcal{D}_2)=\mathcal{W}_{a-1}$ follows the local computation (see Proposition 3.9 of [B]), and the nonnormal locus of $\mathcal{W}_a$ is contained in $\mathcal{W}_{a-1}$ by (4). If $\mathcal{W}_{a-1}$ is nonempty and not equal to the nonnormal locus, there exists a nonempty irreducible component $\mathcal{W}_{a-1}^{\chi_1,\chi_2}$ of $\mathcal{W}_{a-1}$ such that $\phi|_{\mathcal{D}_1(a)}$ is an isomorphism at the generic point of $\mathcal{W}_{a-1}^{\chi_1,\chi_2}$. This is impossible since the fibre has at least two points (one in $\mathcal{D}_1(a-1)\backslash\mathcal{D}_2$ by Lemma 3.1 and another in $\mathcal{D}_1(a)\cap\mathcal{D}_2$). □

Let I_Z denote the ideal sheaf of a closed subscheme Z in a scheme X. When Z is of codimension one (not necessarily a Cartier divisor), we set $\mathcal{O}_X(-Z):=I_Z$. If $\mathcal{L}$ is a line bundle on X and Y is a closed subscheme of X, we denote $\mathcal{L}\otimes I_Z$ and the restriction $I_Z\otimes\mathcal{O}_Y$ of I_Z on Y by $\mathcal{L}(-Z)$ and $\mathcal{O}_Y(-Z)$. We have the straightforward generalizations of [Su, Lemma 4.3 and Proposition 4.1], whose proof we omit.

Lemma 3.5. *Assume given a seminormal variety V with normalization $\sigma\colon\widetilde{V}\to V$. Let the nonnormal locus be W, endowed with its reduced structure. Let $\widetilde{W}$ be the set-theoretic inverse image of W in $\widetilde{V}$, endowed with its reduced structure. Let N be a line bundle on V, and let $\widetilde{N}$ be its pullback to $\widetilde{V}$ ($\widetilde{N}=\sigma^*N$). Suppose $H^0(\widetilde{V},\widetilde{N})\to H^0(\widetilde{W},\widetilde{N})$ is surjective. Then*

(1) *there is an exact sequence*

$$0\longrightarrow H^0(\widetilde{V},\widetilde{N}\otimes I_{\widetilde{W}})\longrightarrow H^0(V,N)\longrightarrow H^0(W,N)\longrightarrow 0;$$

(2) *if $H^1(W,N)\to H^1(\widetilde{W},\widetilde{N})$ is injective, so is $H^1(V,N)\to H^1(\widetilde{V},\widetilde{N})$.*

Lemma 3.6. *The following maps are surjective for any $1\le a\le r$,*

(1) $H^0(\mathcal{D}_1(a),\Theta_{\mathcal{P}})\to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2\cup\mathcal{D}_1(a-1),\Theta_{\mathcal{P}})$;

(2) $H^0(\mathcal{D}_1(a),\Theta_{\mathcal{P}})\to H^0(\mathcal{D}_1(a)\cap\mathcal{D}_2,\Theta_{\mathcal{P}})$.

Lemma 3.6 tells us that the assumption (surjectivity) in Lemma 3.5 is satisfied for the situation $V=\mathcal{W}_a$, $\widetilde{V}=\mathcal{D}_1(a)$, $\sigma=\phi|_{\mathcal{D}_1(a)}$ and $N=\Theta_{\mathcal{U}_X}|_{\mathcal{W}_a}$. Thus we can use Lemma 3.5 to prove the following result.

Proposition 3.2. *We have a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_{X_0},\Theta_{\mathcal{U}_{X_0}})\cong H^0(\mathcal{P},\Theta_{\mathcal{P}}(-\mathcal{D}_2)).$$

Proof. The proof is similar to the proof of Proposition 4.3 of [Su]. □

Lemma 3.7. *Let V be a projective scheme on which a reductive group G acts, $\widetilde{\mathcal{L}}$ be an ample line bundle linearizing the G-action, and V^{ss} be the open subscheme of semistable points. Let V' be a G-invariant closed subscheme of V^{ss} and $\overline{V}'$ its schematic closure in V. Then*

(1) $\overline{V}'^{ss}=V'$, *and $V'/\!/G$ is a closed subscheme of $V^{ss}/\!/G$;*

(2) $H^0(V^{ss},\widetilde{\mathcal{L}})^G=H^0(W,\widetilde{\mathcal{L}})^G$, *where W is an open G-invariant irreducible normal subscheme of V containing V^{ss} and $(\cdot)^{\mathrm{inv}}$ denotes the invariant subspace for an action of G.*

Proof. See Lemma 4.14 and Lemma 4.15 of [NR]. □

Lemma 3.8. *Let V be a normal variety with a G-action, where G is a reductive algebraic group. Suppose a good quotient $\pi: V\to U$ exists. Let $\widetilde{\mathcal{L}}$ be a G-line bundle on V, and suppose it descends to a line bundle $\mathcal{L}$ on U. Let $V''\subset V'\subset V$ be open G-invariant subvarieties of V, such that V' maps onto U and $V''=\pi^{-1}(U'')$ for some nonempty open subset U'' of U. Then any invariant section of $\widetilde{\mathcal{L}}$ on V' extends to V.*

Proof. See Lemma 4.16 of [NR]. □

Proposition 3.3. *Let $\widetilde{\mathcal{R}}_F\subset\mathcal{H}$ be the open set consisting of (E,Q) with E locally free. Then*

$$H^0(\widetilde{\mathcal{R}}^{ss},\Theta_{\widetilde{\mathcal{R}}^{ss}})^G=H^0(\mathcal{H},\Theta_{\mathcal{H}})^G=H^0(\widetilde{\mathcal{R}}_F,\Theta_{\widetilde{\mathcal{R}}_F})^G,$$

where $G=(GL(V_1)\times GL(V_2))\cap SL(V_1\oplus V_2)$.

Proof. The first equality follows from Lemma 3.7, the second equality follows from Lemma 3.8 by taking $V=\widetilde{\mathcal{R}}^{ss}$, $U=\mathcal{P}_{\chi_1,\chi_2}$, $V'=\widetilde{\mathcal{R}}^{ss}\cap\widetilde{\mathcal{R}}_F$ and $U''=\mathcal{P}_{\chi_1,\chi_2}\setminus\{\mathcal{D}_1,\mathcal{D}_2\}$ (one needs Proposition 1.4 to show that U'' is nonempty). □

Lemma 3.9. *Suppose $V\to V/\!/G$ is a good quotient and T is any variety with trivial G-action. Then $V\times T\to V/\!/G\times T$ is a good quotient.*

Proposition 3.4. *Let G_1 and G_2 be reductive algebraic groups acting on the normal projective schemes $\overline{V}_1$ and $\overline{V}_2$ with ample linearizing L_1 and L_2. Suppose that L_1 and L_2 descend to Θ_1 and Θ_2. Then, for any G-invariant open sets $V_1\supset\overline{V}_1^{ss}$ and $V_2\supset\overline{V}_2^{ss}$,*

$$H^0(V_1\times V_2,L_1\otimes L_2)^{G_1\times G_2}=H^0(V_1,L_1)^{G_1}\otimes H^0(V_2,L_2)^{G_2}.$$

Proof. Using Lemma 3.7 and Lemma 3.9, we have

$$\begin{aligned}H^0(V_1\times V_2,L_1\otimes L_2)^{G_1\times G_2}&=\{H^0(V_1\times V_2,L_1\otimes L_2)^{G_1\times\{\mathrm{id}\}}\}^{\{\mathrm{id}\}\times G_2}\\&=H^0(\overline{V}_1^{ss}/\!\!/G_1\times V_2,\Theta_1\otimes L_2)^{\{\mathrm{id}\}\times G_2}\\&=H^0(\overline{V}_1^{ss}/\!\!/G_1\times\overline{V}_2^{ss}/\!\!/G_2,\Theta_1\otimes\Theta_2)\\&=H^0(\overline{V}_1^{ss}/\!\!/G_1,\Theta_1)\otimes H^0(\overline{V}_2^{ss}/\!\!/G_2,\Theta_2)\\&=H^0(V_1,L_1)^{G_1}\otimes H^0(V_2,L_2)^{G_2}.\quad\square\end{aligned}$$

Notation 3.1. For $\mu=(\mu_1,\dots,\mu_r)$ with $0\le\mu_r\le\ldots\le\mu_1\le k-1$, let

$$\{d_i=\mu_{r_i}-\mu_{r_i+1}\}_{i=1}^{l}$$

be the subset of nonzero integers in $\{\mu_i-\mu_{i+1}\}_{i=1}^{r-1}$. Then we define

$$\begin{aligned}&r_i(x_1)=r_i, && d_i(x_1)=d_i, && l_{x_1}=l, && \alpha_{x_1}=\mu_r,\\&r_i(x_2)=r-r_{l-i+1}, && d_i(x_2)=d_{l-i+1}, && l_{x_2}=l, && \alpha_{x_2}=k-\mu_1\end{aligned}$$

and for $j=1,2$, we set

$$\begin{aligned}\vec{a}(x_j)&=\left(\mu_r,\mu_r+d_1(x_j),\dots,\mu_r+\sum_{i=1}^{l_{x_j}-1}d_i(x_j),\mu_r+\sum_{i=1}^{l_{x_j}}d_i(x_j)\right),\\\vec{n}(x_j)&=(r_1(x_j),r_2(x_j)-r_1(x_j),\dots,r_{l_{x_j}}(x_j)-r_{l_{x_j}-1}(x_j)).\end{aligned}$$

We also define

$$\begin{aligned}\chi_1^{\mu}&=\frac{1}{k}\left(\sum_{x\in I_1}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I_1}\alpha_x+rl_1\right)+\frac{1}{k}\sum_{i=1}^{r}\mu_i,\\\chi_2^{\mu}&=\frac{1}{k}\left(\sum_{x\in I_2}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I_2}\alpha_x+rl_2\right)+r-\frac{1}{k}\sum_{i=1}^{r}\mu_i.\end{aligned}$$

One can check that the numbers defined in Notation 3.1 satisfy $(j=1,2)$

$$\sum_{x\in I_j\cup\{x_j\}}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I_j\cup\{x_j\}}\alpha_x+rl_j=k\chi_j^{\mu}. \tag{3.3}$$

Notation 3.2. For the numbers defined in Notation 3.1, let, for $j=1,2$,

$$\mathcal{U}^{\mu}_{X_j} := \mathcal{U}_{X_j}(r, \chi^{\mu}_j, I_j \cup \{x_j\}, \{\vec{n}(x), \vec{a}(x)\}_{x \in I_j \cup \{x_j\}}, k)$$

be the moduli space of s-equivalence classes of semistable parabolic bundles E of rank r on X_j and Euler characteristic $\chi(E)=\chi^{\mu}_j$, together with parabolic structures of type $\{\vec{n}(x)\}_{x\in I\cup\{x_j\}}$ and weights $\{\vec{a}(x)\}_{x\in I\cup\{x_j\}}$ at the points $\{x\}_{x\in I\cup\{x_j\}}$. We define $\mathcal{U}^{\mu}_{X_j}$ to be empty if χ^{μ}_j is not an integer. Let

$$\Theta_{\mathcal{U}^{\mu}_{X_j}} := \Theta(k, l_j, \{\vec{n}(x), \vec{a}(x), \alpha_x\}_{x \in I_j \cup \{x_j\}}, I_j \cup \{x_j\})$$

be the theta line bundle.

Theorem 3.1. *There exists a (noncanonical) isomorphism*

$$H^0(\mathcal{U}_{X_1\cup X_2}, \Theta_{\mathcal{U}_{X_1\cup X_2}}) \cong \bigoplus_{\mu} H^0(\mathcal{U}^{\mu}_{X_1}, \Theta_{\mathcal{U}^{\mu}_{X_1}}) \otimes H^0(\mathcal{U}^{\mu}_{X_2}, \Theta_{\mathcal{U}^{\mu}_{X_2}}),$$

where $\mu=(\mu_1,\dots,\mu_r)$ runs through the integers $0\le\mu_r\le\dots\le\mu_1\le k-1$.

Proof. As in Proposition 3.3, one can show that

$$H^0(\mathcal{P}_{\chi_1,\chi_2}, \Theta_{\mathcal{P}_{\chi_1,\chi_2}}(-\mathcal{D}_2)) = H^0(\widetilde{\mathcal{R}}_F, \Theta_{\widetilde{\mathcal{R}}_F}(-\widehat{\mathcal{D}}_2))^G.$$

Note that $\mathcal{O}_{\widetilde{\mathcal{R}}_F}(-\widehat{\mathcal{D}}_2)=\det\mathcal{E}_{x_2}\otimes(\det\mathcal{Q})^{-1}$ and write $\eta_{x_2}:=(\det\mathcal{E}_{x_2})^{-1}\otimes\det\mathcal{Q}$. We have

$$H^0(\widetilde{\mathcal{R}}_F, \Theta_{\widetilde{\mathcal{R}}_F}(-\widehat{\mathcal{D}}_2))^G = H^0(\mathcal{R}_{1,F}\times\mathcal{R}_{2,F}, \Theta_{\mathcal{R}_{1,F}}\otimes\Theta_{\mathcal{R}_{2,F}}\otimes(\det\mathcal{E}_{x_2})^k\otimes\varrho_*(\eta^{k-1}_{x_2}))^G.$$

Let

$$\mathcal{R}^{\mu}_j := \prod_{x\in I_j\cup\{x_j\}}{}_{\widetilde{\mathbf{Q}}_{1,F}} \operatorname{Flag}_{\vec{n}(x)}(\mathcal{F}^j_x) \xrightarrow{p^{\mu}_j} \mathcal{R}_{j,F},$$

then, by Lemma 4.6 of [Su], we have

$$\varrho_*(\eta^{k-1}_{x_2}) = \bigoplus_{\mu} p^{\mu}_{1*}(\mathcal{L}^{\mu}_1)\otimes p^{\mu}_{2*}(\mathcal{L}^{\mu}_2)$$

where $\mu=(\mu_1,\dots,\mu_r)$ runs through the integers $0\le\mu_r\le\dots\le\mu_1\le k-1$ and

$$\mathcal{L}^{\mu}_1 = (\det\mathcal{E}^1_{x_1})^{\mu_r}\otimes\bigotimes_{i=1}^{l_{x_1}}(\det\mathcal{Q}_{x_1,i})^{d_i(x_1)},$$

$$\mathcal{L}^{\mu}_2 = (\det\mathcal{E}^2_{x_2})^{-\mu_1}\otimes\bigotimes_{i=1}^{l_{x_2}}(\det\mathcal{Q}_{x_2,i})^{d_i(x_2)}$$

are line bundles on $\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu$. By the definition

$$\Theta_{\mathcal{R}_j^\mu}:=(\det R\pi_{\mathcal{R}_j^\mu}\mathcal{E}^j)^k\otimes\bigotimes_{x\in I_j\cup\{x_j\}}\left\{(\det\mathcal{E}_x^j)^{\alpha_x}\otimes\bigotimes_{i=1}^{l_x}(\det\mathcal{Q}_{x,i})^{d_i(x)}\right\}\otimes(\det\mathcal{E}_{y_j}^j)^{l_j},$$

one sees easily that

$$\Theta_{\mathcal{R}_1^\mu}=p_1^{\mu*}(\Theta_{\mathcal{R}_{1,F}})\otimes\mathcal{L}_1^\mu,$$
$$\Theta_{\mathcal{R}_2^\mu}=p_2^{\mu*}(\Theta_{\mathcal{R}_{2,F}}\otimes(\det\mathcal{E}_{x_2})^k)\otimes\mathcal{L}_2^\mu.$$

Thus we have (for any χ_1 and χ_2) the equality

$$H^0(\mathcal{P}_{\chi_1,\chi_2},\Theta_{\mathcal{P}_{\chi_1,\chi_2}}(-\mathcal{D}_2))=\bigoplus_\mu H^0(\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu,\Theta_{\mathcal{R}_1^\mu}\otimes\Theta_{\mathcal{R}_2^\mu})^G.$$

Since $\mathbf{C}^*\times\mathbf{C}^*$ acts trivially on $\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu$, one can see that if

$$H^0(\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu,\Theta_{\mathcal{R}_1^\mu}\otimes\Theta_{\mathcal{R}_2^\mu})^G\neq0,$$

then the χ_j $(j=1,2)$ has to satisfy

$$\sum_{x\in I_j\cup\{x_j\}}\sum_{i=1}^{l_x}d_i(x)r_i(x)+r\sum_{x\in I_j\cup\{x_j\}}\alpha_x+rl_j=k\chi_j.$$

Therefore χ_j has to be χ_j^μ. In this case, $\mathbf{C}^*\times\mathbf{C}^*$ acts trivially on the line bundle,

$$H^0(\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu,\Theta_{\mathcal{R}_1^\mu}\otimes\Theta_{\mathcal{R}_2^\mu})^G=H^0(\mathcal{R}_1^\mu\times\mathcal{R}_2^\mu,\Theta_{\mathcal{R}_1^\mu}\otimes\Theta_{\mathcal{R}_2^\mu})^{SL(V_1)\times SL(V_2)}.$$

Thus, by using Proposition 3.4, we can prove the theorem. □

We end this paper by some remarks. In Notation 1.1, we chose and fixed the ample line bundle $\mathcal{O}(1)$, the theta line bundle and the factorization are generally dependent on this choice. In some cases, although the moduli space itself depends on the choice, the theta bundle and the factorization (also the number of irreducible components of the moduli space) are independent of the choice. For example, when $\chi=0$, $|I|=0$, or the parabolic degree is zero, we have $l_1+l_2=0$. In any case, one can see that $\chi_1^\mu<n_1+r$, thus, for any choice, there are only r components of moduli space contributing to the factorization.

The choice in Notation 1.1 has quite a lot of freedom, it is in general a choice of the partitions of l_1+l_2. In particular, if we are only interested in studying moduli space, we can choose any $\mathcal{O}(1)$.

Corollary 3.1. *There is a choice of $\mathcal{O}(1)$ such that the moduli space $\mathcal{U}_{X_0}$ has r irreducible components and*

$$\mathcal{W}_0 = \emptyset.$$

In particular, when $r=2$, $\mathcal{U}_{X_0}$ has two normal crossing irreducible components.

Proof. One can easily choose $\mathcal{O}(1)$ such that n_1 and n_2 are nonintegers. Thus $n_j<\chi_j<n_j+r$ $(j=1,2)$ has only r possibilities and for each such χ_j there is a nonempty irreducible component by Proposition 1.4. Recall (2.2),

$$n_1+r-\dim Q^{E_2} \le \chi(E_1) \le n_1+\dim Q^{E_1},$$
$$n_2+r-\dim Q^{E_1} \le \chi(E_2) \le n_2+\dim Q^{E_2}.$$

We see that $\dim Q^{E_j} \ge \chi_j - n_j > 0$, which means that

$$\mathcal{D}_1(0) = \mathcal{D}_2(0) = \emptyset.$$

Thus $\mathcal{W}_0=\emptyset$. In particular, when $r=2$, the local model of moduli space at any nonlocally free sheaf is $\mathbf{C}[x,y]/(xy)$, by Lemma 1.4. □

Remark 3.1. When $r=2$ and $\mathcal{O}(1)$ is chosen such that n_1 and n_2 are nonintegers, $\mathcal{P}$ has two disjoint irreducible components $\mathcal{P}_1$ and $\mathcal{P}_2$ and $\mathcal{D}_j \subset \mathcal{P}_j$ $(j=1,2)$ is isomorphic to $\mathcal{W} \subset \mathcal{U}_{X_0}$. Thus $\mathcal{U}_{X_0}$ can be obtained from $\mathcal{P}_1$ and $\mathcal{P}_2$ by identifying $\mathcal{D}_1$ and $\mathcal{D}_2$.

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