Ark. Mat., **41** (2003), 61–84

On a problem of Griffiths: an inversion of Abel's theorem for families of zero-cycles

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Abstract. This paper gives a partial answer to a problem raised by Griffiths in [4], which is a kind of converse of Abel's theorem.

Let Y be a smooth irreducible surface of degree five in the projective space $\mathbf{P}_3$. Let us consider the lines Δ_t of the equations $y=ax+b$ and $z=a'x+b'$, with $t=(a,a',b,b')$. For a generic $t_0\in\mathbf{C}^4$, we can define five analytic maps $P_i: U\to Y$, $i=1,\dots,5$, in a neighborhood U of t_0, such that $\Delta_t\cap Y=\sum_{i=1}^5 P_i(t)$. By Abel's theorem, we have $\sum_{i=1}^5 P_i^*(\omega)=0$ on U for any holomorphic two-form ω on Y. In his article, Griffiths asserts (without giving a proof) the following converse:

Let U be an open subset of $\mathbf{C}^4$, U_i, $i=1,\dots,5$, be five disjoint open subsets of Y and $P_i: U\to U_i$ be five open holomorphic maps. If $\sum_{i=1}^5 P_i^*(\omega)=0$ on U for every holomorphic two-form ω on Y, then for every $t\in U$ the zero-cycle $\sum_{i=1}^5 P_i(t)$ can be written as $\Delta_t\cap Y$, where Δ_t is a straight line.

He concludes that this result may be part of a more general picture.

We did not succeed in proving this assertion without some "uniform position" assumption, defined later; but with this uniform position assumption, we prove an analogous theorem for hypersurfaces of degree $d\geq N+2$ in $\mathbf{P}_N$ for any $N\geq 2$, after an introduction stating the generalized Abel's theorem for families of zero-cycles. In the case $N=2$, i.e., for a family of zero-cycles on a curve, the family is abelian if and only if the corresponding subset in the symmetric product $Y^{(k)}$ is contained in a linear series. So in this case, we show the "general picture", restating Ciliberto's classification of linear series of maximal dimension on plane curves by the use of Gruson–Peskine's numerical character.

Acknowledgments. I want to thank especially G. Henkin, P. Mazet and C. Peskine for their remarks during my Ph.D. studies. I am also grateful to T. Ekedahl for some interesting discussions after my studies, and to the TMR Research Network, who made this possible([1]).

0. Generalization of Abel's theorem for families of zero-cycles

Let Y be a reduced analytic space of pure dimension n. We know the sheaves Ω_Y^q of holomorphic q-forms on Y, and $\mathcal{M}_Y^q$ of meromorphic q-forms on Y. For a meromorphic form ω on Y, we let $\operatorname{Pol}\omega$ denote the closed subset of $w \in Y$ such that $\omega \notin \Omega_{y,w}^q$; we identify ω with its restriction to any dense Zariski open subset of Y. The polar locus of ω, $\operatorname{Pol}\omega$, is thus the smallest closed subset F such that ω is holomorphic on $Y \setminus F$. We also have Barlet's sheaves ω_Y^q of meromorphic forms ω for which the principal value current $[\omega]$ is $\bar{\partial}$-closed (see [1] for definitions and properties). Finally, we have the sheaf $\mathcal{L}_Y^q$ of meromorphic forms ω whose pull-back is holomorphic for some desingularization of Y (a *desingularization* is a proper map from a non-singular $\widetilde{Y}$ to Y, one-to-one over $\operatorname{Reg} Y$). Then we know that it is holomorphic for any desingularization of Y.

Definition 0.1. *Abelian forms* are the sections of ω_Y^q; abelian one-forms on curves are called *abelian differentials. Finite forms* are the sections of $\mathcal{L}_Y^q$.

Let T be a reduced and irreducible analytic space, I be a reduced analytic space of pure dimension $\dim T$, and $\pi: I \to T$ be a proper morphism.

Definition 0.2. The proper morphism π is a *ramified covering* if there is an analytic subset $S \subset T$, without interior point, such that $\pi': \pi^{-1}(T \setminus S) \to T \setminus S$ is an analytic covering, and moreover such that $\pi^{-1}(T \setminus S)$ is dense in I. The constant cardinal of the fiber $\pi^{-1}(t)$, $t \in T \setminus S$, is then called the *degree* of the ramified covering π.

Let $\pi: I \to T$ be a ramified covering and let us consider the set Σ of parameters $t \in T$ over which $\pi_1^{-1}(t)$ is not zero-dimensional (and thus $\Sigma \subset S$).

Let us observe the following fact.

Lemma 0.3. *The set Σ is an analytic subset of codimension at least two in T.*

Proof. Let $I' \subset I$ be the set of points z such that the fiber $\pi_1^{-1}(\pi_1(z))$ at z has dimension ≥ 1. Then we know from Remmert [9] that I' is a closed analytic subset without interior point. Since π_1 is proper, still by Remmert, $\pi_1(I') = \Sigma$ is an analytic subset of T. Let us assume that Σ has an irreducible component T' of codimension one in T. Since the fibers over T' are disjoint of dimension ≥ 1,

([1]) TMR Research Network, ERBFMRXCT 98063.

we deduce that $\pi_1^{-1}(T')$ must have dimension $\geq \dim T'+1=\dim I$. Thus it must contain an irreducible component of I. But the assumption that $\pi_1^{-1}(T\backslash S)$ is dense in I gives us a contradiction. $\square$

We will use the following lemma of Henkin–Passare [6].

Lemma 0.4. *Let T be a reduced and irreducible analytic space of pure dimension. Let ω be a meromorphic q-form on a dense Zariski open subset T' of T. We know from [7] that ω defines a "principal value current" on T', denoted $[\omega]$. Then ω is meromorphic on T if and only if there exists a current μ on T such that the restriction to T' is equal to $[\omega]$. Moreover, if μ is $\bar\partial$-closed, then ω is abelian on T (i.e., the current $[\omega]$ associated to ω as a meromorphic form on T is $\bar\partial$-closed).*

Let us again consider a ramified covering $\phi\colon I\to T$. Let ω be a meromorphic q-form on I; we consider the current μ on T defined by $\mu=\phi_*([\omega])$. Then μ is a current on T of bidegree $(q,0)$, and it is $\bar\partial$-closed on a dense Zariski open subset T' of $\operatorname{Reg} T$, and thus can be represented on T' by a holomorphic q-form ω', by the Dolbeault–Grothendieck isomorphism. From the lemma of Henkin–Passare, we deduce that ω' is a meromorphic q-form on T.

Definition 0.5. This meromorphic q-form is called the *trace* of the meromorphic q-form ω for the ramified covering $\phi\colon I\to T$, and is denoted $\phi_*(\omega)$.

Since ω is holomorphic on $\phi^{-1}(T\backslash\phi(\operatorname{Pol}\omega))$ and thus $\bar\partial$-closed, we deduce that $\phi_*(\omega)$ is $\bar\partial$-closed over $T\backslash\phi(\operatorname{Pol}\omega)$, and thus holomorphic over $T\backslash(\operatorname{Sing} T\cup\phi(\operatorname{Pol}\omega))$. In particular we deduce that $\operatorname{Pol}\phi_*(\omega)\subset\operatorname{Sing}(T)\cup\phi(\operatorname{Pol}\omega)$.

Lemma 0.6. *Let $\phi\colon I\to T$ be a ramified covering. Then the trace of an abelian form ω on I is abelian on T.*

Proof. In fact, this follows from the lemma of Henkin–Passare, and from the fact that on a dense Zariski open subset of Y the current defined by the trace coincides with $\phi_*([\omega])$. $\square$

Let Y be a reduced analytic space of pure dimension. Let us recall that an analytic family of zero-cycles on Y is a holomorphic map $\phi\colon T\to Y^{(k)}$, where $Y^{(k)}$ is the quotient of the cartesian product Y^k by the finite group σ_k permuting the points. Now we will consider an analytic subset I of the cartesian product $T\times Y$, with the morphisms $p_1\colon I\to T$ and $p_2\colon I\to Y$ induced by the canonical projections and such that p_1 is a ramified covering.

Definition 0.7. The set I is a *meromorphic family of zero-cycles* on Y. The *degree* of the family is the degree k of the ramified covering $\pi\colon I\to T$. The family is called *open* if moreover p_1 is open.

Let us recall the following definition of a meromorphic map from [9] (from which a meromorphic function can be viewed as a meromorphic map into $\mathbf{P}_1$).

Definition 0.8. A map $\phi: X \to Y$ defined over a dense Zariski open subset $X \setminus S$ of X is a *meromorphic map* from X to Y if the adherence of the set $G = \{(x, \phi(x)) : x \in X \setminus S\} \subset (X \setminus S) \times Y$ is an analytic subset $G(\phi) \subset X \times Y$, with a proper projection $p_1: G(\phi) \to X$ (so that the graph $G(\phi)$ of ϕ is a meromorphic family of degree one).

Remark 0.9. Notice that another equivalent definition is the following: there exists a proper modification $\mu: X' \to X$ of X and a morphism $\phi': X' \to Y$ such that $\phi' = \phi \circ \mu$.

Let $I \subset T \times Y$ be a meromorphic family of zero-cycles. Then the morphism $p_1: {p_1}^{-1}(T \setminus S) \subset I \to T \setminus S$ is an analytic covering. From this we deduce a holomorphic map $\phi_I: T \setminus S \to Y^{(k)}$. We have the following lemma.

Lemma 0.10. *The map ϕ_I is a meromorphic map from T to $Y^{(k)}$.*

Proof. Let $\pi_i: Y^k \to Y$, $i = 1, \dots, k$, be the natural projections. Then we consider the analytic subset $\bigcap_{i=1}^k (\mathrm{Id}_T \times \pi_i)^{-1}(I) \subset T \times Y^k$. This analytic subset defines a ramified covering with critical locus contained in S. Let I' be the union of the irreducible components for which the k points of Y are generically distinct. This analytic subset is invariant under the action of the finite group on $T \times Y^k$ permuting the points of Y^k, thus it defines an analytic subset $\widetilde{T} \subset T \times Y^{(k)}$ of pure dimension $\dim T$, with a proper natural projection $p_1': \widetilde{T} \to T$. Since p_1' is a ramified covering, the adherence of the graph of $\phi_I: T \setminus S \to Y^{(k)}$ is equal to $\widetilde{T}$. Thus ϕ_I is meromorphic. $\square$

Definition 0.11. The map $\phi_I: T \to Y^{(k)}$ is the *meromorphic map* associated with the meromorphic family $I \subset T \times Y$.

Remark 0.12. Since I is the closure in $T \times Y$ of $p_1^{-1}(T \setminus S)$, we can recover the family from the associated map ϕ_I. Moreover we see that the natural morphism $\tilde{\phi}: \widetilde{T} \to Y^{(k)}$ is an analytic family of zero-cycles.

Now we will define, for a meromorphic family $I \subset T \times Y$, the Abel transform $\mathcal{A}_I(\omega)$ of a meromorphic q-form ω on Y whose polar subset P is such that $T \times P$ contains no irreducible component of I.

Now let $I \subset T \times Y$ be a meromorphic family, with $p_1: I \to T$ and $p_2: I \to Y$ being the natural projections. Let ω be a meromorphic q-form on Y, such that $T \times \mathrm{Pol}\, \omega$ contains no irreducible component of I. Then we make the following definition.

Definition 0.13. The *Abel transform* is defined by $\mathcal{A}_I(\omega)=(p_1)_*(p_2^*(\omega))$.

Let us consider a meromorphic map $\phi: X \to Y$, with graph $G(\phi)\subset X\times Y$. We define the *pull-back* $\phi^*(\omega)$ of a meromorphic form ω on Y, under the assumption that $\phi^{-1}(\mathrm{Pol}\,\omega)$ does not contain X, as the meromorphic form $\phi^*(\omega)=\mathcal{A}_{G(\phi)}(\omega)$.

Lemma 0.14. *Let $t_0\in T\setminus S$. Then we define k holomorphic inverses $p_{1,i}$, $1\le i\le k$, of p_1 in a neighborhood U of t_0, and then k holomorphic maps $P_i(t)=p_2(p_{1,i}(t)): U\to Y$, $1\le i\le k$. We then have $\mathcal{A}_I(\omega)=\sum_{i=1}^k P_i^*(\omega)$ on U.*

Proof. By the definition of the trace $(p_1)_*$, we have $\mathcal{A}_I(\omega)=\sum_{i=1}^k p_{1,i}^*(p_2^*(\omega))$, which equals $\sum_{i=1}^k P_i^*(\omega)$ since $p_{1,i}^*\circ p_2^*=(p_2\circ p_{1,i})^*$. $\square$

Let us consider a meromorphic family of zero-cycles $I\subset T\times Y$. We consider a "meromorphic change of parameters", i.e., a meromorphic map $\phi: T'\to T$, with T' irreducible, and $\phi(T')$ not contained in the critical locus $S\subset T$. We consider a corresponding holomorphic map $\phi'': T''\to T$, with $\phi': T''\to T'$ being a proper modification and $\phi''=\phi\circ\phi'$. Let $I'\subset T'\times Y$ be the set of pairs (t',p) for which we have $(\phi''(t''),p)\in I$ for some $t''\in T''$ with $\phi'(t'')=t'$. Then we can define k holomorphic functions $P_i'(t')=P_i(\phi(t'))$, $i=1,\dots,k$, on a dense Zariski open subset of T'. We can check that I' defines a meromorphic family of zero-cycles of the same degree k.

Definition 0.15. The analytic subset I' is the meromorphic family obtained by the change of parameters $\phi: T'\to T$.

Let ω be a meromorphic q-form on Y such that $T'\times\mathrm{Pol}\,\omega$ does not contain any irreducible component of I'. Then $T\times\mathrm{Pol}\,\omega$ does not contain any irreducible component of I. Since we have $\mathcal{A}_{I'}(\omega)=\sum_{i=1}^k (P_i')^*(\omega)$, with $P_i'(t')=P_i(\phi(t'))$, on a dense Zariski open subset of T', and $\mathcal{A}_I(\omega)=\sum_{i=1}^k P_i^*(\omega)$, we obtain the following proposition.

Proposition 0.16. *We have $\mathcal{A}_{I'}(\omega)=\phi^*(\mathcal{A}_I(\omega))$.*

Remark 0.17. From this we deduce another way to compute the Abel transform of a meromorphic family $I\subset T\times Y$ of degree k, with Y irreducible. Let us consider the "universal family" of zero-cycles of degree k defined by $(t,p)\in Y^{(k)}\star Y$ if and only if $p\in t$, with $Y^{(k)}$ as parameter space, and with the analytic subset $Y^{(k)}\star Y\subset Y^{(k)}\times Y$ as incidence variety. Thus we associate to any meromorphic q-form ω on Y a form $\widetilde{\omega}=\mathcal{A}_{Y^{(k)}\star Y}(\omega)$ on $Y^{(k)}$, which can also be computed in the following way. The quotient morphism $\pi: Y^k\to Y^{(k)}$ is a proper surjective finite morphism of degree $k!$, and thus π^* defines an isomorphism between meromorphic forms on $Y^{(k)}$ and the σ_k-invariant meromorphic forms on Y^k (with inverse $(1/k!)\pi_*$). Then we have $\widetilde{\omega}=(1/k!)\pi_*(\underline{\omega})$, with $\underline{\omega}=\sum_{i=1}^k \pi_i^*(\omega)$, $\pi_1,\dots,\pi_k$ being the natural projections from

Y^k to Y. From this computation, we deduce that if ω is abelian, then $\widetilde{\omega}$ is abelian on $Y^{(k)}$. Now let us consider the map $\phi_I: T \to Y^k$ associated to the family. It can be considered as a meromorphic change of parameters. Thus, the preceding proposition gives us that $\mathcal{A}_I(\omega) = \phi_I^*(\widetilde{\omega})$ on $T \setminus S$.

Now we will show the following result.

Proposition 0.18. *If ω is finite, then $\mathcal{A}_I(\omega)$ is finite.*

Corollary 0.19. *Let us consider a meromorphic map $\phi: X \to Y$, with X irreducible and $\phi^{-1}(\operatorname{Sing} Y) \neq X$. Let ω be a finite form on Y. Then the meromorphic form $\phi^*(\omega)$ is finite on X.*

Proof. We consider the graph $G(\phi) \subset X \times Y$. Then the corollary follows from the identity $\phi^*(\omega) = \mathcal{A}_{G(\phi)}(\omega)$. □

For the proof of Proposition 0.18 we will need the following lemma.

Lemma 0.20. *Let X be an analytic subset in the reduced analytic space Y. If no component of X is contained in* $\operatorname{Sing} Y$, *then the restriction of a finite form on Y to X is a finite form on X.*

Proof. This can easily be shown if we assume the existence of a desingularization $\phi_X: X' \to X$ of X, induced by some desingularization $\phi_Y: Y' \to Y$, with $X' \subset Y'$. In fact, $\phi_Y^*(\omega)$ is then holomorphic on Y', and thus the restriction to X' is holomorphic on X'. But this is equal to $\phi_X^*(\omega|_X)$. Thus $\omega|_X$ is finite on X. The proof is given without this assumption by Kaddar in [8]. □

Proof of Proposition 0.18. Let us consider a desingularization $\phi: T' \to T$ of T. Then from this change of parameters we have the new meromorphic family $I' \subset T' \times Y$, with the projections $p_1': I' \to T'$ and $p_2': I' \to Y$. Let ω be a finite form on Y. Then ω canonically defines a finite form on $T' \times Y$, whose restriction to I' is finite by Lemma 0.20. But this restriction is precisely $(p_2')^*(\omega)$, which is thus finite, and thus $\bar{\partial}$-closed, on I'. Then the trace $(p_1')_*((p_2')^*(\omega))$ is abelian, and thus holomorphic, on T'. Thus $\mathcal{A}_{I'}(\omega) = \phi^*(\mathcal{A}_I(\omega))$ is holomorphic on T', and thus $\mathcal{A}_I(\omega)$ is finite on T. □

Remark 0.21. The trace of a meromorphic form by a ramified covering $\phi: I \to T$, is the Abel transform of the meromorphic family $I' = \{(\phi(x), x) : x \in I\} \subset T \times I$. In particular, we deduce from the proposition that the trace of a finite form by a ramified covering $\phi: X \to Y$ (with Y irreducible and X of pure dimension $\dim Y$) is finite on Y.

Let us observe that the Abel transform of an abelian form is not necessarily abelian, even for an open meromorphic family $I \subset T \times Y$. The simplest example is the following example.

Example 0.22. Let $\phi\colon \widetilde{Y} \to Y$ be a desingularization of an irreducible singular reduced analytic space Y. We consider the family $I \subset \widetilde{Y} \times Y$ defined by $(t, \phi(t))$. Then the Abel transform is just $\phi^*(\omega)$. If the abelian form ω is not finite, then $\mathcal{A}_I(\omega)$ is not holomorphic, and thus not abelian on $\widetilde{Y}$.

We will introduce some supplementary conditions on the meromorphic family, which will imply that the Abel transform of an abelian form is abelian. We need the following definition.

Definition 0.23. The morphism $\phi\colon X \to Y$ is a *submersion* if for every point $p \in X$ there is an open neighborhood U_p of p and an isomorphism $\psi\colon U_p \to F_p \times V_p$, with $V_p = \phi(U_p) \subset Y$, such that if p_2 is the natural projection from $F_p \times V_p$ to V_p, we have $\phi = p_2 \circ \psi$.

Remark 0.24. If X and Y are smooth, this coincides with the classical notion of a submersion (the rank of the differential is maximal) in the holomorphic sense.

Then we have the following lemma from Schwartz [12].

Lemma 0.25. *If the morphism $\phi\colon X \to Y$ is a submersion, then there is a pull-back of currents ϕ^* from Y to X, commuting with $\bar{\partial}$, extending the usual pull-back on smooth forms. Moreover, for ω meromorphic on Y, we have $\phi^*([\omega]) = [\phi^*(\omega)]$. In particular, if ω is abelian, then $\phi^*(\omega)$ is abelian.*

Then we have the following result.

Proposition 0.26. *Let $I \subset T \times Y$ be a family such that $p_2\colon I \to Y$ is a submersion. If ω is abelian on Y, then the Abel transform $\mathcal{A}_I(\omega)$ is abelian on T.*

Proof. Let ω be abelian on Y. From the preceding lemma, the meromorphic form $p_2^*(\omega)$ remains abelian on I. Thus, from Lemma 0.6, the trace $(p_1)_*(p_2^*(\omega))$ is abelian on T. □

Now let $I \subset T \times Y$ be a meromorphic family of zero-cycles of degree k; we thus have the associated meromorphic map $\phi_I\colon T \to Y^{(k)}$. We define a family $(T_p)_{p \in Y}$ of analytic subsets on T by $p_2^{-1}(p) = T_p \times \{p\}$. At a point $t_0 \in T \setminus S$, we thus have k analytic subsets $T_{P_i(t_0)}$, $i = 1, \dots, k$, through t_0, locally defined by $P_i(t) = P_i(t_0)$. We define $T_i(t_0)$ to be the irreducible component of $T_{P_i(t_0)}$ containing $P_i(t_0)$. Then, we have seen that $p_1^{-1}(T_i(t_0)) \subset I$ defines a meromorphic family of zero-cycles, with a fixed point $P_i(t_0)$. We denote by $I_i(t_0)$ the family obtained by removing the component $T_i(t_0) \times \{P_i(t_0)\}$ from $p_1^{-1}(T_i(t_0))$. We call $I_i(t_0) \subset T_i(t_0) \times Y$ *the family obtained from I by fixing the point $P_i(t_0)$*. For this family we have the following lemma.

Lemma 0.27. *If ω is holomorphic at $P_i(t_0)$, then $\mathcal{A}_{I_i(t_0)}(\omega)=\mathcal{A}_I(\omega)|_{T_i(t_0)}$.*

Proof. Let ω be a meromorphic form on Y, holomorphic at $P_i(t_0)$. In a neighborhood of t_0, the Abel transform of ω for I can be computed as $P_1^*(\omega)+\ldots+P_k^*(\omega)$; but since $P_i(t)=P_i(t_0)$ on $T_i(t_0)$, the restriction of $\mathcal{A}_I(\omega)$ to $T_i(t_0)$ is $\sum_{j\neq i} P_j^*(\omega)$, which is precisely equal to $\mathcal{A}_{I_i(t_0)}(\omega)$. $\square$

In the sequel we will consider an algebraic subvariety $Y\subset\mathbf{P}_N$ of pure dimension n. We need the following definition.

Definition 0.28. The meromorphic family $I\subset T\times Y$ is *abelian* if $\mathcal{A}_I(\omega)=0$ for every abelian q-form ω on Y, $q\geq 1$. It is *finite* if $\mathcal{A}_I(\omega)=0$ for every finite q-form ω on Y, $q\geq 1$.

From the preceding results it follows that, if $I\subset T\times Y$ is a meromorphic family and T has no non-zero finite q-forms, $q\geq 1$, then I is a finite family. Moreover, if $p_2\colon I\to Y$ is a submersion and T has no non-zero abelian q-forms, $q\geq 1$, then I is an abelian family. Furthermore, it follows from the preceding lemma that, if I is an abelian family, then the family obtained from I by fixing a point on $\operatorname{Reg} Y$ remains abelian.

We will give two major examples of abelian families of zero-cycles.

0.1. The complete intersections of a fixed multidegree

Let $(k_1,\ldots,k_n)$ be a fixed multidegree, with $n=\dim Y$. We consider the parameter space $T=\mathbf{P}_{N_1}\times\ldots\times\mathbf{P}_{N_n}$, where $\mathbf{P}_{N_i}$ is the projective space parametrizing the homogeneous polynomials of degree k_i; we denote by $Q_{t_i}=0$ the equation associated to $t_i\in\mathbf{P}_{N_i}$.

Then, we define the family $I\subset T\times Y$ by saying that $(t,p)\in T\times Y$ belongs to I if and only if $Q_{t_1}(p)=0,\ldots,Q_{t_n}(p)=0$, where $t=(t_1,\ldots,t_n)$. It is easy to check that $p_1\colon I\to T$ is a ramified covering, and thus defines a meromorphic family. Moreover we have the following result.

Proposition 0.29. *The above family of zero-cycles $I\subset T\times Y$ is abelian.*

Proof. Since T has no non-zero holomorphic q-forms, $q\geq 1$, it suffices to show that $p_2\colon I\to Y$ is a submersion. This can be done as follows. At any point $p_0\in Y$, we consider an open neighborhood U_{p_0} of p_0, and a holomorphic map $M\colon U_{p_0}\to GL_{N+1}(\mathbf{C})$ such that $M(p_0)=\mathrm{Id}$ and $M(p)\cdot p_0=p$. Then, let us consider the transformation $(Q_{t_1},\ldots,Q_{t_n})\mapsto(Q_{t_1}\circ M(p),\ldots,Q_{t_n}\circ M(p))$. This transformation, by projectivization, defines an isomorphism from $p_2^{-1}(p)$ to $p_2^{-1}(p_0)$. More precisely, we have an isomorphism $\phi: p_2^{-1}(U_{p_0}) \subset T \times U_{p_0} \to T_0 \times U_{p_0}$, with $T_0 = p_1(p_2^{-1}(p_0))$, commuting with the projection on the second factor. It is defined by $\phi(t,p) = (t \circ M(p), p)$, where $t \circ M(p)$ is the projectivization of $(Q_{t_1} \circ M(p), \dots, Q_{t_n} \circ M(p))$, considered as an n-tuple of homogeneous polynomials. In particular, $p_2: I \to Y$ is a submersion. □

0.2. The projective transformations of a fixed subvariety

Let $Z \subset \mathbf{P}_N$ be a fixed irreducible subvariety of codimension n. Let further $\mathbf{P}(M_{N+1}(\mathbf{C}))$ be the projective space associated with the non-zero (and possibly non-invertible) projective transformations. We consider the open subset T of $\mathbf{P}(M_{N+1}(\mathbf{C}))$ given by the condition that $t \in \mathbf{P}(M_{N+1}(\mathbf{C}))$ belongs to T if and only if t is well-defined on Z (this means that if $\tilde{t}$ is an associated matrix and $\widetilde{Z}$ the cone over Z, then $\widetilde{Z} \cap \operatorname{Ker} \tilde{t} = \{0\}$).

Then, we consider the following family $I \subset T \times Y$ defined by the condition that $(t,p) \in I$ if and only if $p \in t(Z)$.

Proposition 0.30. *This family of zero-cycles is abelian.*

Proof. First, since the complement of T in $\mathbf{P}(M_{N+1}(\mathbf{C}))$ has codimension at least two, T has no non-zero holomorphic forms. Moreover, the same reasoning as in the previous example allows us to show that $p_2: I \to Y$ is a submersion. Thus, the Abel transform of an abelian q-form, $q \geq 1$, is abelian, and thus zero. □

0.3. Rational families of zero-cycles

Two effective zero-cycles Γ and Γ' on Y of the same degree k are *effectively rationally equivalent* if there is a morphism $\phi: \mathbf{P}_1 \to Y^{(k)}$ such that $\phi(0) = \Gamma$ and $\phi(1) = \Gamma'$. They are *rationally equivalent* if there is some zero-cycle Γ'' such that $\Gamma + \Gamma''$ and $\Gamma' + \Gamma''$ are effectively rationally equivalent. Let us make the following definition.

Definition 0.31. A meromorphic family of zero-cycles $I \subset T \times Y$ is of *rational equivalence* if two generic zero-cycles of the family are rationally equivalent.

It follows from the definition that if T is a smooth compact rationally connected variety, then the family is of rational equivalence. Moreover we have the following result.

Proposition 0.32. *If T is rationally connected, then any meromorphic family $I \subset T \times Y$ is finite.*

Proof. In fact, we know that T has no non-zero holomorphic q-forms, $q \geq 1$, and moreover the Abel transform of a finite form on Y is finite on T. $\square$

Remark 0.33. We do not know if any family of rational equivalence is finite, if Y is of dimension greater than two.

Now we will introduce a notion of uniform position, which generalizes the notion of uniform position for curves ([2]).

First, let us define the dimension of the family. Let us consider the meromorphic map $\phi_I: T \to Y^{(k)}$. Then on a dense Zariski open subset of T, the fiber $\phi_I^{-1}(\phi_I(t))$ has a constant dimension s. We define the *dimension* of the meromorphic family as $\dim T - s$. Thus, a generic $t_0 \in T \setminus S$ has an open neighborhood U_{t_0} over which we can define k analytic maps $P_i: U_{t_0} \to Y$, $i=1, \dots, k$, and such that $\{(P_1(t), \dots, P_k(t)): t \in U_{t_0}\}$ is an analytic subset of Y^k of dimension s, and moreover $T_i(t) \cap U_{t_0}$ is defined by $P_i(t') = P_i(t)$. Then, for a finite subset $I = \{i_1, \dots, i_l\} \subset \{1, \dots, k\}$, we define $n_I(t) = \operatorname{codim}_t \bigcap_{i \in I} T_i(t)$.

Definition 0.34. Let $I \subset T \times Y$ be a meromorphic family of zero-cycles of degree k and dimension s. Then I is in *uniform position* if at a generic point $t_0 \in T \setminus S$, $i \notin I$ and $n_I(t_0) < s$ imply that $n_{I \cup \{i\}}(t_0) > n_I(t_0)$.

We have the following proposition.

Proposition 0.35. *The family $I \subset T \times Y$ of complete intersection zero-cycles of multidegree $(k_1, \dots, k_n)$, with $T = \mathbf{P}_{N_1} \times \dots \times \mathbf{P}_{N_n}$, is in uniform position.*

Proof. In the case of curves ($n=1$), this can be shown in the following way. First, let us show that if $Y \subset \mathbf{P}_N$ is non-degenerate (and thus of degree $d \geq N$), then any N points on a generic hyperplane section $Y \cap H$ generate H.

Let us define the map ϕ from Y^N to the linear subspaces of $\mathbf{P}_N$, which maps N points to the linear subspace they span. Let us consider a point $P_1 \in Y$, and define recursively, for $i \leq N$, $P_i \in Y$ not in the vector space generated by $P_1, \dots, P_{i-1}$. Thus we see that ϕ defines an analytic map from a Zariski open subset of Y^N to $\mathbf{P}_N^*$. We see that this map is defined by a meromorphic map, and thus by an analytic subset $I \subset Y^N \times \mathbf{P}_N^*$, with $p_1: I \to Y^N$ and $p_2: I \to \mathbf{P}_N^*$. If S is an analytic subset of Y^N containing the critical locus, then we define the analytic subset $S' = p_2(p_1^{-1}(S)) \cup Y^*$ of $\mathbf{P}_N^*$, where Y^* is defined as the hyperplanes which cut Y in less than d points. Let $t \in \mathbf{P}_N^* \setminus S'$. Then $H_t \cap Y$ is the set $\{P_1(t), \dots, P_d(t)\}$ of d distinct points. We can choose a loop in $\mathbf{P}_N^* \setminus S'$ such that if we follow the loop analytically, then the points $P_i(t)$ and $P_j(t)$ are exchanged. But we know that any $t \in \mathbf{P}_N^* \setminus S'$ is in $\phi(Y^N \setminus S)$, and thus the hyperplane generated by the N corresponding points is in $\phi(Y^N \setminus S)$. By monodromy, we see that for any $t \in \mathbf{P}_N^* \setminus S'$, any N distinct points in $H_t \cap Y$ generate H_t.

In the general case for curves, we reduce to this case by immersing Y into some projective space $\mathbf{P}_{N'}$, via the linear series defined on Y by the sections with degree k hypersurfaces.

Now we consider the case where Y has arbitrary dimension. Then we consider the parameter space $T=\mathbf{P}_{N_1}\times\ldots\times\mathbf{P}_{N_n}$. At a generic t_0, $T_i(t_0)$, $1\le i\le kd$, define kd linear subspaces of codimension n. Let us assume that for some I, $n_I(t_0)<s$ and $n_{I\cup\{i\}}(t_0)=n_I(t_0)$ with $i\notin I$. The existence of a loop in T permuting the points $P_i(t_0)$ (and thus permuting the leaves $T_i(t_0)$) would then imply that we also have $n_{I\cup\{j\}}(t_0)=n_I(t_0)$ for any $j\notin I$, and thus $n_I(t_0)=n_{\{1,\ldots,kd\}}(t_0)$, which contradicts $n_I(t_0)<s$. $\square$

1. Inversion of Abel's theorem for a hypersurface and the family of lines

Let $Y\subset\mathbf{P}_{n+1}$ be a hypersurface of degree d.

Theorem 1.1. *Let $I\subset T\times Y$ be an open abelian family of zero-cycles of degree d and dimension $2n$, in uniform position. Then the generic zero-cycle $\Gamma_t=P_1(t)+\ldots+P_d(t)\in Y^{(d)}$ is the intersection of Y with a line Δ_t.*

We will use the following lemma, whose proof has been given to me by P. Mazet.

Lemma 1.2. *Let E be a vector space of dimension d, and $E_1,\ldots,E_d$ be d vector subspaces of E. To have a basis $(e_i)_{i=1}^d$ of E satisfying $e_i\in E_i$, it is necessary and sufficient to have*

$$\dim E_I\ge k=\#I$$

for every k such that $1\le k\le d$, and every choice of k distinct integers $I=\{i_1,\ldots,i_k\}\subset\{1,\ldots,d\}$, where $E_I=E_{i_1}+\ldots+E_{i_k}$.

Proof. The necessity is obvious. For the sufficiency, let $N=\sum_{i=1}^d\dim E_i$. We will make induction on N. Let us observe that each E_i must be non-empty. First assume that there exists $k<d$ and k distinct integers $\{i_1,\ldots,i_k\}=I\subset\{1,\ldots,d\}$ such that $\dim E_I=k$. By the induction assumption, we can find

$$e_{i_1}\in E_{i_1},\quad \ldots,\quad e_{i_k}\in E_{i_k},$$

giving a basis of E_I. Moreover, let $\{j_1,\ldots,j_{d-k}\}$ be the complement of $I=\{i_1,\ldots,i_k\}$ in $\{1,\ldots,d\}$, and

$$E'=\frac{E}{E_I},\quad E'_{j_1}=\frac{E_{j_1}}{E_I},\quad \ldots,\quad E'_{j_{d-k}}=\frac{E_{j_{d-k}}}{E_I}.$$

Then the induction assumption applies also to E' and $E'_{j_1}, \dots, E'_{j_{d-k}}$. We can thus obtain a basis of E',

$$e'_{j_1} \in E'_{j_1}, \quad \dots, \quad e'_{j_{d-k}} \in E'_{j_{d-k}}.$$

We can then find for each $e'_{j_l} \in E'_{j_l}$, $1 \le l \le d-k$, some $e_{j_l} \in E_{j_l}$, and then $(e_1, \dots, e_d)$ is a basis of E with $e_i \in E_i$, $1 \le i \le d$.

It remains to investigate the case when $\dim(E_{i_1} + \dots + E_{i_k}) > k$ for every k, $1 \le k < d$ and every choice of integers $\{i_1, \dots, i_k\} = I \subset \{1, \dots, d\}$. Let us replace E_1 by some one-dimensional $F_1 \subset E_1$. We then obtain a new family of subspaces F_1, $F_2 = E_2, \dots, F_d = E_d$. Then for $I = \{i_1, \dots, i_k\}$ let us denote by I' the subset obtained by taking away 1 from I. If I' is non-empty, then $\dim F_I \ge \dim F_{I'} = \dim E_{I'} > \#I' \ge \#I - 1$, and thus $\dim F_I \ge \#I$. If I' is empty, we still have $\dim F_I \ge \#I$. Then, since F_1 is strictly included in E_1, we can apply the induction assumption, and get a basis $(e_i)_{i=1}^d$ of E with $e_i \in F_i \subset E_i$. This concludes the proof of the lemma. □

To prove Theorem 1.1, we will use the following definition and the following classical lemma.

Definition 1.3. Let $\Gamma = P_1 + \dots + P_d$ be a zero-cycle with distinct points on Y. The matrix $M_\Gamma = (m_i(P_j))$, where (m_i) are the monomials of degree $\le d-n-2$ in some affine chart containing Γ, is called a *Brill–Noether matrix* at Γ.

Lemma 1.4. *The function $\phi_\Gamma(l)$ is a strictly increasing function of $l \in \mathbf{N}$, until it is equal to* $\deg \Gamma$.

Proof of Theorem 1.1. The first step is to show that the Hilbert function of Γ, at a generic zero-cycle of the family, is the same as the one of Δ, the intersection of Y with a generic line. Let $t_0 \in T \setminus S$, with the above defined d holomorphic maps $P_i : U \to Y$ in an open neighborhood U of t_0: we put $\Gamma_t = P_1(t) + \dots + P_d(t)$. We choose an affine chart containing Γ_t. Let us show that the last $d-n$ columns of a Brill–Noether matrix M_{Γ_t} are linearly dependent.

We know that on Y, abelian n-forms are in correspondence with polynomials of degree $\le d-n-2$; more precisely, any abelian n-form can be expressed in the form $\omega = (p(x_1, \dots, x_n)/f_y)\, dx_1 \wedge \dots \wedge dx_n$, with $f(x_1, \dots, x_n, y) = 0$ being the equation of Y in some affine chart containing Γ_t. Let us denote the abelian n-forms obtained by taking the monomial m_i instead of $p(x_1, \dots, x_n)$ in the above expression by ω_i.

Then we consider the relations

$$\sum_{j=1}^{d} P_j^*(\omega_i) = \sum_{j=1}^{d} \frac{m_i(P_j)}{f_y(P_j)} P_j^*(dx_1 \wedge \dots \wedge dx_n) = 0.$$

It suffices to choose a matrix (λ_{ki}) in

$$dz_k=\sum_{i=1}^{n}\lambda_{ki}P_k^*(dx_i), \quad 1\le k\le n,$$

such that

$$dz_1\wedge\ldots\wedge dz_n\wedge P_{n+1}^*(dx_1\wedge\ldots\wedge dx_n)\neq 0.$$

The existence of the matrix (λ_{ki}) is a consequence of the preceding lemma, from which it follows that, by the uniform position assumption, if

$$\begin{aligned} E&=\operatorname{Span}\{P_j^*(dx_i):1\le i\le n \text{ and } 1\le j\le d\},\\ E_j&=\operatorname{Span}\{P_j^*(dx_1),\dots,P_j^*(dx_n)\}\subset E,\end{aligned}$$

we can take a basis of E/E_{n+1} by taking $dz_1\in E_1,\dots,dz_n\in E_n$.

Since the choice of the columns is arbitrary, we conclude that the matrix M_{Γ_t} has at most $d-n-1$ linearly independent columns, and thus $\operatorname{rk} M_{\Gamma_t}\le d-n-1$.

But this means that the Hilbert function $\phi_{\Gamma_t}(d-n-2)$ of $\Gamma_t=P_1(t)+\ldots+P_d(t)$ at $d-n-2$ is less or equal to $d-n-1$. Since $\phi_{\Gamma_t}(l)$ is a strictly increasing function of l before reaching the constant value $\deg\Gamma_t=d$, we deduce that $\phi_{\Gamma_t}(1)\le 2$; this means that Γ_t lies on a line. □

Corollary 1.5. *Let Y and Y' be two hypersurfaces of the same degree $d\ge N+2$ in $\mathbf{P}_N$, $N\ge 2$. If $\phi: Y\to Y'$ is an isomorphism, then it is induced by a projective transformation.*

Proof. In fact, the isomorphism ϕ induces a map on zero-cycles; and the d points of a generic line section of Y must transform into the d points of a line section of Y', by the preceding theorem. Thus, we can define (at least locally) a map of the grassmannian of lines into itself; and moreover the lines through $p\in Y$ transform by this map into the lines through $\phi(p)$. But then we know that this map is induced by an automorphism of the grassmannian, which itself must be induced by an automorphism of the ambient projective space. Notice that another proof is given in [5]. □

Remark 1.6. By a slight variation of the proof, we could show the following: if there is an open subset U of Y, *containing a line section*, and a morphism $\phi: U\to Y'$, such that for every abelian ω on Y', $\phi^*(\omega)$ is abelian on Y, then ϕ can be extended to the entire variety Y (and is induced by an automorphism of the projective space). We do not know if the assertion would remain valid if we remove the assumption that U contains a line section.

Remark 1.7. We can show, using a Brill–Noether matrix and Lemma 1.2, the following statement: Let $Y \subset \mathbf{P}_N$ be a non-degenerate smooth algebraic variety of $\mathbf{P}_N$ of dimension n, degree $d=(N-n)(n+1)+2$, and moreover canonical, so that the canonical bundle of Y is isomorphic to $\mathcal{O}_Y(1)$ (we call such varieties *Castelnuovo canonical varieties*, since they are of maximal geometric genus with respect to their degree). Let $I \subset T \times Y$ be an abelian family of zero-cycles of dimension greater than nm in uniform position. Then the generic zero-cycle Γ_t of I is contained in a linear subspace of dimension $N-n$.

2. The case of curves

2.1. General results

Let Y be an irreducible algebraic curve of degree d in $\mathbf{P}_N$. Let $\phi: \widetilde{Y} \to Y$ be the desingularization of Y. Let π be the dimension of the $\mathbf{C}$-vector space of abelian differentials on Y (i.e., global sections of ω_Y^1).

We will identify zero-cycles with support in the regular part of Y, with the corresponding zero-cycles on $\widetilde{Y}$. We know that two zero-cycles $\Gamma, \Gamma' \in \mathrm{Reg}(Y)^{(k)}$ are *rationally equivalent* if there is some rational function r such that the associated zero-cycle in $\widetilde{Y}$ is $(r)=(r)_0-(r)_\infty=\Gamma-\Gamma'$. Then we will say that Γ and Γ' are *linearly equivalent* if moreover this rational function r is holomorphic at the singular points of Y. Let $\Gamma \in \mathrm{Reg}(Y)^{(k)}$. Then the following lemma is a corollary of the theorem of Abel–Rosenlicht ([11]).

Lemma 2.1. *Any zero-cycle rationally equivalent to* Γ *is contained in the fiber* $\phi^{-1}(\phi(\Gamma))$ *at* Γ *of the classical Abel–Jacobi map* $\phi: \widetilde{Y}^{(k)} \to J(Y)=J(\widetilde{Y})$, *given by integration of finite abelian differentials. Any zero-cycle linearly equivalent to* Γ *is contained in the fiber* $(\phi')^{-1}(\phi'(\Gamma))$ *at* Γ *of the generalized Abel–Jacobi map* $\phi': \mathrm{Reg}(Y)^{(k)} \to J'(Y)$, *given by integration of abelian differentials.*

We call the fiber $\phi^{-1}(\phi(\Gamma))$ the *complete rational family through* Γ. We call the fiber $(\phi')^{-1}(\phi'(\Gamma))$ the *complete linear series through* Γ, and let us denote it by $|\Gamma|$. The we have the following lemma.

Lemma 2.2. *A family* $I \subset T \times Y$ *is finite (resp. abelian) if and only if the range of the corresponding map* $\phi_I: T \to Y^{(k)}$ *is contained in a complete rational family (resp. a complete linear series).*

Proof. Let us consider a basis $(\omega_1, \dots, \omega_g)$ of the finite differentials, and let us complete it with $(\omega_{g+1}, \dots, \omega_\pi)$ into a basis of the abelian differentials. In a neighborhood of some zero-cycle with distinct points $P_1+\dots+P_k \in Y^{(k)}$, the fiber of the classical (resp. generalized) Abel–Jacobi map is defined by $z_i = Cte$, with $1 \leq i \leq g$ (resp. $1 \leq i \leq \pi$), where $z_i = \sum_{j=1}^{k} \int_{P_0}^{P_j} \omega_i$. Thus, the condition that the range is contained in the fiber of the classical (resp. generalized) Abel–Jacobi map is equivalent to the conditions $d(z_i \circ \phi_I(t)) = 0$ with $1 \leq i \leq g$ (resp. $1 \leq i \leq \pi$), which can also be written as $\sum_{i=1}^{k} P_j^*(\omega_i) = 0$ for $1 \leq i \leq g$ (resp. $1 \leq i \leq \pi$), where the P_j, $1 \leq j \leq k$, are the locally defined maps over a generic $t \in T$. But these last conditions mean that I is a finite (resp. abelian) family. □

Remark 2.3. For singular curves, we can geometrically describe the difference between complete finite families of zero-cycles (or complete rational families), and complete abelian families (or complete linear series) in the following way. Let k be a sufficiently great integer (such that the curve is l-normal for $l \geq k$). Let H be a hypersurface of degree k, containing Γ, and not meeting $\operatorname{Sing} Y$. If $Y \cap H = \Gamma + \Gamma'$, then the complete linear series through Γ can be obtained by taking the residuals of Γ' in the intersections of Y with the hypersurfaces of degree k containing Γ'. To obtain the greater family of zero-cycles rationally equivalent to Γ, we must take some hypersurface meeting every singular point P_i, with some multiplicity at P_i (such that any holomorphic function annihilating P_i with this multiplicity is holomorphic at P_i). Then we obtain some zero-cycle Γ' (considered as a zero-cycle on the desingularization of Y) such that $H \cap Y = \Gamma + \Gamma'$, and the complete rational family through Γ is obtained by taking the residual of Γ' in the intersections of Y with the hypersurfaces of degree k through Γ'.

Now for a zero-cycle $\Gamma \in \operatorname{Reg}(Y)^{(k)}$ (with support in the regular part of Y), we say that the abelian form ω *annihilates* Γ if $\omega \in H^0(\mathcal{I}_\Gamma \omega_Y^1)$. Then the dimension of the vector space of abelian differentials which annihilate Γ gives the dimension of the complete linear series at Γ, by the following lemma, which is a reformulation of the generalized Riemann–Roch theorem [10].

Lemma 2.4. *We have* $\dim |\Gamma| = k - \pi + h^0(\mathcal{I}_\Gamma \omega_Y^1)$.

Let us recall that a zero-cycle $\Gamma \in \operatorname{Reg}(Y)^{(k)}$ is *special* if the dimension of the complete linear series through Γ is greater that the one at a generic $\Gamma' \in \operatorname{Reg}(Y)^{(k)}$.

We could explicitly describe the abelian differentials on Y as meromorphic differentials on the desingularization Y' (cf. [11]). From this we see that the degree of the poles (i.e., the sum of the order of the poles at the different poles) of an abelian differential is bounded; let δ be this bound. Since the degree of the zero-cycle on Y' associated to an abelian differential on Y is the constant $2g-2$ (where g is the genus of Y'), we see that an abelian differential cannot annihilate a zero-cycle on Y of degree greater than $2g-2+\delta$. Thus we have the following lemma.

Lemma 2.5. *At a zero-cycle* $\Gamma \in \mathrm{Reg}(Y)^{(k)}$ *of degree* $k > 2g-2+\delta$, *we have* $\dim|\Gamma| = k - \pi$.

Now we ask the following question: Which conditions on Y imply that the hyperplane sections define the *unique* linear series on Y of degree d and dimension N?

To have uniqueness, the linear series defined by hyperplane sections must be special, and thus $d \leq 2g-2+\delta$. Moreover the curves have to be linearly normal (i.e., the linear series defined by hyperplane sections have to be complete). The following example shows that some additional conditions on Y are necessary.

Example 2.6. Let Y be a smooth Castelnuovo curve of degree $d \geq 2N$ and genus $\pi(d,N)$ in $\mathbf{P}_N, N \geq 4$. Then the sections of Y by hyperplanes give a complete and special linear series of degree d and dimension N (which is maximal for linear series of degree d), and moreover the unique linear series of degree d and dimension N by Ciliberto [2]. Now take a point $P_0 \in Y$ and let Y' be the projection of Y on a generic $\mathbf{P}_{N-1}$ from the point P_0. The curve Y', isomorphic to Y, is smooth of degree $d-1$. The sections of Y' by hyperplanes in $\mathbf{P}_{N-1}$ still give a complete and special linear series of maximal dimension on Y', but it is not anymore unique. In fact, for each $P \in Y$, we obtain, by taking the hyperplanes through P, such a linear series on Y', and they are distinct for each P.

Ciliberto in [2] gives a sufficient condition for uniqueness, as an inequality on the genus of the curve with respect to the degree. In another paper with Lazarsfeld ([3]), he showed that we have uniqueness if the curve is a smooth complete intersection in $\mathbf{P}_3$ of bidegree $\neq (2,2)$.

Remark 2.7. Another formulation of the uniqueness of linear series of degree d and dimension N on the curve $Y \subset \mathbf{P}_N$ of degree d is that the curve has a unique embedding into $\mathbf{P}_N$, up to projective transformations.

2.2. Plane curves

Let Y be a plane irreducible curve of degree d in $\mathbf{P}_2$. We first introduce the numerical character of a group of points $\Gamma \subset Y$; we then show how we deduce the classification of Ciliberto on linear series of maximal dimension on plane curves from this numerical character (in our statement we do not assume that Y is smooth).

Let $(Y_0{:}Y_1{:}Y_2)$ be homogeneous coordinates such that the point $(0{:}0{:}1)$ does not belong to Y. We can then choose affine coordinates (x,y) with $x = Y_1/Y_0$ and $y = Y_2/Y_0$. The equation of the curve can be written $f(x,y) = y^d + a_1(x)y^{d-1} + \ldots + a_d(x) = 0$, with $a_i(x)$ being a polynomial of degree $\leq i$. Then the abelian differentials on Y can be written explicitly as $\omega = (p(x,y)/\partial_y f)\, dx$, with p being any polynomial in (x,y) of degree $\leq d-3$. A point $P \in Y$ outside infinity is singular if $\partial_y f(P)=0$. Thus an abelian differential is finite at such a singular point P if moreover $p(P)=0$ in such a way that $p/\partial_y f$ remains finite on Y at P.

Let $A=\mathbf{C}[Y_0,Y_1,Y_2]$. We denote by I_Y (resp. I_Γ) the polynomials which annihilate Y (resp. Γ), and let $A_Y=A/I_Y$ (resp. $A_\Gamma=A/A_\Gamma$). Since $I_Y \subset I_\Gamma$, we have the natural surjective map $A_Y \to A_\Gamma$; we denote its kernel by $I_{\Gamma/Y}$.

Let $R=\mathbf{C}[Y_0,Y_1]$ be a graded ring. The graded rings A_Γ and A_Y can be considered as graded R-modules in a natural way. Then $\{1,\dots,y_2^{d-1}\}$, with $y_2=Y_2 \bmod I_Y$, is a system of generators of the R-module A_Y. The surjective morphism

$$\bigoplus_{i=0}^{d-1} R[-i] \longrightarrow A_Y$$

is an isomorphism of graded R-modules from Hilbert's syzygies theorem.

The morphism

$$\nu\colon A_Y \simeq \bigoplus_{i=0}^{d-1} R[-i] \longrightarrow A_\Gamma$$

defined by

$$\nu(a_0(Y_0,Y_1),\dots,a_{d-1}(Y_0,Y_1)) = a_0+a_1x_2+\dots+a_{d-1}x_2^{d-1}$$

is thus a surjective morphism of graded R-modules, with $x_2=y_2 \bmod I_{\Gamma/Y}$. The kernel $I_{\Gamma/Y}$ of this morphism is thus generated (as an R-module) by elements of the form $L_i=\sum_{j=0}^{d-1}\alpha_{ij}y_2^j$. Still from Hilbert's syzygies theorem, we can choose these generators in a way such that there are no relations between these generators in the R-module A_Y. We then have the exact sequence of graded R-modules

$$0 \longrightarrow \bigoplus_{i=0}^{d-1} R[-m_i] \longrightarrow \bigoplus_{i=0}^{d-1} R[-i] \longrightarrow A_\Gamma \longrightarrow 0$$

in which the first morphism is given by the matrix (α_{ij}). From this exact sequence we deduce the Hilbert function of Γ,

$$\phi_\Gamma(l) = \sum_{i=0}^{d-1} (l-i+1)_+ - (l-m_i+1)_+,$$

with $s_+=s$ if $s>0$, and 0 otherwise.

Definition 2.8. The sequence of integers $(m_0, \dots, m_{d-1})$ (put in increasing order) is called the *relative (to Y) numerical character* of Γ.

Remark 2.9. We call it "relative" (to Y), because the definition of Gruson–Peskine (not our definition) assumes that Y is a curve of minimal degree containing Γ.

Lemma 2.10. *The relative numerical character $(m_0, \dots, m_{d-1})$ of Γ satisfies the two conditions $\sum_{i=0}^{d-1}(m_i - i) = \alpha$ and $m_{i+1} \leq m_i + 1$ (it is "without gaps").*

Proof. The first condition comes from the fact that for $l \gg 0$ the Hilbert function of Γ equals $\deg \Gamma$, and one has from the preceding exact sequence

$$\phi_\Gamma(l) = \sum_{i=0}^{d-1} (l-i+1)_+ - (l-m_i+1)_+,$$

and thus $\phi_\Gamma(l)$ is constant, equal to $\sum_{i=0}^{d-1}(m_i - i)$, when $l \geq m_{d-1} - 1$.

The second condition can be shown as follows. The multiplication by y_2 in A_Y defines an endomorphism of $I_{\Gamma/Y}$ and thus we can write

$$y_2 \cdot L_i = \sum_{j=0}^{d-1} \alpha_{ij} L_j,$$

with $\alpha_{ij} \in R(m_i - m_j + 1)$. Suppose for instance that $m_{i+1} \leq m_i + 1$ for $i < k$ and $m_{k+1} > m_k + 1$, with $k \leq d-2$. Then $\alpha_{ij} = 0$ for $0 \leq i \leq k$ and $j > k$; thus, the sub-R-module of $I_{\Gamma/Y}$ generated by $L_0, \dots, L_k$ is stable with respect to multiplication by y_2. By the Cayley–Hamilton theorem, if P is the characteristic polynomial of the matrix $(\alpha_{ij})_{i,j=0}^k$, which has degree $k+1$, we must have $P(y_2)L_i = 0$, $0 \leq i \leq k$. Since Y is irreducible, we deduce that $P(y_2) = 0$. But this impossible, since $k+1 \leq d-1$ and $1, y_2, \dots, y_2^{d-1}$ are independent on R.

Thus the sequence $(m_0, \dots, m_{d-1})$ is without gaps. □

From the expression $\phi_\Gamma(l) = \sum_{i=0}^{d-1}(l-i+1)_+ - (l-m_i+1)_+$ we deduce that $\phi(l)$ is the area between the graphs of $j(i) = m_i$ and of $j'(i) = i$, under the horizontal line $j = l+1$. Thus the monotony of ϕ_Γ implies the third property

$$m_i \geq i,$$

with equality if and only if Γ is empty.

Remark 2.11. If $m_0 \geq d$, then the relative numerical character is the same as the "absolute" numerical character of Gruson–Peskine. If $m_0 < d$, we obtain the absolute numerical character from the relative numerical character, in the following way: In the set of integers $(m_0, \dots, m_{d-1})$ (which is without gaps), we remove (one time!) the integers $m_0, \dots, d-1$. It can be seen that m_0 is the minimal degree of a curve containing Γ and not Y. Moreover, if $m_{d-1} \geq d$, then m_{d-1} is the minimal integer such that ϕ_Γ is constant from $m_{d-1}-1$ onwards.

Lemma 2.12. *Let Γ and Γ' be two groups of points on Y, residual to each other with respect to a curve of degree k, i.e. such that $\Gamma+\Gamma' = H \cap Y$, with H being a curve of degree k which does not meet* Sing Y. *Then between the relative numerical characters m_i and m'_i of Γ and Γ', we have the relation $m_{d-1-i}+m'_i = k+d-1$.*

Proof. Let us consider the exact sequence of graded A_Y-modules

$$0 \longrightarrow I_{\Gamma/Y} \longrightarrow A_Y \longrightarrow A_\Gamma \longrightarrow 0.$$

Apply the functor $\mathrm{Hom}_{A_Y}(\,\cdot\,, I_{(\Gamma+\Gamma')/Y})$ to this exact sequence. We have $I_{(\Gamma+\Gamma')/Y} \simeq A_Y[-k]$. Furthermore, $\mathrm{Hom}_{A_Y}(I_{\Gamma/Y}, I_{(\Gamma+\Gamma')/Y}) = I_{\Gamma'/Y}$, $\mathrm{Hom}_{A_Y}(A_Y, I_{(\Gamma+\Gamma')/Y}) = A_Y[-k]$, and $\mathrm{Hom}_{A_Y}(A_\Gamma, I_{(\Gamma+\Gamma')/Y}) = 0$. We thus obtain the exact sequence of A_Y-modules

$$0 \longrightarrow A_Y[-k] \longrightarrow I_{\Gamma'/Y} \longrightarrow \mathrm{Ext}^1_{A_Y}(A_\Gamma, I_{(\Gamma+\Gamma')/Y}) \longrightarrow 0.$$

The module $\mathrm{Ext}^1_{A_Y}(A_\Gamma, I_{(\Gamma+\Gamma')/Y})$ is exactly the dualizing module $\omega_\Gamma[-d-k+3]$ of A_Γ (because $I_{(\Gamma+\Gamma')/Y} \simeq A_Y[-k]$ and $\omega_Y \simeq A_Y[d-3]$). We thus obtain

$$0 \longrightarrow A_Y[d-3] \longrightarrow I_{\Gamma'/Y}[k+d-3] \longrightarrow \omega_\Gamma \longrightarrow 0.$$

We can also consider the exact sequence of graded R-modules

$$0 \longrightarrow I_{\Gamma/Y} \longrightarrow A_Y \longrightarrow A_\Gamma \longrightarrow 0,$$

with $A_Y \simeq \bigoplus_{i=0}^{d-1} R[-i]$ and $I_{\Gamma/Y} \simeq \bigoplus_{i=0}^{d-1} R[-m_i]$. If we apply $\mathrm{Hom}_R(\,\cdot\,, R[-2])$ to this exact sequence, we obtain

$$0 \longrightarrow \bigoplus_{i=0}^{d-1} R[i-2] \longrightarrow \bigoplus_{i=0}^{d-1} R[m_i-2] \longrightarrow \omega_\Gamma \longrightarrow 0,$$

since $\mathrm{Ext}^1_R(A_\Gamma, R[-2]) \simeq \omega_\Gamma$. We can thus express the Hilbert function of ω_Γ in two different ways, in terms of m_i-2 and in terms of $k+d-3-m'_{d-1-i}$. Moreover, two distinct sets of integers $\{m_i : i=0, \dots, d-1\}$ define two distinct Hilbert functions. Thus the sets $\{m_i-2 : i=0, \dots, d-1\}$ and $\{k+d-3-m'_{d-1-i} : i=0, \dots, d-1\}$ are equal, and since the sequences $(m_i)_{i=0}^{d-1}$ and $(m'_i)_{i=0}^{d-1}$ are in increasing order, we obtain the relation $m_i + m'_{d-1-i} = k+d-1$. $\square$

For any group $\Gamma \subset \mathbf{P}_N$ of points, we have the exact sequence of sheaves

$$0 \longrightarrow \mathcal{I}_\Gamma(l) \longrightarrow \mathcal{O}_{\mathbf{P}_N}(l) \longrightarrow \mathcal{O}_\Gamma(l) \longrightarrow 0$$

from which we get the exact sequence

$$0 \longrightarrow H^0(\mathcal{I}_\Gamma(l)) \longrightarrow H^0(\mathcal{O}_{\mathbf{P}_N}(l)) \longrightarrow H^0(\mathcal{O}_\Gamma(l)) \longrightarrow H^1(\mathcal{I}_\Gamma(l)) \longrightarrow 0$$

and thus $\phi_\Gamma(l) = \deg \Gamma - h^1(\mathcal{I}_\Gamma(l))$, with $\phi_\Gamma(l) = \dim A_\Gamma(l)$ (the Hilbert function of Γ) and $h^1(\mathcal{I}_\Gamma(l)) = \dim H^1(\mathcal{I}_\Gamma(l))$.

Now let $\Gamma \in \mathrm{Reg}(Y)^{(\alpha)}$; we identify Γ with the associated group of points.

Lemma 2.13. *The number $\alpha - \phi_\Gamma(d-3) = h^1(\mathcal{I}_\Gamma(d-3))$ is precisely the dimension of the complete linear series $|\Gamma|$ through Γ, where ϕ_Γ is the Hilbert function of Γ.*

Proof. From the (generalized) Riemann–Roch theorem, we have $\dim |\Gamma| = \alpha - \mathrm{rk}\, M_\Gamma$, where M_Γ is a Brill–Noether matrix at Γ. By the explicit expression of the abelian differentials given above, $\mathrm{rk}\, M_\Gamma$ is also the dimension of the $\mathbf{C}$-vector space of polynomials in (x, y) of degree $\le d-3$ on Γ, and thus $\mathrm{rk}\, M_\Gamma = \phi_\Gamma(d-3)$. Using $\phi_\Gamma(l) = \deg \Gamma - h^1(\mathcal{I}_\Gamma(l))$ we also get $\dim |\Gamma| = h^1(\mathcal{I}_\Gamma(d-3))$. $\square$

Let $\alpha = kd - r$, with $r < d$. Let Δ be the residual of r aligned points in the intersection of Y with a curve of degree k (e.g., take $d-r$ points on a generic linear section, and $(k-1)d$ points on the section of Y with a generic curve of degree $k-1$). Let $A_\Delta = \sum_{i=0}^{\infty} A_\Delta(i)$ be the homogeneous ring of Δ, and ϕ_Δ its Hilbert function. We have the following theorem.

Theorem 2.14. (1) *We have*

$$\phi_\Delta(l) \le \phi_\Gamma(l) \quad \textit{for all } l.$$

(2) *If*

$$\phi_\Delta(l) = \phi_\Gamma(l) \quad \textit{for all } l.$$

then Γ is as Δ the residual of r aligned points in the intersection of Y with a curve of degree k.

(3) *If $\phi_\Delta(i) = \phi_\Gamma(i)$, then*

(i) *if $i \le k + d - r - 3$, then*

$$\phi_\Delta(j) = \phi_\Gamma(j) \quad \textit{for all } j \le i;$$

(ii) *if* $i=k+d-r-2$, *then either*

$$\phi_\Delta(j)=\phi_\Gamma(j) \quad \textit{for all } j\le i$$

or

$$\phi_\Delta(j)=\phi_\Gamma(j) \quad \textit{for all } j\ge i;$$

(iii) *if* $i\ge k+d-r-1$, *then*

$$\phi_\Delta(j)=\phi_\Gamma(j) \quad \textit{for all } j\ge i.$$

Proof. The relative numerical character $(n_0,\dots,n_{d-1})$ of Δ satisfies $n_0=k$, because Δ is not contained in a curve of degree less than k. Moreover, by the preceding lemma we have $n_{d-1}=k+d-1$ if $r=0$, and $n_{d-1}=k+d-2$ if $r>0$. We obtain, from the two conditions $\sum_{i=0}^{d-1}(n_i-i)=\alpha$ and $n_{i+1}\le n_i+1$ that

(1) if $\alpha=kd$ (i.e. $r=0$), then

$$n_0=k, \quad \dots, \quad n_{d-1}=k+d-1;$$

(2) if $r>0$, then

$$\begin{aligned} &n_0=k, & &n_1=k+1, & &\dots, & &n_{d-r-1}=k+d-r-1,\\ &n_{d-r}=k+d-r-1, & &n_{d-r+1}=k+d-r, & &\dots, & &n_{d-1}=k+d-2. \end{aligned}$$

From these values of n_i and from the conditions $\sum_{i=0}^{d-1}(m_i-i)=\alpha$ and $m_{i+1}\le m_i+1$ we deduce that $m_0\ge k=n_0$, and if $m_i<n_i$, then $m_j\le n_j$ for $j\ge i$.

From the explicit expression of the Hilbert functions of Γ and Δ we get

$$\phi_\Gamma(l)-\phi_\Delta(l)=\sum_{i=0}^{d-1}(l-n_i+1)_+-(l-m_i+1)_+.$$

This equation means that $\phi_\Gamma(l)-\phi_\Delta(l)$ is the area between the graphs of $j_\Gamma(i)=m_i$ and of $j_\Delta(i)=n_i$, below the horizontal line $j=l+1$. As $m_0\ge k=n_0$ (from the degree of Γ), and since $m_i<n_i$ implies that $m_j\le n_j$ for all $j\ge i$, the difference $\phi_\Gamma(l)-\phi_\Delta(l)$ is first zero (until $l=k-1$), then increasing and then decreases to zero. Thus we have

$$\phi_\Gamma(i)\ge\phi_\Delta(i) \quad \text{for all } i.$$

More precisely, if the two functions $\phi_\Gamma(i)$ and $\phi_\Delta(i)$ are not equal, the set of integers i for which $\phi_\Gamma(i)-\phi_\Delta(i)\ne 0$ is connected.

Let us assume that $i \leq d+k-r-3$. We can then see that if the graph of j_Γ goes strictly under the graph of j_Δ before i, it must stay strictly under the graph at $i+1$, and thus we would have $\phi_\Gamma(i+1) < \phi_\Delta(i+1)$, which is impossible. Thus we have

$$\phi_\Delta(j) = \phi_\Gamma(j) \quad \text{for all } j \leq i.$$

Let us next assume that $i \geq d+k-r-1$. Then if the graph of j_Γ goes strictly under the graph of j_Δ after i, it must stay strictly under the graph at $i+1$, and thus we would have $\phi_\Gamma(i+1) < \phi_\Delta(i+1)$, which is impossible. Thus we have

$$\phi_\Delta(j) = \phi_\Gamma(j) \quad \text{for all } j \geq i.$$

Let us finally assume that the two graphs coincide everywhere. Then Γ is contained in a curve of degree k. We deduce from the value of m_{d-1} that the residual Γ' of Γ is contained in a line, by Lemma 2.12. This concludes the proof. $\square$

This allows us to restate the classification of Ciliberto [2] for linear series of maximal dimension on plane curves in the case of singular curves. Let us recall the classification, which we state as a corollary of the above theorem on Hilbert functions of a group of points.

Corollary 2.15. *Let $Y \subset \mathbf{P}_2$ be an irreducible algebraic curve of degree d, and $\Gamma \in Y^{(\alpha)}$ be a zero-cycle of degree $\alpha \leq d(d-3)$ supported on* $\operatorname{Reg} Y$. *We denote by $s(\alpha)$ the maximal dimension of a linear series of degree α, and by $|\Gamma|$ the complete linear series through Γ. Let $\alpha = kd-r$, with $r<d$.*

Then $s(\alpha) = \frac{1}{2}k(k+3)-r$ if $r \leq k+1$, and $s(\alpha) = \frac{1}{2}(k-1)(k+2)$ if $r \geq k+1$. Moreover the following are true:

(1) *If $r \leq k$, then* $\dim|\Gamma| = s(\alpha) = \frac{1}{2}k(k+3)-r$ *if and only if Γ is contained in a curve H of degree k, i.e. if there exists (a unique) Γ' of degree r such that $|\Gamma|$ is the residual linear series of Γ' with respect to the curves of degree k through Γ'. Then $|\Gamma|$ has no fixed point.*

(2) *If $r = k+1$, then we have* $\dim|\Gamma| = s(\alpha) = \frac{1}{2}(k-1)(k+2)$ *in (only) two cases. The first case is when Γ contains the intersection Γ''' of Y with a curve of degree $k-1$. In this case, the fixed part of $|\Gamma|$ is $\Gamma'' \subset \Gamma$, the residual of Γ''' in Γ.*

The second case is when Γ is contained in a curve H of degree k, i.e. if there exists a non-aligned Γ' of degree $r = k+1$ (which is then unique) such that $|\Gamma|$ is the residual linear series of Γ' with respect to the curves of degree k through Γ'. In this case, $|\Gamma|$ has no fixed part.

(3) *If $r \geq k+2$, then* $\dim|\Gamma| = s(\alpha) = \frac{1}{2}(k-1)(k+2)$ *if and only if $\Gamma = \Gamma''' + \Gamma''$, where Γ''' is the intersection of Y with a curve of degree $k-1$, and the fixed part of $|\Gamma|$ is $\Gamma'' \subset \Gamma$.*

Proof. Let us assume that $\dim|\Gamma|=\dim|\Delta|$. Then from Lemma 2.13, we have $\phi_\Gamma(d-3)=\phi_\Delta(d-3)$. Assume that $r\leq k+1$. Theorem 2.14 allows us to conclude that $\phi_\Gamma(k)=\phi_\Delta(k)$ for $k\leq d-3$ by the assumption. Thus $m_0=k$ and Γ is contained in a curve X of degree k. We can thus write $X\cap Y=\Gamma+\Gamma'$, with Γ' being a group of points of degree r on Y.

The fact that $|\Gamma|$ has no fixed points if $r\leq k$ comes from the fact that any group of points R of degree $\leq k+1$ *defines independent conditions on curves of degree* k. Thus $\Gamma'+P$ defines independent conditions on curves of degree k for any point P of Γ, and this means that a curve of degree k passing through Γ' can be disjoint from Γ.

If $r>k+1$, let X be a curve of degree m_0 through Γ, and let $X\cap Y=\Gamma+\Gamma'$. By Theorem 2.14, we must have that $m_{d-1}=n_{d-1}=d+k-2$. By the Gorenstein property, we deduce that the numerical character of Γ' satisfies $m'_0=m_0-k+1$, and that Γ' is contained in a curve of degree m_0-k+1. But we have $m_0(m_0-k+1)<\deg\Gamma'=d(m_0-k+1)-(d-r)$, and we deduce that the curve X' of degree m'_0 containing Γ' and X have a common component. By repeating the argument after we have taken out this component, we obtain that X' is contained in X. We deduce that Γ contains the intersection of Y with a curve of degree $m_0-m'_0=k-1$, and that the complete linear series $|\Gamma|$ must contain the $d-r$ other points of Γ (which are contained in $Y\cap X'$) as fixed points. Note that $m_0=k$ if and only if $m'_0=1$.

Let us finally assume that $r=k+1$. There are two possibilities according to Theorem 2.14. If the Hilbert functions are equal after $d-3$, the theorem implies that Γ contains the intersection of Y with a curve of degree $k-1$. If the Hilbert functions are equal before $d-3$ but not after, Γ is on a curve of degree k and the residual Γ' of Γ with respect to this curve is not aligned. Thus $\phi_{\Gamma'+P}(1)=3$ for any point P of Γ. Since the Hilbert function of a group of points increases strictly until it is constant, we must have $\phi_{\Gamma'+P}(k)\geq k+2=r+1$. Thus, for each point P of Γ, $\Gamma'+P$ defines independent conditions on curves of degree k, and this implies that $|\Gamma|$ has no fixed points. $\square$

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Received December 12, 2001
in revised form July 11, 2002

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