Ark. Mat., **38** (2000), 53–75

On weighted positivity and the Wiener regularity of a boundary point for the fractional Laplacian

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Abstract. A sufficient condition for the Wiener regularity of a boundary point with respect to the operator $(-\Delta)^{\mu}$ in $\mathbf{R}^n$, $n\geq 1$, is obtained, for $\mu\in(0,\frac{1}{2}n)\setminus(1,\frac{1}{2}n-1)$. This extends some results for the polyharmonic operator obtained by Maz'ya and Maz'ya–Donchev.

As in the polyharmonic case, the proof is based on a weighted positivity property of $(-\Delta)^{\mu}$, where the weight is a fundamental solution of this operator. It is shown that this property holds for μ as above while there is an interval $[A_n,\frac{1}{2}n-A_n]$, where $A_n\to 1$, as $n\to\infty$, with μ-values for which the property does not hold. This interval is non-empty for $n\geq 8$.

1. Introduction

Wiener's criterion, from 1924, for the regularity of a boundary point states that the condition

$$\sum_{k=0}^{\infty} 2^{(n-2)k}\,\mathrm{cap}(B(2^{-k})\setminus\Omega)=\infty$$

is necessary and sufficient for the regularity of the point $0\in\partial\Omega$ with respect to the Laplace operator in a domain $\Omega\subset\mathbf{R}^n$, $n\geq 2$, [10], [11]. Here $B(r)=\{x\in\mathbf{R}^n:|x|\leq r\}$ and cap denotes the harmonic capacity.

For higher order operators, only a few facts of this type are known, namely some sufficient conditions concerning the polyharmonic operators $(-\Delta)^m$, for certain dimensions. These results are due to Maz'ya [2], [3], for $m=2$, and to Maz'ya–Donchev [6], for $m>2$. The purpose of this paper is to obtain similar results for fractional powers of the Laplacian. This involves extending an interesting integral inequality invented by Maz'ya, the weighted positivity of $(-\Delta)^m$, to the fractional case by using methods different from those in the cited papers.

Now, to be more specific about what is known and what is to be proved, consider the equation

$$(1)\qquad (-\Delta)^\mu u = f \in C_0^\infty(\Omega), \quad u \in H_0^\mu(\Omega),$$

along with the Wiener-type condition

$$(2)\qquad \sum_{k=0}^{\infty} 2^{(n-2\mu)k}\operatorname{cap}_\mu(B(2^{-k})\setminus\Omega) = \infty,$$

and the boundary point regularity condition

$$(3)\qquad u(x)\to 0, \quad \text{as } x\to 0\in\partial\Omega.$$

(The notation is explained in the next section.) If (3) holds for the solution u of (1), for any f, then the point $0\in\partial\Omega$ is said to be regular with respect to the operator $(-\Delta)^\mu$. We notice that (3) is automatically fulfilled if $\mu>\frac{1}{2}n$, by the Sobolev embedding theorem.

We are interested in the implication (2) $\Rightarrow$ (3), when $\mu\in\left[0, \frac{1}{2}n\right]$. The knowledge until now is that this implication holds for those values of μ that are integers and belong to the set

$$(4)\qquad [0,1]\cup\left[\tfrac{1}{2}n-1, \tfrac{1}{2}n\right],$$

and for the additional case $\mu=2$, $n=7$.

The case $\mu=0$ is trivial and the case $\mu=1$ is the sufficiency part of Wiener's criterion. For $\mu\geq 2$, the proofs in [3] and [6] are based on inequalities of the type

$$(5)\qquad \int_{\mathbf{R}^n} ((-\Delta)^\mu u)u\Gamma_\mu\,dx \geq 0, \quad u \text{ real in } C_0^\infty(\mathbf{R}^n),$$

where Γ_μ is the fundamental solution of $(-\Delta)^\mu$. This inequality fails if $\mu\in\left(1, \frac{1}{2}n-1\right)$ is an integer (except if $\mu=2$ and $n=7$), as is also shown in the same papers.

In the present paper we fill the gaps between 0 and 1, and between $\frac{1}{2}n-1$ and $\frac{1}{2}n$, both with respect to the validity of (5) (with an appropriate positive right-hand side instead of 0), and to the validity of the implication (2) $\Rightarrow$ (3). We also extend the non-validity results for the inequality (5). For instance, the necessary condition $2\mu^2>(\mu-1)n$ fills the gaps between the integral points where this inequality does not hold.

Whether the condition (2) is sufficient for the regularity when (5) fails is not known (e.g. in the case $\mu=2$, $n=8$). However, it is interesting that the left and the right intervals in (4) are close to the cases $\mu=0$ respective $\mu>\frac{1}{2}n$, where (3) always holds, and that the same proof, based on (5), works for both these intervals.

The outline of the paper is as follows. The appropriate variant of (5), inequality (8), is established in Section 3 except that a large part of the proof (concerning the right interval) is postponed until Section 4. The method used there involves decomposing the Fourier transform of u in terms of spherical harmonic functions and using properties of the Gegenbauer polynomials. In Section 5 we use a certain integral representation of the bi-gamma function (see formula (29); the author does not know whether this representation is new), and some preliminary results from the previous section, to obtain necessary conditions for the inequality (5).

In Section 6, we combine some technical estimates with the result from Section 3 to obtain certain local estimates which contain the information that leads to the implication (2) $\Rightarrow$ (3). This result is finally obtained in Section 7, where we also provide a more exact pointwise estimate of u near the boundary, see Theorem 17.

2. Notation and preliminaries

For $\mu\in\mathbf{R}$ and u a tempered distribution, the operator $(-\Delta)^\mu$ is defined by

$$((-\Delta)^\mu u)^\wedge(\xi)=|\xi|^{2\mu}\hat{u}(\xi),$$

where $^\wedge$ denotes the Fourier transform with

$$\hat{\phi}(\xi)=\int_{\mathbf{R}^n} e^{-ix\cdot\xi}\phi(x)\,dx, \qquad \text{if } \phi\in C_0^\infty(\mathbf{R}^n).$$

For negative powers of $-\Delta$, Riesz potentials, we have the representation

$$(-\Delta)^{-\alpha/2}u(x)=c_\alpha\int_{\mathbf{R}^n} u(y)|x-y|^{\alpha-n}\,dy,$$

if u is a sufficiently smooth function and e.g. $0<\alpha<n$. Except for this, we will only need the representation (6). We denote the fundamental solution of $(-\Delta)^\mu$ by Γ_μ. Thus $(-\Delta)^\mu\Gamma_\mu=\delta$ and for $0<\mu<\frac{1}{2}n$,

$$\widehat{\Gamma}_\mu(x)=|x|^{-2\mu}, \quad \Gamma_\mu(x)=c_\mu|x|^{2\mu-n},$$

where c_μ is a positive number. We define $\Gamma_0=\delta$.

Let Ω denote a bounded open set in $\mathbf{R}^n$. The space $H_0^\lambda(\Omega)$, $\lambda>0$, is the completion of $C_0^\infty(\Omega)$ in the norm

$$\|u\|_\lambda=\left(\int_{\mathbf{R}^n}|(-\Delta)^{\lambda/2}u|^2\,dx+\int_{\mathbf{R}^n}|u|^2\,dx\right)^{1/2}.$$

We write $\nabla_l u=\{(l!/\alpha!)^{1/2}\partial^\alpha u\}_{|\alpha|=l}$, $\nabla_1 u=\nabla u$ (here α is a multiindex and $\alpha!=\alpha_1!\alpha_2!\dots\alpha_n!$). In the space $C_0^\infty(\Omega)$, the equivalence

$$\|u\|_\lambda\sim\begin{cases}\left(\int_{\mathbf{R}^n}|\nabla_\lambda u|^2\,dx\right)^{1/2}, & \text{if } \{\lambda\}=0,\\ \left(\iint_{\mathbf{R}^{2n}}\frac{|\nabla_{[\lambda]}u(x)-\nabla_{[\lambda]}u(y)|^2}{|x-y|^{n+2\{\lambda\}}}\,dx\,dy\right)^{1/2}, & \text{if } \{\lambda\}\neq 0,\end{cases}$$

holds, where $[\lambda]$ and $\{\lambda\}$ denotes the integral and fractional part of λ respectively. A function in $H_0^\lambda(\Omega)$ will be considered as a function in $H_0^\lambda(\mathbf{R}^n)$, which is zero on the complement of the closure of Ω.

For e an arbitrary compact set and $0<\mu<\frac{1}{2}n$ we define the μ-harmonic capacity as the number

$$\operatorname{cap}_\mu(e)=\inf\{\|u\|_\mu^2 : u\in C_0^\infty(\mathbf{R}^n),\ u=1 \text{ in a neighborhood of } e\}.$$

Positive, unimportant constants, which may vary from place to place, will be denoted by c. We allow such a constant to depend on the dimension and on parameters like μ and λ above (since when considering $(-\Delta)^\mu$, we regard μ as being constant). The notation $a\sim b$ means that a and b are comparable.

Functions are assumed to be real-valued unless it is clear from the context that it should be otherwise.

3. Weighted positivity of $(-\Delta)^\mu$

The following lemma will be used both to prove weighted positivity and in the proof of Lemma 11.

Lemma 1. *If $u,v\in C_0^\infty(\mathbf{R}^n)$ and $0<s<1$ then*

$$u(-\Delta)^s v+v(-\Delta)^s u-(-\Delta)^s(uv)=A_s\int_{\mathbf{R}^n}\frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{n+2s}}\,dy, \tag{6}$$

where $A_s>0$ and $A_s/s(1-s)$ has finite, positive limits as $s\to 0,1$.

Proof. Integrating the right integral with a function $\phi\in C_0^\infty(\mathbf{R}^n)$ and using Parseval's formula, we get

$$\iint_{\mathbf{R}^{2n}}\frac{(u(x)-u(x+y))(v(x)-v(x+y))}{|y|^{n+2s}}\phi(x)\,dx\,dy$$
$$=\frac{1}{(2\pi)^{2n}}\iiint_{\mathbf{R}^{3n}}\frac{(1-e^{-\xi\cdot y})(1-e^{i\eta\cdot y})}{|y|^{n+2s}}\overline{\hat{u}(\xi)}\hat{v}(\eta)\hat{\phi}(\xi-\eta)\,dy\,d\xi\,d\eta.$$

The fact that $|t|^{-2s}\int_{\mathbf{R}^n}(1-\cos(t\cdot y))|y|^{-n-2s}\,dy=A_s^{-1}$ implies that the last integral equals

$$\frac{1}{A_s(2\pi)^{2n}}\iint_{\mathbf{R}^{2n}}(|\eta|^{2s}+|\xi|^{2s}-|\xi-\eta|^{2s})\overline{\hat{u}(\xi)}\hat{v}(\eta)\hat{\phi}(\xi-\eta)\,d\xi\,d\eta$$
$$=\frac{1}{A_s}\int_{\mathbf{R}^n}(u((-\Delta)^s v)\phi+v((-\Delta)^s u)\phi-uv(-\Delta)^s\phi)\,dx.$$

We complete the proof by integrating the last term by parts, so that $(-\Delta)^s$ acts on uv. □

If we take $v=u$ in (6), multiply by Γ_s and integrate we obtain the identity

$$(7)\qquad 2\int_{\mathbf{R}^n}((-\Delta)^s u)u\Gamma_s\,dx=u(0)^2+A_s\iint_{\mathbf{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\Gamma_s(x)\,dx\,dy,$$

valid for $0<s<1$. This is an instance of the weighted positivity property. Corollary 9 shows that if s is replaced by any number μ greater than 1, then, for sufficiently high dimension, the integral on the left can assume negative values. On the other hand, (7) in conjunction with Proposition 10 implies positivity of the integral for μ as in the next lemma. However, the proof in the next section is needed to obtain the appropriate right-hand side.

Lemma 2. *Let $u\in C_0^\infty(\mathbf{R}^n)$, $\frac{1}{2}n-1\le\mu<\frac{1}{2}n$, $0\le\sigma,\tau,l$, where l is an integer and $0<\sigma+\tau+l\le\mu$. Then,*

$$\int_{\mathbf{R}^n}((-\Delta)^\mu u)u\Gamma_\mu\,dx-\frac{u(0)^2}{2}\ge c\left(\int_{\mathbf{R}^n}(-\Delta)^\sigma\nabla_l u\cdot((-\Delta)^\tau\nabla_l u)\Gamma_{\sigma+\tau+l}\,dx-\frac{\varepsilon u(0)^2}{2}\right),$$

where $\varepsilon=1$, if $l=\sigma\tau=0$, and $\varepsilon=0$, otherwise.

If we take $\sigma=s$, $\tau=0$ and $l=0, 1, \dots, m$ in Lemma 2 and then apply Lemma 1, as we did to derive (7), we obtain the following corollary.

Corollary 3. *Let $u\in C_0^\infty(\mathbf{R}^n)$, $\mu\in\left[0,\frac{1}{2}n\right)\setminus\left(1,\frac{1}{2}n-1\right)$, $\mu=m+s$, where m is an integer and $0\le s<1$. If $s>0$ then*

$$\begin{aligned}\int_{\mathbf{R}^n}((-\Delta)^\mu u)u\Gamma_\mu\,dx\ge\tfrac{1}{2}u(0)^2+c\bigg(&\iint_{\mathbf{R}^{2n}}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\Gamma_s(x)\,dx\,dy\\ &+\sum_{l=1}^{m}\int_{\mathbf{R}^n}|\nabla_l u|^2\Gamma_l\,dx\\ &+\iint_{\mathbf{R}^{2n}}\frac{|\nabla_m u(x)-\nabla_m u(y)|^2}{|x-y|^{n+2s}}\Gamma_{m+s}(x)\,dx\,dy\bigg).\end{aligned}\qquad(8)$$

If $s=0$ then the same inequality but without the double integrals holds.

Remark. In the easy case $\mu=1$, (8) becomes an identity for a certain value of c (cf. (7)).

4. Proof of Lemma 2

We need some facts and notation for this section. (Concerning the Gegenbauer polynomials and the spherical harmonic functions, we refer to [1] and [9].)

Let the constant $\gamma=\frac{1}{2}(n-2)$ be associated with $\mathbf{R}^n$. Then the area of the unit sphere $\mathbf{S}^{n-1}$ in $\mathbf{R}^n$ is $\omega=2\pi^{\gamma+1}\Gamma(\gamma+1)^{-1}$. We write $x\in\mathbf{R}^n$ in polar coordinates $r\geq 0$ and $x'\in\mathbf{S}^{n-1}$ as $x=rx'$. The letters j, k, l, m will always denote non-negative integers. Pochhammer's symbol $(\lambda)_m$ is defined by $(\lambda)_0=1$, and

$$(\lambda)_m=\lambda(\lambda+1)\dots(\lambda+m-1)=\Gamma(\lambda+m)/\Gamma(\lambda),$$

if m is a positive integer, Γ denotes the gamma-function.

If $\lambda>0$, we let C_j^λ denote the Gegenbauer polynomial defined by

$$\frac{1}{(1-2xt+t^2)^\lambda}=\sum_{j=0}^{\infty}C_j^\lambda(x)t^j. \tag{9}$$

We will need to know that C_j^λ is a polynomial of degree j which is odd (even) if j is odd (even), and that these polynomials are orthogonal on the interval $[-1,1]$ with respect to the weight function $m_\lambda(t)=(1-t^2)^{\lambda-1/2}$.

For the following theorem, the Funk–Hecke theorem, see [1].

Theorem 4. *Let $n\geq 3$, S_j be any surface harmonic of degree j, $y'\in\mathbf{S}^{n-1}$, and let F be continuous on $[-1,1]$. Then*

$$\int_{\mathbf{S}^{n-1}}F(x'\cdot y')S_j(x')\,dx'=A_jS_j(y')\int_{-1}^{1}F(t)C_j^\gamma(t)m_\gamma(t)\,dt, \tag{10}$$

where $A_j=(4\pi)^\gamma\Gamma(\gamma)j!/(j+2\gamma-1)!$.

The following decomposition of a function $f\in C_0^\infty(\mathbf{R}^n)$ will be useful. Namely, f can be written as a sum

$$f(x)=\sum_{j=0}^{\infty}f_j(r)S_j(r,x') \tag{11}$$

(if $n=1$, then ∞ is to be replaced by 1) converging in the sense of $L^2(\mathbf{R}^n)$, such that $f_j\in C_0^\infty(\mathbf{R}^1)$ with $\operatorname{supp} f_jS_j\subset\operatorname{supp} f$ and for any fixed $r\geq 0$,

$$\int_{\mathbf{S}^{n-1}}S_j(r,x')\overline{S_k(r,x')}\,dx'=\begin{cases}1, & \text{if } j=k,\\ 0, & \text{if } j\neq k.\end{cases} \tag{12}$$

Indeed, for $n=1$ we take

$$S_0(r,1)=S_0(r,-1)=S_1(r,1)=-S_1(r,-1)=1/\sqrt{2}.$$

Then (11) is just a decomposition into even and odd parts. In the case $n=2$ we can use a Fourier series expansion to get, if θ is the argument of x',

$$f(x)=\sum_{j=0}^{\infty} f_j(r)(a_j(r)\cos(j\theta)+b_j(r)\sin(j\theta))=\sum_{j=0}^{\infty} f_j(r)S_j(r,x'),$$

where the functions f_j are chosen so that (12) holds. Similarly, if $n\geq 3$ and $r\geq 0$ is fixed, the function $S_j(r,x')$ is a unit surface harmonic of degree j.

Henceforth, in this section, we let $\lambda=\sigma+\tau+l$ be fixed with $0\leq\lambda<\frac{1}{2}n=\gamma+1$. We introduce the quadratic form

$$I^{\sigma,\tau,l}(f)=\iint_{\mathbf{R}^{2n}} \frac{|x|^{2\sigma}|y|^{2\tau}(x\cdot y)^l}{|x-y|^{2\lambda}} f(x)\overline{f(y)}\,dx\,dy. \tag{13}$$

The relevance of this form in connection with Lemma 2 is seen from the identity

$$\int_{\mathbf{R}^n} (-\Delta)^{\sigma}\nabla_l u\cdot((-\Delta)^{\tau}\nabla_l u)\Gamma_\lambda\,dx=\frac{I^{\sigma,\tau,l}(\hat{u})}{(2\pi)^{2n}}, \tag{14}$$

for a real function $u\in C_0^\infty(\mathbf{R}^n)$. This identity is an immediate consequence of Parseval's formula and the definition of $(-\Delta)^\mu$ and ∇_l. In order to linearize this form (Lemma 5) we need the following quantities.

For $n\geq 3$, $2m+j\geq l$ and $\lambda>0$, let

$$a_{j,m}^{\lambda,l}=A_j\int_{-1}^{1} \nu^l C_{2m+j-l}^{\lambda}(\nu)C_j^{\gamma}(\nu)m_\gamma(\nu)\,d\nu, \tag{15}$$

where A_j is as in (10). It is a simplified special case of formula 2.21.18.15 in Vol. 2 of [8] that

$$a_{j,m}^{\lambda,0}=\omega\frac{(\lambda)_{m+j}(\lambda-\gamma)_m}{(\gamma+1)_{m+j}m!}. \tag{16}$$

We let this extend the definition to cover $n=1$, 2 and $\lambda=0$. Now define the functions

$$\Phi_j^{\sigma,\tau,l}=\sum_{m=m_0}^{\infty} a_{j,m}^{\lambda,l}(\varphi_{2m+j+2\sigma}+\varphi_{2m+j+2\tau}), \tag{17}$$

where m_0 is the smallest non-negative integer with $2m_0+j\geq l$ and

$$\varphi_t(\xi)=\frac{t}{t^2+\xi^2}\text{ for } t>0 \quad\text{and}\quad \varphi_0=\pi\delta.$$

It can be shown that the series converges uniformly for λ as above.

Lemma 5. *If* $f \in C_0^\infty(\mathbf{R}^n)$ *is decomposed as in* (11), $g_j(t)=e^{nt}f_j(e^t)$, *then*

$$\operatorname{Re} I^{\sigma,\tau,l}(f) = \frac{1}{2\pi}\sum_{j=0}^{\infty}\int_{\mathbf{R}^n} |\hat{g}_j(\xi)|^2 \Phi_j^{\sigma,\tau,l}(\xi)\,d\xi. \tag{18}$$

If $n=1$, *the upper limit of the sum should be replaced by* 1.

Proof. We give the proof only for $n\geq 3$. With the new variables introduced by

$$x=e^s x', \quad y=e^t y', \quad p=t-s, \quad \nu=x'\cdot y',$$

the kernel of the form (13) becomes

$$H(x,y)=K(p,\nu)=\nu^l(1-2e^{-|p|}\nu+e^{-2|p|})^{-\lambda}\begin{cases} e^{-(l+2\sigma)|p|}, & \text{if } p\geq 0,\\ e^{-(l+2\tau)|p|}, & \text{if } p\leq 0.\end{cases} \tag{19}$$

We define the functions

$$K_j(p)=A_j\int_{-1}^{1} K(p,\nu)C_j^\gamma(\nu)m_\gamma(\nu)\,d\nu. \tag{20}$$

Now, introducing the operator H, we have

$$\begin{aligned} H(f_jS_j)(y) &= \int_{\mathbf{R}^n} H(x,y)f_j(|x|)S_j(x')\,dx \\ &= \int_{-\infty}^{\infty} g_j(s)\int_{\mathbf{S}^{n-1}} K(p,\nu)S_j(x')\,dx'\,ds = (K_j*g_j)(t)S_j(y'), \end{aligned} \tag{21}$$

where we in the last step used the Funk–Hecke theorem (10). Since the operator H is bounded on $L^2(G)$, for any bounded domain G, this allows us to write

$$\begin{aligned} I^{\sigma,\tau,l}(f) &= (Hf,f) = \sum_{j,k=0}^{\infty}(H(f_jS_j),f_kS_k) = \sum_{j=0}^{\infty}(H(f_jS_j),f_jS_j) \\ &= \sum_{j=0}^{\infty}(K_j*g_j,g_j) = \frac{1}{2\pi}\sum_{j=0}^{\infty}\int_{\mathbf{R}^n}|\hat{g}_j(\xi)|^2\widehat{K}_j(\xi)\,d\xi, \end{aligned} \tag{22}$$

where $(\,\cdot\,,\cdot\,)$ is the L^2-scalar product. To complete the proof we must verify that

$$\Phi_j^{\sigma,\tau,l} = \operatorname{Re}\widehat{K}_j. \tag{23}$$

Expanding the factor $(1-2e^{-|p|}\nu+e^{-2|p|})^{-\lambda}$ in K in terms of the Gegenbauer polynomials (as in (9)), we get for $p>0$,

$$K_j(p)+K_j(-p)=A_j(e^{-2\sigma p}+e^{-2\tau p})\sum_{k=0}^{\infty}e^{-(k+l)p}\int_{-1}^{1}\nu^l C_k^\lambda C_j^\gamma m_\gamma\,d\nu.$$

Since, by the properties of these polynomials, each integral in this formula vanishes unless $k=2m+j-l$ and $m\geq 0$, the formula

$$\int_0^\infty e^{-tp}\cos(\xi p)\,dp=\varphi_t(\xi),\quad t>0,$$

leads to (23). □

Lemma 6. *Under the hypothesis of Lemma* 2

$$\begin{aligned}\Phi_0^{\mu,0,0}-\pi\omega\delta&\geq c(\Phi_0^{\sigma,\tau,l}-\varepsilon\pi\omega\delta),\\ \phi_j^{\mu,0,0}&\geq c\Phi_j^{\sigma,\tau,l}, \qquad j\geq 1,\end{aligned}\tag{24}$$

where $\varepsilon=1$ *if* $l=\sigma\tau=0$, *and* $\varepsilon=0$ *otherwise.*

Proof. We recall that $\lambda=\sigma+\tau+l$ is fixed and notice that $0<\lambda\leq\mu\in[\gamma,\gamma+1)$, by the hypothesis. Since the function $\Phi_j^{\sigma,\tau,l}$ is continuous except if $j=l=\sigma\tau=0$ in which case its singular part is $\pi\omega\delta$, we see that all expressions in (24) are continuous. (We used that μ, $\lambda>0$; notice that $\Phi_0^{0,0,0}=2\pi\omega\delta$.) Thus it is enough to consider points $\xi\neq 0$ and forget about δ-functions.

We first claim that

$$a_{j,m}^{\lambda,l}=\sum_{k=0}^{\min(m,l)}c_k a_{j,m-k}^{\lambda,0},\tag{25}$$

where $|c_k|$ is bounded by a constant only depending on l (and λ). To prove this, first assume that $2l\leq 2m+j$. Then the recursion formula for the Gegenbauer polynomials (see [9]),

$$\nu C_{k-1}^\lambda(\nu)=\frac{k}{2(k+\lambda-1)}C_k^\lambda(\nu)+\frac{k+2\lambda-2}{2(k+\lambda-1)}C_{k-2}^\lambda(\nu),\quad k\geq 2,$$

where we notice that the coefficients are bounded by 1, leads to

$$\nu^l C_{2m+j-l}^\lambda(\nu)=\sum_{k=0}^{l}c_k C_{j+2(m-k)}^\lambda(\nu),$$

which in turn gives (25). If $2l>2m+j$, then the sum in (25) contains the positive term $a_{j,0}^{\lambda,0}$. Since this case occurs only for a limited number of combinations of j and m, we can take $c_k=0$ for $k\neq m$ and c_m so that (25) holds, where c_m is bounded as claimed.

By (25) and the relation $\varphi_s/\varphi_t\leq\max(s/t,t/s)$, we obtain

$$\Phi_j^{\sigma,\tau,l}\leq c\sum_{m=0}^{\infty}|a_{j,m}^{\lambda,0}|\varphi_{2m+j+2\mu}. \tag{26}$$

We can now complete the proof by considering two cases.

(i) If $\gamma\leq\lambda\leq\mu$ then formula (16) shows that $|a_{j,m}^{\lambda,0}|\leq a_{j,m}^{\mu,0}$. Thus (24) follows immediately from (26), with c inverted.

(ii) Now let $0<\lambda<\gamma$ (and thus $\gamma\geq\frac{1}{2}$). This case will follow from the preceding one, once we show that $\Phi_j^{\sigma,\tau,\mu}\leq c\Phi_j^{\gamma,0,0}$. Notice that (16) gives

$$|a_{j,m}^{\lambda,0}|=|a_{0,m}^{\lambda,0}|\frac{(\lambda+m)_j}{(\gamma+1+m)_j}\leq|a_{0,m}^{\lambda,0}|\frac{\lambda+m}{\gamma+m+j}$$

and, together with the definition of $\Phi_j^{\gamma,0,0}$,

$$\Phi_j^{\gamma,0,0}\geq a_{j,0}^{\gamma,0}\varphi_{j+2\gamma}=\frac{\omega\gamma}{\gamma+j}\varphi_{j+2\gamma}\geq\frac{\omega\gamma}{(j+2\gamma)^2+\xi^2}.$$

Writing $b_m=(\lambda+m)a_{0,m}^{\lambda,0}$, we now have

$$|a_{j,m}^{\lambda,0}|\varphi_{2m+j+2\gamma}\leq\frac{2|b_m|}{(2m+j+2\gamma)^2+\xi^2}\leq\frac{2|b_m|\Phi_j^{\gamma,0,0}}{\omega\gamma}.$$

Summing this formula and using (26), with 2γ in place of 2μ, we obtain

$$\Phi_j^{\sigma,\tau,l}\leq c\sum_{m=0}^{\infty}|b_m|\Phi_j^{\gamma,0,0}.$$

The proof is completed by the fact that $\sum_{m=0}^{\infty}|b_m|$ is convergent for the current values of λ and γ. In fact, if $\lambda<\gamma$ (and $\gamma\neq-1,\ -2,\dots$) then

$$\sum_{m=0}^{\infty}b_m=\omega\lambda\sum_{m=0}^{\infty}\frac{(\lambda+1)_m(\lambda-\gamma)_m}{(\gamma+1)_m m!}=\omega\lambda F(\lambda+1,\lambda-\gamma;\gamma+1;1).$$

That is the finite value of a hypergeometric function at the point 1. Furthermore, all but finitely many b_m have the same sign. $\square$

Proof of Lemma 2. By Lemma 6,

$$\frac{1}{2\pi}\sum_{j=0}^{\infty}\int|\hat{g}_j|^2(\Phi_j^{\mu,0,0}-c\Phi_j^{\sigma,\tau,l})\,d\xi\geq\frac{\omega(1-c\varepsilon)|\hat{g}_0(0)|^2}{2},\tag{27}$$

where ε is as in Lemma 2 and c is the same on both sides. Now, let u be the real function from Lemma 2, put $f=\hat{u}$ and let g_j correspond to f as in Lemma 5. We notice that

$$u(0)^2=\frac{\omega|\hat{g}_0(0)|^2}{(2\pi)^{2n}}.$$

This can be verified directly or seen from the relation

$$\left|\int_{\mathbf{R}^n}\hat{u}\,dx\right|^2=I^{0,0,0}(\hat{u})=\omega|\hat{g}_0(0)|^2.$$

In view of this, Lemma 2 follows from (27), Lemma 5 and (14). □

5. Non-positivity

In this section we will find a necessary condition for the inequality

$$\int_{\mathbf{R}^n}((-\Delta)^\mu u)u\Gamma_\mu\,dx\geq 0,\quad u\text{ real in }C_0^\infty(\mathbf{R}^n).\tag{28}$$

By the previous section, (28) is equivalent to $\operatorname{Re} I^{\mu,0,0}\geq 0$, which in turn, by (22), is equivalent to all corresponding $\operatorname{Re}\widehat{K}_j$ being non-negative. The condition will be furnished by a more suitable expansion of $\lim_{\xi\to 0}\operatorname{Re}\widehat{K}_0(\xi)$ than the one we arrive at via (23), from $\lim_{\xi\to 0}\Phi_0^{\mu,0,0}(\xi)$ and the definition of this function.

Lemma 7. *For every $\mu>0$ the identity*

$$\int_0^\infty\left(\frac{1+e^{\mu x}}{(1+e^x)^\mu}-1\right)dx=\psi(1)-\psi(\mu)\tag{29}$$

holds, where $\psi=\Gamma'/\Gamma$ is the bi-gamma function.

Proof. For the following properties of the bi-gamma function, see [1] or [9],

$$\psi(\mu+1)=\frac{1}{\mu}+\psi(\mu),\quad\psi(t)=\log(t)+O(1/t),\text{ as }t\to\infty.$$

Let $\varphi(\mu)$ denote the integral in (29). Since $\varphi(\mu)-\varphi(\mu+1)$ equals

$$\int_0^\infty \frac{e^{-x}+e^{-\mu x}}{(1+e^{-x})^{\mu+1}}\,dx=\frac{1}{\mu}\int_0^\infty\left(\frac{1-e^{-\mu x}}{(1+e^{-x})^{\mu}}\right)'dx=\frac{1}{\mu},$$

we have for each integer $m\geq 0$, if we put $t=m+\mu$, that

$$\psi(t)+\varphi(t)=\psi(\mu)+\varphi(\mu). \tag{30}$$

Define f by

$$f(t)=\log(t)+\int_0^\infty((1+e^{-x})^{-t}-1)\,dx.$$

Then $\psi(t)+\varphi(t)-f(t)\to 0$, as $t\to\infty$, so if $f(t)$ has a limit as $t\to\infty$, then (30) shows that $\psi+\varphi$ is constant, and we obtain (29).

For f we have

$$\begin{aligned}tf'(t)&=1-t\int_0^\infty(1+e^{-x})^{-t}\log(1+e^{-x})\,dx\\&=1-t\int_0^\infty(1+e^{-x})^{-t}e^{-x}\,dx+R(t)=\mathrm{O}(1/t)+R(t),\end{aligned}$$

where R can be estimated according to

$$|R(t)|\leq\frac{t}{2}\int_0^\infty(1+e^{-x})^{-t}e^{-2x}\,dx=\mathrm{O}(1/t).$$

It follows that $f'(t)=\mathrm{O}(1/t^2)$ which implies that $f(t)$ has a limit as $t\to\infty$. This completes the proof. □

The following condition looks complicated, but can be used to derive easier ones, see Corollary 9.

Proposition 8. *The condition*

$$\sum_{m=1}^\infty\left(\frac{(\mu)_m}{m\left(\frac{1}{2}n\right)_m}-\frac{\mu-1}{m(m+\mu-1)}\right)\geq 0 \tag{31}$$

is necessary for (28) *to hold.*

Proof. We shall consider the K_0 corresponding to $\Phi_0^{\mu,0,0}$. From (19), where we have $\lambda=\sigma=\mu$, $l=\tau=0$, and where we write $x=|p|$, $t=\nu$, we get

$$\begin{aligned}K(x,t)+K(-x,t)&=(1+e^{2\mu x})(1-2te^x+e^{2x})^{-\mu}\\&=\frac{1+e^{2\mu x}}{(1+e^{2x})^{\mu}}\sum_{k=0}^\infty\binom{-\mu}{k}\left(\frac{-2te^x}{1+e^{2x}}\right)^k.\end{aligned}$$

(Compare this with the method used in the proof of Lemma 5, where the corresponding factor $(1-2te^x+e^{2x})^{-\mu}$ was expanded in terms of the Gegenbauer polynomials.) Now (20) gives

$$K_0(x)+K_0(-x)=A_0\frac{1+e^{2\mu x}}{(1+e^{2x})^\mu}\sum_{m=0}^{\infty}B_m\left(\frac{2e^x}{1+e^{2x}}\right)^{2m},$$

where $A_0=2\pi^{\gamma+1/2}\Gamma\left(\gamma+\frac{1}{2}\right)^{-1}$ and

$$B_m=\binom{-\mu}{2m}\int_{-1}^{1}t^{2m}m_\gamma(t)\,dt=\binom{-\mu}{2m}\frac{\Gamma\left(m+\frac{1}{2}\right)\Gamma\left(\gamma+\frac{1}{2}\right)}{\Gamma(\gamma+m+1)}.$$

Using the formula

$$\int_0^\infty\frac{1+e^{2\mu x}}{(1+e^{2x})^\mu}\left(\frac{2e^x}{1+e^{2x}}\right)^{2m}dx=2^{2m-1}\frac{\Gamma(m+\mu)\Gamma(m)}{\Gamma(2m+\mu)},\quad m>0,$$

we obtain after simplification

$$\lim_{\xi\to 0}\operatorname{Re}\widehat{K}_0(\xi)=\omega\int_0^\infty\left(\frac{1+e^{2\mu x}}{(1+e^{2x})^\mu}-1\right)dx+\omega\sum_{m=1}^{\infty}\frac{(\mu)_m}{2m(\gamma+1)_m}. \tag{32}$$

By (29) and well-known expansions of $\psi(1)-\psi(\mu)$ we have

$$\int_0^\infty\left(\frac{1+e^{\mu x}}{(1+e^x)^\mu}-1\right)dx=\frac{1}{\mu}-\sum_{m=1}^{\infty}\frac{\mu}{m(m+\mu)}=-\sum_{m=1}^{\infty}\frac{\mu-1}{m(m+\mu-1)}. \tag{33}$$

This gives the condition (31). □

Remark. The first equality in (33) is the special case $\gamma=\mu$ of the identity

$$\int_0^\infty\left(\frac{1+e^{\mu x}}{(1+e^x)^\mu}-1\right)dx+\sum_{m=1}^{\infty}\frac{(\mu)_m}{m(\gamma+1)_m}=\frac{1}{\mu}+\sum_{m=1}^{\infty}\frac{(\mu)_m(\mu-\gamma)_m}{(\gamma+1)_m m!}\left(\frac{1}{m}+\frac{1}{m+\mu}\right), \tag{34}$$

which we obtain by comparing (32) and (23).

Corollary 9. *The condition*

$$2\mu^2>(\mu-1)n \tag{35}$$

is necessary for (28) *to hold.*

This shows that there is an interval $\left[A_n,\frac{1}{2}n-A_n\right]$, which is non-empty for $n\geq 8$ and where $A_n\to 1$, as $n\to\infty$, with μ-values for which (28) does not hold.

Proof. Assume that (35) does not hold. This is equivalent to $a_1 \le b_1$, where we denote the series in (31) by $\sum_{m=1}^{\infty}(a_m - b_m)$. It follows that $\mu<\gamma$ and $\mu^2 \le (\mu-1)(\gamma+1)<\mu\gamma-1$, so the numerator minus the denominator of the right side of

$$\frac{a_m b_{m+1}}{a_{m+1} b_m} = \frac{(\mu+m-1)(\gamma+1+m)}{(\mu+m)^2}$$

is $(\gamma-\mu)(m-1)+\mu\gamma-1-\mu^2>0$. This shows that $a_m<b_m$ for $m>1$ so that (31) does not hold either. $\square$

The following proposition shows that the weighted positivity property of $(-\Delta)^\mu$ is symmetric in μ about the point $\frac{1}{4}n$.

Proposition 10. *Let* $0<\mu<\frac{1}{2}n$. *The inequality* (28) *holds if and only if it holds with* $\frac{1}{2}n-\mu$ *in place of* μ.

Proof. Define v by

$$\hat{v} = \Gamma_\mu (-\Delta)^\mu u.$$

Then, with $\mu' = \frac{1}{2}n-\mu$ and $\check{v}(x)=v(-x)$, we have $\hat{u}=\Gamma_{\mu'}(-\Delta)^{\mu'}\check{v}$ and the integral in (28) becomes

$$\int_{\mathbf{R}^n} \hat{v} u \, dx = \int_{\mathbf{R}^n} v \hat{u} \, dx = \int_{\mathbf{R}^n} ((-\Delta)^{\mu'} v)\bar{v}\Gamma_{\mu'} \, dx$$

($\hat{v}$ real implies $\check{v}=\bar{v}$). Since this integral is real, we see that if (28) holds for μ', it must also hold for μ. (We omit the details about approximating v by functions in $C_0^\infty(\mathbf{R}^n)$.) $\square$

Remark. Both (31) and (35) have the above symmetry. In fact, the series in (31) remains unaffected if μ is replaced by $\mu' = \frac{1}{2}n-\mu$, that is $\lim_{\xi\to 0} \Phi_0^{\mu,0,0}(\xi) = \lim_{\xi\to 0} \Phi_0^{\mu',0,0}(\xi)$. This identity applied to (32), with $\mu=1$, provides a third way to obtain (33).

Another manifestation of the proposition (and the fact that the transformation $u \mapsto v$ in the proof "commutes" with the decomposition (11)) is that $\Phi_j^{\mu,0,0}$ and $\Phi_j^{\mu',0,0}$ seem to have the same zeros.

6. Local estimates

In this section we let $\mu=m+s$, where m is integer and $0<s<1$. The case $s=0$ is much easier, but sometimes needs a slightly different formulation. Being already known (and in fact implicitly contained in the present treatment), we omit this case.

We need some more notation. Let $B_j=\{x\in\mathbf{R}^n:|x|\le 2^{-j}\}$, $C_j=B_{j-1}\setminus B_j$. For $E_j=B_j$, C_j, we introduce the "dimensionless" seminorm

$$|u|_{\lambda,E_j}=\begin{cases}\left(2^{(n-2\lambda)j}\displaystyle\int_{E_j}|\nabla_\lambda u|^2\,dx\right)^{1/2}, & \text{if } \{\lambda\}=0,\\ \left(2^{(n-2\lambda)j}\displaystyle\iint_{E_j\times E_j}\frac{|\nabla_{[\lambda]}u(x)-\nabla_{[\lambda]}u(y)|^2}{|x-y|^{n+2\{\lambda\}}}\,dx\,dy\right)^{1/2}, & \text{if } \{\lambda\}\neq 0,\end{cases}$$

and similarly, for a measurable set E,

$$\langle u\rangle_{\lambda,E}=\begin{cases}\left(\displaystyle\int_E|\nabla_\lambda u|^2\Gamma_\lambda\,dx\right)^{1/2}, & \text{if } \{\lambda\}=0,\\ \left(\displaystyle\iint_{E\times E}\frac{|\nabla_{[\lambda]}u(x)-\nabla_{[\lambda]}u(y)|^2}{|x-y|^{n+2\{\lambda\}}}\Gamma_\lambda(x)\,dx\,dy\right)^{1/2}, & \text{if } \{\lambda\}\neq 0.\end{cases}$$

Thus $|u|_{\lambda,E_j}\sim\langle u\rangle_{\lambda,E_j}$ if $E_j=C_j$ and $\lambda>0$, but not if $E_j=B_j$. We denote the L^2-norm $(\int_E u^2\,dx)^{1/2}$ by $\|u\|_E$. Finally, we introduce the bilinear form

$$Q_\mu(v,w)=\int_{\mathbf{R}^n}((-\Delta)^\mu v)w\Gamma_\mu\,dx.$$

With this notation, the conclusion of Corollary 3 becomes

$$Q_\mu(u,u)\ge\frac{u(0)^2}{2}+c\left(\langle u\rangle_{s,\mathbf{R}^n}^2+\sum_{l=1}^m\langle u\rangle_{l,\mathbf{R}^n}^2+\langle u\rangle_{m+s,\mathbf{R}^n}^2\right).\tag{36}$$

Lemma 11. *Let $\eta\in C_0^\infty(\mathbf{R}^n)$, $\eta(0)=1$ and a neighborhood of* supp $\nabla\eta$ *be contained in C_0. Let $\delta<4s$ and $u\in C_0^\infty(\mathbf{R}^n)$. Then,*

$$Q_\mu(\eta u,\eta u)-Q_\mu(u,\eta^2u)\le c\left(\sum_{k=-\infty}^{\infty}2^{-\delta|k|}\sum_{l=0}^m|u|_{l,C_k}^2+|u|_{m+s,C_0}^2\right).$$

Proof. Writing $\Lambda=(-\Delta)^s$ and $\psi=\eta\Gamma_\mu$, the left-hand side of the inequality can be written as a sum of terms of the form

$$I=\int_{\mathbf{R}^n}((\Lambda\partial^\alpha(\eta u))\partial^\alpha(\psi u)-(\Lambda\partial^\alpha u)\partial^\alpha(\eta\psi u))\,dx,$$

where $|\alpha|=m$. This, in turn, can be written as $I_1+I_2+I_3-I_4$, where

$$I_1=\int_{\mathbf{R}^n}([\Lambda,\eta]\partial^\alpha u)\partial^\alpha(\psi u)\,dx,\quad I_2=\int_{\mathbf{R}^n}([\Lambda,\psi]\partial^\alpha u)[\partial^\alpha,\eta]u\,dx,$$

$$I_3=\int_{\mathbf{R}^n}(\Lambda[\partial^\alpha,\eta]u)[\partial^\alpha,\psi]u\,dx,\quad I_4=\int_{\mathbf{R}^n}(\Lambda\partial^\alpha u)[[\partial^\alpha,\eta],\psi]u\,dx$$

($[A,B]$ denotes the commutator $AB-BA$). In fact, calculating the difference of the integrands, we obtain that $I-(I_1+I_2+I_3-I_4)$ equals

$$\int_{\mathbf{R}^n} (\psi(\partial^\alpha u)\Lambda[\partial^\alpha,\eta]u-\Lambda(\psi\partial^\alpha u)[\partial^\alpha,\eta]u)\,dx=0.$$

Let us consider the above splitting of I in some special cases: If $s=0$ then Λ is the identity so I_1 and I_2 vanish while I_3 and I_4 consist of terms of the type $\int_{\mathbf{R}^n}\xi(\partial^\beta u)\partial^\gamma u\,dx$, where $\xi\in C_0^\infty(C_0)$. Such a term can immediately be estimated by $c(|u|^2_{|\beta|,C_0}+|u|^2_{|\gamma|,C_0})$. If $m\le 1$ then I_4 vanishes and if $m=0$ then only I_1 remains.

We begin by estimating I_1 which we write as I_5+I_6, where

$$I_5=\int_{\mathbf{R}^n}(\Lambda(\eta\partial^\alpha u)-\eta\Lambda\partial^\alpha u-(\partial^\alpha u)\Lambda\eta)\partial^\alpha(\psi u)\,dx,$$
$$I_6=\int_{\mathbf{R}^n}(\Lambda\eta)(\partial^\alpha u)\partial^\alpha(\psi u)\,dx.$$

From Lemma 1 we have $I_5=-A_s\iint_{\mathbf{R}^{2n}} U\,dx\,dy$, where

$$U(x,y)=\frac{(\eta(x)-\eta(y))(\partial^\alpha u(x)-\partial^\alpha u(y))\partial^\alpha(\psi u)(x)}{|x-y|^{n+2s}}.$$

For $kl<0$, the distance between $x\in C_k$ and $y\in C_l$ is comparable with $2^{-\min(k,l)}$. Hence the properties of η imply

$$I_5\le c\biggl(\iint_{C_0\times C_0}|U|\,dx\,dy+\sum_{k,l=-\infty}^{\infty}\iint_{(C_k\times C_l)'}|U|\,dx\,dy\biggr)=c(I_A+I_B),$$

where $(C_k\times C_l)'$ is the set $\{(x,y)\in C_k\times C_l,\ |x-y|\ge c_0 2^{-\min(k,l)}\}$.

The term from I_A containing $\partial^\beta u$ (coming from $\partial^\alpha(\psi u)$) is majorized by

$$c\iint_{\mathbf{R}^{2n}}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|}{|x-y|^{n+2s-1}}|\partial^\beta u(x)|\,dx\,dy$$
$$\le c\biggl(\iint_{\mathbf{R}^{2n}}\frac{|\partial^\beta u(x)|^2}{|x-y|^{n+2s-2}}\,dx\,dy\iint_{\mathbf{R}^{2n}}\frac{|\partial^\alpha u(x)-\partial^\alpha u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy\biggr)^{1/2},$$

where the integrals are taken over $C_0\times C_0$. Since $s<1$, the last expression is less or equal than

$$c|u|_{|\beta|,C_0}|u|_{m+s,C_0}\le c(|u|^2_{|\beta|,C_0}+|u|^2_{m+s,C_0}).$$

This completes the estimate of I_A.

Let us use the notation $[f]_k=\sup_{C_k}|f|$ and $[f]_{k,l}=\sup_{C_k\times C_l}|f(x)-f(y)|$. Then the term from I_B containing $u^{(\beta)}=\partial^\beta u$ can be estimated by

(37)

$$c\sum_{k,l=-\infty}^{\infty} 2^{\min(k,l)(n+2s)}[\eta]_{k,l}[\psi^{(\alpha-\beta)}]_k \times(2^{-nl}\|u^{(\alpha)}\|_{C_k}\|u^{(\beta)}\|_{C_k}+2^{-n(k+l)/2}\|u^{(\alpha)}\|_{C_l}\|u^{(\beta)}\|_{C_k}).$$

Now, writing $a_k^\sigma=2^{k(n/2-|\sigma|)}\|u^{(\sigma)}\|_{C_k}$, we have $a_k^\sigma\le|u|_{|\sigma|,C_k}$ and we only need to verify that the coefficients for $a_k^\alpha a_k^\beta$ and $a_l^\alpha a_k^\beta$ are less than $c2^{-d(|k|+|l|)}$, for some $d\ge 2s$, since then the inequality

$$\sum_{k,l=-\infty}^{\infty} 2^{-d(|k|+|l|)}a_m^\alpha a_k^\beta\le c\sum_{k=-\infty}^{\infty} 2^{-\delta|k|}((a_k^\alpha)^2+(a_k^\beta)^2),$$

where m can be either k or l, gives us the desired result. We notice that

$$[\eta]_{k,l}[\psi^{(\alpha-\beta)}]_k\le c2^{k(n-|\alpha|-|\beta|-2s)},$$

where $c=0$ unless $kl\le 0\le k$. Now we can calculate the coefficients in two cases:

(i) $l\le 0\le k$. The coefficient for $a_k^\alpha a_k^\beta$ is majorized by

$$c2^{k(n-|\alpha|-|\beta|-2s-n/2+|\alpha|-n/2+|\beta|)}2^{l(n+2s-n)}=c2^{-2s(k-l)},$$

so in this case we can take $d=2s$. Similarly, for $a_l^\alpha a_k^\beta$, we can take $d=2s+|\alpha|$.

(ii) $k=0\le l$. Here we get $d=n$ and $d=n-|\alpha|$, respectively. Since for any other k, l the coefficients in (37) vanish, we have completed the estimate of I_5.

For I_2, we proceed in the same manner as for I_1 and write $I_2=I_7+I_8$. The corresponding I_A is handled identically. As for I_B we switch η and ψ in (37), noticing that

$$[\psi]_{k,l}\le c2^{\max(k,l)(n-2|\alpha|-2s)}$$

and that η must be differentiated so that we only need to consider $k=0$. The result is: (i) $l\le 0=k$. For $a_k^\alpha a_k^\beta$ and $a_l^\alpha a_k^\beta$ we can take $d=2s$ and $d=2s+|\alpha|$, respectively. (ii) $k=0\le l$. Here we can take $d=2s+2|\alpha|$ and $d=2s+|\alpha|$, respectively.

The estimate of I_6 follows easily, with $d=2s$, from the fact that $\Lambda\eta$ is bounded. For the integral

$$I_8=\int_{\mathbf{R}^n}(\Lambda\psi)(\partial^\alpha u)[\partial^\alpha,\eta]u\,dx$$

the result is immediate, since $\Lambda\psi$ is bounded on the support of the derivatives of η.

In the estimate of I_3 and I_4 we write $\Lambda = I_{2(1-s)}(-\Delta)$, where $I_{2(1-s)} = (-\Delta)^{s-1}$ denotes the Riesz potential of order $2(1-s)$. Integrating by parts, like

$$\int_{\mathbf{R}^n} (I_{2(1-s)}(-\Delta)v)w\,dx = \begin{cases} -\displaystyle\int_{\mathbf{R}^n} I_{2(1-s)} \nabla v \cdot \nabla w\,dx & \text{for } I_3, \\ \displaystyle\int_{\mathbf{R}^n} v I_{2(1-s)}(-\Delta)w\,dx & \text{for } I_4, \end{cases}$$

we see that it suffices to consider the integrals

$$\int_{\mathbf{R}^n} I_{2(1-s)}(\xi\partial^\beta u)(\partial^{\alpha-\gamma+\sigma}\psi)\partial^\gamma u\,dx \quad \text{and} \quad \int_{\mathbf{R}^n} I_{2(1-s)}(\xi\partial^\beta u)\partial^\alpha u\,dx,$$

where $|\sigma|=1$, $0\le\beta\le\alpha$, $0\le\gamma\le\alpha$, $\xi\in C_0^\infty(C_0)$. Using the easily proved operator norm estimate

$$\|I_{2(1-s)}\|_{L_2(C_0)\to L_2(C_k)} \le c2^{\min(k,0)(n-2(1-s))}2^{-kn/2},$$

we obtain for the first integral that, for $a_0^\beta a_k^\gamma$ and $k\ge 0$, we can take $d=2s+|\alpha|-1$. (When this integral occurs, $|\alpha|\ge 1$.) The second integral is clearly better for $k\ge 0$. For $a_0^\beta a_k^\alpha$ and $k\le 0$ one gets $d=2s+|\alpha|-2$. (Here we note that $I_4=0$, if $|\alpha|\le 1$). □

For $u\in H_0^\mu(\Omega)\cap\mathbf{C}^\infty(\Omega)$, $\delta\in(2s,4s)$, we introduce the quantities

$$A_j(u) = \sup_{p\in\Omega\cap B_j} u(p)^2 + \sum_{k=j}^{\infty}\left(\sum_{l=1}^{m} |u|_{l,B_k}^2 + |u|_{m+s,B_k}^2\right),$$

$$N_j(u) = \sum_{k=-\infty}^{j}\left(2^{-\delta(j-k)}\sum_{l=0}^{m}|u|_{l,B_k}^2\right) + |u|_{m+s,B_j}^2.$$

We will write $A_j = A_j(u)$ and $N_j = N_j(u)$. Notice that N_j is finite, since it can be estimated by means of $\|u\|_\mu^2$.

The following lemma gives a pointwise estimate without any requirements on the boundary of the domain Ω.

Lemma 12. *Let $\mu=m+s$ be as in Corollary 3 and let $u\in H_0^\mu(\Omega)$ satisfy $(-\Delta)^\mu u=0$ on $\Omega\cap B_j$. Then,*

$$A_{j+1} \le cN_j. \tag{38}$$

Proof. For simplicity we prove this only for $\mu=s\in(0,1)$. The general case is almost identical. Let $v\in C_0^\infty(\mathbf{R}^n)$ and let η be as in Lemma 11. Combining that lemma with (36) applied to ηv, we obtain,

$$v(0)^2+\sum_{k=1}^{\infty} 2^{-2sk}|v|_{0,C_k}^2+\langle v\rangle_{s,B_0}^2 \le v(0)^2+c\langle \eta v\rangle_{s,\mathbf{R}^n}^2 \le c_1 Q_s(\eta v,\eta v)$$

$$\le c_1 Q_s(v,\eta^2 v)+c\biggl(\sum_{k=-\infty}^{\infty} 2^{-\delta|k|}|v|_{0,C_k}^2+|v|_{s,C_0}^2\biggr).$$

(The first step is only a simple estimate.) Since $\delta>2s$ we can take k_1 so large that the sum on the left majorizes the sum of all terms where $k>k_1$ on the right. If we then replace $v(x)$ by $v(2^{-j-2}x+p)$, $p\in B_{j+1}$, and make a change of variables in the integrals that define Q_s and the seminorms, we get

$$v(p)^2+\langle v_p\rangle_{s,B_{j+2}}^2 \le c_1 Q_s(v_p,\eta_{j+2}^2 v_p)+c\biggl(\sum_{k=-\infty}^{j+k_1} 2^{-\delta|k-j|}|v_p|_{0,C_{k+2}}^2+|v_p|_{s,C_{j+2}}^2\biggr),$$

where $v_p(x)=v(x+p)$ and $\eta_{j+2}(x)=\eta(2^{j+2}x)$. Now let $v\in C_0^\infty(\Omega)$ tend to u in the norm of $H_0^\mu(\mathbf{R}^n)$. Then the sum and the last term on the right tend to the same expression with u in place of v. The fact that $(-\Delta)^s u_p=0$, which in particular makes u_p smooth on $\operatorname{supp}(\eta_{j+2}^2 v_p)$, implies that Q_s on the right tends to zero. If we, in addition, let $v=u$ in a neighborhood of p, then we can replace v by u also on the left.

Now, by replacing C_{k+2} and C_{j+2} by B_k and B_j, respectively, and enlarging the constant c, we can replace u_p by u and $j+k_1$ by j on the right to obtain

$$u(p)^2+\langle u_p\rangle_{s,B_{j+2}}^2+|u|_{s,B_{j+1}}^2 \le c\biggl(\sum_{k=-\infty}^{j} 2^{-\delta(j-k)}|u|_{0,B_k}^2+|u|_{s,B_j}^2\biggr). \tag{39}$$

(We have also added the term $|u|_{s,B_{j+1}}^2$ to the left, which is possible thanks to the last term on the right.) Finally, the easily proved inequality,

$$\sum_{k=j}^{\infty} |u|_{\lambda,B_k}^2 \le c\langle u\rangle_{\lambda,B_j}^2, \quad \lambda>0,$$

shows that, for $m=0$, A_{j+1} is majorized by a constant times the supremum over all $p\in B_{j+1}$ of the left-hand side of (39). Since, for $m=0$, the right-hand side of (39) equals cN_j, the proof is complete. □

The following inequality is proved in a similar way as Theorem 10.1.2 in [4]. See also Lemma 2.3 in [7].

Lemma 13. *Let* $u \in H_0^\mu(\Omega) \cap C^\infty(\Omega)$. *Then*

$$|u|_{0,B_j}^2 \le \frac{c}{\gamma_j}\left(\sum_{l=1}^{m} |u|_{l,B_j}^2 + |u|_{m+s,B_j}^2\right), \tag{40}$$

where

$$\gamma_j = 2^{(n-2\mu)j}\operatorname{cap}_\mu(B_j \setminus \Omega).$$

Lemma 14. *Let* $\mu = m+s$ *be as in Corollary* 3 *and let* $u \in H_0^\mu(\Omega)$ *satisfy* $(-\Delta)^\mu u = 0$ *on* $\Omega \cap B_{j_0}$. *Let* $j_1 \ge 1$. *Then, for* $j \ge j_0$,

$$A_{j+j_1} \le c 2^{-\delta(j+j_1)}\left(2^{\delta j_0} N_{j_0} + \sum_{k=j_0+1}^{j-1} 2^{\delta k} A_k\right) + \frac{c_1}{\gamma_j}(A_j - A_{j+1}), \tag{41}$$

where c_1, *but not* c, *depends on* j_1.

Proof. Since the kth term from N_j can be estimated by $2^{-\delta(j-k)} A_k$ and since $|u|_{\lambda,B_{j+1}}^2 \le 2^{n-2\lambda} |u|_{\lambda,B_j}^2$, we see that N_{j+j_1-1} is majorized by

$$2^{-\delta(j+j_1-1)}\left(2^{\delta j_0} N_{j_0} + \sum_{k=j_0+1}^{j-1} 2^{\delta k} A_k\right) + c_1'\left(\sum_{l=0}^{m} |u|_{l,B_j}^2 + |u|_{m+s,B_j}^2\right).$$

Using Lemma 13 on $|u|_{0,B_j}^2$ and then the boundedness from below of $(\gamma_j^{-1})_{j=-\infty}^{\infty}$ it follows that N_{j+j_1-1} is less than the righ-hand side of (41). The assertion now follows from Lemma 12. □

7. Regularity of a boundary point

The remark about the case $s=0$ in the beginning of the previous section does also apply here.

Definition 15. The point $O \in \partial\Omega$ is said to be regular with respect to $(-\Delta)^\mu$ if the solution of the equation

$$(-\Delta)^\mu u = f \in C_0^\infty(\Omega), \quad u \in H_0^\mu(\Omega), \tag{42}$$

satisfies $u(x) \to 0$, as $x \to O$.

It is shown in [5] that for $\mu = 1$, regularity in the above sense is equivalent to the Wiener regularity.

Since $\sup_{\Omega \cap B_j} u(p)^2 \le A_j$, the following lemma shows that divergence of the series $\sum_{k=-\infty}^{\infty} \gamma_k$ is sufficient for regularity of the point $0 \in \partial\Omega$.

Lemma 16. *Let μ be as in Corollary 3 and let $u \in H_0^\mu(\Omega)$ satisfy $(-\Delta)^\mu u = 0$ on $\Omega \cap B_{j_0}$. Then,*

$$A_j \le cN_{j_0} 2^{-c' \sum_{k=j_0}^{j-1} \gamma_k}, \quad j \ge j_0 + 1. \tag{43}$$

Proof. To simplify notation in the proof, we redefine A_{j_0} as

$$A_{j_0} := N_{j_0}.$$

Since $A_{j+j_1} \le A_{j+1}$, it then follows from (41) that

$$\left(1 + \frac{\gamma_j}{c_1}\right) A_{j+j_1} \le \frac{\gamma_j c_0}{c_1} \sum_{k=j_0}^{j-1} 2^{-\delta(j+j_1-k)} A_k + A_j, \quad j \ge j_0. \tag{44}$$

Before continuing the proof we choose the numbers c, c' and j_1. Take j_1 and $d_1 > 0$ so that

$$2c_0 \sum_{k=j_0}^{j-1} 2^{(c'\gamma - \delta)(j+j_1-k)} \le 1, \quad c' \le d_1, \tag{45}$$

where γ majorizes all γ_j. Again due to the boundedness of $(\gamma_j)_{j=-\infty}^{\infty}$, there is a $d_2 > 0$, and then since $\gamma_{j+1} \le 2^{n-2\mu}\gamma_j$, a number $d_3 > 0$ such that

$$\left(1 + \frac{\gamma_j}{2c_1}\right)^{-1} \le 2^{-d_2\gamma_j} \le 2^{-c' \sum_{k=j}^{j+j_1-1} \gamma_k}, \quad c' \le d_3. \tag{46}$$

Now choose $c' = \min(d_1, d_3)$. By Lemma 12, the fact that $(A_j)_{j=-\infty}^{\infty}$ decreases and $(\gamma_j)_{j=-\infty}^{\infty}$ is bounded, and by the redefinition of A_{j_0}, we can take c so that (43) is satisfied for $j_0 \le j \le j_0 + j_1 - 1$.

We now complete the proof by an induction step from j to $j + j_1$. First assume that the following inequality holds,

$$A_{j+j_1} \le 2c_0 \sum_{k=j_0}^{j-1} 2^{-\delta(j+j_1-k)} A_k. \tag{47}$$

Then by the induction hypothesis,

$$\begin{aligned} A_{j+j_1} &\le 2c_0 c N_{j_0} \sum_{k=j_0}^{j-1} 2^{-\delta(j+j_1-k)} 2^{-c' \sum_{l=j_0}^{k-1} \gamma_l} \\ &\le 2c_0 c N_{j_0} 2^{-c' \sum_{j_0}^{j+j_1-1} \gamma_k} \sum_{k=j_0}^{j-1} 2^{(c'\gamma - \delta)(j+j_1-k)}. \end{aligned}$$

Combining this with (45), we obtain (43) with $j+j_1$ in place of j. If on the other hand (47) does not hold, then (44) gives

$$A_{j+j_1} \le \Bigl(1+\frac{\gamma_j}{2c_1}\Bigr)$$

which together with (46) and the induction hypothesis again leads to (43) for $j+j_1$. □

To formulate our theorem, we need the following continuous versions of some earlier introduced quantities. Let $B(r)=\{x\in\mathbf{R}^n:|x|\le r\}$. Put

$$\gamma(r)=r^{2\mu-n}\operatorname{cap}_\mu(B(r)\setminus\Omega).$$

Let $\mu=m+s$ and $\delta\in(2s,4s)$ be as before and define

$$N(u,r)=\sum_{l=0}^{m} r^\delta \int_{\mathbf{R}^n} |\nabla_l u|^2 (r+|x|)^{2l-n-\delta}\,dx \\ +r^{2(m+s)-n}\iint_{B(r)\times B(r)} \frac{|\nabla_m u(x)-\nabla_m u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy.$$

Thus $\gamma(2^{-j})=\gamma_j$ and we easily obtain $N(u,2^{-j})\sim N_j(u)$. As for $N_j(u)$, $N(u,r)$ is finite.

Theorem 17. *Let $\mu\in\bigl(0,\frac12 n\bigr)\setminus\bigl(1,\frac12 n-1\bigr)$ and let $u\in H_0^\mu(\Omega)$ satisfy $(-\Delta)^\mu u=0$ on $\Omega\cap B_{r_0}$. Then, for $r\le\frac12 r_0$,*

$$\sup_{p\in\Omega\cap B(r)} u(p)^2 \le cN(u,r_0)\exp\biggl(-c'\int_r^{r_0}\frac{\gamma(\varrho)}{\varrho}\,d\varrho\biggr). \tag{48}$$

Proof. By the inequality

$$2^{-c'\sum_{k=j_0}^{j-1}\gamma_k} \le \exp\biggl(-c''\int_{2^{-j}}^{2^{-j_0}}\frac{\gamma(\varrho)}{\varrho}\,d\varrho\biggr)$$

and the equivalence of the continuous and discrete versions of N, (48) follows from Lemma 16, for r_0 and r of the form $r_0=2^{-j_0}$ and $r=2^{-j}$. From this, the assertion follows for arbitrary $r\le\frac12 r_0$. Finally, a transformation $x\mapsto tx$ with $t\in[1,2)$, shows that also r_0 can be arbitrary. □

Remark. It is shown in [5] that, for μ an admissible integer, a similar estimate holds, where $N(u,r_0)$ is replaced by the mean value of u^2 over the annulus $B(r_0)\setminus B\bigl(\frac12 r_0\bigr)$.

Acknowledgement. The author wishes to thank V. Maz'ya for useful discussions and for giving me access to an early version of [5].

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Received May 28, 1998

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