A class of hyponormal operators and weak*-continuity of hermitian operators

Kirsti Mattila

We will first in this paper consider a class of hyponormal operators which we call *-hyponormal operators. We give an example of a hyponormal operator which is not *-hyponormal. It follows from a theorem of Ackermans, van Eijndhoven and Martens [1] that subnormal operators on a Hilbert space are *-hyponormal. We prove a generalized Fuglede—Putnam theorem and some other results for these operators.

We will also prove some results on the following problem which was mentioned in [4]:

Problem (1). *Let T be a bounded linear operator on a Banach space X. If $T^* = H+iK$ for some hermitian operators H and K on X^*, is it true that $T=H_0+iK_0$ for some hermitian operators H_0 and K_0 on X?*

It is known that if T^* is normal, then T is normal (Behrends [4]). We show that (1) is true if T^* is a *-hyponormal operator with a weakly compact commutator. Finally we prove that if X is a dualoid space (in particular a dual space) or a C^*-algebra with a unit element, then (1) is true for all operators T such that $T^*= H+iK$.

Let X be a complex Banach space and X^* the dual space of X. We denote by $B(X)$ the space of all bounded linear operators on X. If X and Y are two Banach spaces, then $B(X, Y)$ is the space of all bounded linear operators from X to Y. A *normal* operator on X is an operator which can be written in the form $H+iK$ where H and K are commuting hermitian operators on X. We will only be concerned with bounded operators. The adjoint of an operator $T\in B(X)$ is hermitian if and only if T is hermitian (see [6, § 9] or [7, § 17]). We refer to [6] and [7] for basic facts about numerical ranges and hermitian operators.

Research supported in part by the Swedish Natural Science Research Council.

1. *-hyponormal operators

In the following definition H and K are hermitian operators.

Definition 1. *An operator* $T\in B(X)$ *is called*
(i) *hyponormal if* $T=H+iK$ *and* $i(HK-KH)\geqq 0$.
(ii) **-hyponormal if* $T=H+iK$ *and the inequality*

$$(*)\qquad \|e^{zT}e^{-\bar{z}\bar{T}}\|\leqq 1,$$

where $\bar{T}$ *is the operator* $H-iK$, *holds for all complex numbers z.*

Normal operators are obviously *-hyponormal. In Proposition 1 we give some sufficient conditions implying that the restriction of a *-hyponormal operator to an invariant subspace is *-hyponormal. If the space is a Hilbert space, it follows that the restriction of every *-hyponormal operator to a closed invariant subspace is *-hyponormal. In particular, subnormal operators on a Hilbert space are *-hyponormal. This was proved in [1].

Proposition 1. *Let P be a projection on X with* $\|P\|=1$ *and let N be a *-hyponormal (or normal) operator on X such that*

$$NPX\subset PX \quad\text{and}\quad \bar{N}(I-P)X\subset (I-P)X.$$

Then the operator $N|_{PX}$ *is *-hyponormal.*

Proof. Let $N=H+iK$ and let $T=N|_{PX}$. Let A and B be the operators on PX defined by

$$Ay=PHy \quad\text{and}\quad By=PKy.$$

Then A and $B\in B(PX)$ and $T=A+iB$. The operators A and B are hermitian. To see that, let $y\in PX$ with $\|y\|=1$ and let $f\in(PX)^*$ with $\|f\|=f(y)=1$. By the Hahn—Banach theorem there is a functional $g\in X^*$ such that $\|g\|=1$ and $g|_{PX}=f$. We have

$$f(Ay)=g(PHy)=(P^*g)(Hy).$$

Since $(P^*g)(y)=g(Py)=f(y)=1$ and $\|P^*g\|\leqq 1$ it follows that $\|P^*g\|=(P^*g)(y)=1$. Since H is hermitian we conclude that A is hermitian. Similarly B is hermitian.

Since $P\bar{N}P=P\bar{N}$ and $PNP=NP$, we have

$$\bar{T}^jT^ky=(P\bar{N})^jN^ky=P\bar{N}^jN^ky,$$

whenever $y\in PX$ and j and k are non-negative integers. Therefore

$$\|e^{zT}e^{-\bar{z}\bar{T}}y\|=\|Pe^{zN}e^{-\bar{z}\bar{N}}y\|\leqq\|P\|\,\|y\|\leqq\|y\|$$

for every $z\in\mathbf{C}$ and $y\in PX$. This implies $(*)$.

Proposition 2. *A *-hyponormal operator is hyponormal.*

Proof. Assume that T is *-hyponormal and $T=H+iK$. We have for all complex numbers z

$$1 \geqq \|e^{zT}e^{-\bar{z}T}e^{-zT}e^{\bar{z}T}\| = \|I-|z|^2A\|+r(z),$$

where $A=\bar{T}T-T\bar{T}$ and $|r(z)|\leqq M|z|^3$ for some $M>0$ if $|z|\leqq 1$. Given $\mu\in B(X)^*$ with $\|\mu\|=\mu(I)=1$, it follows that

$$\left|1-|z|^2\mu(A)\right| \leqq 1+M|z|^3 \quad (|z| \leqq 1).$$

Since $A=2i(HK-KH)$, A is hermitian [6, Lemma 5.4] and therefore $\mu(A)$ is a real number. We now have

$$-\mu(A) \leqq Mt \quad (0 < t \leqq 1).$$

Thus $\mu(A)\geqq 0$. It follows that $i(HK-KH)\geqq 0$.

Remark 1. It is well-known that an operator S on a Hilbert space $\mathscr{H}$ is hyponormal if and only if

$$\|\bar{S}x\| \leqq \|Sx\| \quad \text{for all} \quad x\in\mathscr{H}.$$

(Indeed we have $\|Sx\|^2-\|\bar{S}x\|^2=(\bar{S}Sx, x)-(S\bar{S}x, x)=((\bar{S}S-S\bar{S})x, x)$). The condition $(*)$ in Definition 1 can be written

$$\|e^{zT}x\| \leqq \|e^{\bar{z}T}x\| \quad \text{for all} \quad x\in X \quad \text{and} \quad z\in\mathbf{C}.$$

If T is an operator on a Hilbert space, the conjugate of e^{zT} is $e^{\bar{z}T}$. Hence we have:

(α) T is *-hyponormal if and only if
e^{zT} is hyponormal for all complex numbers z.

Also we have:

(β) T is normal if and only if
e^{zT} is normal for all complex numbers z.

These relations are not in general true in Banach spaces as the following example shows.

Let H be a hermitian operator such that the spectrum of H is $\{-1, 0, 1\}$ and H^2 is not hermitian. For example, if P is a hermitian projection on a Hilbert space $\mathscr{H}$ and $P\neq 0$, $P\neq I$, then the operator $S\mapsto PS-SP$ on $B(\mathscr{H})$ has these properties [3]. Then $H^3=H$ by the spectral mapping theorem [8, Theorem 7.4(iv)] and by [7, Theorem 27.3]. Now there are real coefficients a and $b\neq 0$ such that

$$e^H = I+aH+bH^2.$$

Since H^2 is not hermitian it is not equal to $A+iB$ for any hermitian operators A and B by [5, (2.12)]. Therefore e^H is neither normal nor hyponormal. Thus (α) and (β) do not hold.

We now give an example of a hyponormal operator which is not *-hyponormal.

Example. Let l_2 be the Hilbert space of all complex sequences $\{\alpha_n\}_{n=0}^{\infty}$ such that the series $\Sigma|\alpha_n|^2$ converges and let U be the unilateral shift on l_2 defined by

$$U(\alpha_0, \alpha_1, \ldots) = (0, \alpha_0, \alpha_1, \ldots).$$

The operator $T=\bar{U}+2U$ is hyponormal. By results of Ito and Wong [14, Remark 4] T is not subnormal. There are vectors x and numbers z such that

$$\|e^{zT}x\| > \|e^{zT}x\|.$$

This can be seen by a direct calculation taking for example $z=0.6$ and $x=\{\alpha_n\}$ where $\alpha_0=1$ and $\alpha_2=-4$ and otherwise $\alpha_n=0$. It follows that T is not *-hyponormal.

In the case of a normal operator the result of the following theorem is included in [10]. The proof in [10] is different from the next proof. If T is only assumed to be hyponormal and X is strictly c-convex, then the conclusion of the following theorem is also true by [16, Theorem 2.4].

Theorem 3. *If T is *-hyponormal and $Tx=0$ for some $x\in X$, then $\bar{T}x=0$.*

Proof. Assume that $Tx=0$. Let $f\in X^*$. Then the function $g(z)=f(e^{zT}x)$ is entire. Since

$$|f(e^{zT}x)| = |f(e^{zT}e^{\bar{z}T}x)| \leqq \|f\|\,\|x\|,$$

g is bounded. By Liouville's theorem g is constant. Thus $g(z)\equiv g(0)$. We conclude that

$$f\big((e^{zT}-I)x\big) = 0 \quad \text{for all} \quad z\in\mathbf{C} \quad \text{and for all} \quad f\in X^*.$$

This implies by the Hahn—Banach theorem that $(e^{zT}-I)\,x=0$ for all z. Taking the derivative at $z=0$ we obtain $\bar{T}x=0$.

Remark 2. If T is *-hyponormal and $Tx=\lambda x$ for some $\lambda\in\mathbf{C}$ and $x\in X$, then $\bar{T}x=\bar{\lambda}x$ since $T-\lambda I$ is also *-hyponormal.

From Theorem 3 we obtain an extension of the Fuglede—Putnam theorem. There are several extensions of this theorem for hyponormal operators on a Hilbert space. For further references see [16].

Theorem 4. *Let T be a *-hyponormal operator on Y and U a *-hyponormal operator on X. If $TS=S\bar{U}$ for some $S\in B(X, Y)$, then $\bar{T}S=SU$.*

Proof. We will show that the operator

$$\delta(S) = TS - S\bar{U}$$

is a *-hyponormal operator on $B(X, Y)$. The result then follows from Theorem 3. Note that $\bar{\delta}$ is the operator $S \mapsto \bar{T}S - SU$.

Given $A \in B(Y)$ and $B \in B(X)$, let

$$l(S) = AS, \quad r(S) = SB \quad (S \in B(X, Y))$$

and let $d = l - r$. Since l and r commute, we have ([6, Theorem 3.2])

$$e^{l-r} = e^l e^{-r}.$$

Hence

$$e^d(S) = e^l(e^{-r}(S)) = e^A S e^{-B} \quad (S \in B(X, Y)). \tag{2}$$

Using (2) and the assumption that T and U are *-hyponormal it follows that for all $z \in \mathbf{C}$ and $S \in B(X, Y)$

$$\|e^{z\bar{\delta}} e^{-\bar{z}\delta} S\| = \|e^{zT} e^{-\bar{z}T} S e^{\bar{z}\bar{U}} e^{-zU}\| \leqq \|S\|.$$

This completes the proof.

Corollary 5. *Assume that T is a *-hyponormal operator on X and Y is a subspace of X such that the following conditions hold:*

(i) *Y is a Banach space with respect to a norm $|\,.\,|$ on Y and there is a constant M such that $\|y\| \leqq M|y|$ for all $y \in Y$.*

(ii) *$TY \subset Y$, $T|_Y$ is bounded and there are hermitian operators A and B on Y such that $T|_Y = A + iB$ and the operator $A - iB$ is *-hyponormal.*

Then $\bar{T}Y \subset Y$.

Proof. Let $T_1 = T|_Y$. The inclusion $j\colon Y \to X$ is bounded by the assumption (i). Since $Tj = jT_1$ it follows from Theorem 4 that $\bar{T}j = j\bar{T}_1$. Hence $\bar{T}Y \subset Y$.

2. On the weak*-continuity of hermitian operators

The following theorem was proved in [4] for normal operators. For the case when X is a dualoid space (see Definition 2 below) or a C^*-algebra with unit more general results will be proved in Theorems 7 and 9.

In the proofs of the following two theorems we shall make use of the canonical projection on the third dual of X. If i_X is the canonical embedding of X into X^{**}, then $P = i_{X^*} \circ i_X^*$ is a projection on X^{***} whose range is $\widehat{(X^*)}$ and whose kernel is $(\hat{X})^\perp$ ($\hat{X}$ is the canonical image of X and $(\hat{X})^\perp$ is the annihilator of $\hat{X}$ in X^{***}). Note that $\|P\| = 1$.

Theorem 6. *Assume that an operator* $T \in B(X)$ *has the following properties:*
(i) $T^* = H + iK$, *where H and K are hermitian, and* T^* *is* **-hyponormal.*
(ii) $HK - KH$ *is weakly compact.*

Then $T = H_0 + iK_0$ *for some hermitian operators* H_0 *and* K_0 *on* X *and* $\bar{T}$ *is* **-hyponormal.*

Proof. Let P be the projection with norm one on X^{***} such that $PX^{***} = \widehat{(X^*)}$ and $\operatorname{Ker}(P) = (\hat{X})^{\perp}$. It is obvious that T^{***} commutes with P and the space PX^{***} is invariant for H^{**} and K^{**}. Let $Z = (\hat{X})^{\perp} = (I-P)X^{***}$. Let A and B be the operators on Z defined by

$$Az = (I-P)H^{**}z, \quad Bz = (I-P)K^{**}z.$$

Then A, $B \in B(Z)$ and $T^{***}|_Z = A + iB$.

We will show that A is hermitian with respect to an equivalent norm on Z. Since $(I-P)H^{**}P = 0$, we have for every $z \in Z$ and $k = 1, 2, \ldots$

$$A^k z = (I-P)(H^{**})^k z.$$

Therefore,

$$\|e^{itA}z\| = \|z + (I-P)(e^{itH^{**}} - I)z\| = \|Pz + (I-P)e^{itH^{**}}z\| \leqq (\|P\| + \|I-P\|)\|z\|$$

for every $z \in Z$ and $t \in \mathbf{R}$. By [6, Lemma 10.3] there is an equivalent norm on Z such that A is hermitian with respect to this norm. The same is true for B.

Let $C = HK - KH$. Since C is weakly compact, it follows that $C^{**}X^{***} \subset \widehat{(X^*)}$. Thus $(I-P)C^{**} = 0$. This implies, since $(I-P)H^{**}P = 0$ and $(I-P)K^{**}P = 0$, that for every $z \in Z$

$$(AB - BA)z = (I-P)H^{**}(I-P)K^{**}z - (I-P)K^{**}(I-P)H^{**}z$$
$$= (I-P)H^{**}K^{**}z - (I-P)K^{**}H^{**}z = (I-P)C^{**}z = 0.$$

Hence $AB = BA$. By a theorem of Lumer [7, Lemma 33.8] there is an equivalent norm $|\cdot|$ on Z such that A and B are hermitian with respect to this norm.

By applying Corollary 5 to the operator T^{***} and the space Z provided with the norm $|\cdot|$ we obtain $(\overline{T^{***}})Z \subset Z$. This implies that $H^*\hat{X} \subset \hat{X}$ and $K^*\hat{X} \subset \hat{X}$. We define operators H_0 and K_0 on X by

$$H_0 x = i_X^{-1}(H^*\hat{x}), \quad K_0 x = i_X^{-1}(K^*\hat{x}).$$

Then $H_0^* = H$ and $K_0^* = K$. It follows that H_0 and K_0 are hermitian and $T = H_0 + iK_0$. Since T^* is *-hyponormal and $(\bar{T})^* = (\overline{T^*})$ we have

$$\|e^{zT}e^{-\bar{z}T}\| = \|e^{-\bar{z}(T)^*}e^{zT^*}\| \leqq 1.$$

Thus $\bar{T}$ is *-hyponormal.

We do not know whether the condition $T^*=H+iK$ always implies that H and K are weak *-continuous. We will show that this is true if X is a dualoid space or a C^*-algebra with unit. A dualoid space was defined in [11] as follows:

Definition 2. *A Banach space X is called a dualoid space, if there is a projection of norm one on X^{**} whose range is $\hat{X}$.*

For example all dual spaces and $L_1(0, 1)$ are dualoid spaces. If K is a compact and extremally disconnected space (=a stonian space), then $C(K)$ is a $\mathscr{P}_1$-space [13] and hence a dualoid space. There are stonian spaces K such that $C(K)$ is not isomorphic to any dual space (see [20, § 4] or [18, § 3.9]).

Theorem 7. *Let X be a dualoid Banach space and let T be an operator in $B(X)$ such that $T^*=H+iK$ where H and K are hermitian operators on X^*. Then there are hermitian operators H_0 and K_0 on X such that $T=H_0+iK_0$.*

Proof. Let P be a projection of norm one on X^{**} whose range is $\hat{X}$ and let i_X be the canonical embedding of X into X^{**}. The operators

$$A = i_X^{-1}PH^*i_X \quad \text{and} \quad B = i_X^{-1}PK^*i_X$$

are bounded linear operators on X. Since $T^{**}\hat{X}\subset\hat{X}$, we have

$$i_XT = T^{**}i_X = PT^{**}i_X = PH^*i_X+iPK^*i_X.$$

It follows that $T=A+iB$. It remains to show that A and B are hermitian. Since

$$A^{**}\hat{x} = PH^*\hat{x} \quad (x\in X)$$

we can show in the same way as in the proof of Proposition 1 that the operator $A^{**}|_{\hat{X}}$ is hermitian. But

$$\|e^{itA}x\| = \|e^{itA^{**}}\hat{x}\| = \|\hat{x}\| = \|x\|$$

for all $t\in\mathbf{R}$ and $x\in X$ and therefore A is hermitian. Similarly B is hermitian.

We shall finally prove that the result of Theorem 7 is also true for all C^*-algebras which have a unit. There are C^*-algebras which are not dualoid spaces, for example c_0. Such are more generally all infinite dimensional C^*-algebras which are separable or which are ideals in their second duals. This follows from the next proposition.

Proposition 8. *Let A be a C^*-algebra such that $\hat{A}$ is complemented in A^{**}.*
(i) *Then $A\supset l_\infty$ or A is finite dimensional.*
(ii) *If $\hat{A}$ is also an ideal of A^{**}, then A is finite dimensional.*

Proof. (i) Assume that $A\not\supset c_0$. Then the identity operator on A is weakly compact by [2, Theorem 4.2]. Thus A is reflexive and by a result of Ogasawara [17, Theorem 2] A is finite dimensional.

If $A \supset c_0$, then $A \supset l_\infty$ by a theorem of Rosenthal [19, Corollary 1.5].

(ii) If $\hat{A}$ is an ideal of A^{**}, then $\hat{A}$ is an M-ideal of A^{**} [22, Proposition 5.2]. It follows from [12, Corollary 3.6(c)] that A is reflexive. Then, by [17, Theorem 2], A is finite dimensional.

Remark 3. Let A be a C^*-algebra. Then by [23] $\hat{A}$ is an ideal of A^{**} if and only if A is dual in the sense defined by Klaplansky [15]. By Proposition 8 a C^*-algebra which is dual in this sense is not complemented in its second dual, in particular it is not isomorphic to a dual space, unless it is finite dimensional.

Theorem 9. *Let A be a C^*-algebra with a unit element. If $T \in B(A)$ and $T^* = H + iK$ for some hermitian operators H and K on A^*, then there are hermitian operators H_0 and K_0 on A such that $T = H_0 + iK_0$.*

Proof. The space A^{**} with the Arens product is a W^*-algebra with unit [8], [9]. Given $u \in A^{**}$, let Δ_u be the inner derivation

$$\Delta_u(x) = ux - xu \quad \text{for all} \quad x \in A^{**}.$$

If $\Delta_u(\hat{A}) \subset \hat{A}$, then $\Delta_{\bar{u}}(\hat{A}) \subset \hat{A}$, since $(\bar{a}) = (\hat{\bar{a}})$ for every $a \in A$ (see [8, Theorem 38.19]).

There are hermitian elements h, h', k and k' in A^{**} such that the hermitian operators H^* and K^* can be written

$$H^* = L_h + \Delta_{h'}, \quad K^* = L_k + \Delta_{k'}$$

where L_h and L_k are left multiplication operators on A^{**}. This follows from the results of Sinclair [21, Remark 3.5] and Sakai and Kadison [18, Corollary 8.6.6]. Since $T^{**}\hat{A} \subset \hat{A}$ and A has a unit, we conclude that $h + ik \in \hat{A}$. Thus $h \in \hat{A}$ and $k \in \hat{A}$. We also have

$$L_h + iL_k = L_c^{**}$$

for some $c \in A$. Now

$$T^{**} - L_c^{**} = \Delta_{h'+ik'}.$$

From the beginning of the proof it follows that

$$\Delta_{h'}(\hat{A}) \subset \hat{A} \quad \text{and} \quad \Delta_{k'}(\hat{A}) \subset \hat{A}.$$

Therefore $H^*\hat{A} \subset \hat{A}$ and $K^*\hat{A} \subset \hat{A}$. This implies that H and K are weak*-continuous operators on A^* which completes the proof.

Remark 4. Let A be a C^*-algebra such that $\hat{A}$ is not an ideal of A^{**}. We will show that there are hermitian operators on A^* which are not weak*-continuous. Since $\hat{A}$ is a self-adjoint subspace of A^{**}, it follows that $\hat{A}$ is not a right (nor a left) ideal of A^{**}. Let F be an element of A^{**} such that $\hat{A}F \not\subset \hat{A}$. Notice that A^{**} has a unit element even if A does not have one ([8, Corollary 29.8 and Lemma 39.14]). We can assume that F is a hermitian element of A^{**}. The right multiplication R_F is then a hermitian operator on A^{**} and it is the adjoint of the left multiplication L_F on A^*. The operator L_F is hermitian and it is not weak*-continuous.

References

1. Ackermans, S. T. M., van Eijndhoven, S. J. L. and Martens, F. J. L., On almost commuting operators. *Indag. Math.* **45** (1983), 385—391.
2. Akemann, C. A., Dodds P. G. and Gamlen, J. L. B., Weak compactness in the dual space of a C^*-algebra. *J. Functional Analysis* **10** (1972), 446—450.
3. Anderson, J. and Foias, C., Properties which normal operators share with normal derivations and related operators. *Pacific J. Math.* **61** (1975), 313—325.
4. Behrends, E., Normal operators and multipliers on complex Banach spaces and a symmetry property of L^1-predual spaces. *Israel J. Math.* **47** (1984), 23—28.
5. Berkson, E., Hermitian projections and orthogonality in Banach spaces. *Proc. London Math. Soc.* (3) **24** (1972), 101—118.
6. Bonsall, F. F. and Duncan, J., *Numerical ranges of operators on normed spaces and of elements of normed algebras.* London Math. Soc. Lecture Note Series 2, Cambridge 1971.
7. Bonsall, F. F. and Duncan, J., *Numerical ranges II.* London Math. Soc. Lecture Note Series 10, Cambridge 1973.
8. Bonsall, F. F. and Duncan, J., *Complete normed algebras.* Springer-Verlag, Berlin—Heidelberg—New York 1973.
9. Civin, P. and Yood, B., The second conjugate space of a Banach algebra as an algebra. *Pacific J. Math.* **11** (1961), 847—870.
10. Dowson, H. R., Gillespie, T. A. and Spain, P. G., A commutativity theorem for hermitian operators. *Math. Ann.* **220** (1976), 215—217.
11. Godefroy, G., Étude des projections de norme 1 de E'' sur E. Unicité de certains préduaux. Applications. *Ann. Inst. Fourier* (Grenoble) **29** (1979), 53—70.
12. Harmand, P. and Lima, Å., Banach spaces which are M-ideals in their biduals. *Trans. Amer. Math. Soc.* **283** (1984), 253—264.
13. Hasumi, M., The extension property of complex Banach spaces. *Tôhoku Math. J.* **10** (1958), 135—142.
14. Ito, T. and Wong, T. K., Subnormality and quasinormality of Toeplitz operators. *Proc. Amer. Math. Soc.* **34** (1972), 157—164.
15. Kaplansky, I., The structure of certain operator algebras. *Trans. Amer. Math. Soc.* **70** (1951), 219—255.
16. Mattila, K., Complex strict and uniform convexity and hyponormal operators. *Math. Proc. Cambridge Philos. Soc.* **96** (1984), 483—493.
17. Ogasawara, T., Finite-dimensionality of certain Banach algebras. *J. Sci Hiroshima Univ. Ser. A.* **17** (1954), 359—364.
18. Pedersen, G. K., *C^*-algebras and their automorphism groups.* Academic Press, London—New York—San Francisco 1979.
19. Rosenthal, H. P., On relatively disjoint families of measures, with some applications to Banach space theory. *Studia Math.* **37** (1970), 13—36.

20. Rosenthal, H. P., On injective Banach spaces and the spaces $L^\infty(\mu)$ for finite measures μ. *Acta Math.* **124** (1970), 205—248.
21. Sinclair, A. M., Jordan homomorphisms and derivations on semisimple Banach algebras. *Proc. Amer. Math. Soc.* **24** (1970), 209—214.
22. Smith, R. R. and Ward, J. D., M-ideal structure in Banach algebras. *J. Functional Analysis* **27** (1978), 337—349.
23. Tomiuk, B. J. and Wong, P. K., Arens product and duality in B^*-algebras. *Proc. Amer. Math. Soc.* **25** (1970), 529—535.

Received August 1, 1985

Department of Mathematics
University of Stockholm
Box 6701
S—113 85 Stockholm
Sweden

Current address:

Department of Mathematics
Royal Institute of Technology
S—10044 Stockholm
Sweden